



## Definition of Error Analysis:

The numerical solution of engineering problems involves repeated operations. In this section we shall examine the question of errors in terms of their source and how they may affect the solution of engineering problem. In general, error may be related to mathematical formulation and or simplification. Measured physical parameters and computation errors can arise from the following operations and limitations of computation.

### Significant Figures Error:

The significant figures of a number are those digits that carry meaning contributing to its precision. Some examples using these rules:  $1 + 1.1 = 2$ , 1 is significant to the ones place, 1.1 is significant to the tenths place. Of the two, the least accurate is the ones place. The answer cannot have any significant figures past the ones place.

### Round-Off Error or (Rounding Error):

A round-off error is the difference between the calculated approximation of a number and its exact mathematical value.

### Truncation Error:

Truncation error is the error made by truncating an infinite sum and approximating it by a finite sum. For instance, if we approximate the sine function by the first two non-zero term of its Taylor series, as in  $\sin(x) = x - \frac{1}{6}x^3$  for small  $x$ , the resulting error is a truncation error. It is present even with infinite-precision arithmetic, because it is caused by truncation of the infinite Taylor series to form the algorithm.

$$\text{Absolute Error} = \text{Calculated value} - \text{True value}$$

$$\text{Relative Error} = \frac{\text{Absolute Error}}{\text{True Value}}$$

$$\text{Percentage Error} = \text{Relative Error} * 100$$

## Operation of Error Analysis:

### 1. Addition:

$$\text{value} = (\bar{x} + e_x) + (\bar{y} + e_y)$$

$$\text{Absolute error } (e_{x+y}) = e_x + e_y$$



$$\text{Relative Error} = \frac{\text{Absolute Error}}{\text{True Value}} = \frac{e_x + e_y}{\bar{x} + \bar{y}}$$

$$\text{Max. Relative Error} = \frac{\text{Absolute Error}}{\text{True Value}} = \frac{e_x + e_y}{\bar{x} + \bar{y}}$$

## 2. Subtraction:

$$\text{value} = (\bar{x} + e_x) - (\bar{y} + e_y)$$

$$\text{Absolute error } (e_{x-y}) = e_x - e_y$$

$$\text{Relative Error} = \frac{\text{Absolute Error}}{\text{True Value}} = \frac{e_x - e_y}{\bar{x} - \bar{y}}$$

$$\text{Max. Relative Error} = \frac{\text{Absolute Error}}{\text{True Value}} = \frac{e_x + e_y}{\bar{x} + \bar{y}}$$

## 3. Multiplication:

$$\text{value} = (\bar{x} + e_x) \cdot (\bar{y} + e_y) = \bar{x} \cdot \bar{y} + \bar{x} \cdot e_y + \bar{y} \cdot e_x + e_x \cdot e_y$$

The value of  $(e_x \cdot e_y)$  is very small = 0

$$\text{Absolute error } (e_{x \cdot y}) = \bar{x} \cdot e_y + \bar{y} \cdot e_x$$

$$\text{Relative Error} = \frac{\text{Absolute Error}}{\text{True Value}} = \frac{\bar{x} \cdot e_y + \bar{y} \cdot e_x}{\bar{x} \cdot \bar{y}} = \frac{e_x}{\bar{x}} + \frac{e_y}{\bar{y}}$$

$$\text{Max. Relative Error} = \frac{\text{Absolute Error}}{\text{True Value}} = \frac{\bar{x} \cdot e_y + \bar{y} \cdot e_x}{\bar{x} \cdot \bar{y}} = \frac{|e_x|}{\bar{x}} + \frac{|e_y|}{\bar{y}}$$

## 4. Division:

$$\text{value} = \frac{\bar{x} + e_x}{\bar{y} + e_y} = \frac{\bar{x} + e_x}{\bar{y}} \left(1 + \frac{e_y}{\bar{y}}\right)^{-1} = \frac{\bar{x}}{\bar{y}} \left(\frac{e_x}{\bar{x}} - \frac{e_y}{\bar{y}}\right)$$

$$\text{Absolute error } (e_{x/y}) = \frac{\bar{x}}{\bar{y}} \left(\frac{e_x}{\bar{x}} - \frac{e_y}{\bar{y}}\right)$$

$$\text{Max. Absolute error } (e_{x/y}) = \frac{\bar{x}}{\bar{y}} \left(\frac{|e_x|}{\bar{x}} + \frac{|e_y|}{\bar{y}}\right)$$

$$\text{Relative Error} = \frac{\text{Absolute Error}}{\text{True Value}} = \frac{e_x / e_y}{\bar{x} / \bar{y}} = \frac{e_x}{\bar{x}} - \frac{e_y}{\bar{y}}$$

$$\text{Max. Relative Error} = \frac{\text{Absolute Error}}{\text{True Value}} = \frac{e_x / e_y}{\bar{x} / \bar{y}} = \frac{|e_x|}{\bar{x}} + \frac{|e_y|}{\bar{y}}$$



**Example 1:**

If  $x = 25.4 \pm 0.05$  and  $y = 12.37 \pm 0.005$  Determine the maximum values of absolute & relative errors in all arithmetic operations?

**Solution:**

1. Addition:

$$\text{Max. Absolute error } (e_{x+y}) = e_x + e_y = 0.05 + 0.005 = 0.055$$

$$\text{Max. Relative Error} = \frac{\text{Absolute Error}}{\text{True Value}} = \frac{e_x + e_y}{\bar{x} + \bar{y}} = \frac{0.055}{25.4 + 12.37} = 0.00146$$

2. Subtraction:

$$\text{Max. Absolute error } (e_{x-y}) = e_x + e_y = 0.05 + 0.005 = 0.055$$

$$\text{Relative Error} = \frac{\text{Absolute Error}}{\text{True Value}} = \frac{e_x - e_y}{\bar{x} - \bar{y}} = \frac{0.045}{25.4 - 12.37} = 0.00119$$

$$\text{Max. Relative Error} = \frac{\text{Absolute Error}}{\text{True Value}} = \frac{e_x + e_y}{\bar{x} + \bar{y}} = \frac{0.055}{25.4 + 12.37} = 0.00146$$

3. Multiplication:

$$\text{Max. Absolute error } (e_{x \cdot y}) = \bar{x} \cdot e_y + \bar{y} \cdot e_x = 25.4 * 0.005 + 12.37 * 0.05 = 0.7455$$

$$\text{Relative Error} = \frac{\text{Absolute Error}}{\text{True Value}} = \frac{e_x \cdot e_y}{\bar{x} \cdot \bar{y}} = \frac{e_x}{\bar{x}} + \frac{e_y}{\bar{y}} = \frac{0.05}{25.4} + \frac{0.005}{12.37} = 0.00237$$

$$\text{Max. Relative Error} = \frac{\text{Absolute Error}}{\text{True Value}} = \frac{e_x \cdot e_y}{\bar{x} \cdot \bar{y}} = \frac{|e_x|}{\bar{x}} + \frac{|e_y|}{\bar{y}} = \frac{0.05}{25.4} + \frac{0.005}{12.37} = 0.00237$$

4. Division:

$$\text{Absolute error } (e_{x/y}) = \frac{\bar{x}}{\bar{y}} \left( \frac{e_x}{\bar{x}} - \frac{e_y}{\bar{y}} \right) = \frac{25.4}{12.37} \left( \frac{0.05}{25.4} - \frac{0.005}{12.37} \right) = 0.00321$$

$$\text{Max. Absolute error } (e_{x/y}) = \frac{\bar{x}}{\bar{y}} \left( \frac{|e_x|}{\bar{x}} + \frac{|e_y|}{\bar{y}} \right) = \frac{25.4}{12.37} \left( \frac{0.05}{25.4} + \frac{0.005}{12.37} \right) = 0.00487$$

$$\text{Relative Error} = \frac{\text{Absolute Error}}{\text{True Value}} = \frac{e_x / e_y}{\bar{x} / \bar{y}} = \frac{e_x}{\bar{x}} - \frac{e_y}{\bar{y}} = \frac{0.05}{25.4} - \frac{0.005}{12.37} = 0.00156$$



$$\text{Max. Relative Error} = \frac{\text{Absolute Error}}{\text{True Value}} = \frac{e_x / e_y}{\bar{x} / \bar{y}} = \frac{|e_x|}{\bar{x}} + \frac{|e_y|}{\bar{y}} = \frac{0.05}{25.4} + \frac{0.005}{12.37} = 0.00237$$

**Example 2:**

What will be the approximate values between which  $\left(\frac{1.362(7.54-13.2)}{47}\right)$  can be expected to lie, assuming the maximum possible error in each of the significant figures?

**Solution**

$$\frac{1.362(7.54 - 13.2)}{47} = \frac{z(x - y)}{w}$$

$$\text{Error} = \frac{0.0005(0.005-0.05)}{0.5}$$

$$\text{Max. Absolute error } (e_{x-y}) = e_s = e_x + e_y = 0.005 + 0.05 = 0.055$$

$$\text{Absolute error } (e_{z.s}) = e_m = \bar{z} \cdot e_s \pm \bar{s} \cdot e_z = 1.362 * 0.055 \pm 5.66 * 0.005 = 0.07208$$

$$\text{Max. Absolute error } \left(\frac{e_m}{w}\right) = e_d = \frac{\bar{m}}{\bar{w}} \left(\frac{|e_m|}{\bar{m}} + \frac{|e_w|}{\bar{w}}\right) = -0.164 \left(\frac{0.07208}{7.70892} + \frac{0.5}{47}\right) = 0.00328$$

$$\text{Max. Relative error } (e_{m/w}) = e_d = \left(\frac{|e_m|}{\bar{m}} + \frac{|e_w|}{\bar{w}}\right) = \left(\frac{0.07208}{7.70892} + \frac{0.5}{47}\right) = 0.01999$$

**Definition of Matrix**

A matrix is a set of variables or constants arranged in rows and columns in a rectangular or square array.

**1. Column Matrix:**

- A matrix which has only one column is called a Column Matrix.
- Sum of two column matrices of same dimension is also a column matrix of the same dimension.
- If a row matrix is multiplied with a column matrix having the same number of elements, then the resultant has one element only.



Example

$$\begin{bmatrix} 21 \\ 17 \\ 9 \end{bmatrix}$$

The above matrix has three rows but only one column, so it is a column matrix.

## 2. Row Matrix:

- Row Matrix is a matrix with only one row.
- Sum of two row matrices of same dimension is also a row matrix of the same dimension.
- The dimension of a row matrix is given as  $1 \times n$  where  $n$  is the number of columns.

Example:  $[3 \ 5 \ 1]$  is a row matrix as it has only one row.

## 3. A null matrix or zero matrix:

A null matrix is a matrix with all its entries being zero. Some examples of zero matrices are

$$O_{K \times m, n} = \begin{bmatrix} 0_K & 0_K & \dots & 0_K \\ 0_K & 0_K & \dots & 0_K \\ \vdots & \vdots & \ddots & \vdots \\ 0_K & 0_K & \dots & 0_K \end{bmatrix}_{m \cdot n}$$

## 4. Identity Matrix

- A square matrix in which all the main diagonal elements are 1's and all the remaining elements are 0's is called an Identity Matrix.
- Identity Matrix is also called as Unit Matrix or Elementary Matrix.
- Identity Matrix is denoted with the letter " $I_{n \times n}$ ", where  $n \times n$  represents the order of the matrix.
- One of the important properties of identity matrix is:  $A \times I_{n \times n} = A$ , where  $A$  is any square matrix of order  $n \times n$ .

Example

$$I_1 = [1], \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \dots, \quad I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$



### 5. A diagonal matrix:

A diagonal matrix is a matrix (usually a square matrix) in which the entries outside the main diagonal ( $\searrow$ ) are all zero. The diagonal entries themselves may or may not be zero. Thus, the matrix  $D = (d_{ij})$  with  $n$  columns and  $n$  rows is diagonal if:

$$d_{i,j} = 0 \text{ if } i \neq j \quad \forall i, j \in \{1, 2, \dots, n\}$$

For example, the following matrix is diagonal:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

### 6. A symmetric matrix:

A symmetric matrix is a square matrix that is equal to its transpose. Formally, matrix  $A$  is symmetric if

$$A = A^T.$$

Because equal matrices have equal dimensions, only square matrices can be symmetric. The entries of a symmetric matrix are symmetric with respect to the main diagonal. So if the entries are written as  $A = (a_{ij})$ , then  $a_{ij} = a_{ji}$ , for all indices  $i$  and  $j$ . The following  $3 \times 3$  matrix is symmetric:

$$\begin{bmatrix} 1 & 7 & 3 \\ 7 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix}.$$

### 7. Lower Triangular Matrix or (Left Triangular Matrix)

This is a square matrix in which all elements *above* the diagonal are *zero* and the diagonal as well as the lower - diagonal elements can take on any value.

$$L = \begin{bmatrix} l_{1,1} & & & & 0 \\ l_{2,1} & l_{2,2} & & & \\ l_{3,1} & l_{3,2} & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ l_{n,1} & l_{n,2} & \cdots & l_{n,n-1} & l_{n,n} \end{bmatrix}$$



This matrix is lower triangular

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 8 & 0 \\ 4 & 9 & 7 \end{bmatrix}$$

### 8. Upper Triangular Matrix or (Right Triangular Matrix):

This is a square matrix in which all elements *below* the diagonal are *zero* and the diagonal as well as the above - diagonal elements can take on any value.

$$U = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & \dots & u_{1,n} \\ & u_{2,2} & u_{2,3} & \dots & u_{2,n} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & u_{n-1,n} \\ 0 & & & & u_{n,n} \end{bmatrix}$$

**Examples:** this matrix is upper triangular and this matrix

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

### 9. Transpose Matrix:

The transpose of a matrix  $\mathbf{A}$  is another matrix  $\mathbf{A}^T$  (also written  $\mathbf{A}'$ ,  $\mathbf{A}^{tr}$ ,  ${}^t\mathbf{A}$  or  $\mathbf{A}^t$ ) created by any one of the following equivalent actions:

- reflect  $\mathbf{A}$  over its main diagonal (which runs from top-left to bottom-right) to obtain  $\mathbf{A}^T$
- write the rows of  $\mathbf{A}$  as the columns of  $\mathbf{A}^T$
- write the columns of  $\mathbf{A}$  as the rows of  $\mathbf{A}^T$

Formally, the  $i$  th row,  $j$  th column element of  $\mathbf{A}^T$  is the  $j$  th row,  $i$  th column element of  $\mathbf{A}$ :

$$[\mathbf{A}^T]_{ij} = [\mathbf{A}]_{ji}$$

If  $\mathbf{A}$  is an  $m \times n$  matrix then  $\mathbf{A}^T$  is an  $n \times m$  matrix.



