

CHAPTER

3

Steady-State Conduction— Multiple Dimensions

3-1 | INTRODUCTION

In Chapter 2 steady-state heat transfer was calculated in systems in which the temperature gradient and area could be expressed in terms of one space coordinate. We now wish to analyze the more general case of two-dimensional heat flow. For steady state with no heat generation, the Laplace equation applies.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \quad [3-1]$$

assuming constant thermal conductivity. The solution to this equation may be obtained by analytical, numerical, or graphical techniques.

The objective of any heat-transfer analysis is usually to predict heat flow or the temperature that results from a certain heat flow. The solution to Equation (3-1) will give the temperature in a two-dimensional body as a function of the two independent space coordinates x and y . Then the heat flow in the x and y directions may be calculated from the Fourier equations

$$q_x = -kA_x \frac{\partial T}{\partial x} \quad [3-2]$$

$$q_y = -kA_y \frac{\partial T}{\partial y} \quad [3-3]$$

These heat-flow quantities are directed either in the x direction or in the y direction. The total heat flow at any point in the material is the resultant of the q_x and q_y at that point. Thus the total heat-flow vector is directed so that it is perpendicular to the lines of constant temperature in the material, as shown in Figure 3-1. So if the temperature distribution in the material is known, we may easily establish the heat flow.

3-2 | MATHEMATICAL ANALYSIS OF TWO-DIMENSIONAL HEAT CONDUCTION

We first consider an analytical approach to a two-dimensional problem and then indicate the numerical and graphical methods that may be used to advantage in many other problems.



It is worthwhile to mention here that analytical solutions are not always possible to obtain; indeed, in many instances they are very cumbersome and difficult to use. In these cases numerical techniques are frequently used to advantage. For a more extensive treatment of the analytical methods used in conduction problems, the reader may consult References 1, 2, 10, and 11.

Consider the rectangular plate shown in Figure 3-2. Three sides of the plate are maintained at the constant temperature T_1 , and the upper side has some temperature distribution impressed upon it. This distribution could be simply a constant temperature or something more complex, such as a sine-wave distribution. We shall consider both cases.

To solve Equation (3-1), the separation-of-variables method is used. The essential point of this method is that the solution to the differential equation is assumed to take a product form

$$T = XY \quad \text{where} \quad \begin{aligned} X &= X(x) \\ Y &= Y(y) \end{aligned} \quad [3-4]$$

The boundary conditions are then applied to determine the form of the functions X and Y . The basic assumption as given by Equation (3-4) can be justified *only* if it is possible to find a solution of this form that satisfies the boundary conditions.

Figure 3-1 | Sketch showing the heat flow in two dimensions.

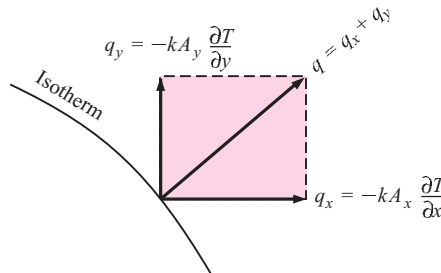
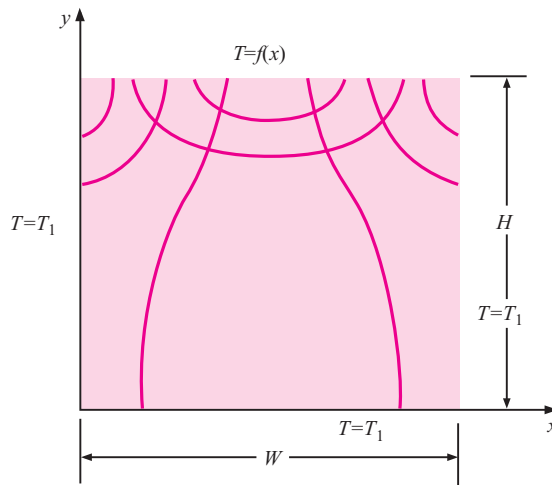


Figure 3-2 | Isotherms and heat flow lines in a rectangular plate.





First consider the boundary conditions with a sine-wave temperature distribution impressed on the upper edge of the plate. Thus

$$\begin{aligned} T &= T_1 && \text{at } y = 0 \\ T &= T_1 && \text{at } x = 0 \\ T &= T_1 && \text{at } x = W \end{aligned} \quad [3-5]$$

$$T = T_m \sin\left(\frac{\pi x}{W}\right) + T_1 \quad \text{at } y = H$$

where T_m is the amplitude of the sine function. Substituting Equation (3-4) in (3-1) gives

$$-\frac{1}{X} \frac{d^2 X}{dx^2} = \frac{1}{Y} \frac{d^2 Y}{dy^2} \quad [3-6]$$

Observe that each side of Equation (3-6) is independent of the other because x and y are independent variables. This requires that each side be equal to some constant. We may thus obtain two ordinary differential equations in terms of this constant,

$$\frac{d^2 X}{dx^2} + \lambda^2 X = 0 \quad [3-7]$$

$$\frac{d^2 Y}{dy^2} - \lambda^2 Y = 0 \quad [3-8]$$

where λ^2 is called the *separation constant*. Its value must be determined from the boundary conditions. Note that the form of the solution to Equations (3-7) and (3-8) will depend on the sign of λ^2 ; a different form would also result if λ^2 were zero. The only way that the correct form can be determined is through an application of the boundary conditions of the problem. So we shall first write down all possible solutions and then see which one fits the problem under consideration.

For $\lambda^2 = 0$:

$$\begin{aligned} X &= C_1 + C_2 x \\ Y &= C_3 + C_4 y \\ T &= (C_1 + C_2 x)(C_3 + C_4 y) \end{aligned} \quad [3-9]$$

This function cannot fit the sine-function boundary condition, so the $\lambda^2 = 0$ solution may be excluded.

For $\lambda^2 < 0$:

$$\begin{aligned} X &= C_5 e^{-\lambda x} + C_6 e^{\lambda x} \\ Y &= C_7 \cos \lambda y + C_8 \sin \lambda y \\ T &= (C_5 e^{-\lambda x} + C_6 e^{\lambda x})(C_7 \cos \lambda y + C_8 \sin \lambda y) \end{aligned} \quad [3-10]$$

Again, the sine-function boundary condition cannot be satisfied, so this solution is excluded also.

For $\lambda^2 > 0$:

$$\begin{aligned} X &= C_9 \cos \lambda x + C_{10} \sin \lambda x \\ Y &= C_{11} e^{-\lambda y} + C_{12} e^{\lambda y} \\ T &= (C_9 \cos \lambda x + C_{10} \sin \lambda x)(C_{11} e^{-\lambda y} + C_{12} e^{\lambda y}) \end{aligned} \quad [3-11]$$

Now, it is possible to satisfy the sine-function boundary condition; so we shall attempt to satisfy the other conditions. The algebra is somewhat easier to handle when the substitution

$$\theta = T - T_1$$



is made. The differential equation and the solution then retain the same form in the new variable θ , and we need only transform the boundary conditions. Thus

$$\begin{aligned} \theta &= 0 && \text{at } y = 0 \\ \theta &= 0 && \text{at } x = 0 \\ \theta &= 0 && \text{at } x = W \\ \theta &= T_m \sin \frac{\pi x}{W} && \text{at } y = H \end{aligned} \quad [3-12]$$

Applying these conditions, we have

$$0 = (C_9 \cos \lambda x + C_{10} \sin \lambda x)(C_{11} + C_{12}) \quad [a]$$

$$0 = C_9(C_{11}e^{-\lambda y} + C_{12}e^{\lambda y}) \quad [b]$$

$$0 = (C_9 \cos \lambda W + C_{10} \sin \lambda W)(C_{11}e^{-\lambda y} + C_{12}e^{\lambda y}) \quad [c]$$

$$T_m \sin \frac{\pi x}{W} = (C_9 \cos \lambda x + C_{10} \sin \lambda x)(C_{11}e^{-\lambda H} + C_{12}e^{\lambda H}) \quad [d]$$

Accordingly,

$$\begin{aligned} C_{11} &= -C_{12} \\ C_9 &= 0 \end{aligned}$$

and from (c),

$$0 = C_{10}C_{12} \sin \lambda W (e^{\lambda y} - e^{-\lambda y})$$

This requires that

$$\sin \lambda W = 0 \quad [3-13]$$

Recall that λ was an undetermined separation constant. Several values will satisfy Equation (3-13), and these may be written

$$\lambda = \frac{n\pi}{W} \quad [3-14]$$

where n is an integer. The solution to the differential equation may thus be written as a sum of the solutions for each value of n . This is an infinite sum, so that the final solution is the infinite series

$$\theta = T - T_1 = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{W} \sinh \frac{n\pi y}{W} \quad [3-15]$$

where the constants have been combined and the exponential terms converted to the hyperbolic function. The final boundary condition may now be applied:

$$T_m \sin \frac{\pi x}{W} = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{W} \sinh \frac{n\pi H}{W}$$

which requires that $C_n = 0$ for $n > 1$. The final solution is therefore

$$T = T_m \frac{\sinh(\pi y/W)}{\sinh(\pi H/W)} \sin\left(\frac{\pi x}{W}\right) + T_1 \quad [3-16]$$

The temperature field for this problem is shown in Figure 3-2. Note that the heat-flow lines are perpendicular to the isotherms.



We now consider the set of boundary conditions

$$\begin{aligned} T &= T_1 & \text{at } y = 0 \\ T &= T_1 & \text{at } x = 0 \\ T &= T_1 & \text{at } x = W \\ T &= T_2 & \text{at } y = H \end{aligned}$$

Using the first three boundary conditions, we obtain the solution in the form of Equation (3-15):

$$T - T_1 = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{W} \sinh \frac{n\pi y}{W} \quad [3-17]$$

Applying the fourth boundary condition gives

$$T_2 - T_1 = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{W} \sinh \frac{n\pi H}{W} \quad [3-18]$$

This is a Fourier sine series, and the values of the C_n may be determined by expanding the constant temperature difference $T_2 - T_1$ in a Fourier series over the interval $0 < x < W$. This series is

$$T_2 - T_1 = (T_2 - T_1) \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} + 1}{n} \sin \frac{n\pi x}{W} \quad [3-19]$$

Upon comparison of Equation (3-18) with Equation (3-19), we find that

$$C_n = \frac{2}{\pi} (T_2 - T_1) \frac{1}{\sinh(n\pi H/W)} \frac{(-1)^{n+1} + 1}{n}$$

and the final solution is expressed as

$$\frac{T - T_1}{T_2 - T_1} = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} + 1}{n} \sin \frac{n\pi x}{W} \frac{\sinh(n\pi y/W)}{\sinh(n\pi H/W)} \quad [3-20]$$

An extensive study of analytical techniques used in conduction heat transfer requires a background in the theory of orthogonal functions. Fourier series are one example of orthogonal functions, as are Bessel functions and other special functions applicable to different geometries and boundary conditions. The interested reader may consult one or more of the conduction heat-transfer texts listed in the references for further information on the subject.

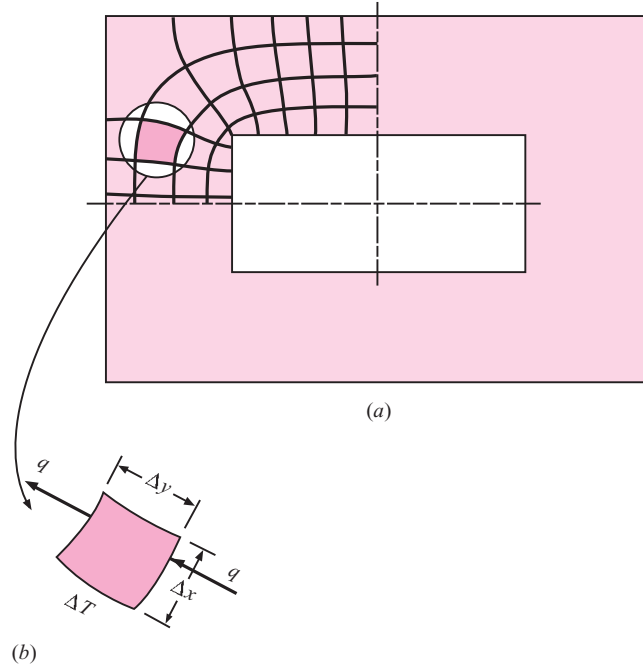
3-3 | GRAPHICAL ANALYSIS

Consider the two-dimensional system shown in Figure 3-3. The inside surface is maintained at some temperature T_1 , and the outer surface is maintained at T_2 . We wish to calculate the heat transfer. Isotherms and heat-flow lanes have been sketched to aid in this calculation. The isotherms and heat-flow lanes form groupings of curvilinear figures like that shown in Figure 3-3b. The heat flow across this curvilinear section is given by Fourier's law, assuming unit depth of material:

$$q = -k \Delta x(1) \frac{\Delta T}{\Delta y} \quad [3-21]$$



Figure 3-3 | Sketch showing element used for curvilinear-square analysis of two-dimensional heat flow.



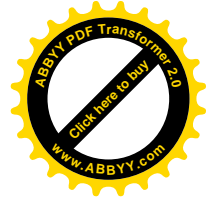
This heat flow will be the same through each section within this heat-flow lane, and the total heat flow will be the sum of the heat flows through all the lanes. If the sketch is drawn so that $\Delta x \cong \Delta y$, the heat flow is proportional to the ΔT across the element and, since this heat flow is constant, the ΔT across each element must be the same within the same heat-flow lane. Thus the ΔT across an element is given by

$$\Delta T = \frac{\Delta T_{\text{overall}}}{N}$$

where N is the number of temperature increments between the inner and outer surfaces. Furthermore, the heat flow through each lane is the same since it is independent of the dimensions Δx and Δy when they are constructed equal. Thus we write for the total heat transfer

$$q = \frac{M}{N} k \Delta T_{\text{overall}} = \frac{M}{N} k (T_2 - T_1) \tag{3-22}$$

where M is the number of heat-flow lanes. So, to calculate the heat transfer, we need only construct these curvilinear-square plots and count the number of temperature increments and heat-flow lanes. Care must be taken to construct the plot so that $\Delta x \approx \Delta y$ and the lines are perpendicular. For the corner section shown in Figure 3-3a the number of temperature increments between the inner and outer surfaces is about $N = 4$, while the number of heat-flow lanes for the corner section may be estimated as $M = 8.2$. The total number of heat-flow lanes is four times this value, or $4 \times 8.2 = 32.8$. The ratio M/N is thus $32.8/4 = 8.2$ for the whole wall section. This ratio will be called the *conduction shape factor* in subsequent discussions.



The accuracy of this method is dependent entirely on the skill of the person sketching the curvilinear squares. Even a crude sketch, however, can frequently help to give fairly good estimates of the temperatures that will occur in a body. An electrical analogy may be employed to sketch the curvilinear squares, as discussed in Section 3-9.

The graphical method presented here is mainly of historical interest to show the relation of heat-flow lanes and isotherms. It may not be expected to be used for the solution of many practical problems.

3-4 | THE CONDUCTION SHAPE FACTOR

In a two-dimensional system where only two temperature limits are involved, we may define a conduction shape factor S such that

$$q = kS \Delta T_{\text{overall}} \quad [3-23]$$

The values of S have been worked out for several geometries and are summarized in Table 3-1. A very comprehensive summary of shape factors for a large variety of geometries is given by Rohsenow [15] and Hahne and Grigull [17]. Note that the inverse hyperbolic cosine can be calculated from

$$\cosh^{-1} x = \ln(x \pm \sqrt{x^2 - 1})$$

For a three-dimensional wall, as in a furnace, separate shape factors are used to calculate the heat flow through the edge and corner sections, with the dimensions shown in Figure 3-4. When all the interior dimensions are greater than one-fifth of the wall thickness,

$$S_{\text{wall}} = \frac{A}{L} \quad S_{\text{edge}} = 0.54D \quad S_{\text{corner}} = 0.15L$$

where

A = area of wall

L = wall thickness

D = length of edge

Note that the shape factor per unit depth is given by the ratio M/N when the curvilinear-squares method is used for calculations.

Figure 3-4 | Sketch illustrating dimensions for use in calculating three-dimensional shape factors.