## Rules for Combined Matrix Operation

Given the matrices  $[A]$ ,  $[B]$ ,  $[C]$  & the scalar  $(K)$  and assuming that the various operations are summarized below:

### Addition and Subtraction

1.  $[A] + [B] = [B] + [A]$
2.  $[A] \pm [0] = [A]$
3.  $[A] + [B] + [C] = [A] + [[B] + [C]] = [[A] + [B]] + [C]$
4.  $K[A] \pm [B] = K[A] \pm K[B]$
5.  $[A] + [A] = [A] [[I] + [I]] = [A][I](2) = 2 * [A]$
6.  $[[A] \pm [B]]^t = [A]^t \pm [B]^t$

### Multiplication and Sums

1.  $[A][I] = [I][A] = [A]$
2.  $K[A][B] = [KA][B] = [A][KB]$
3.  $K[A][[B] + [C]] = K[A][B] + K[A][C]$
4.  $K[[A] + [B]][C] = K[A][C] + K[B][C]$
5.  $[[A] + [B]]^2 = [A]^2 + 2[A][B] + [B]^2$
6.  $[[A] - [B]][[A] + [B]] = [A]^2 + [A][B] - [B][A] + [B]^2$
7.  $K[A][B][C] = [KA][B][C] = [A][KB][C] = [A][B][KC]$
8.  $[[A][B]]^t = [B]^t[A]^t$
9.  $[[A][B][C]]^t = [C]^t[B]^t[A]^t$
10.  $[A][A]^{-1} = [A]^{-1}[A] = [I]$
11.  $[[A][B]]^{-1} = [B]^{-1}[A]^{-1}$
12.  $[KA]^{-1} = \left(\frac{1}{K}\right)[A]^{-1}$
13.  $[[A]^{-1}]^t = [[A]^t]^{-1}$
14.  $[[A]^n]^{-1} = [[A]^{-1}]^n$





**Area of Matrix:**

In general the area of any region can be found by subdivided the region into many triangle and assign node number.

**1. In the region:**

Equation below can be applied to find the area of a region includes the origin of the coordinate system

$$A_r = \frac{1}{2} \left[ \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix} + \dots + \begin{vmatrix} x_{n-1} & y_{n-1} \\ x_n & y_n \end{vmatrix} \right]$$

**2. Out the region:**

For region, that not includes the origin of the coordinate system. The following equation can be applied to find the area

$$A_r = \frac{1}{2} \left[ \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} + \dots + \begin{vmatrix} x_1 & y_1 & 1 \\ x_{n-1} & y_{n-1} & 1 \\ x_{n-2} & y_{n-2} & 1 \end{vmatrix} \right]$$

**Volume of Matrix:**

**1. In the region:**

$$V = \frac{1}{6} * \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_2 \end{vmatrix}$$

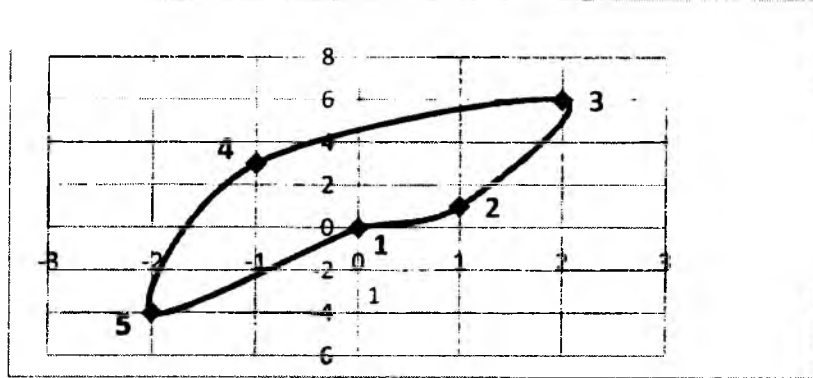
**2. Out the region:**

$$V = \frac{1}{6} * \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_2 & 1 \end{vmatrix}$$

**Example 1:**

Determine the area enclosed by the following coordinate:

X	0	1	2	-1	-2
Y	0	1	6	3	-4



**Solution:**

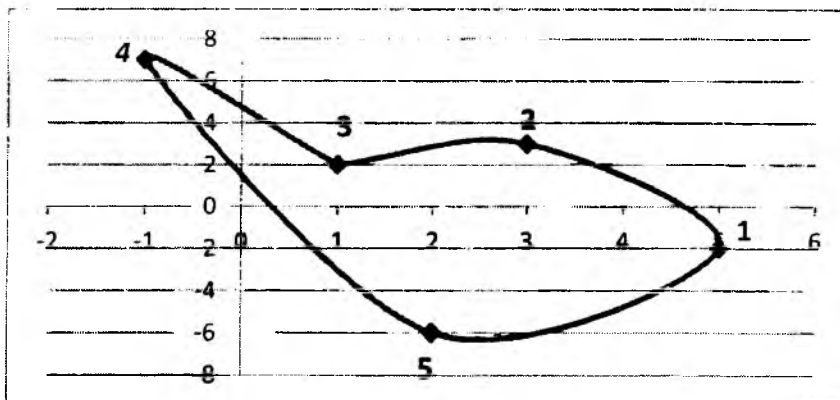
$$2A = \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 2 & 6 \end{vmatrix} + \begin{vmatrix} 2 & 6 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} -1 & 3 \\ -2 & -4 \end{vmatrix} + \begin{vmatrix} -2 & -4 \\ 0 & 0 \end{vmatrix} = 0 + 4 + 12 + 0 = 26$$

$$\therefore A = \frac{26}{2} = 13$$

**Example 2:**

Determine the area enclosed by the following coordinate:

X	1	3	5	2	-1
y	2	3	-2	-6	7



**Solution:**

Node	1	2	3	4	5
X	5	3	1	-1	2
Y	-2	3	2	7	-6



$$\begin{aligned}
 2A &= \begin{vmatrix} 5 & -2 & 1 \\ 3 & 3 & 1 \\ 1 & 2 & 1 \end{vmatrix} + \begin{vmatrix} 5 & -2 & 1 \\ 1 & 2 & 1 \\ -1 & 7 & 1 \end{vmatrix} + \begin{vmatrix} 5 & -2 & 1 \\ -1 & 7 & 1 \\ 2 & -6 & 1 \end{vmatrix} = 5 \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} - 3 \begin{vmatrix} -2 & 1 \\ 2 & 1 \end{vmatrix} \\
 &+ \begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix} + 5 \begin{vmatrix} 2 & 1 \\ 7 & 1 \end{vmatrix} - \begin{vmatrix} -2 & 1 \\ 7 & 1 \end{vmatrix} + (-1) \begin{vmatrix} -2 & 1 \\ 2 & 1 \end{vmatrix} + 5 \begin{vmatrix} 7 & 1 \\ -6 & 1 \end{vmatrix} \\
 &- (-1) \begin{vmatrix} -2 & 1 \\ -6 & 1 \end{vmatrix} + 2 \begin{vmatrix} -2 & 1 \\ 7 & 1 \end{vmatrix} \\
 &= 5 + 12 - 5 - 25 + 9 + 4 + 65 + 4 - 18 = 51
 \end{aligned}$$

$$\therefore A = \frac{51}{2} = 25.5$$

**/\* Numerical Methods used to solve the first category of Linear Algebraic Equation \*/**



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### Cramer's Rule Method

Consider the following two nonhomogeneous equations:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Pre multiplying by the transport matrix of the cofactors of the coefficient matrix given

$$\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Or

$$\begin{bmatrix} a_{11} a_{22} - a_{12} a_{21} & 0 \\ 0 & a_{11} a_{22} - a_{12} a_{21} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{22} b_1 - a_{12} b_2 \\ a_{11} b_2 - a_{21} b_1 \end{bmatrix}$$

$$x_1 = \frac{a_{22} b_1 - a_{12} b_2}{a_{11} a_{22} - a_{12} a_{21}} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$



$$x_2 = \frac{a_{11} b_2 - a_{21} b_1}{a_{11} a_{22} - a_{12} a_{21}} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

In general the solution of a set linear algebraic equation by this method is given as:

$$x_j = \frac{\det[A_j]}{\det[A]} = \frac{|A_j|}{|A|}; \quad j = 1, 2, \dots, n$$

Where:

$|A|$  is the determinant of the coefficient matrix  $[a_{ij}]$

$|A_j|$  is the determinant of the coefficient matrix with its  $j$ th column replaced by the  $[b_i]$  vector.

**Algorithm for CRAMMER'S RULE METHOD:**

Step1: Form coefficient matrix  $[a_{ij}]$  and the right hand vector  $[b_i]$

Step2: Evaluate  $\det.[A]$  using upper Triangle method or pivotal condensation. If  $\det[A]=0$ , then the set of equations has no solution. Otherwise proceed to the next step.

Step3: Evaluate  $\det.[A_j]$  by replacing the  $j$ th column of matrix  $[A]$  with the column vector  $[b]$ .

Step4: Repeat step 3 for  $j=1, 2, \dots, n$ .

Step5: Solve for the unknowns  $x_j = \frac{\det[A_j]}{\det[A]}; \dots \dots \dots j = 1, 2, \dots, n$ .

**Example 1:** Use Cramer's Rule to solve:

$$2x - 3y = 5$$

$$x + y = :$$

**Solution**

Forming the various matrices:  $[A] = \begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}, [B] = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$

$$[A][x] = [b]$$

$$[A_1] = \begin{bmatrix} 5 & -3 \\ 5 & 1 \end{bmatrix}, [A_2] = \begin{bmatrix} 2 & 5 \\ 1 & 5 \end{bmatrix}$$

$$\det. [A] = 2 * 1 + 3 * 1 = 5$$



$$\det. [A_1] = 5 * 1 - (-3) * 5 = 20$$

$$\det. [A_2] = 2 * 5 - (5) * 1 = 5$$

$$x_1 = \frac{\det[A_1]}{\det[A]} = \frac{|A_1|}{|A|} = \frac{20}{5} = 4$$

$$x_2 = \frac{\det[A_2]}{\det[A]} = \frac{|A_2|}{|A|} = \frac{5}{5} = 1$$

**Example 2:** Use Cramer's Rule to solve:

$$3x + 4y - 3z = 5$$

$$3x - 2y + 4z = 7$$

$$3x + 2y - z = 3$$

**Solution:**

$$[A] = \begin{bmatrix} 3 & 4 & -3 \\ 3 & -2 & 4 \\ 3 & 2 & -1 \end{bmatrix}, [b] = \begin{bmatrix} 5 \\ 7 \\ 3 \end{bmatrix}$$

$$[A][x] = [b]$$

$$\begin{bmatrix} 3 & 4 & -3 \\ 3 & -2 & 4 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 3 \end{bmatrix}$$

$$[A_1] = \begin{bmatrix} 5 & 4 & -3 \\ 7 & -2 & 4 \\ 3 & 2 & -1 \end{bmatrix}, [A_2] = \begin{bmatrix} 3 & 5 & -3 \\ 3 & 7 & 4 \\ 3 & 3 & -1 \end{bmatrix}, [A_3] = \begin{bmatrix} 3 & 4 & 5 \\ 3 & -2 & 7 \\ 3 & 2 & 3 \end{bmatrix}$$

$$\begin{aligned} \det[A] &= \begin{vmatrix} 3 & 4 & -3 \\ 3 & -2 & 4 \\ 3 & 2 & -1 \end{vmatrix} = 3 * \begin{vmatrix} -2 & 4 \\ 2 & -1 \end{vmatrix} - 3 * \begin{vmatrix} 4 & -3 \\ 2 & -1 \end{vmatrix} + 3 * \begin{vmatrix} 4 & -3 \\ -2 & 4 \end{vmatrix} \\ &= 3(-6) - 3.2 + 3.10 = 6 \end{aligned}$$

$$\begin{aligned} \det[A_1] &= \begin{vmatrix} 5 & 4 & -3 \\ 7 & -2 & 4 \\ 3 & 2 & -1 \end{vmatrix} = 5 * \begin{vmatrix} -2 & 4 \\ 2 & -1 \end{vmatrix} - 7 * \begin{vmatrix} 4 & -3 \\ 2 & -1 \end{vmatrix} + 3 * \begin{vmatrix} 4 & -3 \\ -2 & 4 \end{vmatrix} \\ &= 5(-6) - 7.2 + 3.10 = -14 \end{aligned}$$

$$\begin{aligned} \det[A_2] &= \begin{vmatrix} 3 & 5 & -3 \\ 3 & 7 & 4 \\ 3 & 3 & -1 \end{vmatrix} = 3 * \begin{vmatrix} 7 & 4 \\ 3 & -1 \end{vmatrix} - 3 * \begin{vmatrix} 5 & -3 \\ 3 & -1 \end{vmatrix} + 3 * \begin{vmatrix} 5 & -3 \\ 7 & 4 \end{vmatrix} \\ &= 3(-19) - 3 * (-4) + 3.41 = 54 \end{aligned}$$



$$\det[A_3] = \begin{vmatrix} 3 & 4 & 5 \\ 3 & -2 & 7 \\ 3 & 2 & 3 \end{vmatrix} = 3 * \begin{vmatrix} -2 & 7 \\ 2 & 3 \end{vmatrix} - 3 * \begin{vmatrix} 4 & 5 \\ 2 & 3 \end{vmatrix} + 3 * \begin{vmatrix} 4 & 5 \\ -2 & 7 \end{vmatrix}$$

$$= 3(-20) - 3.2 + 3.38 = 48$$

$$x_1 = \frac{\det[A_1]}{\det[A]} = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 5 & 4 & -3 \\ 7 & -2 & 4 \\ 3 & 2 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 4 & -3 \\ 3 & -2 & 4 \\ 3 & 2 & -1 \end{vmatrix}} = \frac{-14}{6} = \frac{-7}{3} = -2.33333$$

$$x_2 = \frac{\det[A_2]}{\det[A]} = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 3 & 5 & -3 \\ 3 & 7 & 4 \\ 3 & 3 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 4 & -3 \\ 3 & -2 & 4 \\ 3 & 2 & -1 \end{vmatrix}} = \frac{54}{6} = 9$$

$$x_3 = \frac{\det[A_3]}{\det[A]} = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} 3 & 4 & 5 \\ 3 & -2 & 7 \\ 3 & 2 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 4 & -3 \\ 3 & -2 & 4 \\ 3 & 2 & -1 \end{vmatrix}} = \frac{48}{6} = 8$$



## SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS

### INTRODECTION:

We come across, very often, simultaneous linear algebraic equations for its solutions, especially, in the field of science and engineering. In lower classes, we have solved such equations by Cramer's rule or by matrix methods. These methods become tedious when the number of unknown in the system is large. After the availability of computers, we go to numerical methods which are suited for computer operations.