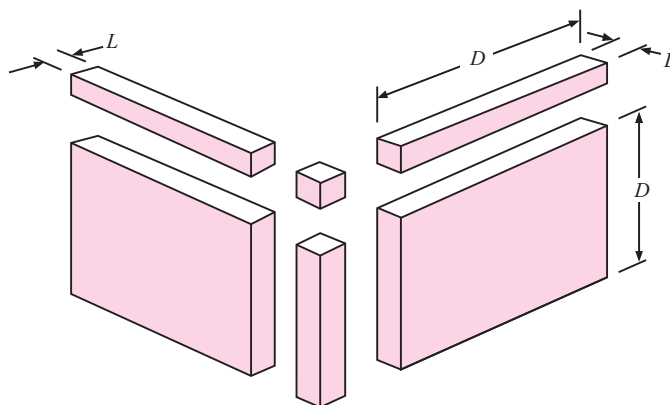




Table 3-1 | Conduction shape factors, summarized from References 6 and 7.

Note: For buried objects the temperature difference is $\Delta T = T_{\text{object}} - T_{\text{far field}}$. The far-field temperature is taken the same as the isothermal surface temperature for semi-infinite media.

Physical system	Schematic	Shape factor	Restrictions
Isothermal cylinder of radius r buried in semi-infinite medium having isothermal surface		$\frac{2\pi L}{\cosh^{-1}(D/r)}$ $\frac{2\pi L}{\ln(D/r)}$	$L \gg r$ $L \gg r$ $D > 3r$
Isothermal sphere of radius r buried in infinite medium		$4\pi r$	
Isothermal sphere of radius r buried in semi-infinite medium having isothermal surface $\Delta T = T_{\text{surf}} - T_{\text{far field}}$		$\frac{4\pi r}{1 - r/2D}$	
Conduction between two isothermal cylinders of length L buried in infinite medium		$\frac{2\pi L}{\cosh^{-1}\left(\frac{D^2 - r_1^2 - r_2^2}{2r_1 r_2}\right)}$	$L \gg r$ $L \gg D$
Row of horizontal cylinders of length L in semi-infinite medium with isothermal surface		$S = \frac{2\pi L}{\ln\left[\left(\frac{1}{\pi r}\right) \sinh(2\pi D/l)\right]}$	$D > 2r$
Buried cube in infinite medium, L on a side		$8.24L$	



Table 3-1 | (Continued).

Physical system	Schematic	Shape factor	Restrictions
Isothermal cylinder of radius r placed in semi-infinite medium as shown		$\frac{2\pi L}{\ln(2L/r)}$	$L \gg 2r$
Isothermal rectangular parallelepiped buried in semi-infinite medium having isothermal surface		$1.685L \left[\log \left(1 + \frac{b}{a} \right) \right]^{-0.59} \left(\frac{b}{c} \right)^{-0.078}$	See Reference 7
Plane wall		$\frac{A}{L}$	One-dimensional heat flow
Hollow cylinder, length L		$\frac{2\pi L}{\ln(r_o/r_i)}$	$L \gg r$
Hollow sphere		$\frac{4\pi r_o r_i}{r_o - r_i}$	
Thin horizontal disk buried in semi-infinite medium with isothermal surface		$\frac{4\pi r}{\pi/2 - \tan^{-1}(r/2D)}$	$D = 0$ $D \gg 2r$ $D/2r > 1$ $\tan^{-1}(r/2D)$ in radians
Hemisphere buried in semi-infinite medium $\Delta T = T_{\text{sphere}} - T_{\text{far field}}$		$2\pi r$	
Isothermal sphere buried in semi-infinite medium with insulated surface		$\frac{4\pi r}{1 + r/2D}$	
Two isothermal spheres buried in infinite medium		$\frac{4\pi r_2}{\frac{r_2}{r_1} \left[1 - \frac{(r_1/D)^4}{1 - (r_2/D)^2} \right] - \frac{2r_2}{D}}$	$D > 5r_{\text{max}}$



Table 3-1 | (Continued).

Physical system	Schematic	Shape factor	Restrictions
Thin rectangular plate of length L , buried in semi-infinite medium having isothermal surface		$\frac{\pi W}{\ln(4 W/L)}$ $\frac{2\pi W}{\ln(4 W/L)}$ $\frac{2\pi W}{\ln(2\pi D/L)}$	$D = 0$ $W > L$ $D \gg W$ $W > L$ $W \gg L$ $D > 2W$
Parallel disks buried in infinite medium		$\frac{4\pi r}{\left[\frac{\pi}{2} - \tan^{-1}(r/D)\right]}$	$D > 5r$ $\tan^{-1}(r/D)$ in radians
Eccentric cylinders of length L		$\frac{2\pi L}{\cosh^{-1}\left(\frac{r_1^2 + r_2^2 - D^2}{2r_1 r_2}\right)}$	$L \gg r_2$
Cylinder centered in a square of length L		$\frac{2\pi L}{\ln(0.54 W/r)}$	$L \gg W$
Horizontal cylinder of length L centered in infinite plate		$\frac{2\pi L}{\ln(4D/r)}$	
Thin horizontal disk buried in semi-infinite medium with adiabatic surface $\Delta T = T_{\text{disk}} - T_{\text{far field}}$		$\frac{4\pi r}{\pi/2 + \tan^{-1}(r/2D)}$	$D/2r > 1$ $\tan^{-1}(r/2D)$ in radians



Buried Pipe

EXAMPLE 3-1

A horizontal pipe 15 cm in diameter and 4 m long is buried in the earth at a depth of 20 cm. The pipe-wall temperature is 75°C, and the earth surface temperature is 5°C. Assuming that the thermal conductivity of the earth is 0.8 W/m · °C, calculate the heat lost by the pipe.

■ Solution

We may calculate the shape factor for this situation using the equation given in Table 3-1. Since $D < 3r$,

$$S = \frac{2\pi L}{\cosh^{-1}(D/r)} = \frac{2\pi(4)}{\cosh^{-1}(20/7.5)} = 15.35 \text{ m}$$

The heat flow is calculated from

$$q = kS\Delta T = (0.8)(15.35)(75 - 5) = 859.6 \text{ W} \quad [2933 \text{ Btu/h}]$$

Cubical Furnace

EXAMPLE 3-2

A small cubical furnace 50 by 50 by 50 cm on the inside is constructed of fireclay brick [$k = 1.04 \text{ W/m} \cdot \text{°C}$] with a wall thickness of 10 cm. The inside of the furnace is maintained at 500°C, and the outside is maintained at 50°C. Calculate the heat lost through the walls.

■ Solution

We compute the total shape factor by adding the shape factors for the walls, edges, and corners:

Walls:
$$S = \frac{A}{L} = \frac{(0.5)(0.5)}{0.1} = 2.5 \text{ m}$$

Edges:
$$S = 0.54D = (0.54)(0.5) = 0.27 \text{ m}$$

Corners:
$$S = 0.15L = (0.15)(0.1) = 0.015 \text{ m}$$

There are six wall sections, twelve edges, and eight corners, so that the total shape factor is

$$S = (6)(2.5) + (12)(0.27) + (8)(0.015) = 18.36 \text{ m}$$

and the heat flow is calculated as

$$q = kS\Delta T = (1.04)(18.36)(500 - 50) = 8.592 \text{ kW} \quad [29,320 \text{ Btu/h}]$$

Buried Disk

EXAMPLE 3-3

A disk having a diameter of 30 cm and maintained at a temperature of 95°C is buried at a depth of 1.0 m in a semi-infinite medium having an isothermal surface temperature of 20°C and a thermal conductivity of 2.1 W/m · °C. Calculate the heat lost by the disk.

■ Solution

This is an application of the conduction shape factor relation $q = kS \Delta T$. Consulting Table 3-1 we find a choice of three relations for S for the geometry of a disk buried in a semi-infinite medium with an isothermal surface. Clearly, $D \neq 0$ and D is not large compared to $2r$, so the relation we select for the shape factor is for the case $D/2r > 1.0$:

$$S = \frac{4\pi r}{[\pi/2 - \tan^{-1}(r/2D)]}$$



Note that this relation differs from the one for an insulated surface by the minus sign in the denominator. Inserting $r = 0.15$ m and $D = 1.0$ m we obtain

$$S = \frac{4\pi(0.15)}{[\pi/2 - \tan^{-1}(0.15/2)]} = \frac{4\pi(0.15)}{[\pi/2 - 0.07486]} = 1.26 \text{ m}$$

For buried objects the shape factor is based on $\Delta T = T_{\text{object}} - T_{\text{far field}}$. The far-field temperature is taken as the isothermal surface temperature, and the heat lost by the disk is therefore

$$q = kS\Delta T = (2.1)(1.26)(95 - 20) = 198.45 \text{ W}$$

EXAMPLE 3-4

Buried Parallel Disks

Two parallel 50-cm-diameter disks are separated by a distance of 1.5 m in an infinite medium having $k = 2.3$ W/m · °C. One disk is maintained at 80°C and the other at 20°C. Calculate the heat transfer between the disks.

■ Solution

This is a shape-factor problem and the heat transfer may be calculated from

$$q = kS\Delta T$$

where S is obtained from Table 3-1 as

$$S = \frac{4\pi r}{[\pi/2 - \tan^{-1}(r/D)]} \quad \text{for } D > 5r$$

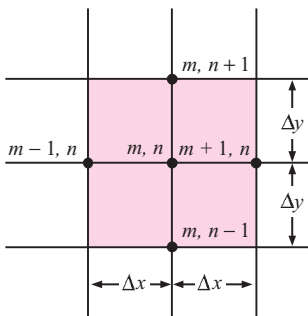
With $r = 0.25$ m and $D = 1.5$ m we obtain

$$S = \frac{4\pi(0.25)}{[\pi/2 - \tan^{-1}(0.25/1.5)]} = \frac{4\pi(0.25)}{[\pi/2 - 0.1651]} = 2.235$$

and

$$q = kS \Delta T = (2.3)(2.235)(80 - 20) = 308.4 \text{ W}$$

Figure 3-5 | Sketch illustrating nomenclature used in two-dimensional numerical analysis of heat conduction.



3-5 | NUMERICAL METHOD OF ANALYSIS

An immense number of analytical solutions for conduction heat-transfer problems have been accumulated in the literature over the past 150 years. Even so, in many practical situations the geometry or boundary conditions are such that an analytical solution has not been obtained at all, or if the solution has been developed, it involves such a complex series solution that numerical evaluation becomes exceedingly difficult. For such situations the most fruitful approach to the problem is one based on finite-difference techniques, the basic principles of which we shall outline in this section.

Consider a two-dimensional body that is to be divided into equal increments in both the x and y directions, as shown in Figure 3-5. The nodal points are designated as shown, the m locations indicating the x increment and the n locations indicating the y increment. We wish to establish the temperatures at any of these nodal points within the body, using Equation (3-1) as a governing condition. Finite differences are used to approximate differential increments in the temperature and space coordinates; and the smaller we choose these finite increments, the more closely the true temperature distribution will be approximated.



The temperature gradients may be written as follows:

$$\begin{aligned} \left. \frac{\partial T}{\partial x} \right]_{m+1/2,n} &\approx \frac{T_{m+1,n} - T_{m,n}}{\Delta x} \\ \left. \frac{\partial T}{\partial x} \right]_{m-1/2,n} &\approx \frac{T_{m,n} - T_{m-1,n}}{\Delta x} \\ \left. \frac{\partial T}{\partial y} \right]_{m,n+1/2} &\approx \frac{T_{m,n+1} - T_{m,n}}{\Delta y} \\ \left. \frac{\partial T}{\partial y} \right]_{m,n-1/2} &\approx \frac{T_{m,n} - T_{m,n-1}}{\Delta y} \\ \left. \frac{\partial^2 T}{\partial x^2} \right]_{m,n} &\approx \frac{\left. \frac{\partial T}{\partial x} \right]_{m+1/2,n} - \left. \frac{\partial T}{\partial x} \right]_{m-1/2,n}}{\Delta x} = \frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2} \\ \left. \frac{\partial^2 T}{\partial y^2} \right]_{m,n} &\approx \frac{\left. \frac{\partial T}{\partial y} \right]_{m,n+1/2} - \left. \frac{\partial T}{\partial y} \right]_{m,n-1/2}}{\Delta y} = \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2} \end{aligned}$$

Thus the finite-difference approximation for Equation (3-1) becomes

$$\frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2} + \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2} = 0$$

If $\Delta x = \Delta y$, then

$$T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} = 0 \quad [3-24]$$

Since we are considering the case of constant thermal conductivity, the heat flows may all be expressed in terms of temperature differentials. Equation (3-24) states very simply that the net heat flow into any node is zero at steady-state conditions. In effect, the numerical finite-difference approach replaces the continuous temperature distribution by fictitious heat-conducting rods connected between small nodal points that do not generate heat.

We can also devise a finite-difference scheme to take heat generation into account. We merely add the term \dot{q}/k into the general equation and obtain

$$\frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2} + \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2} + \frac{\dot{q}}{k} = 0$$

Then for a square grid in which $\Delta x = \Delta y$,

$$T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} + \frac{\dot{q}(\Delta x)^2}{k} - 4T_{m,n} = 0 \quad [3-24a]$$

To utilize the numerical method, Equation (3-24) must be written for each node within the material and the resultant system of equations solved for the temperatures at the various nodes. A very simple example is shown in Figure 3-6, and the four equations for nodes 1, 2, 3, and 4 would be

$$\begin{aligned} 100 + 500 + T_2 + T_3 - 4T_1 &= 0 \\ T_1 + 500 + 100 + T_4 - 4T_2 &= 0 \end{aligned}$$

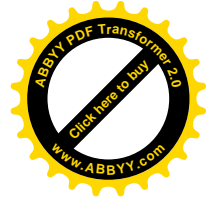
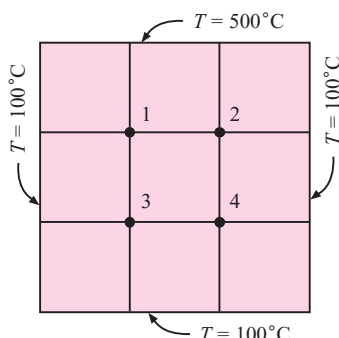


Figure 3-6 | Four-node problem.



$$100 + T_1 + T_4 + 100 - 4T_3 = 0$$

$$T_3 + T_2 + 100 + 100 - 4T_4 = 0$$

These equations have the solution

$$T_1 = T_2 = 250^\circ\text{C} \quad T_3 = T_4 = 150^\circ\text{C}$$

Of course, we could recognize from symmetry that $T_1 = T_2$ and $T_3 = T_4$ and would then only need two nodal equations,

$$100 + 500 + T_3 - 3T_1 = 0$$

$$100 + T_1 + 100 - 3T_3 = 0$$

Once the temperatures are determined, the heat flow may be calculated from

$$q = \sum k \Delta x \frac{\Delta T}{\Delta y}$$

where the ΔT is taken at the boundaries. In the example the heat flow may be calculated at either the 500°C face or the three 100°C faces. If a sufficiently fine grid is used, the two values should be very nearly the same. As a matter of general practice, it is usually best to take the arithmetic average of the two values for use in the calculations. In the example, the two calculations yield:

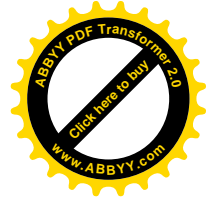
500°C face:

$$q = -k \frac{\Delta x}{\Delta y} [(250 - 500) + (250 - 500)] = 500k$$

100°C face:

$$q = -k \frac{\Delta y}{\Delta x} [(250 - 100) + (150 - 100) + (150 - 100) + (150 - 100) + (150 - 100) + (250 - 100)] = -500k$$

and the two values agree in this case. The calculation of the heat flow in cases in which curved boundaries or complicated shapes are involved is treated in References 2, 3, and 15.



When the solid is exposed to some convection boundary condition, the temperatures at the surface must be computed differently from the method given above. Consider the boundary shown in Figure 3-7. The energy balance on node (m, n) is

$$-k \Delta y \frac{T_{m,n} - T_{m-1,n}}{\Delta x} - k \frac{\Delta x}{2} \frac{T_{m,n} - T_{m,n+1}}{\Delta y} - k \frac{\Delta x}{2} \frac{T_{m,n} - T_{m,n-1}}{\Delta y} = h \Delta y (T_{m,n} - T_{\infty})$$

If $\Delta x = \Delta y$, the boundary temperature is expressed in the equation

$$T_{m,n} \left(\frac{h \Delta x}{k} + 2 \right) - \frac{h \Delta x}{k} T_{\infty} - \frac{1}{2} (2T_{m-1,n} + T_{m,n+1} + T_{m,n-1}) = 0 \quad [3-25]$$

An equation of this type must be written for each node along the surface shown in Figure 3-7. So when a convection boundary condition is present, an equation like (3-25) is used at the boundary and an equation like (3-24) is used for the interior points.

Equation (3-25) applies to a plane surface exposed to a convection boundary condition. It will not apply for other situations, such as an insulated wall or a corner exposed to a convection boundary condition. Consider the corner section shown in Figure 3-8. The energy balance for the corner section is

$$-k \frac{\Delta y}{2} \frac{T_{m,n} - T_{m-1,n}}{\Delta x} - k \frac{\Delta x}{2} \frac{T_{m,n} - T_{m,n-1}}{\Delta y} = h \frac{\Delta x}{2} (T_{m,n} - T_{\infty}) + h \frac{\Delta y}{2} (T_{m,n} - T_{\infty})$$

If $\Delta x = \Delta y$,

$$2T_{m,n} \left(\frac{h \Delta x}{k} + 1 \right) - 2 \frac{h \Delta x}{k} T_{\infty} - (T_{m-1,n} + T_{m,n-1}) = 0 \quad [3-26]$$

Other boundary conditions may be treated in a similar fashion, and a convenient summary of nodal equations is given in Table 3-2 for different geometrical and boundary situations. Situations *f* and *g* are of particular interest since they provide the calculation equations that may be employed with curved boundaries, while still using uniform increments in Δx and Δy .

Figure 3-7 | Nomenclature for nodal equation with convective boundary condition.

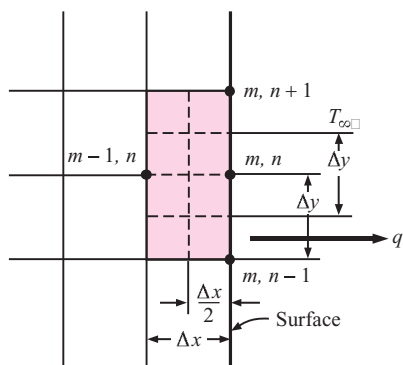


Figure 3-8 | Nomenclature for nodal equation with convection at a corner section.

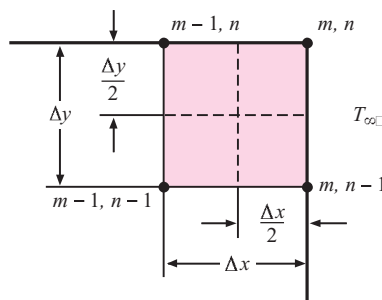


Table 3-2 | Summary of nodal formulas for finite-difference calculations. (Dashed lines indicate element volume.)[†]

Physical situation	Nodal equation for equal increments in x and y (second equation in situation is in form for Gauss-Seidel iteration)
<p>(a) Interior node</p>	$0 = T_{m+1,n} + T_{m,n+1} + T_{m-1,n} + T_{m,n-1} - 4T_{m,n}$ $T_{m,n} = (T_{m+1,n} + T_{m,n+1} + T_{m-1,n} + T_{m,n-1})/4$
<p>(b) Convection boundary node</p>	$0 = \frac{h\Delta x}{k} T_{\infty} + \frac{1}{2}(2T_{m-1,n} + T_{m,n+1} + T_{m,n-1}) - \left(\frac{h\Delta x}{k} + 2\right) T_{m,n}$ $T_{m,n} = \frac{T_{m-1,n} + (T_{m,n+1} + T_{m,n-1})/2 + \text{Bi} T_{\infty}}{2 + \text{Bi}}$ $\text{Bi} = \frac{h\Delta x}{k}$
<p>(c) Exterior corner with convection boundary</p>	$0 = 2\frac{h\Delta x}{k} T_{\infty} + (T_{m-1,n} + T_{m,n-1}) - 2\left(\frac{h\Delta x}{k} + 1\right) T_{m,n}$ $T_{m,n} = \frac{(T_{m-1,n} + T_{m,n-1})/2 + \text{Bi} T_{\infty}}{1 + \text{Bi}}$ $\text{Bi} = \frac{h\Delta x}{k}$
<p>(d) Interior corner with convection boundary</p>	$0 = 2\frac{h\Delta x}{k} T_{\infty} + 2T_{m-1,n} + T_{m,n+1} + T_{m+1,n} + T_{m,n-1} - 2\left(3 + \frac{h\Delta x}{k}\right) T_{m,n}$ $T_{m,n} = \frac{\text{Bi} T_{\infty} + T_{m,n+1} + T_{m-1,n} + (T_{m+1,n} + T_{m,n-1})/2}{3 + \text{Bi}}$ $\text{Bi} = \frac{h\Delta x}{k}$



Table 3-2 | (Continued).

Physical situation	Nodal equation for equal increments in x and y (second equation in situation is in form for Gauss-Seidel iteration)
<p>(e) Insulated boundary</p>	$0 = T_{m,n+1} + T_{m,n-1} + 2T_{m-1,n} - 4T_{m,n}$ $T_{m,n} = (T_{m,n+1} + T_{m,n-1} + 2T_{m-1,n})/4$
<p>(f) Interior node near curved boundary[‡]</p>	$0 = \frac{2}{b(b+1)} T_2 + \frac{2}{a+1} T_{m+1,n} + \frac{2}{b+1} T_{m,n-1} + \frac{2}{a(a+1)} T_1 - 2 \left(\frac{1}{a} + \frac{1}{b} \right) T_{m,n}$
<p>(g) Boundary node with convection along curved boundary—node 2 for (f) above[§]</p>	$0 = \frac{b}{\sqrt{a^2+b^2}} T_1 + \frac{b}{\sqrt{c^2+1}} T_3 + \frac{a+1}{b} T_{m,n} + \frac{h\Delta x}{k} (\sqrt{c^2+1} + \sqrt{a^2+b^2}) T_\infty$ $- \left[\frac{b}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{c^2+1}} + \frac{a+1}{b} + (\sqrt{c^2+1} + \sqrt{a^2+b^2}) \frac{h\Delta x}{k} \right] T_2$

[†] Convection boundary may be converted to insulated surface by setting $h = 0$ ($Bi = 0$).

[‡] This equation is obtained by multiplying the resistance by $4/(a+1)(b+1)$.

[§] This relation is obtained by dividing the resistance formulation by 2.

Nine-Node Problem

EXAMPLE 3-5

Consider the square of Figure Example 3-5. The left face is maintained at 100°C and the top face at 500°C, while the other two faces are exposed to an environment at 100°C:

$$h = 10 \text{ W/m}^2 \cdot \text{°C} \quad \text{and} \quad k = 10 \text{ W/m} \cdot \text{°C}$$

The block is 1 m square. Compute the temperature of the various nodes as indicated in Figure Example 3-5 and the heat flows at the boundaries.

■ Solution

The nodal equation for nodes 1, 2, 4, and 5 is

$$T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} = 0$$

The equation for nodes 3, 6, 7, and 8 is given by Equation (3-25), and the equation for 9 is given by Equation (3-26):

$$\frac{h\Delta x}{k} = \frac{(10)(1)}{(3)(10)} = \frac{1}{3}$$

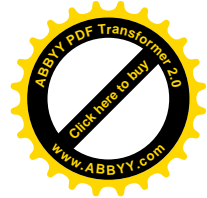
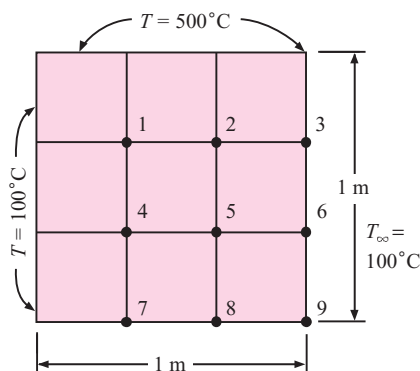


Figure Example 3-5 | Nomenclature for Example 3-5.



The equations for nodes 3 and 6 are thus written

$$2T_2 + T_6 + 567 - 4.67T_3 = 0$$

$$2T_5 + T_3 + T_9 + 67 - 4.67T_6 = 0$$

The equations for nodes 7 and 8 are given by

$$2T_4 + T_8 + 167 - 4.67T_7 = 0$$

$$2T_5 + T_7 + T_9 + 67 - 4.67T_8 = 0$$

and the equation for node 9 is

$$T_6 + T_8 + 67 - 2.67T_9 = 0$$

We thus have nine equations and nine unknown nodal temperatures. We shall discuss solution techniques shortly, but for now we just list the answers:

Node	Temperature, °C
1	280.67
2	330.30
3	309.38
4	192.38
5	231.15
6	217.19
7	157.70
8	184.71
9	175.62

The heat flows at the boundaries are computed in two ways: as conduction flows for the 100 and 500°C faces and as convection flows for the other two faces. For the 500°C face, the heat flow into the face is

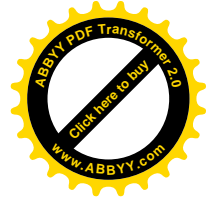
$$q = \sum k \Delta x \frac{\Delta T}{\Delta y} = (10) \left[500 - 280.67 + 500 - 330.30 + (500 - 309.38) \left(\frac{1}{2} \right) \right]$$

$$= 4843.4 \text{ W/m}$$

The heat flow out of the 100°C face is

$$q = \sum k \Delta y \frac{\Delta T}{\Delta x} = (10) \left[280.67 - 100 + 192.38 - 100 + (157.70 - 100) \left(\frac{1}{2} \right) \right]$$

$$= 3019 \text{ W/m}$$



The convection heat flow *out* the right face is given by the convection relation

$$\begin{aligned}
 q &= \sum h \Delta y (T - T_{\infty}) \\
 &= (10) \left(\frac{1}{3}\right) \left[309.38 - 100 + 217.19 - 100 + (175.62 - 100) \left(\frac{1}{2}\right) \right] \\
 &= 1214.6 \text{ W/m}
 \end{aligned}$$

Finally, the convection heat flow *out* the bottom face is

$$\begin{aligned}
 q &= \sum h \Delta x (T - T_{\infty}) \\
 &= (10) \left(\frac{1}{3}\right) \left[(100 - 100) \left(\frac{1}{2}\right) + 157.70 - 100 + 184.71 - 100 + (175.62 - 100) \left(\frac{1}{2}\right) \right] \\
 &= 600.7 \text{ W/m}
 \end{aligned}$$

The total heat flow out is

$$q_{\text{out}} = 3019 + 1214.6 + 600.7 = 4834.3 \text{ W/m}$$

This compares favorably with the 4843.4 W/m conducted into the top face. A solution of this example using the Excel spreadsheet format is given in Appendix D.

Solution Techniques

From the foregoing discussion we have seen that the numerical method is simply a means of approximating a continuous temperature distribution with the finite nodal elements. The more nodes taken, the closer the approximation; but, of course, more equations mean more cumbersome solutions. Fortunately, computers and even programmable calculators have the capability to obtain these solutions very quickly.

In practical problems the selection of a large number of nodes may be unnecessary because of uncertainties in boundary conditions. For example, it is not uncommon to have uncertainties in *h*, the convection coefficient, of ±15 to 20 percent.

The nodal equations may be written as

$$\begin{aligned}
 a_{11}T_1 + a_{12}T_2 + \dots + a_{1n}T_n &= C_1 \\
 a_{21}T_1 + a_{22}T_2 + \dots &= C_2 \\
 a_{31}T_1 + \dots &= C_3 \\
 \dots & \\
 a_{n1}T_1 + a_{n2}T_2 + \dots + a_{nn}T_n &= C_n
 \end{aligned} \tag{3-27}$$

where T_1, T_2, \dots, T_n are the unknown nodal temperatures. By using the matrix notation

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \\ a_{31} & & \dots & \\ \dots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \quad [C] = \begin{bmatrix} C_1 \\ C_2 \\ \cdot \\ \cdot \\ C_n \end{bmatrix} \quad [T] = \begin{bmatrix} T_1 \\ T_2 \\ \cdot \\ \cdot \\ T_n \end{bmatrix}$$

Equation (3-27) can be expressed as

$$[A][T] = [C] \tag{3-28}$$



and the problem is to find the inverse of [A] such that

$$[T] = [A]^{-1}[C] \tag{3-29}$$

Designating [A]⁻¹ by

$$[A]^{-1} = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & \dots \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix}$$

the final solutions for the unknown temperatures are written in expanded form as

$$\begin{aligned} T_1 &= b_{11}C_1 + b_{12}C_2 + \dots + b_{1n}C_n \\ T_2 &= b_{21}C_1 + \dots \\ &\dots \dots \dots \\ T_n &= b_{n1}C_1 + b_{n2}C_2 + \dots + b_{nn}C_n \end{aligned} \tag{3-30}$$

Clearly, the larger the number of nodes, the more complex and time-consuming the solution, even with a high-speed computer. For most conduction problems the matrix contains a large number of zero elements so that some simplification in the procedure is afforded. For example, the matrix notation for the system of Example 3-5 would be

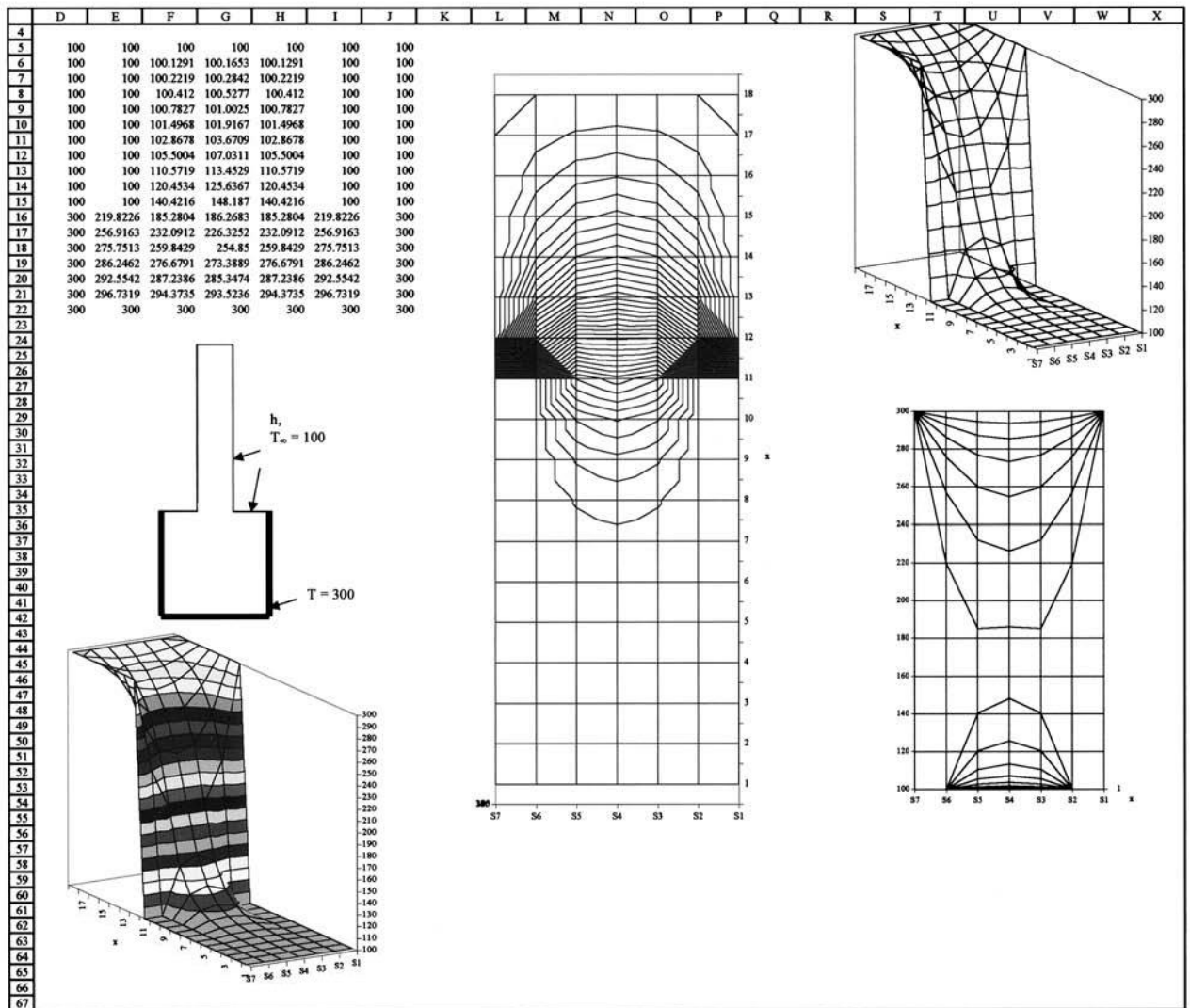
$$\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -4.67 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & -4.67 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & -4.67 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 & -4.67 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -2.67 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{bmatrix} = \begin{bmatrix} -600 \\ -500 \\ -567 \\ -100 \\ 0 \\ -67 \\ -167 \\ -67 \\ -67 \end{bmatrix}$$

We see that because of the structure of the equations the coefficient matrix is very sparse. For this reason iterative methods of solution may be very efficient. The Gauss-Seidel iteration method is probably the most widely used for solution of these equations in heat transfer problems, and we shall discuss that method in Section 3-7.