The **Linear algebraic equations** and **n unknown variables** of the form:

$$80x_1 - 20x_2 - 20x_3 = 20$$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$-20x_1 + 40x_2 - 20x_3 = 20$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$-20x_1 - 20x_2 + 130x_3 = 20$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{nn}x_n = b_n$$

Systems of Linear algebraic equations can be expressed very conveniently in terms of Matrix notation.

Systems of Linear algebraic equations arise in **many different types of problems**, for example:

1. Network problems (e.g., electrical networks).
2. Fitting approximation functions.
3. Systems of finite difference equations that arise in the numerical solution of differential equations.

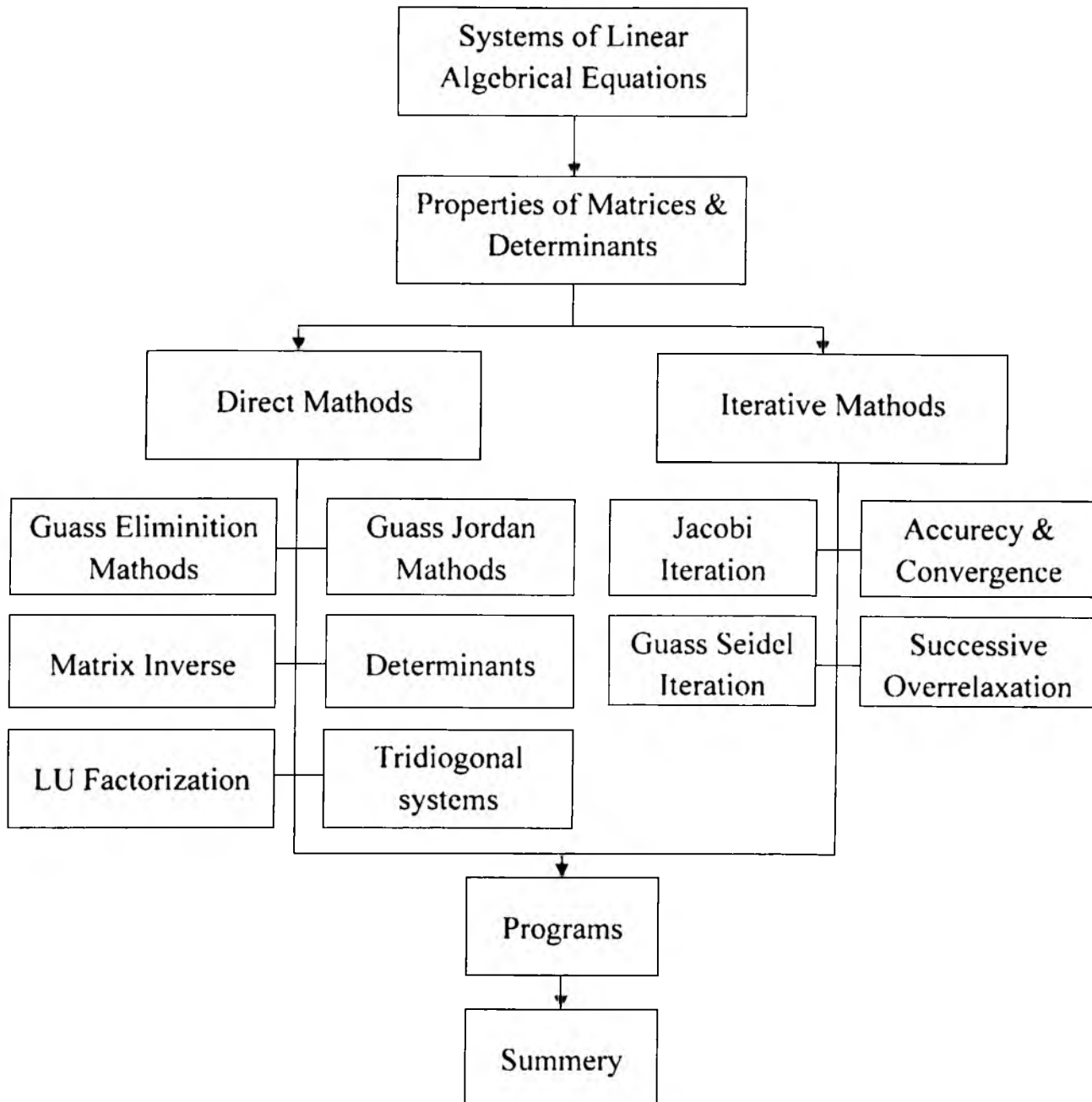
There are **two fundamentally different approaches** for solving systems of linear algebraic equations:

1. Direct elimination methods.
2. Iterative methods.





Therefore, the organization for solving these systems as the following:





## Solution of Simultaneous Linear Algebraic Equations (Direct Methods)

### 1. Gauss – Elimination Method (Direct Methods):

This is a direct method based on the elimination of the unknowns by combining equations such that the  $n$  equations in  $n$  unknowns are reduced to an equivalent upper triangular system which could be solved by back substitution.

Consider the  $n$  linear equation in  $n$  unknowns, viz.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

The system is equivalent to  $AX=B$

Where:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Form the augmented matrix from the matrices

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & \vdots & b_2 \\ \dots & \dots & \dots & \dots & \vdots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} & \vdots & b_n \end{bmatrix}$$

Dividing the first equation (row) by  $a_{11}$  gives:

$$\begin{bmatrix} 1 & a_{12}^- & \dots & a_{1n}^- & \vdots & b_1^- \\ a_{21} & a_{22} & \dots & a_{2n} & \vdots & b_2 \\ \dots & \dots & \dots & \dots & \vdots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} & \vdots & b_n \end{bmatrix}$$

Multiplying the first row by  $-a_{i1}$  for  $i=2, \dots, n$  then adding to the  $i$ th row to eliminates the first column except the elements in first row.



$$\begin{bmatrix} 1 & a_{12}^- & \dots & a_{1n}^- & \vdots & b_1^- \\ 0 & a_{22}^- & \dots & a_{2n}^- & \vdots & b_2^- \\ \dots & \dots & \dots & \dots & \vdots & \dots \\ 0 & a_{n2}^- & \dots & a_{nn}^- & \vdots & b_n^- \end{bmatrix}$$

Divide the second row by  $a_{22}^-$  gives:

$$\begin{bmatrix} 1 & a_{12}^- & \dots & a_{1n}^- & \vdots & b_1^- \\ 0 & 1 & \dots & a_{2n}^- & \vdots & b_2^- \\ \dots & \dots & \dots & \dots & \vdots & \dots \\ 0 & a_{n2}^- & \dots & a_{nn}^- & \vdots & b_n^- \end{bmatrix}$$

Multiplying the second row by  $-a_{i2}^-$  for  $i = 3, \dots, n$  then adding to the  $i$ th row to eliminate the second column except the elements in first & second row.

$$\begin{bmatrix} 1 & a_{12}^- & \dots & a_{1n}^- & \vdots & b_1^- \\ 0 & 1 & \dots & a_{2n}^- & \vdots & b_2^- \\ \dots & \dots & \dots & \dots & \vdots & \dots \\ 0 & 0 & \dots & a_{nn}^- & \vdots & b_n^- \end{bmatrix}$$

This process is continued until the last row (row  $n$ ) becomes in the following form  $(0 \ 0 \ \dots \ 1 \ : \ c)$

Note: you have to avoid a zero divisor in the position in each stage by interchanging rows.

Once the process is completed the matrix becomes as the following triangular form:

$$\begin{bmatrix} 1 & u_{12} & \dots & u_{1n} & \vdots & c_1 \\ 0 & 1 & \dots & u_{2n} & \vdots & c_2 \\ \dots & \dots & \dots & \dots & \vdots & \dots \\ 0 & 0 & \dots & 1 & \vdots & c_n \end{bmatrix}$$

The unknowns are then determined by back substitution i.e.;

$$x_n = c_n$$

$$x_{n-1} = c_{n-1} - u_{n-1} * x_n$$

⋮

$$x_1 = c_1 - u_{12} * x_2 - \dots - u_{1n} * x_n$$



**Example 1:** Solve the following systems by (Gauss- Elimination) method:

$$x+2y + z =3$$

$$2x +3 y + 3z=10$$

$$3x-y+2z=13$$

**Solution**

The given system is equivalent to

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ 13 \end{bmatrix}$$

$$(A, B) = \left[ \begin{array}{ccc|cc} 1 & 2 & 1 & 1 & 3 \\ 2 & 3 & 3 & 3 & 10 \\ 3 & -1 & 2 & 2 & 13 \end{array} \right]$$

Now, we will make the matrix a upper triangular

$$\left[ \begin{array}{ccc|cc} 1 & 2 & 1 & 1 & 3 \\ 2 & 3 & 3 & 3 & 10 \\ 3 & -1 & 2 & 2 & 13 \end{array} \right] \xrightarrow{\begin{array}{l} R_2-2R_1 \rightarrow R_2 \\ R_3-3R_1 \rightarrow R_3 \end{array}}$$

$$\left[ \begin{array}{ccc|cc} 1 & 2 & 1 & 1 & 3 \\ 0 & -1 & 1 & 1 & 4 \\ 0 & -7 & -1 & -1 & 4 \end{array} \right] \xrightarrow{\begin{array}{l} R_2 / -1 \rightarrow R_2 \\ R_3 + 7R_2 \rightarrow R_3 \end{array}}$$

Now take  $b_{22} = -1$  as the pivot and  $b_{32}$  as make as zero.

$$\left[ \begin{array}{ccc|cc} 1 & 2 & 1 & 1 & 3 \\ 0 & 1 & -1 & 1 & 4 \\ 0 & 0 & -8 & -8 & 24 \end{array} \right] \xrightarrow{R_3 / -8 \rightarrow R_3} \left[ \begin{array}{ccc|cc} 1 & 2 & 1 & 1 & 3 \\ 0 & 1 & -1 & 1 & 4 \\ 0 & 0 & 1 & 1 & 3 \end{array} \right]$$

From this, we get:

$$x+2y+z=3,$$

$$y-z=-4,$$

$$z=3$$

$$\therefore y = -4 + z = -4 + 3 = -1$$

$$\therefore x = 3 - 2y - z = 3 - 2 * -1 - 3 = 2$$



**Example 2:** Solve the following systems by (Gauss- Elimination) method:

$$x+2y+3z=6$$

$$2x+3y-z=7$$

$$3x+2y+9z=14$$

**Solution**

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & -1 \\ 3 & 2 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 14 \end{bmatrix}$$

$$(A, B) = \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & 3 & -1 & 7 \\ 3 & 2 & 9 & 14 \end{array} \right] \begin{array}{l} R_2-2R_1 \rightarrow R_2 \\ R_3-3R_1 \rightarrow R_3 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -1 & -7 & -5 \\ 0 & -4 & 0 & -4 \end{array} \right] R_3-4R_2 \rightarrow R_3$$

$$= \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -1 & -7 & -5 \\ 0 & 0 & 28 & 16 \end{array} \right]$$

$$28z=16, z = \frac{16}{28} = 0.57142$$

$$-y-7z=-5, y=5-7z=5-7(0.57142)=1.00006$$

$$x + 2y + 3z = 6, x = 6 - 2y - 3z = 6-2(1.00006) + 3(0.57142) = 2.28562$$



**Example 2:** Solve the following systems by (Gauss- Elimination) method:

$$x+2y+3z=6$$

$$2x+3y-z=7$$

$$3x+2y+9z=14$$

**Solution**

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & -1 \\ 3 & 2 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 14 \end{bmatrix}$$

$$(A, B) = \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & 3 & -1 & 7 \\ 3 & 2 & 9 & 14 \end{array} \right] \xrightarrow{\begin{array}{l} R_2-2R_1 \rightarrow R_2 \\ R_3-3R_1 \rightarrow R_3 \end{array}}$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -1 & -7 & -5 \\ 0 & -4 & 0 & -4 \end{array} \right] \xrightarrow{R_3-4R_2 \rightarrow R_3}$$

$$= \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -1 & -7 & -5 \\ 0 & 0 & 28 & 16 \end{array} \right]$$

$$28z=16, z = \frac{16}{28} = 0.57142$$

$$-y-7z=-5, y=5-7z=5-7(0.57142)=1.00006$$

$$x + 2y + 3z = 6, x = 6 - 2y - 3z = 6-2(1.00006) + 3(0.57142) = 2.28562$$

## 2. Gauss – Jordan Elimination Method (Direct Method):

This method is a modification of the above Gauss elimination method. In this method, the coefficient matrix A of the system AX= B is brought to a diagonal matrix or unit matrix by making the matrix A not only upper triangular but also lower triangular by making all elements above the leading diagonal of A also as zero. By this way, the system AX=B will reduce to the form:

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & b_1^* \\ 0 & 1 & 0 & \vdots & b_2^* \\ \dots & \dots & \dots & \vdots & \dots \\ 0 & 0 & 1 & \vdots & b_3^* \end{bmatrix}$$

Form above:

$$X_1 = b_1^*, \dots, X_2 = b_2^*, \dots X_n = b_n^*$$



**Note:** By this method, the values of  $X_1, X_2, \dots, X_n$  are get immediately without using the process of back substitution.