Software Packages for Solution of Equations

Several software packages are available for solution of simultaneous equations, including MathCAD (22), TK Solver (23), Matlab (24), and Microsoft Excel (25, 26, 27). The spreadsheet grid of Excel is particularly adaptable to formulation, solution, and graphical displays associated with the nodal equations. Details of the use of Excel as a tool for such problems are presented in Appendix D for both steady-state and transient conditions.

Figure 3-9 | General conduction node.



An example of an Excel worksheet is shown in Figure 3-9, which includes a schematic of a protruding fin cooled by a convection environment, numerical solution displayed to match the geometric configuration, and four graphical presentations of the results. While one might regard this presentation as graphical overkill, it does illustrate a variety of options available in the Excel format. Several examples are discussed in detail in Appendix D, including the effects of heat sources and radiation boundary conditions.

Other methods of solution include a transient analysis carried through to steady state (see Chapter 4), direct elimination (Gauss elimination [9]), or more sophisticated iterative techniques [12]. An Excel spreadsheet solution treating Example 3-5 as a transient problem carried through to steady state is given Appendix D.



3-6 | NUMERICAL FORMULATION IN TERMS OF RESISTANCE ELEMENTS

Up to this point we have shown how conduction problems can be solved by finite-difference approximations to the differential equations. An equation is formulated for each node and the set of equations solved for the temperatures throughout the body. In formulating the equations we could just as well have used a resistance concept for writing the heat transfer between nodes. Designating our node of interest with the subscript i and the adjoining nodes with subscript j , we have the general-conduction-node situation shown in Figure 3-10. At steady state the net heat input to node i must be zero or

$$q_i + \sum_j \frac{T_j - T_i}{R_{ij}} = 0 \tag{3-31}$$

where q_i is the heat delivered to node i by heat generation, radiation, etc. The R_{ij} can take the form of convection boundaries, internal conduction, etc., and Equation (3-31) can be set equal to some residual for a relaxation solution or to zero for treatment with matrix and iterative methods.

No new information is conveyed by using a resistance formulation, but some workers may find it convenient to think in these terms. When a numerical solution is to be performed that takes into account property variations, the resistance formulation is particularly useful. In addition, there are many heat-transfer problems where it is convenient to think of convection and radiation boundary conditions in terms of the thermal resistance they impose on the system. In such cases the relative magnitudes of convection, radiation, and conduction resistances may have an important influence on the behavior of the thermal model. We shall examine different boundary resistances in the examples. It will be clear that one will want to increase thermal resistances when desiring to impede the heat flow and decrease the thermal resistance when an increase in heat transfer is sought. In some cases the term *thermal impedance* is employed as a synonym for thermal resistance, following this line of thinking.

For convenience of the reader Table 3-3 lists the resistance elements that correspond to the nodes in Table 3-2. Note that all resistance elements are for unit depth of material and $\Delta x = \Delta y$. The nomenclature for the table is that R_{m+} refers to the resistance on the positive x side of node (m, n) , R_{n-} refers to the resistance on the negative y side of node (m, n) , and so on.

Figure 3-10 | General conduction node.

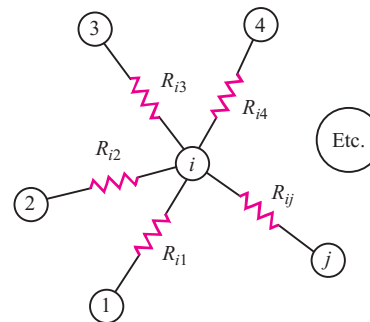




Table 3-3 | Resistances for nodes of Table 3-2 $\Delta x = \Delta y, \Delta z = 1$.

Physical situation	R_{m+}	R_{m-}	R_{n+}	R_{n-}	ΔV
(a) Interior node	$\frac{1}{k}$	$\frac{1}{k}$	$\frac{1}{k}$	$\frac{1}{k}$	$(\Delta x)^2$
(b) Convection boundary	$\frac{1}{h \Delta x}$	$\frac{1}{k}$	$\frac{2}{k}$	$\frac{2}{k}$	$\frac{(\Delta x)^2}{2}$
(c) Exterior corner, convection	$\frac{2}{h \Delta x}$	$\frac{2}{k}$	$\frac{2}{h \Delta x}$	$\frac{2}{k}$	$\frac{(\Delta x)^2}{4}$
(d) Interior corner, convection [†]	$\frac{2}{k}$	$\frac{1}{k}$	$\frac{1}{k}$	$\frac{2}{k}$	$\frac{3(\Delta x)^2}{4}$
(e) Insulated boundary	∞	$\frac{1}{k}$	$\frac{2}{k}$	$\frac{2}{k}$	$\frac{(\Delta x)^2}{2}$
(f) Interior node near curved boundary	$\frac{2}{(b+1)k}$ to node ($m+1, n$)	$\frac{2a}{(b+1)k}$ to node 1	$\frac{2b}{(a+1)k}$ to node 2	$\frac{2}{(a+1)k}$ to node ($m, n-1$)	$0.25(1+a)(1+b)(\Delta x)^2$
(g) Boundary node with curved boundary node 2 for (f) above	$R_{23} = \frac{2\sqrt{c^2+1}}{bk}$ $R_{21} = \frac{2\sqrt{a^2+b^2}}{bk}$ $R_{2-\infty} = \frac{2}{h\Delta x(\sqrt{c^2+1} + \sqrt{a^2+b^2})}$ $R_{n-} = \frac{2b}{k(a+1)}$ to node (m, n)				$\Delta V = 0.125[(2+a)+c](\Delta x)^2$

[†] Also $R_{\infty} = 1/h\Delta x$ for convection to T_{∞} .

The resistance formulation is also useful for numerical solution of complicated three-dimensional shapes. The volume elements for the three common coordinate systems are shown in Figure 3-11, and internal nodal resistances for each system are given in Table 3-4. The nomenclature for the (m, n, k) subscripts is given in Table 3-3, and the plus or minus sign on the resistance subscripts designates the resistance in a positive or negative direction from the central node (m, n, k). The elemental volume ΔV is also indicated for each coordinate system. We note, of course, that in a practical problem the coordinate increments are frequently chosen so that $\Delta x = \Delta y = \Delta z$, etc., and the resistances are simplified.

3-7 | GAUSS-SEIDEL ITERATION

When the number of nodes is very large, an iterative technique may frequently yield a more efficient solution to the nodal equations than a direct matrix inversion. One such method is called the *Gauss-Seidel iteration* and is applied in the following way. From Equation (3-31) we may solve for the temperature T_i in terms of the resistances and temperatures of the adjoining nodes T_j as

$$T_i = \frac{q_i + \sum_j (T_j/R_{ij})}{\sum_j (1/R_{ij})} \quad [3-32]$$

The Gauss-Seidel iteration makes use of the difference equations expressed in the form of Equation (3-32) through the following procedure.

1. An initial set of values for the T_i is assumed. This initial assumption can be obtained through any expedient method. For a large number of nodes to be solved on a computer the T_i 's are usually assigned a zero value to start the calculation.

Figure 3-11 | Volume of resistance elements: (a) cartesian, (b) cylindrical, and (c) spherical coordinate systems.

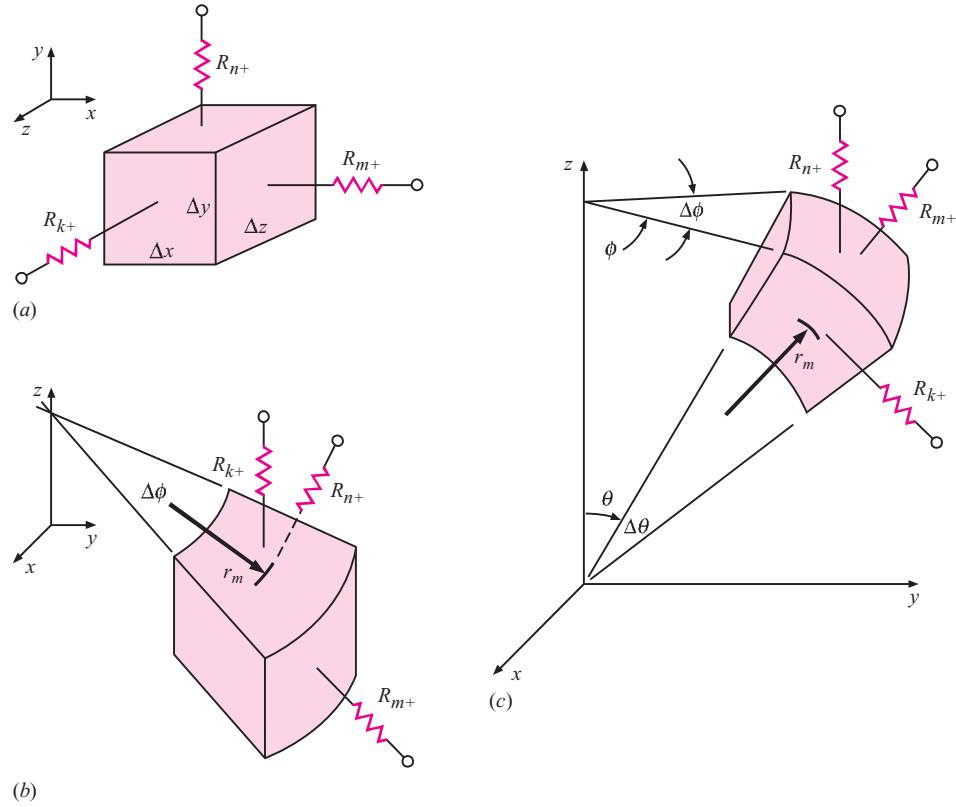


Table 3-4 | Internal nodal resistances for different coordinate systems.

	Cartesian	Cylindrical	Spherical
Nomenclature for increments	x, m	r, m	r, m
	y, n	ϕ, n	ϕ, n
	z, k	z, k	θ, k
Volume element ΔV	$\Delta x \Delta y \Delta z$	$r_m \Delta r \Delta \phi \Delta z$	$r_m^2 \sin \theta \Delta r \Delta \phi \Delta \theta$
R_{m+}	$\frac{\Delta x}{\Delta y \Delta z k}$	$\frac{\Delta r}{(r_m + \Delta r/2) \Delta \phi \Delta z k}$	$\frac{\Delta r}{(r_m + \Delta r/2)^2 \sin \theta \Delta \phi \Delta \theta k}$
R_{m-}	$\frac{\Delta x}{\Delta y \Delta z k}$	$\frac{\Delta r}{(r_m - \Delta r/2) \Delta \phi \Delta z k}$	$\frac{\Delta r}{(r_m - \Delta r/2)^2 \sin \theta \Delta \phi \Delta \theta k}$
R_{n+}	$\frac{\Delta y}{\Delta x \Delta z k}$	$\frac{r_m \Delta \phi}{\Delta r \Delta z k}$	$\frac{\Delta \phi \sin \theta}{\Delta r \Delta \theta k}$
R_{n-}	$\frac{\Delta y}{\Delta x \Delta z k}$	$\frac{r_m \Delta \phi}{\Delta r \Delta z k}$	$\frac{\Delta \phi \sin \theta}{\Delta r \Delta \theta k}$
R_{k+}	$\frac{\Delta z}{\Delta x \Delta y k}$	$\frac{\Delta z}{r_m \Delta \phi \Delta r k}$	$\frac{\Delta \theta}{\sin(\theta + \Delta \theta/2) \Delta r \Delta \phi k}$
R_{k-}	$\frac{\Delta z}{\Delta x \Delta y k}$	$\frac{\Delta z}{r_m \Delta \phi \Delta r k}$	$\frac{\Delta \theta}{\sin(\theta - \Delta \theta/2) \Delta r \Delta \phi k}$



2. Next, the new values of the nodal temperatures T_i are calculated according to Equation (3-32), *always using the most recent values of the T_j .*
3. The process is repeated until successive calculations differ by a sufficiently small amount. In terms of a computer program, this means that a test may be inserted to stop the calculations when

$$|T_{i_{n+1}} - T_{i_n}| \leq \delta \quad \text{for all } T_i$$

where δ is some selected constant and n is the number of iterations. Alternatively, a nondimensional test may be selected such that

$$\epsilon \geq \left| \frac{T_{i_{n+1}} - T_{i_n}}{T_{i_n}} \right|$$

Obviously, the smaller the value of δ , the greater the calculation time required to obtain the desired result. The reader should note, however, that the *accuracy* of the solution to the physical problem is not dependent on the value of δ alone. This constant governs the accuracy of the solution to the set of difference equations. The solution to the physical problem also depends on the selection of the increment Δx .

As we noted in the discussion of solution techniques, the matrices encountered in the numerical formulations are very sparse; they contain a large number of zeros. In solving a problem with a large number of nodes it may be quite time-consuming to enter all these zeros, and the simple form of the Gauss-Seidel equation may be preferable.

Nodal Equations for $\Delta x = \Delta y$

For nodes with $\Delta x = \Delta y$ and no heat generation, the form of Equation (3-32) has been listed as the second equation in segments of Table 3-2. The nondimensional group

$$\frac{h\Delta x}{k} = \text{Bi}$$

is called the *Biot number*. Note that equations for convection boundaries may be converted to insulated boundaries by simply setting $\text{Bi} = 0$ in the respective formula.

Heat Sources and Boundary Radiation Exchange

To include heat generation or radiation heat transfer in the nodal equations for $\Delta x = \Delta y$, one need only add a term

$$q_i/k$$

to the numerator of each of the equations. For heat sources

$$q_i = \dot{q}\Delta V$$

where \dot{q} is the heat generated per unit volume and ΔV is the volume of the respective node. Note that the volume elements are indicated in Table 3-2 by dashed lines. For an interior node $\Delta V = \Delta x\Delta y$, for a plane convection boundary $\Delta V = (\Delta x/2)\Delta y$, for an exterior corner $\Delta V = (\Delta x/2)(\Delta y/2)$, etc.

For radiation exchange at boundary node,

$$q_i = q''_{\text{rad},i} \times \Delta A$$

where ΔA is the surface area of the node exposed to radiation, and $q''_{\text{rad},i}$ is the *net radiation transferred to node i per unit area* as determined by the methods of Chapter 8.



For the common case of a surface exposed to a large enclosure at radiation temperature of T_r , the net radiation *to the surface* per unit area is given by Equation (1-12),

$$q''_{\text{rad},i} = \sigma \varepsilon_i (T_r^4 - T_i^4)$$

where ε_i is the emissivity of node i and all temperatures must be expressed in degrees absolute.

3-8 | ACCURACY CONSIDERATIONS

We have already noted that the finite-difference approximation to a physical problem improves as smaller and smaller and smaller increments of Δx and Δy are used. But we have not said how to estimate the accuracy of this approximation. Two basic approaches are available.

1. Compare the numerical solution with an analytical solution for the problem, if available, or an analytical solution for a similar problem.
2. Choose progressively smaller values of Δx and observe the behavior of the solution. If the problem has been correctly formulated and solved, the nodal temperatures should converge as Δx becomes smaller. It should be noted that computational round-off errors increase with an increase in the number of nodes because of the increased number of machine calculations. This is why one needs to observe the convergence of the solution.

It can be shown that the error of the finite-difference approximation to $\partial T/\partial x$ is of the order of $(\Delta x/L)^2$ where L is some characteristic body dimension.

Analytical solutions are of limited utility in checking the accuracy of a numerical model because most problems that will need to be solved by numerical methods either do not have an analytical solution at all or, if one is available, it may be too cumbersome to compute.

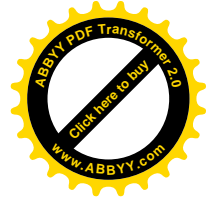
Energy Balance as Check on Solution Accuracy

In discussing solution techniques for nodal equations, we stated that an accurate solution of these equations does not ensure an accurate solution to the physical problem. In many cases the final solution is in serious error simply because the problem was not formulated correctly at the start. No computer or convergence criterion can correct this kind of error. One way to check for formulation errors is to perform some sort of energy balance using the final solution. The nature of the balance varies from problem to problem but for steady state it always takes the form of energy in equals energy out. If the energy balance does not check within reasonable limits, there is a likelihood that the problem has not been formulated correctly. Perhaps a constant is wrong here or there, or an input data point is incorrect, a faulty computer statement employed, or one or more nodal equations incorrectly written. If the energy balance does check, one may then address the issue of using smaller values of Δx to improve accuracy.

In the examples we present energy balances as a check on problem formulation.

Accuracy of Properties and Boundary Conditions

From time to time we have mentioned that thermal conductivities of materials vary with temperature; however, over a temperature range of 100 to 200°C the variation is not great (on the order of 5 to 10 percent) and we are justified in assuming constant values to simplify problem solutions. Convection and radiation boundary conditions are particularly notorious for their nonconstant behavior. Even worse is the fact that for many practical problems the



basic uncertainty in our knowledge of convection heat-transfer coefficients may not be better than ± 25 percent. Uncertainties of surface-radiation properties of ± 10 percent are not unusual at all. For example, a highly polished aluminum plate, if allowed to oxidize heavily, will absorb as much as 300 percent more radiation than when it was polished.

These remarks are not made to alarm the reader, but rather to show that selection of a large number of nodes for a numerical formulation does not necessarily produce an accurate solution to the physical problem; we must also examine uncertainties in the boundary conditions. At this point the reader is ill-equipped to estimate these uncertainties. Later chapters on convection and radiation will clarify the matter.

Gauss-Seidel Calculation

EXAMPLE 3-6

Apply the Gauss-Seidel technique to obtain the nodal temperatures for the four nodes in Figure 3-6.

Solution

It is useful to think in terms of a resistance formulation for this problem because all the connecting resistances between the nodes in Figure 3-6 are equal; that is,

$$R = \frac{\Delta y}{k\Delta x} = \frac{\Delta x}{k\Delta y} = \frac{1}{k} \tag{a}$$

Therefore, when we apply Equation (3-32) to each node, we obtain ($q_i = 0$)

$$T_i = \frac{\sum_j k_j T_j}{\sum_j k_j} \tag{b}$$

Because each node has four resistances connected to it and k is assumed constant,

$$\sum_j k_j = 4k$$

and

$$T_i = \frac{1}{4} \sum_j T_j \tag{c}$$

We now set up an iteration table as shown and use initial temperature assumptions of 300 and 200°C. Equation (c) is then applied repeatedly until satisfactory convergence is achieved. In the table, five iterations produce convergence with 0.13 degree. To illustrate the calculation, we can note the two specific cases below:

$$(T_2)_{n=1} = \frac{1}{4}(500 + 100 + T_4 + T_1) = \frac{1}{4}(500 + 100 + 200 + 275) = 268.75$$

$$(T_3)_{n=4} = \frac{1}{4}(100 + T_1 + T_4 + 100) = \frac{1}{4}(100 + 250.52 + 150.52 + 100) = 150.26$$

Number of iterations n	T_1	T_2	T_3	T_4
0	300	300	200	200
1	275	268.75	168.75	159.38
2	259.38	254.69	154.69	152.35
3	251.76	251.03	151.03	150.52
4	250.52	250.26	150.26	150.13
5	250.13	250.07	150.07	150.03

Note that in computing $(T_3)_{n=4}$ we have used the most recent information available to us for T_1 and T_4 .



EXAMPLE 3-7 Numerical Formulation with Heat Generation

We illustrate the resistance formulation in cylindrical coordinates by considering a 4.0-mm-diameter wire with uniform heat generation of 500 MW/m^3 . The outside surface temperature of the wire is 200°C , and the thermal conductivity is $19 \text{ W/m} \cdot ^\circ\text{C}$. We wish to calculate the temperature distribution in the wire. For this purpose we select four nodes as shown in Figure Example 3-7a. We shall make the calculations per unit length, so we let $\Delta z = 1.0$. Because the system is one-dimensional, we take $\Delta\phi = 2\pi$. For all the elements Δr is chosen as 0.5 mm. We then compute the resistances and volume elements using the relations from Table 3-4, and the values are given below. The computation of R_{m+} for node 4 is different from the others because the heat-flow path is shorter. For node 4, r_m is 1.75 mm, so the positive resistance extending to the known surface temperature is

$$R_{m+} = \frac{\Delta r/2}{(r_m + \Delta r/4) \Delta\phi \Delta z k} = \frac{1}{15\pi k}$$

The temperature equation for node 4 is written as

$$T_4 = \frac{2749 + 6\pi k T_3 + 15\pi k (200)}{21\pi k}$$

where the 200 is the known outer surface temperature.

Node	r_m , mm	R_{m+} , $^\circ\text{C/W}$	R_{m-} , $^\circ\text{C/W}$	$\Delta V = r_m \Delta r \Delta\phi \Delta z$, μm^3	$q_i = \dot{q} \Delta V$, W
1	0.25	$\frac{1}{2\pi k}$	∞	0.785	392.5
2	0.75	$\frac{1}{4\pi k}$	$\frac{1}{2\pi k}$	2.356	1178
3	1.25	$\frac{1}{6\pi k}$	$\frac{1}{4\pi k}$	3.927	1964
4	1.75	$\frac{1}{15\pi k}$	$\frac{1}{6\pi k}$	5.498	2749

Figure Example 3-7a | Example Schematic.

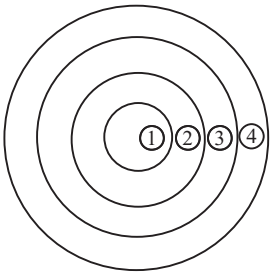
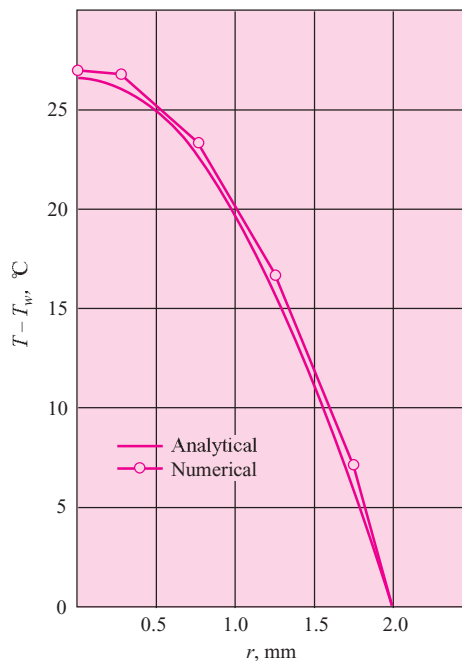
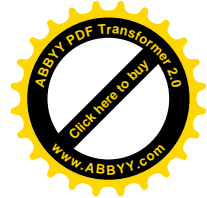


Figure Example 3-7b | Comparison of analytical and numerical solution.





A summary of the values of $\sum(1/R_{ij})$ and T_i according to Equation (3-32) is now given to be used in a Gauss-Seidel iteration scheme.

Node	$\sum \frac{1}{R_{ij}}, \text{W}/^\circ\text{C}$	$T_i = \frac{q_i + \sum(T_j/R_{ij})}{\sum(1/R_{ij})}$
1	$2\pi k = 119.38$	$T_1 = 3.288 + T_2$
2	$6\pi k = 358.14$	$T_2 = 3.289 + \frac{1}{3}T_1 + \frac{2}{3}T_3$
3	$10\pi k = 596.90$	$T_3 = 3.290 + 0.4T_2 + 0.6T_4$
4	$21\pi k = 1253.50$	$T_4 = 2.193 + \frac{2}{7}T_3 + 142.857$

Thirteen iterations are now tabulated:

Iteration <i>n</i>	Node temperature, °C			
	<i>T</i> ₁	<i>T</i> ₂	<i>T</i> ₃	<i>T</i> ₄
0	240	230	220	210
1	233.29	227.72	220.38	208.02
2	231.01	227.21	218.99	207.62
3	230.50	226.12	218.31	207.42
4	229.41	225.30	217.86	207.30
5	228.59	224.73	217.56	207.21
6	228.02	224.34	217.35	207.15
7	227.63	224.07	217.21	207.11
8	227.36	223.88	217.11	207.08
9	227.17	223.75	217.04	207.06
10	227.04	223.66	216.99	207.04
11	226.95	223.60	216.95	207.04
12	226.89	223.55	216.93	207.03
13	226.84	223.52	216.92	207.03
Analytical	225.904	222.615	216.036	206.168
Gauss-Seidel check	225.903	222.614	216.037	206.775
Exact solution of nodal equations	226.75	223.462	216.884	207.017

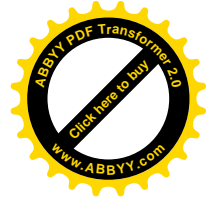
We may compare the iterative solution with an exact calculation which makes use of Equation (2-25a):

$$T - T_w = \frac{\dot{q}}{4k}(R^2 - r^2)$$

where T_w is the 200°C surface temperature, $R = 2.0$ mm, and r is the value of r_m for each node. The analytical values are shown following iteration 13, and then a Gauss-Seidel check is made on the analytical values. There is excellent agreement on the first three nodes and somewhat less on node 4. Finally, the exact solutions to the nodal equations are shown for comparison. These are the values the iterative scheme would converge to if carried far enough. In this limit the analytical and numerical calculations differ by a constant factor of about 0.85°C, and this difference results mainly from the way in which the surface resistance and boundary condition are handled. A smaller value of Δr near the surface would produce better agreement. A graphical comparison of the analytical and numerical solutions is shown in Figure Example 3-7b.

The total heat loss from the wire may be calculated as the conduction through R_{m+} at node 4. Then

$$q = \frac{T_4 - T_w}{R_{m+}} = 15\pi k(207.03 - 200) = 6.294 \text{ kW/m} \quad [6548 \text{ Btu/h} \cdot \text{ft}]$$



This must equal the total heat generated in the wire, or

$$q = \dot{q}V = (500 \times 10^6)\pi(2 \times 10^{-3})^2 = 6.283 \text{ kW/m} \quad [6536 \text{ Btu/h} \cdot \text{ft}]$$

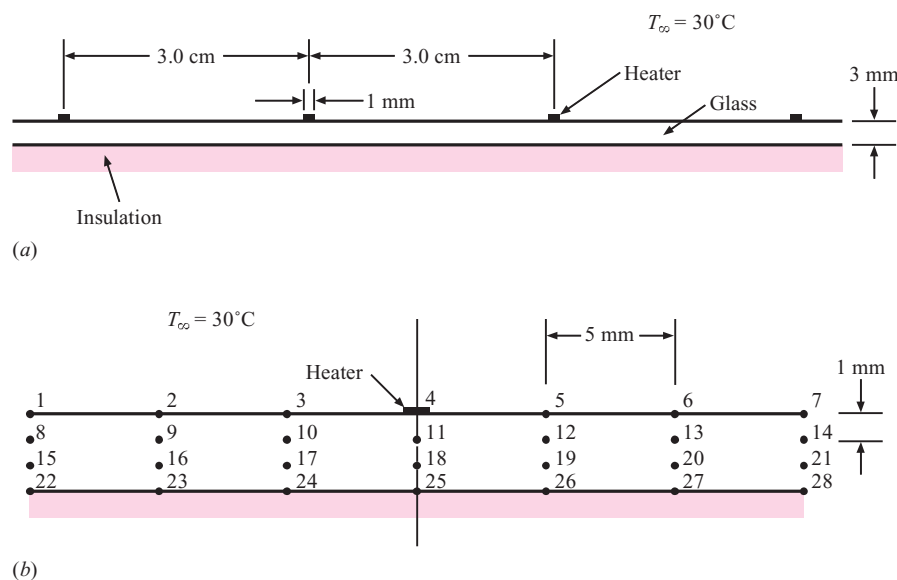
The difference between the two values results from the inaccuracy in determination of T_4 . Using the exact solution value of 207.017°C would give a heat loss of 6.2827 kW. For this problem the exact value of heat flow is 6.283 kW because the heat-generation calculation is independent of the finite-difference formulation.

Heat Generation with Nonuniform Nodal Elements

EXAMPLE 3-8

A layer of glass [$k = 0.8 \text{ W/m} \cdot ^\circ\text{C}$] 3 mm thick has thin 1-mm electric conducting strips attached to the upper surface, as shown in Figure Example 3-8. The bottom surface of the glass is insulated, and the top surface is exposed to a convection environment at 30°C with $h = 100 \text{ W/m}^2 \cdot ^\circ\text{C}$. The strips generate heat at the rate of 40 or 20 W per meter of length. Determine the steady-state temperature distribution in a typical glass section, using the numerical method for both heat-generation rates.

Figure Example 3-8 | (a) Physical system, (b) nodal boundaries.



■ Solution

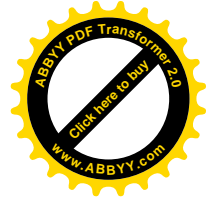
The nodal network for a typical section of the glass is shown in the figure. In this example we have *not* chosen $\Delta x = \Delta y$. Because of symmetry, $T_1 = T_7$, $T_2 = T_6$, etc., and we only need to solve for the temperatures of 16 nodes. We employ the resistance formulation. As shown, we have chosen $\Delta x = 5 \text{ mm}$ and $\Delta y = 1 \text{ mm}$. The various resistances may now be calculated:

Nodes 1, 2, 3, 4:

$$\frac{1}{R_{m+}} = \frac{1}{R_{m-}} = \frac{k(\Delta y/2)}{\Delta x} = \frac{(0.8)(0.001/2)}{0.005} = 0.08$$

$$\frac{1}{R_{n+}} = hA = (100)(0.005) = 0.5$$

$$\frac{1}{R_{n-}} = \frac{k\Delta x}{\Delta y} = \frac{(0.8)(0.005)}{0.001} = 4.0$$



Nodes 8, 9, 10, 11, 15, 16, 17, 18:

$$\frac{1}{R_{m+}} = \frac{1}{R_{m-}} = \frac{k\Delta y}{\Delta x} = \frac{(0.8)(0.001)}{0.005} = 0.16$$

$$\frac{1}{R_{n+}} = \frac{1}{R_{n-}} = \frac{k\Delta x}{\Delta y} = 4.0$$

Nodes 22, 23, 24, 25:

$$\frac{1}{R_{m+}} = \frac{1}{R_{m-}} = \frac{k(\Delta y/2)}{\Delta x} = 0.08$$

$$\frac{1}{R_{n+}} = \frac{k\Delta x}{\Delta y} = 4.0$$

$$\frac{1}{R_{n-}} = 0 \quad (\text{insulated surface})$$

The nodal equations are obtained from Equation (3-31) in the general form

$$\sum (T_j/R_{ij}) + q_i - T_i \sum (1/R_{ij}) = 0$$

Only node 4 has a heat-generation term, and $q_i = 0$ for all other nodes. From the above resistances we may calculate the $\sum (1/R_{ij})$ as

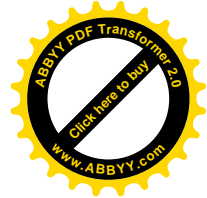
Node	$\sum (1/R_{ij})$
1, 2, 3, 4	4.66
8, . . . , 18	8.32
22, 23, 24, 25	4.16

For node 4 the equation is

$$(2)(0.08)T_3 + 4.0T_5 + (0.5)(30) + q_4 - 4.66T_4 = 0$$

The factor of 2 on T_3 occurs because $T_3 = T_5$ from symmetry. When all equations are evaluated and the solution obtained, the following temperatures result:

Node temperature, °C	q/L , W/m	
	20	40
1	31.90309	33.80617
2	32.78716	35.57433
3	36.35496	42.70993
4	49.81266	69.62532
8	32.10561	34.21122
9	33.08189	36.16377
10	36.95154	43.90307
11	47.82755	65.65510
15	32.23003	34.46006
16	33.26087	36.52174
17	37.26785	44.53571
18	46.71252	63.42504
22	32.27198	34.54397
23	33.32081	36.64162
24	37.36667	44.73333
25	46.35306	62.70613



The results of the model and calculations may be checked by calculating the convection heat lost by the top surface. Because all the energy generated in the small heater strip must eventually be lost by convection (the bottom surface of the glass is insulated and thus loses no heat), we know the numerical value that the convection should have. The convection loss at the top surface is given by

$$\begin{aligned} q_c &= \sum h_i A_i (T_i - T_\infty) \\ &= (2)(100) \left[\frac{\Delta x}{2} (T_1 - T_\infty) + \Delta x (T_2 + T_3 - 2T_\infty) + \frac{\Delta x}{2} (T_4 - T_\infty) \right] \end{aligned}$$

The factor of 2 accounts for both sides of the section. With $T_\infty = 30^\circ\text{C}$ this calculation yields

$$\begin{aligned} q_c &= 19.999995 && \text{for } q/L = 20 \text{ W/m} \\ q_c &= 40.000005 && \text{for } q/L = 40 \text{ W/m} \end{aligned}$$

Obviously, the agreement is excellent.