**Example 1:** Solve the following systems by (Gauss- Jordan) method:

$$8x - 2y + z = 9$$

$$x + 3y + z = 13$$

$$3x + y - 5z = 17$$

**Solution**

$$\begin{bmatrix} 8 & -2 & 1 \\ 1 & 3 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 17 \end{bmatrix}$$

$$(A, B) = \begin{bmatrix} 8 & -2 & 1 & 9 \\ 1 & 3 & 1 & 13 \\ 3 & 1 & -5 & 17 \end{bmatrix} \xrightarrow{\text{change row 1 \& 2}}$$

$$\begin{bmatrix} 1 & 3 & 1 & 13 \\ 8 & -2 & 1 & 9 \\ 3 & 1 & -5 & 17 \end{bmatrix} \xrightarrow{\text{change row 2 \& 3}}$$

$$\begin{bmatrix} 1 & 3 & 1 & 13 \\ 3 & 1 & -5 & 17 \\ 8 & -2 & 1 & 9 \end{bmatrix} \xrightarrow{\begin{matrix} R_2 - 3R_1 \rightarrow R_2 \\ R_3 - 8R_1 \rightarrow R_3 \end{matrix}} \begin{bmatrix} 1 & 3 & 1 & 13 \\ 0 & -8 & -8 & -22 \\ 0 & -26 & -7 & -95 \end{bmatrix} \xrightarrow{\begin{matrix} R_2 \rightarrow -8 \\ R_3 \rightarrow -8 \end{matrix}}$$

$$\begin{bmatrix} 1 & 3 & 1 & 13 \\ 0 & 1 & 1 & 2.75 \\ 0 & -26 & -7 & -95 \end{bmatrix} \xrightarrow{\begin{matrix} R_1 - 3R_2 \rightarrow R_1 \\ R_3 + 26R_2 \rightarrow R_3 \end{matrix}}$$

$$\begin{bmatrix} 1 & 0 & -2 & 4.75 \\ 0 & 1 & 1 & 2.75 \\ 0 & 0 & 19 & -23.5 \end{bmatrix} \xrightarrow{\begin{matrix} R_3 \rightarrow \frac{R_3}{19} \\ R_1 \rightarrow \frac{R_1}{19} \end{matrix}}$$

$$\begin{bmatrix} 1 & 0 & -2 & 4.75 \\ 0 & 1 & 1 & 2.75 \\ 0 & 0 & 1 & -1.23684 \end{bmatrix} \xrightarrow{\begin{matrix} R_1 + 2R_3 \rightarrow R_1 \\ R_2 - R_3 \rightarrow R_2 \end{matrix}} \begin{bmatrix} 1 & 0 & 0 & 2.27632 \\ 0 & 1 & 0 & 3.98684 \\ 0 & 0 & 1 & -1.23684 \end{bmatrix}$$

$$X = 2.27632, y = 3.98684, z = -1.23684$$



**Example 2:** Solve the following systems by (Gauss- Jordan) method;

$$x + 3y - 7z = 10$$

$$6x + y - z = 5$$

$$2x - 2y + z = 8$$

**Solution**

$$\begin{bmatrix} 1 & 3 & -7 \\ 6 & 1 & -1 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ 8 \end{bmatrix}$$

$$(A, B) = \left[ \begin{array}{ccc|c} 1 & 3 & -7 & 10 \\ 6 & 1 & -1 & 5 \\ 2 & -2 & 1 & 8 \end{array} \right] \begin{array}{l} R_2 - 3R_1 \rightarrow R_2 \\ R_3 - 8R_1 \rightarrow R_3 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 3 & -7 & 10 \\ 0 & -17 & 41 & -55 \\ 0 & -8 & -15 & -12 \end{array} \right] \begin{array}{l} R_2 \\ -17 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 3 & -7 & 10 \\ 0 & 1 & -2.41176 & 3.23529 \\ 0 & -8 & -15 & -12 \end{array} \right] \begin{array}{l} R_1 - 3R_2 \rightarrow R_1 \\ R_3 + 8R_2 \rightarrow R_3 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0.23528 & 0.29413 \\ 0 & 1 & -2.41176 & 3.23629 \\ 0 & 0 & -4.29408 & 13.88232 \end{array} \right] \begin{array}{l} R_3 \\ -4.29408 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0.23528 & 0.29413 \\ 0 & 1 & -2.41176 & 3.23629 \\ 0 & 0 & 1 & 3.23289 \end{array} \right] \begin{array}{l} R_1 - 0.23528R_3 \rightarrow R_1 \\ R_2 + 2.41176R_3 \rightarrow R_2 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 1.05476 \\ 0 & 1 & 0 & -4.56166 \\ 0 & 0 & 1 & -3.23289 \end{array} \right]$$

$$X = 1.05476, y = -4.56166, z = -3.23289$$





## *The solution of Numerical Algebraic and Transcendental Equations*

### Graphical Methods:

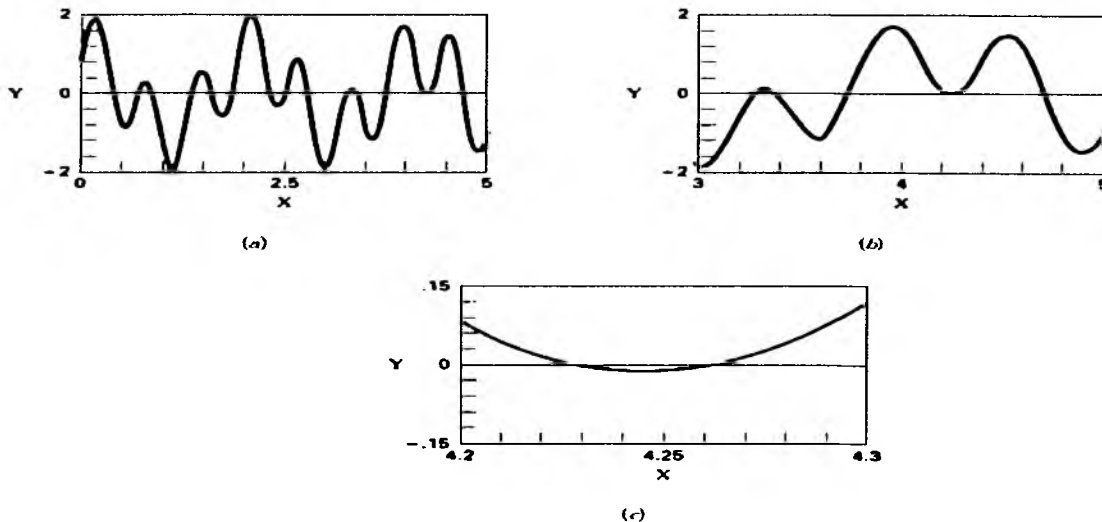
A simple method for obtaining an estimate of the root of the equation  $f(x) = 0$  is to make a plot of the function and observe where it crosses the  $x$  axis. This point, which represents the  $x$  value for which  $f(x) = 0$ , provides a rough approximation of the root.

**Example 1:** Use the graphical approach to determine the locate roots of equations. The function  $f(x) = \sin 10x + \cos 3x$ , has several roots over the range  $x = 0$  to  $x = 5$ . Use computer graphics to gain insight in to the behavior of this function.

**Solution:** Packages such as Excel and MATLAB software can be used to generate plots.

Figure 5.4a is a plot of  $f(x)$  from  $x = 0$  to  $x = 5$ . This plot suggests the presence of several roots, including a possible double root at about  $x = 4.2$  where  $f(x)$  appears to be tangent to has several roots over the range  $x = 0$  to  $x = 5$ .

The progressive enlargement of  $f(x) = \sin 10x + \cos 3x$  by the computer. Such interactive graphics permits the analyst to determine that two distinct roots exist between  $x = 4.2$  and  $x = 4.3$



A more detailed picture of the behavior of  $f(x)$  is obtained by changing the plotting range from  $x = 3$  to  $x = 5$ , as shown in Fig. 5.4b. Finally, in Fig. 5.4c, the vertical scale is narrowed further to  $f(x) = -0.15$  to  $f(x) = 0.15$  and the horizontal scale is narrowed to  $x = 4.2$  to  $x = 4.3$ . This plot shows clearly that a double root does not exist in this region and that in fact there are two distinct roots at about  $x = 4.23$  and  $x = 4.26$ .



### The Bisection Method (or Interval Halving Method):

The *bisection method*, which is alternatively called binary chopping, interval halving, is one type of incremental search method in which the interval is always divided in half. If a function changes sign over an interval, the function value at the midpoint is evaluated. The location of the root is then determined as lying at the midpoint of the subinterval within which the sign change occurs. The process is repeated to obtain refined estimates.

When applying the graphical technique, you have observed that  $f(x)$  changed sign on opposite sides of the root. In general, if  $f(x)$  is real and continuous in the interval from  $x_L$  to  $x_R$  and  $f(x_L)$  and  $f(x_R)$  have opposite signs, that is,  $f(x_L)f(x_R) < 0$ , then there is at least one real root between  $x_L$  and  $x_R$ . *Incremental search methods* capitalize on this observation by locating an interval where the function changes sign. Then the location of the sign change (and consequently, the root) is identified more precisely by dividing the interval into a number of subintervals.

#### Algorithm of Bisection Method:

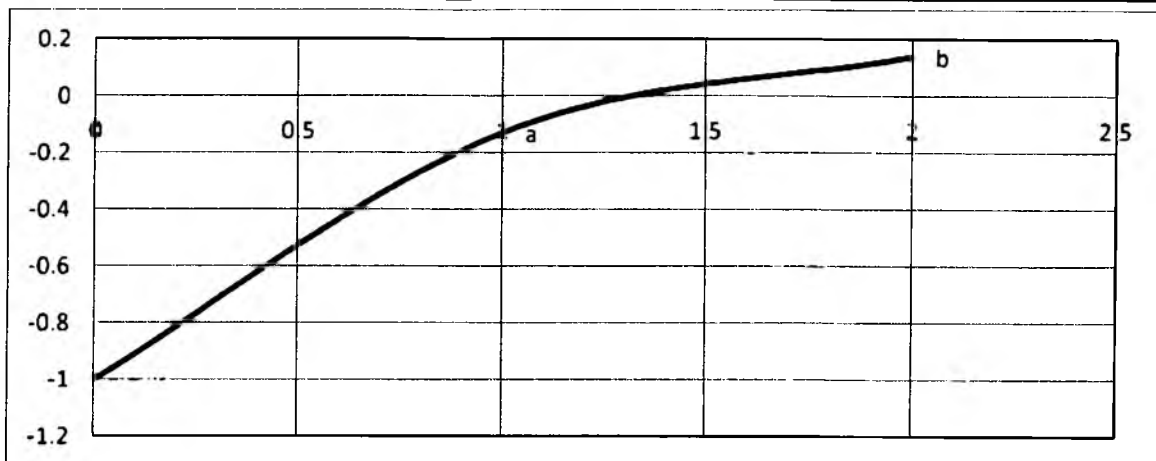
- Step1. Choose left  $x_L$  and right  $x_R$  guesses for the root such that the function changes sign over the interval. This can be checked by ensuring that  $f(x_L)f(x_R) < 0$ .
- Step2. An estimate of the root  $x_{i+1}$  is determined by  $x_R$
- Step3. Make the following evaluations to determine in which subinterval the root lies:
  - (a) If  $f(x_L)f(x_R) < 0$ , the root lies in the left subinterval. Set  $x_{i+1} = x_L$  and return to step 2.
  - (b) If  $f(x_L)f(x_R) > 0$ , the root lies in the right subinterval. Therefore, set  $x_{i+1} = x_R$  and return to step 2.
  - (c) If  $f(x_L)f(x_R) = 0$ , the root equals  $x_{i+1}$ ; terminate the computation.

**Example 1:** Find the positive root of the following equation by (Bisection) method,  $f(x) = \sin(\frac{x}{2\pi}) - \cos^2(x)$  to 3 dec. ?

#### Solution

$$f(x) = \sin(\frac{x}{2\pi}) - \cos^2(x)$$

X	0	1	2
F(x)	-1	-0.13	0.14



The root lies between 1 & 2

$$f(0) = \sin\left(\frac{x}{2\pi}\right) - \cos^2(x) = \sin\left(\frac{0}{2\pi}\right) - \cos^2(0) = -1$$

$$f(1) = \sin\left(\frac{x}{2\pi}\right) - \cos^2(x) = \sin\left(\frac{1}{2\pi}\right) - \cos^2(1) = -0.13$$

$$f(2) = \sin\left(\frac{x}{2\pi}\right) - \cos^2(x) = \sin\left(\frac{2}{2\pi}\right) - \cos^2(2) = 0.14$$

$$\text{Bisection} = \frac{a + b}{2}$$

$$\frac{1 + 2}{2} = 1.5$$

$$f(1.5) = \sin\left(\frac{x}{2\pi}\right) - \cos^2(x) = \sin\left(\frac{1.5}{2\pi}\right) - \cos^2(1.5) = 0.231$$

The root lies between 1 & 1.5

$$\frac{1 + 1.5}{2} = 1.25$$

$$f(1.25) = \sin\left(\frac{x}{2\pi}\right) - \cos^2(x) = \sin\left(\frac{1.25}{2\pi}\right) - \cos^2(1.25) = 0.098$$

The root lies between 1 & 1.25

$$\frac{1 + 1.25}{2} = 1.125$$

$$f(1.125) = \sin\left(\frac{x}{2\pi}\right) - \cos^2(x) = \sin\left(\frac{1.125}{2\pi}\right) - \cos^2(1.125) = -0.007$$

The root lies between 1.125 & 1.25

$$\frac{1.125 + 1.25}{2} = 1.188$$



$$f(1.188) = \sin\left(\frac{x}{2\pi}\right) - \cos^2(x) = \sin\left(\frac{1.188}{2\pi}\right) - \cos^2(1.188) = 0.048$$

The root lies between 1.125 & 1.188

$$\frac{1.125 + 1.188}{2} = 1.157$$

$$f(1.157) = \sin\left(\frac{x}{2\pi}\right) - \cos^2(x) = \sin\left(\frac{1.157}{2\pi}\right) - \cos^2(1.157) = 0.021$$

The root lies between 1.125 & 1.157

$$\frac{1.125 + 1.157}{2} = 1.141$$

$$f(1.141) = \sin\left(\frac{x}{2\pi}\right) - \cos^2(x) = \sin\left(\frac{1.141}{2\pi}\right) - \cos^2(1.141) = 0.007$$

The root lies between 1.125 & 1.141

$$\frac{1.125 + 1.141}{2} = 1.133$$

$$f(1.133) = \sin\left(\frac{x}{2\pi}\right) - \cos^2(x) = \sin\left(\frac{1.133}{2\pi}\right) - \cos^2(1.133) = 0$$

The Approximation root is = 1.133

### Or another solution

The root lies between 1 & 2

$$f(0) = \sin\left(\frac{x}{2\pi}\right) - \cos^2(x) = \sin\left(\frac{0}{2\pi}\right) - \cos^2(0) = -1$$

$$f(1) = \sin\left(\frac{x}{2\pi}\right) - \cos^2(x) = \sin\left(\frac{1}{2\pi}\right) - \cos^2(1) = -0.13$$

$$f(2) = \sin\left(\frac{x}{2\pi}\right) - \cos^2(x) = \sin\left(\frac{2}{2\pi}\right) - \cos^2(2) = 0.14$$

$$\text{Bisection} = \frac{a + b}{2}$$

$$\frac{1 + 2}{2} = 1.5$$

$$f(1.5) = \sin\left(\frac{x}{2\pi}\right) - \cos^2(x) = \sin\left(\frac{1.5}{2\pi}\right) - \cos^2(1.5) = 0.231$$

The root lies between 1 & 1.5



I	$x_a$	$f(x_a)$	$x_b$	$f(x_b)$	Root lies between a,b	$x_{i+1}$
0	1	-0.13	2	0.14	1,2	1.5
1	1	-0.13	1.5	0.231	1.1.5	1.25
2	1	-0.13	1.25	0.098	1,1.25	1.125
3	1.125	-0.0007	1.25	0.098	1.125,1.25	1.188
4	1.125	-0.0007	1.188	0.048	1.125,1.188	1.157
5	1.125	-0.0007	1.157	0.021	1.125,1.157	1.141
6	1.125	-0.0007	1.141	0.007	1.125,1.141	1.133

$$f(1.133) = \sin\left(\frac{x}{2\pi}\right) - \cos^2(x) = \sin\left(\frac{1.133}{2\pi}\right) - \cos^2(1.133) = 0$$

The Approximation root is = 1.133

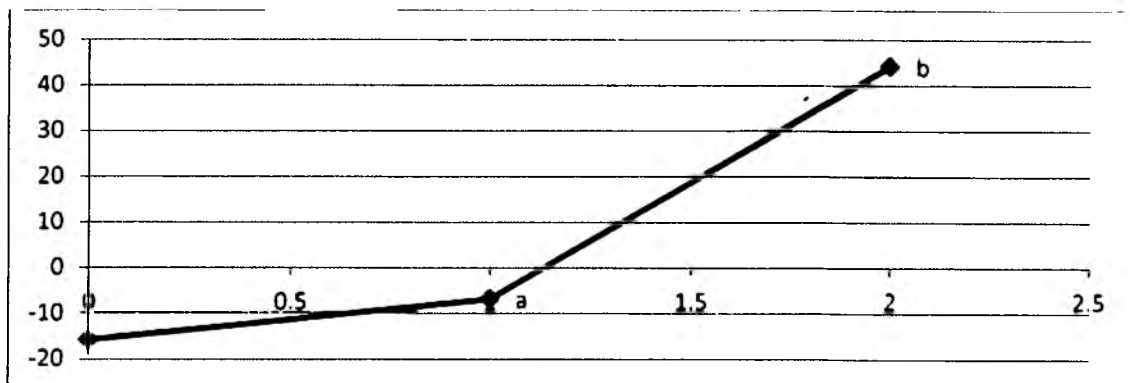
**Example 2:** Find the positive root of the following equation by (Bisection) method,

$$f(x) = 6x^3 + 3x^2 - 16 \text{ to } 2 \text{ dec.}?$$

**Solution**

$$f(x) = 6x^3 + 3x^2 - 16$$

X	0	1	2
F(x)	-16	-7	44



The root lies between 1 & 2



$$\text{Bisection} = \frac{a + b}{2}$$

$$\frac{1 + 2}{2} = 1.5$$

$$f(1.5) = 6 * 1.5^3 + 3 * 1.5^2 - 16 = 11$$

The root lies between 1&1.5

$$\frac{1 + 1.5}{2} = 1.25$$

$$f(1.25) = 6 * 1.25^3 + 3 * 1.25^2 - 16 = 0.40$$

The root lies between 1 & 1.25

$$\frac{1 + 1.25}{2} = 1.12$$

$$f(1.12) = 6 * 1.12^3 + 3 * 1.12^2 - 16 = -3.80$$

The root lies between 1.12 & 1.25

$$\frac{1.12 + 1.25}{2} = 1.18$$

$$f(1.12) = 6 * 1.12^3 + 3 * 1.12^2 - 16 = -1.96$$

The root lies between 1.18 & 1.25

$$\frac{1.18 + 1.25}{2} = 1.21$$

$$f(1.21) = 6 * 1.21^3 + 3 * 1.21^2 - 16 = -0.97$$

The root lies between 1.21 & 1.25

$$\frac{1.21 + 1.25}{2} = 1.23$$

$$f(1.23) = 6 * 1.23^3 + 3 * 1.23^2 - 16 = -0.29$$

The root lies between 1.23 & 1.25

$$\frac{1.23 + 1.25}{2} = 1.24$$

$$f(1.24) = 6 * 1.24^3 + 3 * 1.24^2 - 16 = 0.05$$

The root lies between 1.23 & 1.24



---

$$\frac{1.23 + 1.24}{2} = 1.23$$

$$f(1.23) = 6 * 1.23^3 + 3 * 1.23^2 - 16 = -0.29$$

The root lies between 1.23 & 1.24

$$\frac{1.23 + 1.24}{2} = 1.23$$

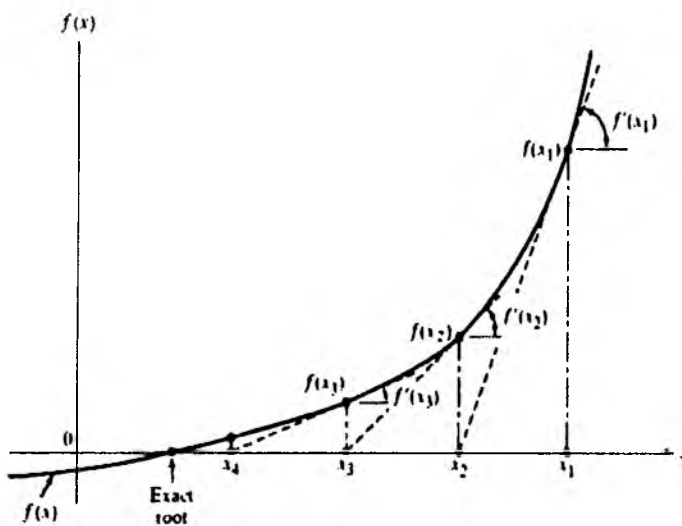
$$f(1.23) = 6 * 1.23^3 + 3 * 1.23^2 - 16 = -0.29$$

The better approximation root is = 1.23



### Newton-Raphson First Method or (Tangent Method):

This method converges faster than the other methods described before. The basis of this method is an initial value close to the root must first be estimated from the sketch of the function in question & evaluate the derivative of the function & that is why this method is known as the method of "Tangent".



The slope of the tangent is  $(\tan \theta)$ , but also the slope can be defined as the first derivative of the function evaluated at  $X_i$ .

$$f'(x_i) = \text{slope} = \tan\theta = \frac{f(x_i)}{x_i - x_{i+1}}$$

$$X_{i+1} = X_i - \frac{f(x_i)}{f'(x_i)}$$

The condition for convergence of (N.R. 1<sup>st</sup> M.) is:





$$\left| \frac{f(x_i)\bar{f}(x_i)}{[\bar{f}(x_i)]^2} \right| < 1$$

**Algorithm of Newton –Raphson First Method:**

Step 1: Sketch the function in equation & determine an initial approximation for the root.

$$x_0 = x_i$$

Step 2: Determine the 1<sup>st</sup> derivative of  $f(x_i)$ , then evaluate at  $x=x_i$ . If  $f'(x_i) = 0$ , then go to step 1.

Step 3: Establish tolerance (T) value for the function  $T \geq |f(x_i)|$  for root  $x_i \cong root$ .

Step 4: Compute new approximation for the root

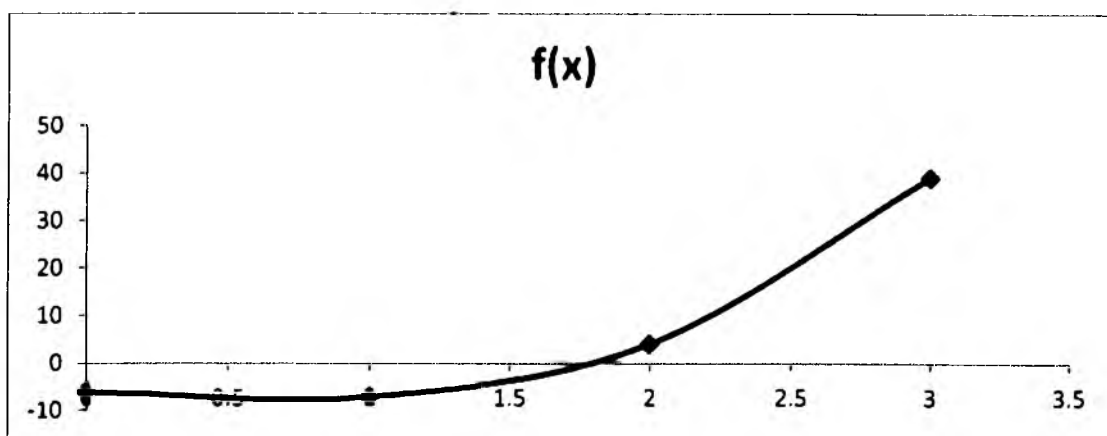
$$X_{i+1} = X_i - \frac{f(x_i)}{f'(x_i)}$$

Step 5: Check tolerance (T). If  $T \geq |f(x_{i+1})|$ , then end, otherwise set  $X_i = X_{i+1}$  and go to step4.

**Example 1:** Find the positive root of  $f(x) = 2x^3 - 3x - 6 = 0$  by Newton – Raphson method correct to five decimal places.

**Solution:**

<b>X</b>	0	1	2	3
<b>f(x)</b>	-6	-7	+4	+39



A root lies between 1 & 2.

Let  $x_0 = x_i = 2$ .

$$f(2) = 2(2)^3 - 3(2) - 6 = +4$$



$$f(x_i) = 6x^2 - 3$$

$$f(2) = 6(2)^2 - 3 = +21$$

$$X_1 = X_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{4}{21} = 1.80952$$

$$\text{Let } x_i = x_{i+1} = x_1 = 1.80952.$$

$$f(1.80952) = 2(1.80952)^3 - 3(1.80952) - 6 = 0.42148$$

$$f(x_i) = 6x^2 - 3$$

$$f'(1.80952) = 6(1.80952)^2 - 3 = 16.64617.$$

$$X_2 = X_1 - \frac{f(x_1)}{f'(x_1)} = 1.80952 - \frac{0.42148}{16.64617} = 1.78420$$

$$\text{Let } x_i = x_{i+1} = x_2 = 1.78420.$$

$$f(1.78420) = 2(1.78420)^3 - 3(1.78420) - 6 = 0.00693$$

$$f(x_i) = 6x^2 - 3$$

$$f'(1.78420) = 6(1.78420)^2 - 3 = 16.10021.$$

$$X_3 = X_2 - \frac{f(x_2)}{f'(x_2)} = 1.78420 - \frac{0.00693}{16.10021} = 1.78376$$

$$\text{Let } x_i = x_{i+1} = x_3 = 1.78376.$$

$$f(1.78376) = 2(1.78376)^3 - 3(1.78376) - 6 = -0.00014$$

$$f(x_i) = 6x^2 - 3$$

$$f'(1.78376) = 6(1.78376)^2 - 3 = 16.09079$$

$$X_4 = X_3 - \frac{f(x_3)}{f'(x_3)} = 1.78376 - \frac{-0.00014}{16.09079} = 1.78376$$

i	$x_i$	$f(x_i)$	$f'(x_i)$	$x_{i+1}$
0	2	4	21	1.80952
1	1.80952	0.42148	16.64617	1.78420
2	1.78420	0.00693	16.10021	1.78376
3	1.78376	-0.00014	16.09079	1.78376

The better approximation root is 1.78376.



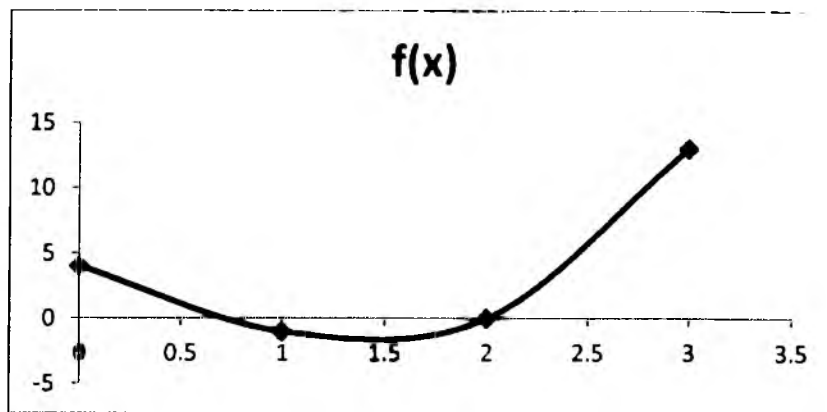
**Example 2:** Using N.R. 1<sup>st</sup> method, find the root between 0 & 1 of  $x^3 = 6x - 4$  correct to three decimal places.

Solution:

$$f(x_i) = x^3 - 6x + 4.$$

$$\bar{f}(x_i) = 3x^2 - 6.$$

<b>X</b>	0	1	2	3
<b>F(x)</b>	+4	-1	0	+13



The first root is between 0 & 1

$$\text{Let } x_i = x_{i+1} = x_0 = 1$$

$$f(x_0) = x^3 - 6x + 4 = -1$$

$$\bar{f}(x_0) = 3x^2 - 6 = -3$$

$$X_1 = X_0 - \frac{f(x_0)}{\bar{f}(x_0)} = 1 - \frac{-1}{-3} = 0.666$$

<b>i</b>	<b><math>x_i</math></b>	<b><math>f(x_i)</math></b>	<b><math>\bar{f}(x_i)</math></b>	<b><math>x_{i+1}</math></b>
0	1	-1	-3	0.666
1	0.666	0.299	-4.669	0.730
2	0.730	0.009	-4.401	0.732
3	0.732	0	-4.392	0.732

The first root=0.732.



The 2'nd root is between 1 & 3, from the sketch of function.

$$\text{Let } x_i = x_{i+1} = x_0 = 3$$

$$f(x_0) = x^3 - 6x + 4 = 13$$

$$\bar{f}(x_0) = 3x^2 - 6 = 21$$

$$X_1 = X_0 - \frac{f(x_0)}{\bar{f}(x_0)} = 3 - \frac{13}{21} = 2.380$$

i	$x_i$	$f(x_i)$	$\bar{f}(x_i)$	$x_{i+1}$
0	3	13	21	2.380
1	2.380	3.201	10.993	2.088
2	2.088	0.575	7.079	2.006
3	2.006	0.036	6.072	2.000
4	2.000	0	6	2.000

The 2'nd exact root = 2.



### Newton – Raphson second method:

The Newton- Raphson 1'st method doesn't always converge to a root. The second Newton-Raphson method converges faster & move stable than all the methods discussed before.

The general equation presenting this method is given by:

$$x_{i+1} = x_i + \left( \frac{\bar{f}(x_i)}{2\bar{f}(x_i)} - \frac{\bar{f}(x_i)}{f(x_i)} \right)^{-1}$$

**Note:** An initial approximation  $x_i$  for the root must first be assumed or graphically determined for this method as in N-R first method.

### Algorithm for Newton – Raphson 2'nd method:-

Step1: Sketch the function  $f(x_i)$  in equation.

Step 2: Determine an initial approximation for the root.

$$x_0 = x_i$$

Step 3: Determine the 1'st & 2'nd derivative of  $f(x_i)$ , then evaluate at  $x=x_i$ . If  $\dot{f}(x_i) = 0$  or  $\ddot{f}(x_i) = 0$ , then go to step 1.



$$\left| \frac{f(x_i)\bar{f}(x_i)}{[\bar{f}(x_i)]^2} \right| < 1$$

Step 4: Compute new approximation for the root

$$x_{i+1} = x_i + \left( \frac{\bar{f}(x_i)}{2\bar{f}(x_i)} - \frac{\bar{f}(x_i)}{f(x_i)} \right)^{-1}$$

Step 5: Check tolerance (T). If  $T \geq |f(x_{i+1})|$ , then end, otherwise set  $X_i = X_{i+1}$  and go to step 4.