Composite Material with Nonuniform Nodal Elements

EXAMPLE 3-9

A composite material is embedded in a high-thermal-conductivity material maintained at 400°C as shown in Figure Example 3-9a. The upper surface is exposed to a convection environment at 30°C with $h = 25 \text{ W/m}^2 \cdot ^\circ\text{C}$. Determine the temperature distribution and heat loss from the upper surface for steady state.

■ Solution

For this example we choose nonsquare nodes as shown in Figure Example 3-9b. Note also that nodes 1, 4, 7, 10, 13, 14, and 15 consist of *two* materials. We again employ the resistance formulation.

For node 1:

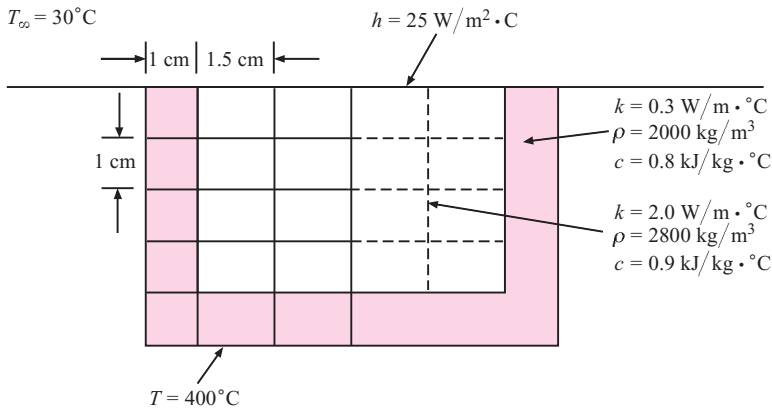
$$\begin{aligned} \frac{1}{R_{m+}} &= \frac{kA}{\Delta x} = \frac{(2.0)(0.005)}{0.015} = 0.6667 \\ \frac{1}{R_{m-}} &= \frac{kA}{\Delta x} = \frac{(0.3)(0.005)}{0.01} = 0.15 \\ \frac{1}{R_{n+}} &= hA = (25)(0.005 + 0.0075) = 0.3125 \\ \frac{1}{R_{n-}} &= \left(\frac{kA}{\Delta y} \right)_L + \left(\frac{kA}{\Delta y} \right)_R = \frac{(0.3)(0.005) + (2.0)(0.0075)}{0.01} = 1.65 \end{aligned}$$

For nodes 4, 7, 10:

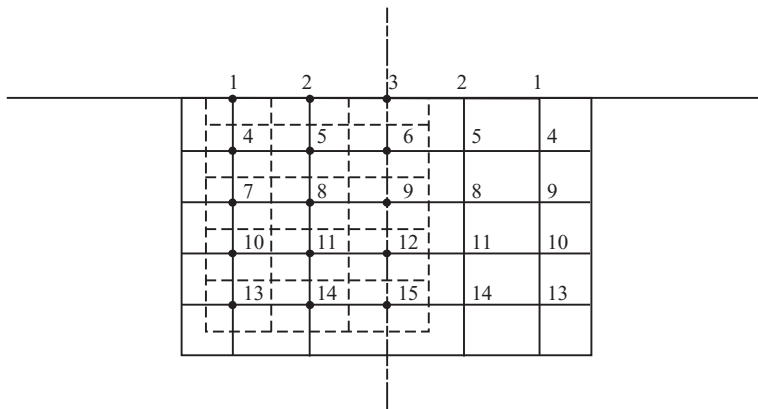
$$\begin{aligned} \frac{1}{R_{m+}} &= \frac{(2.0)(0.01)}{0.015} = 1.3333 \\ \frac{1}{R_{m-}} &= \frac{(0.3)(0.01)}{0.01} = 0.3 \\ \frac{1}{R_{n+}} &= \frac{1}{R_{n-}} = 1.65 \end{aligned}$$



Figure Example 3-9 | (a) Physical system, (b) nodal boundaries.



(a)



(b)

For node 13:

$$\frac{1}{R_{m+}} = \frac{(2.0)(0.005) + (0.3)(0.005)}{0.015} = 0.76667$$

$$\frac{1}{R_{m-}} = \frac{(0.3)(0.01)}{0.01} = 0.3$$

$$\frac{1}{R_{n+}} = 1.65$$

$$\frac{1}{R_{n-}} = \frac{(0.3)(0.0075) + (0.3)(0.005)}{0.01} = 0.375$$

For nodes, 5, 6, 8, 9, 11, 12:

$$\frac{1}{R_{m+}} = \frac{1}{R_{m-}} = \frac{(2.0)(0.01)}{0.015} = 1.3333$$

$$\frac{1}{R_{n+}} = \frac{1}{R_{n-}} = \frac{(2.0)(0.015)}{0.01} = 3.0$$

For nodes 2, 3:

$$\frac{1}{R_{m+}} = \frac{1}{R_{m-}} = \frac{(2.0)(0.005)}{0.015} = 0.6667$$



$$\frac{1}{R_{n+}} = hA = (2.5)(0.015) = 0.375$$

$$\frac{1}{R_{n-}} = 3.0$$

For nodes 14, 15:

$$\frac{1}{R_{m+}} = \frac{1}{R_{m-}} = \frac{(2.0)(0.005) + (0.3)(0.005)}{0.015} = 0.76667$$

$$\frac{1}{R_{n+}} = 3.0$$

$$\frac{1}{R_{n-}} = \frac{(0.3)(0.015)}{0.01} = 0.45$$

We shall use Equation (3-32) for formulating the nodal equations. For node 1, $\sum(1/R_{ij}) = 2.7792$, and we obtain

$$T_1 = \frac{1}{2.7792} [(400)(0.15) + (30)(0.3125) + T_2(0.6667) + 1.65T_4]$$

For node 3, $\sum(1/R_{ij}) = 4.7083$, and the nodal equation is

$$T_3 = \frac{1}{4.7083} [T_2(0.6667)(2) + 3.0T_6 + (0.375)(30)]$$

The factor of 2 on T_2 occurs because of the mirror image of T_2 to the right of T_3 .

A similar procedure is followed for the other nodes to obtain 15 nodal equations with the 15 unknown temperatures. These equations may then be solved by whatever computation method is most convenient. The resulting temperatures are:

$T_1 = 254.956$	$T_2 = 247.637$	$T_3 = 244.454$
$T_4 = 287.334$	$T_5 = 273.921$	$T_6 = 269.844$
$T_7 = 310.067$	$T_8 = 296.057$	$T_9 = 291.610$
$T_{10} = 327.770$	$T_{11} = 313.941$	$T_{12} = 309.423$
$T_{13} = 343.516$	$T_{14} = 327.688$	$T_{15} = 323.220$

The heat flow out the top face is obtained by summing the convection loss from the nodes:

$$q_{\text{conv}} = \sum hA_i(T_i - T_\infty)$$

$$= (2)(25)[(0.0125)(254.96 - 30) + (0.015)(247.64 - 30)$$

$$+ (0.0075)(244.45 - 30)]$$

$$= 382.24 \text{ W per meter of depth}$$

As a check on this value, we can calculate the heat conducted in from the 400°C surface to nodes 1, 4, 7, 10, 13, 14, and 15:

$$q_{\text{cond}} = \sum kA_i \frac{\Delta T}{\Delta x}$$

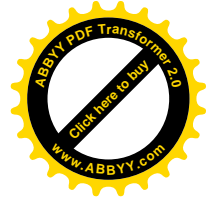
$$q_{\text{cond}} = 2 \frac{0.3}{0.01} [(0.005)(400 - 254.96) + (0.01)(400 - 287.33) + (0.01)(400 - 310.07)$$

$$+ (0.01)(400 - 327.77) + (0.0225)(400 - 343.52) + (0.015)(400 - 327.69)$$

$$+ (0.0075)(400 - 323.22)]$$

$$= 384.29 \text{ W per meter of depth}$$

The agreement is excellent.



Radiation Boundary Condition

EXAMPLE 3-10

A 1-by-2-cm ceramic strip [$k = 3.0 \text{ W/m} \cdot ^\circ\text{C}$, $\rho = 1600 \text{ kg/m}^3$, and $c = 0.8 \text{ kJ/kg} \cdot ^\circ\text{C}$] is embedded in a high-thermal-conductivity material, as shown in Figure Example 3-10, so that the sides are maintained at a constant temperature of 900°C . The bottom surface of the ceramic is insulated, and the top surface is exposed to a convection and radiation environment at $T_\infty = 50^\circ\text{C}$; $h = 50 \text{ W/m}^2 \cdot ^\circ\text{C}$, and the radiation heat loss is calculated from

$$q = \sigma A \epsilon (T^4 - T_\infty^4)$$

where

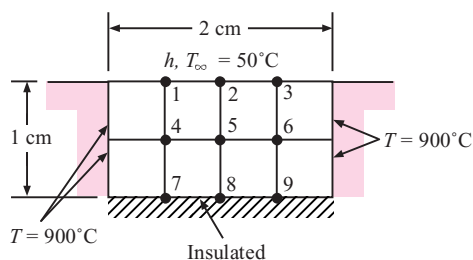
$A =$ surface area

$$\sigma = 5.669 \times 10^{-8} \text{ W/m}^2 \cdot ^\circ\text{K}^4$$

$$\epsilon = 0.7$$

Solve for the steady-state temperature distribution of the nodes shown and the rate of heat loss. The radiation temperatures are in degrees Kelvin.

Figure Example 3-10



■ Solution

We shall employ the resistance formulation and note that the radiation can be written as

$$q = \sigma \epsilon A (T^4 - T_\infty^4) = \frac{T - T_\infty}{R_{\text{rad}}} \tag{a}$$

$$\frac{1}{R_{\text{rad}}} = \sigma \epsilon A (T^2 + T_\infty^2)(T + T_\infty) \tag{b}$$

From symmetry $T_1 = T_3$, $T_4 = T_6$, and $T_7 = T_9$, so we have only six unknown nodes. The resistances are now computed:

Nodes 1, 2:

$$\frac{1}{R_{m+}} = \frac{1}{R_{m-}} = \frac{kA}{\Delta x} = \frac{(3.0)(0.0025)}{0.005} = 1.5 \quad \frac{1}{R_{n-}} = \frac{(3.0)(0.005)}{0.005} = 3.0$$

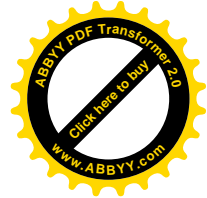
$$\frac{1}{R_{n+, \text{conv}}} = hA = (50)(0.005) = 0.25 \tag{c}$$

$$\frac{1}{R_{n+, \text{rad}}} = \sigma \epsilon A (T^2 + T_\infty^2)(T + T_\infty)$$

The radiation term introduces nonlinearities and will force us to employ an iterative solution.

Nodes 4, 5:

$$\text{All } \frac{1}{R} = \frac{kA}{\Delta x} = \frac{(3.0)(0.005)}{0.005} = 3.0$$



Nodes 7, 8:

$$\frac{1}{R_{m+}} = \frac{1}{R_{m-}} = 1.5 \quad \frac{1}{R_{n+}} = 3.0$$

Because the bottom surface is insulated, $1/R_{n-} = 0$. We now use Equation (3-32)

$$T_i = \frac{\sum(T_j/R_{ij})}{\sum(1/R_{ij})} \quad [3-33]$$

and tabulate:

Node	$\sum(1/R_{ij})$
1	$6.25 + 1/R_{\text{rad}}$
2	$6.25 + 1/R_{\text{rad}}$
4	12
5	12
7	6
8	6

Our nodal equations are thus expressed in degrees Kelvin because of the radiation terms and become

$$T_1 = \frac{1}{\sum(1/R_{ij})} [1.5T_2 + 3T_4 + (1.5)(1173) + (323)(0.25) + \sigma\epsilon(0.005)(T_1^2 + 323^2)(T_1 + 323)(323)]$$

$$T_2 = \frac{1}{\sum(1/R_{ij})} [1.5T_1(2) + 3T_5 + (323)(0.25) + \sigma\epsilon(0.005)(T_2^2 + 323^2)(T_2 + 323)(323)]$$

$$T_4 = \frac{1}{12} [(1173)(3.0) + 3T_1 + 3T_7 + 3T_5] \quad T_5 = \frac{1}{12} [2T_4(3.0) + 3T_2 + 3T_8]$$

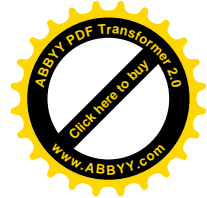
$$T_7 = \frac{1}{6} [(1173)(1.5) + 3T_4 + 1.5T_8] \quad T_8 = \frac{1}{6} [2T_7(1.5) + 3T_5]$$

The radiation terms create a very nonlinear set of equations. The computational algorithm we shall use is outlined as follows:

1. Assume $T_1 = T_2 = 1173$ K.
2. Compute $1/R_{\text{rad}}$ and $\sum(1/R_{ij})$ for nodes 1 and 2 on the basis of this assumption.
3. Solve the set of equations for T_1 through T_8 .
4. Using new values of T_1 and T_2 , recalculate $1/R_{\text{rad}}$ values.
5. Solve equations again, using new values.
6. Repeat the procedure until answers are sufficiently convergent.

The results of six iterations are shown in the table. As can be seen, the convergence is quite rapid. The temperatures are in kelvins.

Iteration	T_1	T_2	T_4	T_5	T_7	T_8
1	990.840	944.929	1076.181	1041.934	1098.951	1070.442
2	1026.263	991.446	1095.279	1068.233	1113.622	1090.927
3	1019.879	982.979	1091.827	1063.462	1110.967	1087.215
4	1021.056	984.548	1092.464	1064.344	1111.457	1087.901
5	1020.840	984.260	1092.347	1064.182	1111.367	1087.775
6	1020.879	984.313	1092.369	1064.212	1111.384	1087.798



As a practical matter, the iterations would be carried out using a commercial software package (such as those mentioned in References 22–27) and only the final set of values would be displayed on the computer.

At this point we may note that in a practical problem the value of ϵ will only be known within a tolerance of several percent, and thus there is nothing to be gained by carrying the solution to unreasonable limits of accuracy.

The heat loss is determined by calculating the radiation and convection from the top surface (nodes 1, 2, 3):

$$\begin{aligned}
 q_{\text{rad}} &= \sum \sigma \epsilon A_i (T_i^4 - 323^4) \\
 &= (5.669 \times 10^{-8})(0.7)(0.005)[(2)(1020.88^4 - 323^4) + 984.313^4 - 323^4] \\
 &= 610.8 \text{ W/m depth} \\
 q_{\text{conv}} &= \sum h A_i (T_i - 323) \\
 &= (50)(0.005)[(2)(1020.88 - 323) + 984.313 - 323] = 514.27 \text{ W} \\
 q_{\text{total}} &= 610.8 + 514.27 = 1125.07 \text{ W/m depth}
 \end{aligned}$$

This can be checked by calculating the conduction input from the 900°C surfaces:

$$\begin{aligned}
 q_{\text{cond}} &= \sum k A_i \frac{\Delta T}{\Delta x} \\
 &= \frac{(2)(3.0)}{0.005} [(0.0025)(1173 - 1020.879) + (0.005)(1173 - 1092.369) \\
 &\quad + (0.0025)(1173 - 1111.384)] \\
 &= 1124.99 \text{ W/m depth}
 \end{aligned}$$

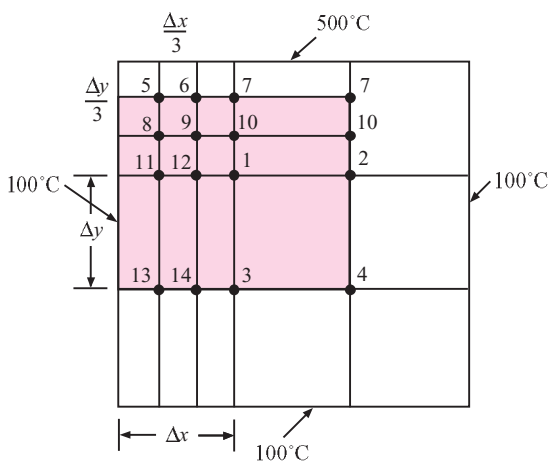
The agreement is excellent.