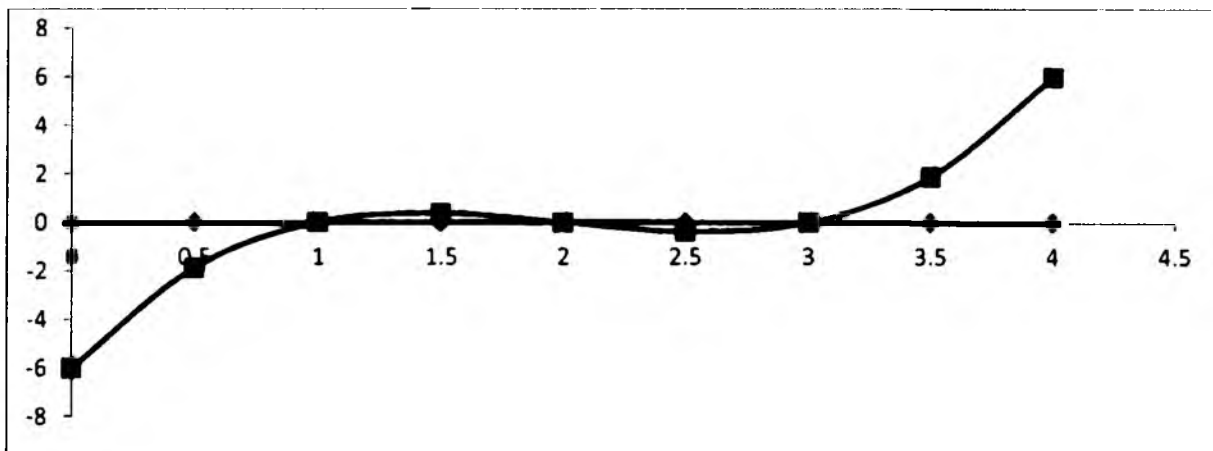
**Example 1:** Determine all of the roots to the following nonlinear algebraic equation using (2'nd N-R method).

$$f(x) = x^3 - 6x^2 + 11x - 6$$

**Solution:**

Step1:

X	0	0.5	1	1.5	2	2.5	3	3.5	4
F(x)	-6	-1.875	0	0.375	0	-0.375	0	1.875	6



The first root between 0.5 & 1.5

Step 2: Take initial value  $x_i = 0.5$

$$x_0 = x_i = 0.5$$

Step 3:

$$\bar{f}(x_i) = 3x_i^2 - 12x_i + 11$$



$$\bar{f}(x_i) = 6x_i - 12$$

$$\text{Step 4: } x_{i+1} = x_i + \left( \frac{\bar{f}(x_i)}{2\bar{f}(x_i)} - \frac{\bar{f}(x_i)}{f(x_i)} \right)^{-1}$$

i	$x_i$	$f(x_i)$	$\bar{f}(x_i)$	$\bar{\bar{f}}(x_i)$	$x_{i+1}$
0	0.5	-1.875	5.75	-9	0.93781
1	0.93781	-0.136206	2.38469	-6.37309	0.99964
2	0.99964	-0.00072	2.00216	-6.00216	0.99999
3	0.99999	-0.00002	2.00006	-6.00006	0.99999

The exact root = 0.99999

The second root between 1.5 & 2.5

Step 2: Take initial value  $x_i = 1.5$

$$x_0 = x_i = 1.5$$

Step 3:

$$\bar{f}(x_i) = 3x_i^2 - 12x_i + 11$$

$$\bar{\bar{f}}(x_i) = 6x_i - 12$$

$$\text{Step 4: } x_{i+1} = x_i + \left( \frac{\bar{\bar{f}}(x_i)}{2\bar{\bar{f}}(x_i)} - \frac{\bar{\bar{f}}(x_i)}{\bar{f}(x_i)} \right)^{-1}$$

I	$x_i$	$f(x_i)$	$\bar{f}(x_i)$	$\bar{\bar{f}}(x_i)$	$x_{i+1}$
0	1.5	-10.625	-0.25	-3	1.66732
1	1.66732	0.29586	-0.66797	-1.99608	1.93385
2	1.93385	0.06586	-0.98687	-0.39687	1.99970
3	1.99970	0.00029	-0.99999	-0.00178	1.99999
4	1.99999	0.00001	-0.99999	-0.00005	2
5	2	0	-1	0	2

The exact root = 2



The third root between 2.5 & 3.5

Step 2: Take initial value  $x_i = 2.5$

$$x_0 = x_i = 2.5$$

Step 3:

$$\bar{f}(x_i) = 3x_i^2 - 12x_i + 11$$

$$\bar{\bar{f}}(x_i) = 6x_i - 12$$

$$\text{Step 4: } x_{i+1} = x_i + \left( \frac{\bar{f}(x_i)}{2\bar{f}(x_i)} - \frac{\bar{\bar{f}}(x_i)}{\bar{f}(x_i)} \right)^{-1}$$

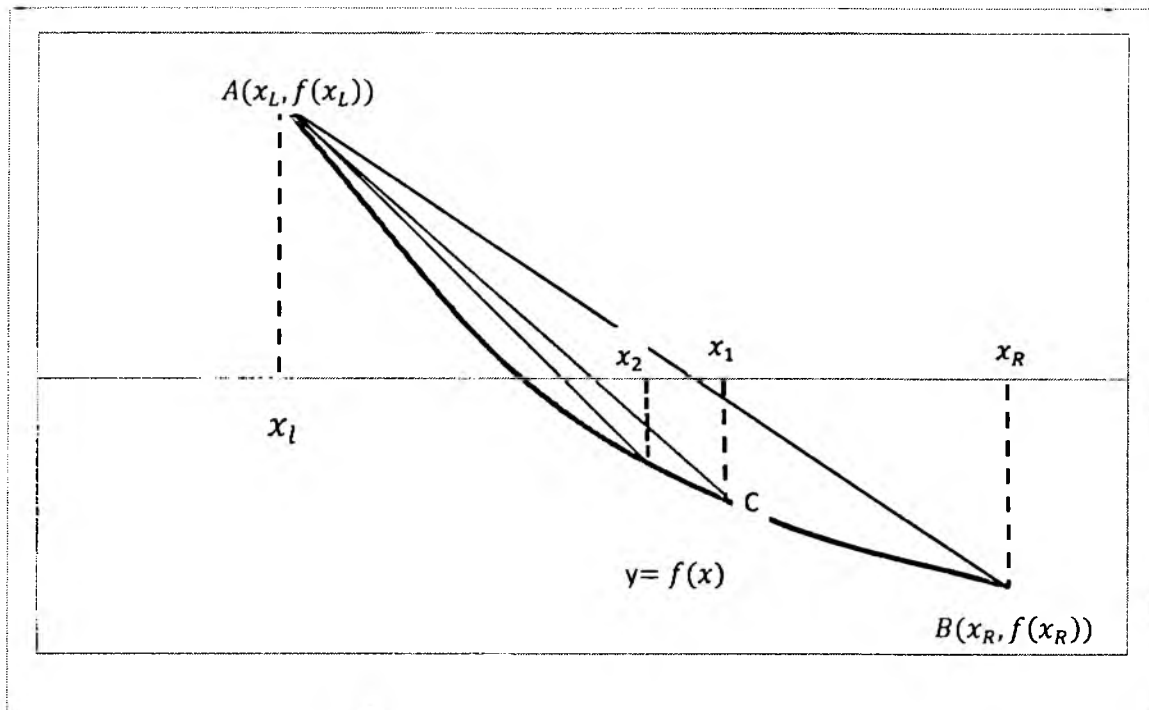
I	$x_i$	$f(x_i)$	$\bar{f}(x_i)$	$\bar{\bar{f}}(x_i)$	$x_{i+1}$
0	2.5	-0.375	-0.25	3	2.35
1	2.35	-0.30712	-0.6325	2.1	2.08114
2	2.08114	-0.08060	-0.98024	0.48689	2.00056
3	2.00056	-0.00055	-0.99999	0.00336	2.00000
4	2.00000	0	-1	0.00005	2.00000

The exact root = 2.00000



### Geometrical interpretation (False Position method or False method):

If A  $(x_L, f(x_L))$  and B  $(x_R, f(x_R))$  are two points on  $y = f(x)$  such that  $f(x_L)$  and  $f(x_R)$  are opposite in sign, then the chord AB meets x – axis at  $x = x_1$ . This  $x_1$  is the approximate root of  $f(x) = 0$ . Now  $c(x_1, f(x_1))$  is on the curve.



If  $f(x_L) \cdot f(x_1) < 0$ , join the chord AC which cuts x – axis at  $x = x_2$ . Then  $x_2$  is the second approximate root of  $f(x) = 0$ . This process is continued until we get the root to the desired accuracy.

The order of convergence of False method is 1.618 .(This may be assumed).

**Example 1:** Use the (False -Position) method to find the root of  $(x - 2\sin(x) = 0)$ , that is located in the interval  $(1, 2)$ ?

$f(x) = x - 2\sin(x)$ , the root lies between 1 & 2

$$x_L = 1, f(x_L) = x_L - 2\sin(x_L) = 1 - 2\sin(1) = -0.68294$$

$$x_R = 2, f(x_R) = x_R - 2\sin(x_R) = 2 - 2\sin(2) = 0.18140$$

$$x_m = x_L - f(x_L) \frac{x_R - x_L}{f(x_R) - f(x_L)} = 1 - (-0.68294) \frac{2 - 1}{0.18140 - (-0.68294)} = 1.79012$$



I	$x_L$	$f(x_L)$	$x_R$	$f(x_R)$	$x_m$
0	1	-0.68294	2	0.18140	1.79012
1	1.79012	-0.16196	2	0.18140	1.88911
2	1.88911	-0.01041	2	0.18140	1.89512
3	1.89512	-0.00006	2	0.18140	1.89547
4	1.89547	-0.00003	2	0.18140	1.89548
5	1.89548	-0.00002	2	0.18140	1.89549
6	1.89549	0	2	0.18140	1.89549

The better approximation root is = 1.89549

**Example 2:** The following function is known to have a root in the interval (1,2) ,  $f(x) = e^x - 2x - 2$  using (False -Position) method to estimate the actual root, to 3 decimal?

$$f(x) = e^x - 2x - 2$$

$$x_L = 1, x_R = 2$$

$$x_m = x_L - f(x_L) \frac{x_R - x_L}{f(x_R) - f(x_L)}$$

$$f(x_L) = e^{x_L} - 2x_L - 2 = e^1 - 2 - 2 = -1.2817$$

$$f(x_R) = e^{x_R} - 2x_R - 2 = e^2 - 2 * 2 - 2 = 1.3890$$

$$x_m = 1 - (-1.2817) \frac{2 - 1}{(1.3890) - (-1.2817)} = 1.4799$$

I	$x_L$	$f(x_L)$	$x_R$	$f(x_R)$	$x_m$
0	1	-1.2817	2	1.3890	1.4799
1	1.4799	-0.5672	2	1.3890	1.6307
2	1.6307	-0.1539	2	1.3890	1.6675
3	1.6675	-0.0360	2	1.3890	1.6759
4	1.6759	-0.0081	2	1.3890	1.6777
5	1.6777	-0.0021	2	1.3890	1.6781
6	1.6781	-0.0008	2	1.3890	1.6782
7	1.6782	-0.0004	2	1.3890	1.6782

( $x_6 = x_7$ ), The better approximation root is = 1.6782



## The Secant Method:

A potential problem in implementing the Newton – Raphson method is the evolution of the derivative. Although this is not inconvenient for polynomials and many other functions, there are certain functions whose derivatives may be extremely difficult or inconvenient to evaluate. For these cases, the derivative can be approximated by a backward finite divided difference, as in Fig. (1)

$$\dot{f}(x_i) \cong \frac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i} \quad \dots \quad \text{Equ. (1)}$$

Figure (1)

Graphical depiction of the Secant method, this technique is similar the Newton – Raphson technique in the sense that an estimate of the root is predicated by extrapolating a tangent of the function to the x axis. However, the secant method uses a difference rather than a derivative to estimate the slope.

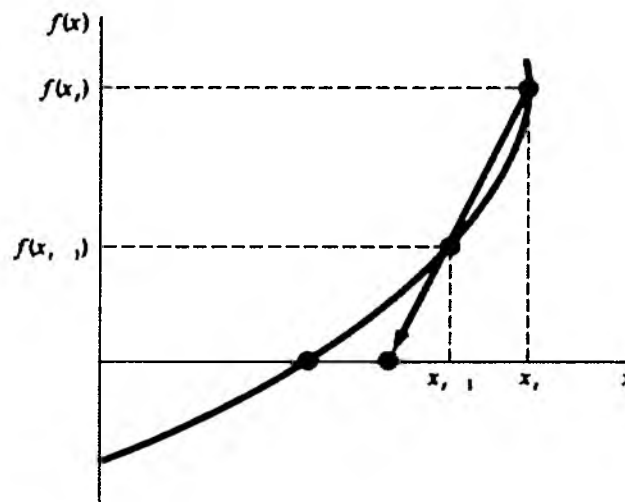


Figure (1)

The approximation can be substituted into equation (1) to yield the following iterative equation:

$$(x_{i+1}) \cong \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)} \quad \dots \quad \text{Equ. (2)}$$

Equation (2) is the formula for the secant method. Notes that the approach requires two initial estimates of x, however, because f(x) is not required to change signs between the estimates, it is not classified as a bracketing method.



**Example 1:**

Use the secant method to estimate the root of  $f(x) = e^{-x} - x$ . Start with initial estimates of  $x_{i-1} = 0$  &  $x_i = 1$ ?

Solution:

$$(x_{i+1}) \cong \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$$

For i=0

$$x_{-1} = 0, f(x_{-1}) = e^{-x_{-1}} - x_{-1} = e^0 - 0 = 1$$

$$x_0 = 1, f(x_0) = e^{-x_0} - x_0 = e^{-1} - 1 = -0.63212$$

$$x_1 = x_0 - \frac{f(x_0)(x_{-1} - x_0)}{f(x_{-1}) - f(x_0)} = 1 - \frac{-0.63212(0 - 1)}{1 - (-0.63212)} = 0.61270$$

For i=1

$$x_0 = 1, f(x_0) = e^{-x_0} - x_0 = e^{-1} - 1 = -0.63212$$

$$x_1 = 0.61270, f(x_1) = e^{-x_1} - x_1 = e^{-0.61270} - 0.61270 = -0.07081$$

$$x_2 = x_1 - \frac{f(x_1)(x_0 - x_1)}{f(x_0) - f(x_1)} = 0.61270 - \frac{-0.07081(1 - 0.61270)}{-0.63212 - (-0.07081)} = 0.56384$$

For i=2

$$x_1 = 0.61270, f(x_1) = e^{-x_1} - x_1 = e^{-0.61270} - 0.61270 = -0.07081$$

$$x_2 = 0.56384, f(x_2) = e^{-x_2} - x_2 = e^{-0.56384} - 0.56384 = 0.00518$$

$$x_3 = x_2 - \frac{f(x_2)(x_1 - x_2)}{f(x_1) - f(x_2)} = 0.56384 - \frac{0.00518(0.61270 - 0.56384)}{-0.07081 - 0.00518} = 0.56717$$

For i=3

$$x_2 = 0.56384, f(x_2) = e^{-x_2} - x_2 = e^{-0.56384} - 0.56384 = 0.00518$$

$$x_3 = 0.56717, f(x_3) = e^{-x_3} - x_3 = e^{-0.56717} - 0.56717 = -0.00004$$

$$x_4 = x_3 - \frac{f(x_3)(x_2 - x_3)}{f(x_2) - f(x_3)} = 0.56717 - \frac{-0.00004(0.56384 - 0.56717)}{0.00518 - (-0.00004)} = 0.56714$$



I	$x_{i-1}$	$f(x_{i-1})$	$x_i$	$f(x_i)$	$x_{i+1}$
0	0	1	1	-0.63212	0.61270
1	1	-0.63212	0.61270	-0.07081	0.56384
2	0.61270	-0.07081	0.56384	0.00518	0.56717
3	0.56384	0.00518	0.56717	-0.00004	0.56714
4	0.56717	-0.00004	0.56714	0.00001	0.56715
5	0.56714	0.00001	0.56715	-0.00001	<del>0.56715</del>

The root = 0.56715

***The difference Between the Secant and False –Position Methods:***

Note the similarity between the secant method and false –position method. For example, equation of secant and equation of false position are identical on a term by term basis. Both use two initial estimates to compute an approximation of the slope of the function that is used to project to the s axis for a new estimate of the root. However, a critical difference between the methods is how one of the initial values is replaced by the new estimate. Recall that in the false position method the latest estimate of the root replaces whichever of the original values yielded a function value with the same sign as  $f(x_i)$ . Consequently, the two estimates always bracket the root. Therefore, for all practical purposes, the method always converges because the root is kept within the bracket. In contrast, the secant method replaces the values in strict sequence. With the new value  $x_{i+1}$  replacing  $x_i$  and  $x_{i-1}$ . As a result, the two values can sometimes lie on the same side of the root. For certain cases, this can lead to divergence.

Figure(2): Comparison of the false-position and the secant methods. The first iterations (a) and (b) for both techniques are identical. However, for the second iterations (c) and (d), the points used differ. As a consequence, the secant method can diverge, as indicated in (d).



I	$x_{i-1}$	$f(x_{i-1})$	$x_i$	$f(x_i)$	$x_{i+1}$
0	0	1	1	-0.63212	0.61270
1	1	-0.63212	0.61270	-0.07081	0.56384
2	0.61270	-0.07081	0.56384	0.00518	0.56717
3	0.56384	0.00518	0.56717	-0.00004	0.56714
4	0.56717	-0.00004	0.56714	0.00001	0.56715
5	0.56714	0.00001	0.56715	-0.00001	0.56715

The root = 0.56715

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For the *secant method* and the sequential criterion for replacing estimates results in As in Fig. 2d, the approach is divergent:

Iteration	$x_{i-1}$	$f(x_{i-1})$	$x_i$	$f(x_i)$	$x_{i+1}$
0	0.5	-0.69315	5.0	1.60944	1.8546
1	5	1.60944	1.8546	0.61769	-0.10438

Although the secant method may be divergent, when it converges it usually does so at a quicker rate than the false-position method. For instance, Fig. 6.9 demonstrates the superiority of the secant method in this regard. The inferiority of the false-position method is due to one end staying fixed to maintain the bracketing of the root. This property, which is an advantage in that it prevents divergence, is a shortcoming with regard to the rate of convergence; it makes the finite-difference estimate a less-accurate approximation of the derivative.

**Modified Secant Method:**

Rather than using two arbitrary values to estimate the derivative, an alternative approach involves a fractional perturbation of the independent variable to estimate  $f'(x_i)$ ,

$$f'(x_i) \cong \frac{f(x_i + \delta x_i) - f(x_i)}{\delta x_i}$$

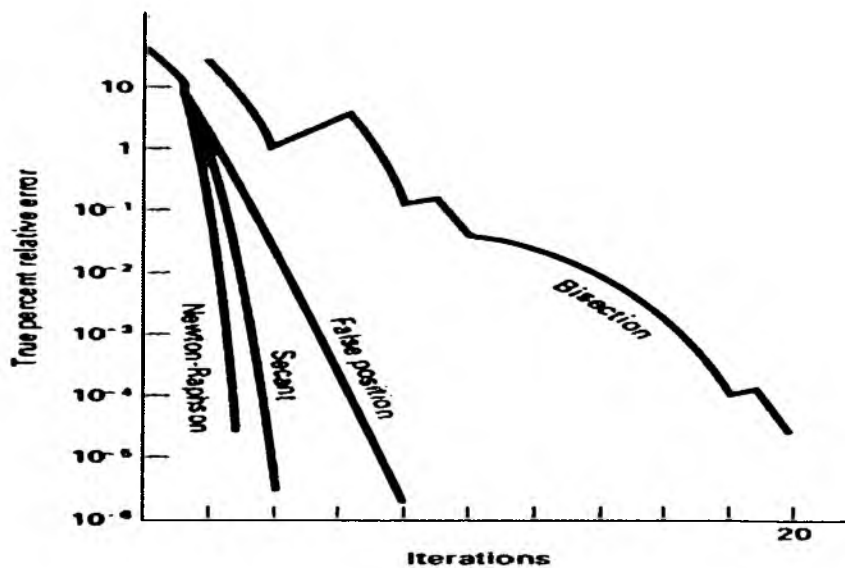


Figure (3)



**Modified Secant Method:**

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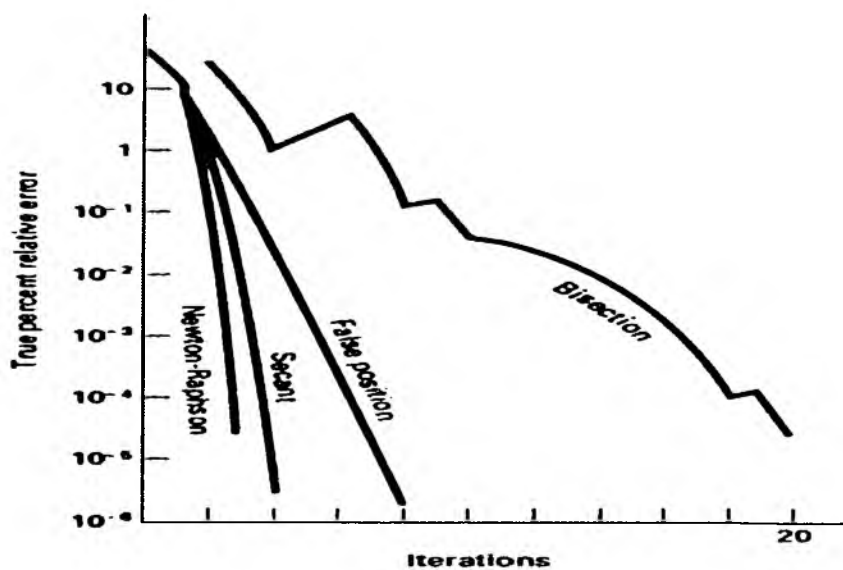


Figure (3)





Where  $\delta$  = a small perturbation fraction. This approximation can be substituted into equation above to yield the following iterative equation:

$$x_{i+1} = x_i - \frac{\delta_{x_i} * f(x_i)}{f(\delta_{x_i} + x_i) - f(x_i)}$$

**Example 3:** Use the modified secant method to estimate the root of  $f(x) = e^{-x} - x$  Use a value of 0.01 for  $\delta$  and start with  $x_0 = 1$ ?

Solution

**For i=0:**

$$x_0 = 1, f(x_0) = e^{-x_0} - x_0 = e^{-1} - 1 = -0.63212$$

$$\delta_{x_0} = 0.01$$

$$\delta_{x_0} = \delta_{x_0} * x_0 = 0.01 * 1 = 0.01$$

$$\delta_{x_0} + x_0 = 0.01 + 1 = 1.01, f(\delta_{x_0} + x_0) = f(1.01) = e^{-1.01} - 1.01 = -0.64578$$

$$x_1 = x_0 - \frac{\delta_{x_0} * f(x_0)}{f(\delta_{x_0} + x_0) - f(x_0)} = 1 - \frac{0.01 * -0.63212}{-0.64578 - (-0.63212)} = 0.53725$$

**For i=1:**

$$x_1 = 0.53725, f(x_1) = e^{-x_1} - x_1 = e^{-0.53725} - 0.53725 = 0.04710$$

$$\delta_{x_1} = 0.01$$

$$\delta_{x_1} = \delta_{x_1} * x_1 = 0.01 * 0.53725 = 0.00537$$

$$\delta_{x_1} + x_1 = 0.00537 + 0.53725 = 0.54262$$

$$f(\delta_{x_1} + x_1) = f(0.54262) = e^{-0.54262} - 0.54262 = 0.03860$$

$$x_2 = x_1 - \frac{\delta_{x_1} * f(x_1)}{f(\delta_{x_1} + x_1) - f(x_1)} = 0.53725 - \frac{0.00537 * 0.04710}{0.03860 - 0.04710} = 0.56701$$

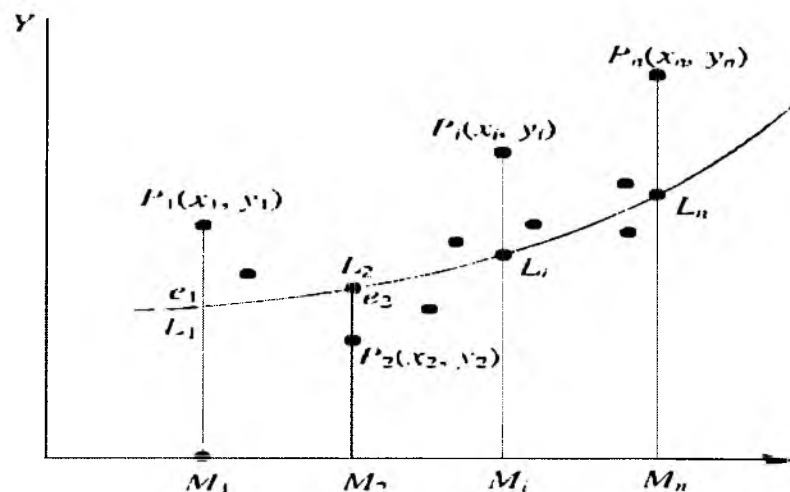
I	$x_i$	$f(x_i)$	$\delta_{x_i} = \delta_{x_0} * x_i$	$\delta_{x_i} + x_i$	$f(\delta_{x_i} + x_i)$	$x_{i+1}$
0	1	-0.63212	0.01	1.01	-0.64578	0.53725
1	0.53725	0.04710	0.00537	0.54262	0.03860	0.56701
2	0.56701	0.00021	0.00567	0.57268	-0.00867	0.56714
3	0.56714	0.00001	0.00567	0.57281	-0.00887	0.56715
4	0.56715	-0.00001	0.00567	0.57282	-0.00889	0.56714
5	0.56714	0.00001	0.00567	0.57281	-0.00887	0.56715

The choice of a proper value for  $\delta$  is not automatic. If  $\delta$  is too small, the method can be swamped by round-off error caused by subtractive cancellation in the denominator of Eq. (6.8). If it is too big, the technique can become inefficient and even divergent. However, if chosen correctly, it provides a nice alternative for cases where evaluating the derivative is difficult and developing two initial guesses is inconvenient.



## PRINCIPLE OF LEAST SQUARES

The graphical method has the drawback in that the straight line drawn may not be unique but principle of least squares provides a unique set of values to the constants and hence suggests a curve of best fit to the given data. The method of least square is probably the most systematic procedure to fit a unique curve through the given data points.



We will consider some of the best fitting curves of the type:

1. A straight line.
2. A second degree curve.
3. The exponential curve  $y = ae^{bx}$ .
4. The curve  $y = ax^n$ .

### 1. Fitting a straight line by the method of least squares:

Let  $(x_i, y_i), i = 0, 1, 2, \dots, n$  be the  $n$  sets of observations and let the related relation by  $y = ax + b$ . Now we have to select  $a$  and  $b$  so that the straight line is the best fit to the data.

As explained earlier, the residual at  $x = x_i$  is

$$d_i = y_i - f(x_i) = y_i - (ax_i + b), i = 1, 2, \dots, n$$

$$E = \sum_{i=1}^n d_i^2 = \sum_{i=1}^n [y_i - (ax_i + b)]^2$$

By the principle of least squares,  $E$  is minimum.



$$\frac{\partial E}{\partial a} = 0 \text{ and } \frac{\partial E}{\partial b} = 0$$

i.e.,  $2 \sum [y_i - (ax_i + b)] (-x_i) = 0$  &  $2 \sum [y_i - (ax_i + b)] (-1) = 0$

i.e.,  $\sum_{i=1}^n (x_i y_i - ax_i^2 - bx_i) = 0$  &  $\sum_{i=1}^n (y_i - ax_i - b) = 0$

i.e.,  $a \sum_{i=1}^n x_i^2 + b \sum_{i=1}^n x_i = \sum_{i=1}^n x_i y_i$  .... (eq.1)

And  $a \sum_{i=1}^n x_i + nb = \sum_{i=1}^n y_i$  ..... (eq.2)

Since,  $x_i, y_i$  are known, equations (1) & (2) give two equations in a & b. Solve for a & b from (1) & (2) & obtain the best fit  $y = ax + b$ .

**Note:**

- Equations (1) & (2) are called normal equations.
- Dropping suffix i from (1) & (2), the normal equations are

$$a \sum x + nb = \sum y \text{ \& } a \sum x^2 + b \sum x = \sum xy$$

Which are get taking  $\sum$  on both sides of  $y = ax + b$  & also taking  $\sum$  on both sides after multiplying by x both sides of  $y = ax + b$ .

- Transformation like  $X = \frac{x-a}{h}, Y = \frac{y-b}{h}$  reduce the linear equation  $y = ax + b$  to the form  $Y = AX + B$ . Hence, a linear fit is another linear fit in both systems of coordinates.