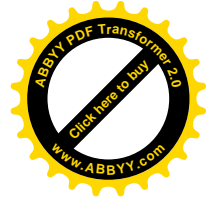
Use of Variable Mesh Size

EXAMPLE 3-11

One may use a variable mesh size in a problem with a finer mesh to help in regions of large temperature gradients. This is illustrated in Figure Example 3-11, in which Figure 3-6 is redrawn with a fine mesh in the corner. The boundary temperatures are the same as in Figure 3-6. We wish to calculate the nodal temperatures and compare with the previous solution. Note the symmetry of the problem: $T_1 = T_2$, $T_3 = T_4$, etc.

Figure Example 3-11



**■ Solution**

Nodes 5, 6, 8, and 9 are internal nodes with $\Delta x = \Delta y$ and have nodal equations in the form of Equation (3-24). Thus,

$$\begin{aligned}600 + T_6 + T_8 - 4T_5 &= 0 \\500 + T_5 + T_7 + T_9 - 4T_6 &= 0 \\100 + T_5 + T_9 + T_{11} - 4T_8 &= 0 \\T_8 + T_6 + T_{10} + T_{12} - 4T_9 &= 0\end{aligned}$$

For node 7 we can use a resistance formulation and obtain

$$\begin{aligned}1/R_{7-6} &= k \\1/R_{7-500^\circ} &= \frac{k(\Delta x/6 + \Delta x/2)}{\Delta y/3} = 2k \\1/R_{7-10} &= 2k\end{aligned}$$

and we find

$$1000 + T_6 + 2T_{10} - 5T_7 = 0$$

Similar resistances are obtained for node 10.

$$\begin{aligned}1/R_{10-9} &= k \\1/R_{10-7} = 2k &= 1/R_{10-1}\end{aligned}$$

so that

$$2T_7 + T_9 + 2T_1 - 5T_{10} = 0$$

For node 1,

$$\begin{aligned}1/R_{1-12} &= \frac{k(\Delta y/6 + \Delta y/2)}{\Delta x/3} = 2k \\1/R_{1-3} &= \frac{k(\Delta x/6 + \Delta x/2)}{\Delta y} = 2k/3 \\1/R_{1-10} &= 2k\end{aligned}$$

and the nodal equation becomes

$$3T_{12} + 3T_{10} + T_3 - 7T_1 = 0$$

For node 11,

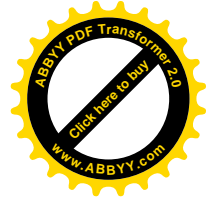
$$\begin{aligned}1/R_{11-100^\circ} = 1/R_{11-12} &= \frac{k(\Delta y/6 + \Delta y/2)}{\Delta x/3} = 2k \\1/R_{11-8} &= k \\1/R_{11-13} &= \frac{k(\Delta x/3)}{\Delta y} = k/3\end{aligned}$$

and the nodal equation becomes

$$600 + 6T_{12} + 3T_8 + T_{13} - 16T_{11} = 0$$

Similarly, the equation for node 12 is

$$3T_9 + 6T_{11} + 6T_1 + T_{14} - 16T_{12} = 0$$



For node 13,

$$1/R_{13-100^\circ} = \frac{k \Delta y}{\Delta x/3} = 3k = 1/R_{13-14}$$

$$1/R_{13-11} = 1/R_{13-100} = k/3$$

and we obtain

$$1000 + 9T_{14} + T_{11} - 20T_{13} = 0$$

Similarly for node 14,

$$100 + 9T_{13} + 9T_3 + T_{12} - 20T_{14} = 0$$

Finally, from resistances already found, the nodal equation for node 3 is

$$200 + 9T_{14} + 2T_1 - 13T_3 = 0$$

We choose to solve the set of equations by the Gauss-Seidel iteration technique and thus write them in the form $T_i = f(T_j)$. The solution was set up on a computer with all initial values for the T_i 's taken as zero. The results of the computations are shown in the following table.

Node	Number of iterations				
	2	10	20	30	50
1	59.30662	232.6668	247.1479	247.7605	247.7875
2	59.30662	232.6668	247.1479	247.7605	247.7875
3	50.11073	139.5081	147.2352	147.5629	147.5773
4	50.11073	139.5081	147.2352	147.5629	147.5773
5	206.25	288.358	293.7838	294.0129	294.023
6	248.75	359.025	366.9878	367.3243	367.3391
7	291.45	390.989	398.7243	399.0513	399.0657
8	102.9297	200.5608	208.4068	208.7384	208.753
9	121.2334	264.2423	275.7592	276.2462	276.2677
10	164.5493	302.3108	313.5007	313.974	313.9948
11	70.95459	156.9976	164.3947	164.7076	164.7215
12	73.89051	203.6437	214.5039	214.9634	214.9836
13	70.18905	115.2635	119.2079	119.3752	119.3826
14	62.82942	129.8294	135.6246	135.8703	135.8811

Again, the results of the various sets of iterations are shown merely to illustrate the rapidity of convergence. In actual practice only the final set of values would be displayed on the computer. Note that these solutions for $T_1 = T_2 = 247.79^\circ\text{C}$ and $T_3 = T_4 = 147.58^\circ\text{C}$ are somewhat below the values of 250°C and 150°C obtained when only four nodes were employed, but only modestly so.

Three-Dimensional Numerical Formulation

EXAMPLE 3-12

To further illustrate the numerical formulation, consider the simple three-dimensional block shown in Figure Example 3-12a. The block has dimensions of $3 \times 4 \times 4$ cm with the front surface exposed to a convection environment with $T_\infty = 10^\circ\text{C}$ and $h = 500 \text{ W/m}^2 \cdot ^\circ\text{C}$. The four sides are maintained constant at 100°C and the back surface is insulated. We choose $\Delta x = \Delta y = \Delta z = 1$ cm and set up the nodes as shown. The front surface has nodes 11, 12, 13, 14, 15, 16; the next Δz nodes are 21, 22, 23, 24, 25, 26 and so on. We shall use the resistance formulation in the form of Equation (3-32) to set up the nodal equations.

Figure Example 3-12a | Schematic.

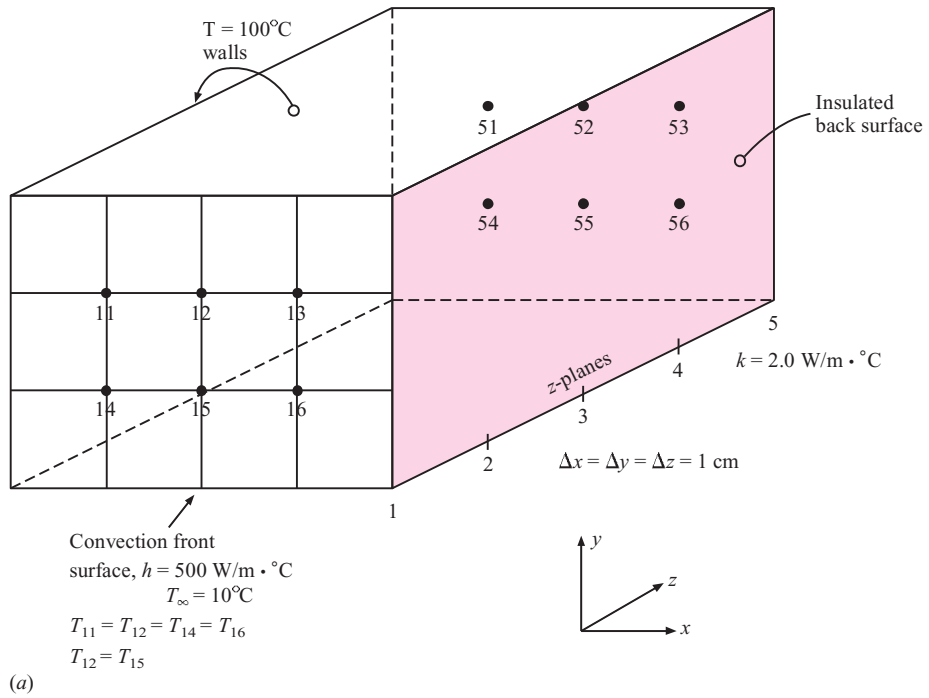
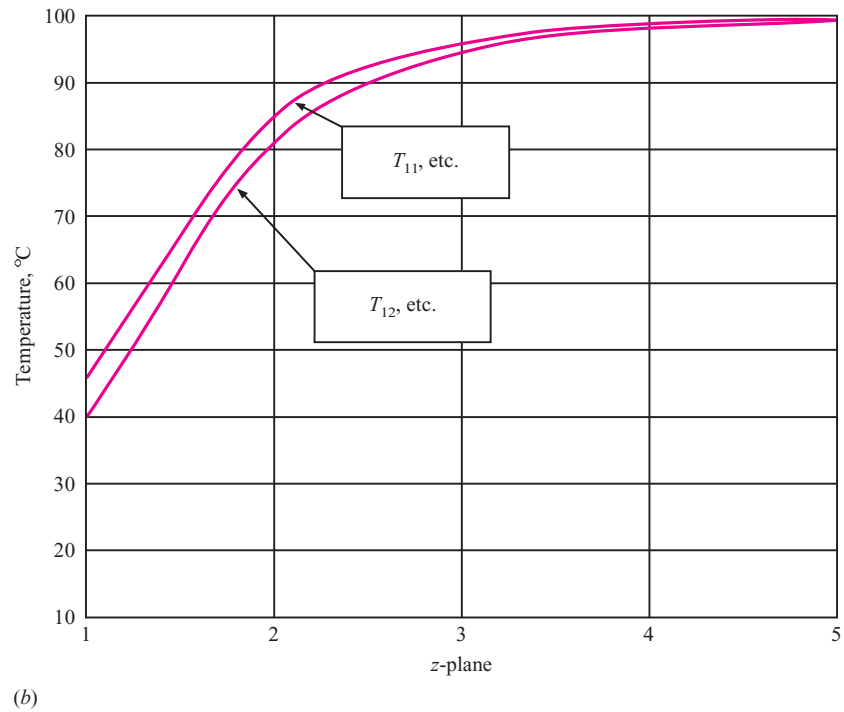


Figure Example 3-12b | Results.





■ **Solution**

All of the interior nodes for z -planes 2, 3, 4 have resistances of

$$1/R = k A / \Delta x = (2)(0.01)^2 / 0.01 = 0.02 = 1/R_{11-21} = 1/R_{21-22}, \text{ etc.}$$

The surface conduction resistances for surface z -plane 1 are

$$1/R_{11-12} = k A / \Delta x = (2)(0.01/2) / 0.01 = 0.01 = 1/R_{11-14}, \text{ etc.}$$

The surface convection resistances are

$$1/R_{11-\infty} = h A = (500)(0.01)^2 = 0.05$$

For surface nodes like 11 the $\sum(1/R_{ij})$ term in Equation (3-32) becomes

$$\sum(1/R_{11-j}) = (4)(0.01) + 0.02 + 0.05 = 0.11$$

while, for interior nodes, we have

$$\sum(1/R_{21-j}) = (6)(0.02) = 0.12$$

For the insulated back surface nodes

$$\sum(1/R_{51-j}) = (4)(0.01) + (0.02) = 0.06$$

There are 30 nodes in total; 6 in each z -plane. We could write the equations for all of them but prefer to take advantage of the symmetry of the problem as indicated in the figure. Thus,

$$T_{11} = T_{13} = T_{14} = T_{16} \quad \text{and} \quad T_{12} = T_{15}, \text{ etc.}$$

We may then write the surface nodal equations as

$$T_{11} = [0.05 T_{\infty} + 0.02 T_{21} + (0.01)(100 + 100 + T_{14} + T_{12})] / 0.11$$

$$T_{12} = [0.05 T_{\infty} + 0.02 T_{22} + (0.01)(100 + T_{11} + T_{15} + T_{13})] / 0.11$$

Inserting $T_{\infty} = 10$ and simplifying we have

$$T_{11} = (2.5 + 0.02 T_{21} + 0.01 T_{12}) / 0.1$$

$$T_{12} = (1.5 + 0.02 T_{22} + 0.02 T_{11}) / 0.1$$

Following the same procedure for the other z -planes we obtain

$$T_{21} = (200 + T_{11} + T_{31} + T_{22}) / 5$$

$$T_{22} = (100 + T_{12} + T_{32} + T_{21}) / 5$$

$$T_{31} = (200 + T_{21} + T_{41} + T_{32}) / 5$$

$$T_{32} = (100 + T_{22} + T_{42} + T_{31}) / 5$$

$$T_{41} = (200 + T_{31} + T_{51} + T_{42}) / 5$$

$$T_{42} = (100 + T_{32} + T_{52} + 2 T_{41}) / 5$$

$$T_{51} = (2 + 0.02 T_{41} + 0.01 T_{52}) / 0.05$$

$$T_{52} = (1 + 0.02 T_{42} + 0.02 T_{51}) / 0.05$$

Solving the 10 equations gives the following results for the temperatures in each z -plane.

z-plane	Node 1	Node 2
1	45.9	40.29
2	84.36	80.57
3	95.34	93.83
4	98.49	97.93
5	99.16	98.94



Figure Example 3-12b gives a graphical display of the results, and the behavior is as expected. The temperature drops as the cooled front surface is approached. Node 2 is cooled somewhat more than node 1 because it is in contact with only a single 100° surface.

■ Comments

While this is a rather simple three-dimensional example, it has illustrated the utility of the resistance formulation in solving such problems. As with two-dimensional systems, variable mesh sizes, heat generation, and variable boundary conditions can be accommodated with care and patience.

Remarks on Computer Solutions

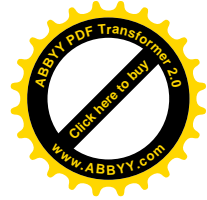
It should be apparent by now that numerical methods and computers give the engineer powerful tools for solving very complex heat-transfer problems. Many large commercial software packages are available, and new ones appear with increasing regularity. One characteristic common to almost *all* heat-transfer software is a requirement that the user *understand* something about the subject of heat transfer. Without such an understanding it can become very easy to make gross mistakes and never detect them at all. We have shown how energy balances are one way to check the validity of a computer solution. Sometimes common sense also works well. We know, for example, that a plate will cool faster when air is blown across the plate than when exposed to still air. Later, in Chapters 5 through 7, we will see how to quantify these effects and will be able to anticipate what influence they may have on a numerical solution to a conduction problem. A similar statement can be made pertaining to radiation boundary conditions, which will be examined in Chapter 8. These developments will give the reader a “feel” for what the effects of various boundary conditions should be and insight about whether the numerical solution obtained for a problem appears realistic. Up to now, boundary conditions have been stipulated quantities, but experienced heat-transfer practitioners know that they are seldom easy to determine in the real world.

3-9 | ELECTRICAL ANALOGY FOR TWO-DIMENSIONAL CONDUCTION

Steady-state electric conduction in a homogeneous material of constant resistivity is analogous to steady-state heat conduction in a body of similar geometric shape. For two-dimensional electric conduction the Laplace equation applies:

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} = 0$$

where E is the electric potential. A very simple way of solving a two-dimensional heat-conduction problem is to construct an electrical analog and experimentally determine the geometric shape factors for use in Equation (3-23). One way to accomplish this is to use a commercially available paper that is coated with a thin conductive film. This paper may be cut to an exact geometric model of the two-dimensional heat-conduction system. At the appropriate edges of the paper, good electrical conductors are attached to simulate the temperature boundary conditions on the problem. An electric-potential difference is then impressed on the model. It may be noted that the paper has a very high resistance in



comparison with the conductors attached to the edges, so that a constant-potential condition can be maintained at the region of contact.

Once the electric potential is impressed on the paper, an ordinary voltmeter may be used to plot lines of constant electric potential. With these constant-potential lines available, the flux lines may be easily constructed since they are orthogonal to the potential lines. These equipotential and flux lines have precisely the same arrangement as the isotherms and heat-flux lines in the corresponding heat-conduction problem. The shape factor is calculated immediately using the method which was applied to the curvilinear squares.

It may be noted that the conducting-sheet analogy is not applicable to problems where heat generation is present; however, by addition of appropriate resistances, convection boundary conditions may be handled with little trouble. Schneider [2] and Ozisik [10] discuss the conducting-sheet method, as well as other analogies for treating conduction heat-transfer problems, and Kayan [4, 5] gives a detailed discussion of the conducting-sheet method. Because of the utility of numerical methods, analogue techniques for solution of heat-transfer problems are largely of historical interest.