**Example 1:**

By the method of least squares find the straight line to the data given below:

x	5	10	15	20	25
y	16	19	23	26	30

**Solution:**

Let the straight line be  $y = ax + b$

The normal equations are  $a \sum x + 5b = \sum y$  ..... (eq.1)

$$a \sum x^2 + b \sum x = \sum xy$$
 ..... (eq.2)

To calculate  $\sum x, \sum x^2, \sum y, \sum xy$  we form below the table.



	x	y	x <sup>2</sup>	xy
	5	16	25	80
	10	19	100	190
	15	23	225	345
	20	26	400	520
	25	30	625	750
<b>Total</b>	<b>75</b>	<b>114</b>	<b>1375</b>	<b>1885</b>

The normal equations are  $75a+5b=114$  ..... (eq.1)

$$1375a+75b=1885 \quad \text{..... (eq.2)}$$

Eliminate b, multiply (1) by 15

$$1125a+75b=1710 \quad \text{..... (eq.3)}$$

(eq.2) – (eq.3) gives,  $250a=175$  or  $a=0.7$ , hence  $b=12.3$

Hence, the best fitting line is  $y=0.7x+12.3$

$$\text{Let } X = \frac{x-x_{mid}}{h} = \frac{x-15}{5}, Y = \frac{y-y_{mid}}{h} = \frac{y-23}{5}$$

Let the line in the new variable by  $Y=AX+B$

	x	y	X	X <sup>2</sup>	Y	XY
	5	16	-2	4	-1.4	2.8
	10	19	-1	1	-0.8	0.8
	15	23	0	0	0	0
	20	26	1	1	0.6	0.6
	25	30	2	4	1.4	2.8
<b>Total</b>			<b>0</b>	<b>10</b>	<b>-0.2</b>	<b>7</b>

The normal equations are  $A \sum X + 5B = \sum Y$  ..... (eq.4)

$$A \sum X^2 + B \sum X = \sum XY \quad \text{..... (eq.5)}$$

Therefore,  $-5B = -0.2 \rightarrow B = -0.04$

$$10A = 7 \rightarrow A = 0.7$$

The equations  $Y=0.7X - 0.04$

$$\text{i.e. } \frac{y-23}{5} = 0.7 \left( \frac{x-15}{5} \right) - 0.04 \rightarrow y - 23 = 0.7x - 10.5 - 0.2$$

$$\text{i.e. } y=0.7x+33.3$$

Which is the same equation as seen before.



**Example 2:**

Fit a straight line to the data given below. Also estimate the value of y at x=2.5

x	0	1	2	3	4
y	1	1.8	3.3	4.5	6.3

**Solution:**

Let the best fit be  $y = ax + b$  ..... (eq.1)

The normal equations are  $a \sum x + 5b = \sum y$  ..... (eq.2)

$a \sum x^2 + b \sum x = \sum xy$  ..... (eq.3)

We prepare the table for easy use.

	x	y	$x^2$	xy
	0	1	0	0
	1	1.8	1	1.8
	2	3.3	4	6.6
	3	4.5	9	13.5
	4	6.3	16	25.2
<b>Total</b>	<b>10</b>	<b>16.9</b>	<b>30</b>	<b>47.1</b>

Substituting in (eq.2) and (eq.3), we get,

$$10a + 5b = 16.9$$

$$30a + 10b = 47.1$$

Solving eq.(2)-eq.(1), we get,  $a = 1.33$ ,  $b = 0.72$

Hence, the equation is  $y = 1.33x + 0.72$

$$y \text{ (at } x=2.5) = 1.33(2.5) + 0.72 = 4.045$$

**Example 3:**

By proper transformation, convert the relation  $y = a + bxy$  to a linear form & find the equation to fit the data.

x	-4	1	2	3
y	4	6	10	8

**Solution:**

Let  $X = xy$ ,



The equation becomes  $y=a+bX$ .

The normal equations are  $a \sum X + b \sum X^2 = \sum Xy$  ..... (eq.1)

$4a + b \sum X = \sum y$  ..... (eq.2)

And

	<b>x</b>	<b>y</b>	<b>X</b>	<b>X<sup>2</sup></b>	<b>Xy</b>
	-4	4	-16	256	-64
	1	6	6	36	36
	2	10	20	400	200
	3	8	24	576	192
<b>Total</b>		<b>28</b>	<b>34</b>	<b>1268</b>	<b>364</b>

Substituting in (eq.2) and (eq.3), we get,

$$34a+1268b=364$$

$$4a+34b=28$$

Solving, we get,  $a=5.90605$ ,  $b= 0.12870$

Therefore, the equation is  $y = 5.90605 + 0.12870X$

i.e.  $y= 5.90605 + 0.12870Xy$

Using this equation we get  $y(1-0.12870x)=5.90605$

i.e.,  $y = \frac{5.90605}{1-0.12870x}$

We tabulate the value to verify:

<b>x</b>	-4	1	2	3
<b>y</b>	3.89890	6.77843	7.95320	9.61054

**Note:** if we take  $u = \frac{1}{xy}$ ,  $v = \frac{1}{x}$  we get

$v=ax + b$ . Taking this as linear, we get

$$a=10.5, b=-0.13$$

That is  $y=10.5-0.13xy$

i.e.  $y = \frac{10.5}{1+0.13x}$



---

x	-4	1	2	3
y	21.875	92.9203	8.33333	7.55396

The values of y are far away from the given values. Perhaps, the selection of the form is not correct.



## 2. Fitting a parabola or fitting a second degree curve by the method of least squares:

Let  $(x_i, y_i)$ ,  $i=0, 1, 2, \dots, n$  be the  $n$  sets of observations of two related variables  $x$  &  $y$ . Let  $y = ax^2 + bx + c$  be the equation which fits them best.

Now, we have to find the constants  $a, b, c$ .

For any  $x = x_i$ , the expected value of  $y$  is  $ax_i^2 + bx_i + c$  & the corresponding observe value is  $y_i$ .

The residual  $d_i = y_i - (ax_i^2 + bx_i + c)$

Let  $E$  denote the sum of the squares of the residuals.

That is,  $E = \sum_{i=1}^n [y_i - (ax_i^2 + bx_i + c)]^2$

By the principle of least squares,  $E$  is minimum for best values  $a, b, c$ .

$$\frac{\partial E}{\partial a} = 0, \frac{\partial E}{\partial b} = 0, \frac{\partial E}{\partial c} = 0$$

Differentiating  $E$ , partially w. r. t.  $a, b, c$  & equating to zero, we get,

$$\sum_{i=1}^n 2[y_i - (ax_i^2 + bx_i + c)](-x_i)^2 = 0$$

$$\sum_{i=1}^n 2[y_i - (ax_i^2 + bx_i + c)](-x_i) = 0$$

$$\sum_{i=1}^n 2[y_i - (ax_i + bx_i + c)](-1) = 0$$

Simplifying, we get

$$a \sum x_i^4 + b \sum x_i^3 + c \sum x_i^2 = \sum x_i^2 y_i$$

$$a \sum x_i^3 + b \sum x_i^2 + c \sum x_i = \sum x_i y_i$$

$$a \sum x_i^2 + b \sum x_i + nc = \sum y_i$$





The three equations (1), (2), (3) give the values of a, b, c.

Substitution these values of a, b, c in  $y = ax^2 + bx + c$ , we get the result.

**Note:** To obtain the normal equations, we remember the following:

- In  $y = ax^2 + bx + c$ , take  $\sum$  on both sides.
- Multiply by x both sides & then take  $\sum$  on both sides.
- Multiply both sides by  $x^2$  & then take  $\sum$  on both sides.

**Example 4:**

Fit a parabola, by the method of least squares, to the following data, also estimate y at x=6.

x	1	2	3	4	5
y	5	12	26	60	97

**Solution:**

Then, the normal equations are  $a \sum x^2 + b \sum x + nc = \sum y$  .. (eq.1)

$$a \sum x^3 + b \sum x^2 + c \sum x = \sum xy \quad \dots(\text{eq.2})$$

$$a \sum x^4 + b \sum x^3 + c \sum x^2 = \sum x^2y \quad \dots(\text{eq.3})$$

We form the table.

	x	y	$x^2$	$x^3$	$x^4$	xy	$x^2y$
	1	5	1	1	1	5	5
	2	12	4	8	16	24	48
	3	26	9	27	81	78	234
	4	60	16	64	256	240	960
	5	97	25	125	625	485	2425
<b>Total</b>	<b>15</b>	<b>200</b>	<b>55</b>	<b>225</b>	<b>979</b>	<b>832</b>	<b>3672</b>

Hence, the equations (1), (2) & (3) becomes,

$$55a+15b+5c=200 \quad \dots (\text{eq.4})$$

$$225a+55b+15c=832 \quad \dots (\text{eq.5})$$

$$979a+225b+55c=3672 \quad \dots (\text{eq.6})$$

Solving we get, a=5.7143, b=-11.0858 & c=10.4001



Hence, the parabola is  $y=5.7143x^2-11.0858x+10.4001$

$$y(x=6)=149.6001$$

**Example 5:**

Fit a second degree parabola to the data:

x	1929	1930	1931	1932	1933	1934	1935
y	353	357	358	358	361	362	362

**Solution:**

Let  $X = \frac{x-x_{mid}}{h} = \frac{x-1932}{1} = x - 1932, Y = \frac{y-358}{1} = y - 358$

Let  $Y= ax^2+ bX+ C$  be the best fit.

The normal's equations are

$$a \sum X^2 + b \sum X + nc = \sum y \quad \dots (eq.1)$$

$$a \sum X^3 + b \sum X^2 + c \sum X = \sum Xy \quad \dots (eq.2)$$

$$a \sum X^4 + b \sum X^3 + c \sum X^2 = \sum X^2y \quad \dots (eq.3)$$

Calculation Table:

	x	y	X	Y	X <sup>2</sup>	X <sup>3</sup>	X <sup>4</sup>	XY	X <sup>2</sup> Y
	1929	353	-3	-5	9	-27	81	15	-45
	1930	357	-2	-1	4	-8	16	2	-4
	1931	358	-1	0	1	-1	1	0	0
	1932	358	0	0	0	0	0	0	0
	1933	361	1	3	1	1	1	3	3
	1934	362	2	4	4	8	16	8	16
	1935	362	3	4	9	27	81	12	36
<b>Total</b>			<b>0</b>	<b>5</b>	<b>28</b>	<b>0</b>	<b>196</b>	<b>40</b>	<b>6</b>

Hence, the normal equations becomes,

$$28a+7c=5 \quad \dots (eq.4)$$

$$28b=40 \quad \dots (eq.5)$$

$$196a+28c=5 \quad \dots (eq.6)$$

$$b=1.42857, a=-0.17857, c=1.42857$$



The equation is  $Y = -0.17857X^2 + 1.42857X + 1.42857$

i.e.  $y - 358 = -0.17857(x - 1932)^2 + 1.42857(x - 1932) + 1.42857$

i.e.  $y = -0.17857x^2 + 691.42305x - 669293.2364$

**Example 6:**

Fit a second degree parabola to the following data, taking y as depended variable:

x	1	2	3	4	5	6	7	8	9
y	2	6	7	8	7	11	11	10	9

**Solution:**

Let  $X = x - \bar{x} = x - 5$ ,  $Y = y - 7$

Let  $Y = aX^2 + bX + c$  be the best fit.

	x	y	X	Y	X <sup>2</sup>	X <sup>3</sup>	X <sup>4</sup>	XY	X <sup>2</sup> Y
	1	2	-4	-5	16	-64	256	20	-80
	2	6	-3	-1	9	-27	81	3	-9
	3	7	-2	0	4	-8	16	0	0
	4	8	-1	1	1	-1	1	-1	1
	5	7	0	0	0	0	0	0	0
	6	11	1	4	1	1	1	4	4
	7	11	2	4	4	8	16	8	16
	8	10	3	3	9	27	81	9	27
	9	9	4	2	16	64	256	8	32
<b>Total</b>			<b>0</b>	<b>8</b>	<b>60</b>	<b>0</b>	<b>708</b>	<b>51</b>	<b>-9</b>

Hence, the normal equations are

$$708a + 60c = -9 \quad \dots \text{(eq.4)}$$

$$60b = 51 \quad \dots \text{(eq.5)}$$

$$60a + 9c = 8 \quad \dots \text{(eq.6)}$$

Solving we get,  $b = 0.85$ ,  $a = -0.20238$ ,  $c = 2.23809$

Hence the equation is  $Y = -0.20238X^2 + 0.85X + 2.23809$

i.e.  $y - 7 = -0.20238(x - 5)^2 + 0.85(x - 5) + 2.23809$

i.e.  $y = -0.20238x^2 + 2.8738x - 0.07141$



### 3. Fitting an exponential curve:

Let  $(x_i, y_i)$ ,  $i = 0, 1, 2, \dots, n$  be the  $n$  sets of observations of related data & let  $y = ae^{bx}$  be the best fit the data. Then taking logarithm on both sides,

$$\log_{10} y = \log_{10} a + x \log_{10} b$$

$$Y = A + Bx \text{ where } Y = \log_{10} y, A = \log_{10} a, B = \log_{10} b$$

This being linear in  $x$  &  $Y$ , we can find  $A, B$  since  $x$  &  $Y = \log_{10} y$  are known. From  $A, B$ , we can get  $a, b$  & hence  $y = ae^{bx}$  is found out.

### 4. Fitting a curve of the form $y = ax^b$ :

Since  $y = ax^b$ ,  $\log_{10} y = \log_{10} a + b \log_{10} x$

$$Y = A + bX \text{ where } Y = \log_{10} y, A = \log_{10} a, X = \log_{10} x$$

Again using this linear fit, we find  $A, b$ .

Hence  $a, b$  are known. Thus  $y = ax^b$  is found out.

### Example 8:

From the table given below, find the best values of  $a$  &  $b$  in the law  $y = ae^{bx}$  by the method of least squares.

x	0	5	8	12	20
y	3.0	1.5	1.0	0.55	0.18

**Solution:**  $y = ae^{bx}$

$$\log_{10} y = \log_{10} a + bx \log_{10} e$$

i.e.  $Y = A + Bx \dots (\text{eq.1})$

The normal equations are

$$B \sum x + 5A = \sum Y \dots (\text{eq.2})$$

$$B \sum x^2 + A \sum x = \sum xY \dots (\text{eq.3})$$

	x	y	Y	$x^2$	xY
	0	3.0	0.47712	0	0
	5	1.5	0.17609	25	0.88045
	8	1.0	0	64	0
	12	0.55	-0.25964	144	-3.11568
	20	0.18	-0.74473	400	-14.8946



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Total	45		-0.35116	633	-17.12983
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Using equations (2) & (3),

$$5A+45B=-0.35116$$

$$45A+633B=-17.12983$$

Solving we get  $A=0.48120$ ,  $B=-0.06127$

$$a = 10^A = 3.02831$$

$$bx \log_{10} e = B = -0.06127$$

$$b = -0.06127x \log_e 10 = -0.14108$$

Hence, the curve is  $y = 3.02831 e^{-0.14108 x}$