3-10 | SUMMARY

There is a myriad of analytical solutions for steady-state conduction heat-transfer problems available in the literature. In this day of computers most of these solutions are of small utility, despite their exercise in mathematical facilities. This is not to say that we cannot use the results of past experience to anticipate answers to new problems. But, most of the time, the problem a person wants to solve can be attacked directly by numerical techniques, *except* when there is an easier way to do the job. As a summary, the following suggestions are offered:

1. When tackling a two- or three-dimensional heat-transfer problem, *first* try to reduce it to a one-dimensional problem. An example is a cylinder with length much larger than its diameter.
2. If possible, select a simple shape-factor model that may either exactly or approximately represent the physical situation. See comments under items 4 and 5.
3. Seek some simple analytical solutions but, if solutions are too complicated, go directly to the numerical techniques.
4. In practical problems, recognize that convection and radiation boundary conditions are subject to large uncertainties. This means that, in most practical situations, undue concern over accuracy of solution to numerical nodal equations is unjustified.
5. In general, approach the solution in the direction of simple to complex, and make use of checkpoints along the way.

REVIEW QUESTIONS

1. What is the main assumption in the separation-of-variables method for solving Laplace's equation?
2. Define the conduction shape factor.
3. What is the basic procedure in setting up a numerical solution to a two-dimensional conduction problem?



4. Once finite-difference equations are obtained for a conduction problem, what methods are available to effect a solution? What are the advantages and disadvantages of each method, and when would each technique be applied?
5. Investigate the computer software packages that are available at your computer center for solution of conduction heat-transfer problems.

LIST OF WORKED EXAMPLES

- 3-1 Buried pipe
- 3-2 Cubical furnace
- 3-3 Buried disk
- 3-4 Buried parallel disks
- 3-5 Nine-node problem
- 3-6 Gauss-Seidel calculation
- 3-7 Numerical formulation with heat generation
- 3-8 Heat generation with nonuniform nodal elements
- 3-9 Composite material with nonuniform nodal elements
- 3-10 Radiation boundary condition
- 3-11 Use of variable mesh size
- 3-12 Three-dimensional numerical formulation

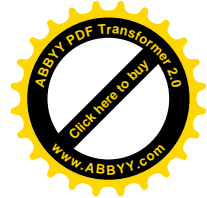
PROBLEMS

- 3-1 Beginning with the separation-of-variables solutions for $\lambda^2 = 0$ and $\lambda^2 < 0$ [Equations (3-9) and (3-10)], show that it is not possible to satisfy the boundary conditions for the constant temperature at $y = H$ with either of these two forms of solution. That is, show that, in order to satisfy the boundary conditions

$$\begin{aligned} T &= T_1 & \text{at } y = 0 \\ T &= T_1 & \text{at } x = 0 \\ T &= T_1 & \text{at } x = W \\ T &= T_2 & \text{at } y = H \end{aligned}$$

either a trivial or physically unreasonable solution results when either Equation (3-9) or (3-10) is used.

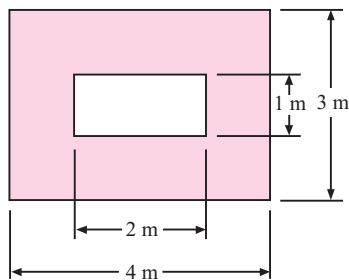
- 3-2 Write out the first four nonzero terms of the series solutions given in Equation (3-20). What percentage error results from using only these four terms at $y = H$ and $x = W/2$?
- 3-3 A horizontal pipe having a surface temperature of 67°C and diameter of 25 cm is buried at a depth of 1.2 m in the earth at a location where $k = 1.8 \text{ W/m} \cdot ^\circ\text{C}$. The earth surface temperature is 15°C . Calculate the heat lost by the pipe per unit length.
- 3-4 A 6.0-cm-diameter pipe whose surface temperature is maintained at 210°C passes through the center of a concrete slab 45 cm thick. The outer surface temperatures of the slab are maintained at 15°C . Using the flux plot, estimate the heat loss from the pipe per unit length. Also work using Table 3-1.
- 3-5 A 2.5-cm-diameter pipe carrying condensing steam at 101 kPa passes through the center of an infinite plate having a thickness of 5 cm. The plate is exposed to room



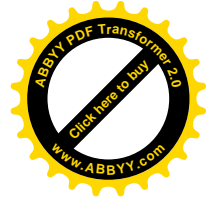
air at 27°C with a convection coefficient of 5.1 W/m² · °C on both sides. The plate is composed of an insulation material having $k = 0.1 \text{ W/m} \cdot ^\circ\text{C}$. Calculate the heat lost by the steam pipe per meter of length.

- 3-6 A heavy-wall tube of Monel, 2.5-cm ID and 5-cm OD, is covered with a 2.5-cm layer of glass wool. The inside tube temperature is 300°C, and the temperature at the outside of the insulation is 40°C. How much heat is lost per foot of length? Take $k = 11 \text{ Btu/h} \cdot \text{ft} \cdot ^\circ\text{F}$ for Monel.
- 3-7 A symmetrical furnace wall has the dimensions shown in Figure P3-7. Using the flux plot, obtain the shape factor for this wall.

Figure P3-7

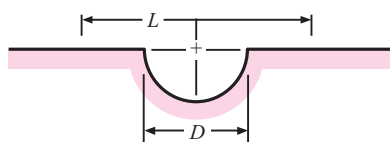


- 3-8 A furnace of 70- by 60- by 90-cm inside dimensions is constructed of a material having a thermal conductivity of 0.5 Btu/h · ft · °F. The wall thickness is 6 in. The inner and outer surface temperatures are 500 and 100°F, respectively. Calculate the heat loss through the furnace wall.
- 3-9 A cube 35 cm on each external side is constructed of fireclay brick. The wall thickness is 5.0 cm. The inner surface temperature is 500°C, and the outer surface temperature is 80°C. Compute the heat flow in watts.
- 3-10 Two long cylinders 8.0 and 3.0 cm in diameter are completely surrounded by a medium with $k = 1.4 \text{ W/m} \cdot ^\circ\text{C}$. The distance between centers is 10 cm, and the cylinders are maintained at 200 and 35°C. Calculate the heat-transfer rate per unit length.
- 3-11 A 10-cm-diameter sphere maintained at 30°C is buried in the earth at a place where $k = 1.2 \text{ W/m} \cdot ^\circ\text{C}$. The depth to the centerline is 24 cm, and the earth surface temperature is 0°C. Calculate the heat lost by the sphere.
- 3-12 A 20-cm-diameter sphere is totally enclosed by a large mass of glass wool. A heater inside the sphere maintains its outer surface temperature at 170°C while the temperature at the outer edge of the glass wool is 20°C. How much power must be supplied to the heater to maintain equilibrium conditions?
- 3-13 A large spherical storage tank, 2 m in diameter, is buried in the earth at a location where the thermal conductivity is 1.5 W/m · °C. The tank is used for the storage of an ice mixture at 0°C, and the ambient temperature of the earth is 20°C. Calculate the heat loss from the tank.
- 3-14 A 2.5-cm-diameter pipe carrying condensing steam at 101 kPa passes through the center of a square block of insulating material having $k = 0.04 \text{ W/m} \cdot ^\circ\text{C}$. The block is 5 cm on side and 2 m long. The outside of the block is exposed to room air at 27°C and a convection coefficient of $h = 5.1 \text{ W/m}^2 \cdot ^\circ\text{C}$. Calculate the heat lost by the steam pipe.



- 3-15** The solid shown in Figure P3-15 has the upper surface, including the half-cylinder cutout, maintained at 100°C . At a large depth in the solid the temperature is 300 K ; $k = 1\text{ W/m}\cdot^\circ\text{C}$. What is the heat transfer at the surface for the region where $L = 30\text{ cm}$ and $D = 10\text{ cm}$?

Figure P3-15



- 3-16** In certain locales, power transmission is made by means of underground cables. In one example an 8.0-cm-diameter cable is buried at a depth of 1.3 m, and the resistance of the cable is $1.1 \times 10^{-4}\ \Omega/\text{m}$. The surface temperature of the ground is 25°C , and $k = 1.2\text{ W/m}\cdot^\circ\text{C}$ for earth. Calculate the maximum allowable current if the outside temperature of the cable cannot exceed 110°C .
- 3-17** A copper sphere 4.0 cm in diameter is maintained at 70°C and submerged in a large earth region where $k = 1.3\text{ W/m}\cdot^\circ\text{C}$. The temperature at a large distance from the sphere is 12°C . Calculate the heat lost by the sphere.
- 3-18** Two long, eccentric cylinders having diameters of 20 and 5 cm, respectively, are maintained at 100 and 20°C and separated by a material with $k = 2.5\text{ W/m}\cdot^\circ\text{C}$. The distance between centers is 5.5 cm. Calculate the heat transfer per unit length between the cylinders.
- 3-19** Two pipes are buried in the earth and maintained at temperatures of 200 and 100°C . The diameters are 9 and 18 cm, and the distance between centers is 40 cm. Calculate the heat-transfer rate per unit length if the thermal conductivity of earth at this location is $1.1\text{ W/m}\cdot^\circ\text{C}$.
- 3-20** A hot sphere having a diameter of 1.5 m is maintained at 300°C and buried in a material with $k = 1.2\text{ W/m}\cdot^\circ\text{C}$ and outside surface temperature of 30°C . The depth of the centerline of the sphere is 3.75 m. Calculate the heat loss.
- 3-21** A scheme is devised to measure the thermal conductivity of soil by immersing a long electrically heated rod in the ground in a vertical position. For design purposes, the rod is taken as 2.5 cm in diameter with a length of 1 m. To avoid improper alteration of the soil, the maximum surface temperature of the rod is 55°C while the soil temperature is 10°C . Assuming a soil conductivity of $1.7\text{ W/m}\cdot^\circ\text{C}$, what are the power requirements of the electric heater in watts?
- 3-22** Two pipes are buried in an insulating material having $k = 0.8\text{ W/m}\cdot^\circ\text{C}$. One pipe is 10 cm in diameter and carries a hot fluid at 300°C while the other pipe is 2.8 cm in diameter and carries a cool fluid at 15°C . The pipes are parallel and separated by a distance of 12 cm on centers. Calculate the heat-transfer rate between the pipes per meter of length.
- 3-23** At a certain location the thermal conductivity of the earth is $1.5\text{ W/m}\cdot^\circ\text{C}$. At this location an isothermal sphere having a temperature of 5°C and a diameter of 2.0 m is buried at a centerline depth of 5.0 m. The earth temperature is 25°C . Calculate the heat gained by the sphere.
- 3-24** Two parallel pipes are buried very deep in the earth at a location where they are in contact with a rock formation having $k = 3.5\text{ W/m}\cdot^\circ\text{C}$. One pipe has a diameter



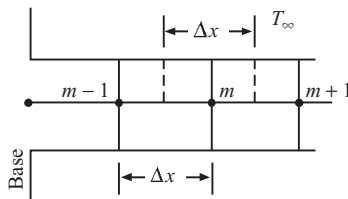
- of 20 cm and carries a hot fluid at 120°C while the other pipe has a diameter of 40 cm and carries a cooler fluid at 20°C . The distance between centers of the pipes is 1.0 m and both pipes are very long in respect to their diameters and spacing. Calculate the conduction heat transfer between the two pipes per unit length of pipe. Express as W/m length.
- 3-25** Steam pipes are sometimes carelessly buried in the earth without insulation. Consider a 10-cm pipe carrying steam at 150°C buried at a depth of 23 cm to centerline. The buried length is 100 m. Assuming that the earth thermal conductivity is $1.2 \text{ W/m} \cdot ^{\circ}\text{C}$ and the surface temperature is 15°C , estimate the heat lost from the pipe.
- 3-26** A hot steam pipe, 5 cm in diameter and carrying steam at 150°C , is placed in the center of a 15-cm-thick slab of lightweight structural concrete. The outside of the concrete slab is exposed to a convection environment that maintains the top and bottom of the sheet at 20°C . Calculate the heat lost per unit length of pipe.
- 3-27** Seven 1.0-cm-diameter tubes carrying steam at 100°C are buried in a semi-infinite medium having a thermal conductivity of $1.2 \text{ W/m} \cdot ^{\circ}\text{C}$ and surface temperature of 25°C . The depth to the centerline of the tubes is 5 cm and the spacing between centers is 3 cm. Calculate the heat lost per unit length for each tube.
- 3-28** Two parallel pipes 5 cm and 10 cm in diameter are totally surrounded by loosely packed asbestos. The distance between centers for the pipes is 20 cm. One pipe carries steam at 110°C while the other carries chilled water at 3°C . Calculate the heat lost by the hot pipe per unit length.
- 3-29** A long cylinder has its surface maintained at 135°C and is buried in a material having a thermal conductivity of $15.5 \text{ W/m} \cdot ^{\circ}\text{C}$. The diameter of the cylinder is 3 cm and the depth to its centerline is 5 cm. The surface temperature of the material is 46°C . Calculate the heat lost by the cylinder per meter of length.
- 3-30** A 2.5-m-diameter sphere contains a mixture of ice and water at 0°C and is buried in a semi-infinite medium having a thermal conductivity of $0.2 \text{ W/m} \cdot ^{\circ}\text{C}$. The top surface of the medium is isothermal at 30°C and the sphere centerline is at a depth of 8.5 m. Calculate the heat lost by the sphere.
- 3-31** An electric heater in the form of a 50- by-100-cm plate is laid on top of a semi-infinite insulating material having a thermal conductivity of $0.74 \text{ W/m} \cdot ^{\circ}\text{C}$. The heater plate is maintained at a constant temperature of 120°C over all its surface, and the temperature of the insulating material a large distance from the heater is 15°C . Calculate the heat conducted into the insulating material.
- 3-32** A thin isothermal disk, having a diameter of 1.8 cm, is maintained at 40°C and buried in a semi-infinite medium at a depth of 2 cm. The medium has a thermal conductivity of $0.8 \text{ W/m} \cdot ^{\circ}\text{C}$ and its surface is maintained at 15°C . Calculate the heat lost by the disk.
- 3-33** Two parallel pipes, each having a diameter of 5 cm, carry steam at 120°C and chilled water at 5°C , respectively, and are buried in an infinite medium of fiberglass blanket ($k = 0.04 \text{ W/m} \cdot ^{\circ}\text{C}$). Plot the heat transfer between the pipes per unit length as a function of the centerline spacing between the pipes.
- 3-34** A small furnace has inside dimensions of 60 by 70 by 80 cm with a wall thickness of 5 cm. Calculate the overall shape factor for this geometry.
- 3-35** A 15-cm-diameter steam pipe at 150°C is buried in the earth near a 5-cm pipe carrying chilled water at 5°C . The distance between centers is 15 cm and the thermal conductivity of the earth at this location may be taken as $0.7 \text{ W/m} \cdot ^{\circ}\text{C}$. Calculate the heat lost by the steam pipe per unit length.



- 3-36 Derive an equation equivalent to Equation (3-24) for an interior node in a three-dimensional heat-flow problem.
- 3-37 Derive an equation equivalent to Equation (3-24) for an interior node in a one-dimensional heat-flow problem.
- 3-38 Derive an equation equivalent to Equation (3-25) for a one-dimensional convection boundary condition.
- 3-39 Considering the one-dimensional fin problems of Chapter 2, show that a nodal equation for nodes along the fin in the Figure P3-39 may be expressed as

$$T_m \left[\frac{hP(\Delta x)^2}{kA} + 2 \right] - \frac{hP(\Delta x)^2}{kA} T_\infty - (T_{m-1} + T_{m+1}) = 0$$

Figure P3-39



- 3-40 Show that the nodal equation corresponding to an insulated wall shown in Figure P3-40 is

$$T_{m,n+1} + T_{m,n-1} + 2T_{m-1,n} - 4T_{m,n} = 0$$

Figure P3-40

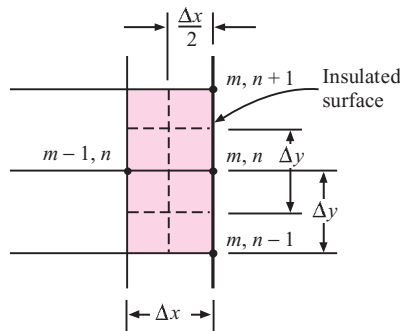
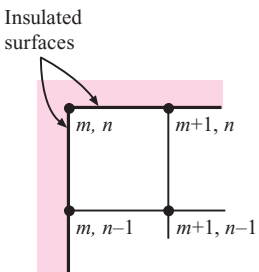
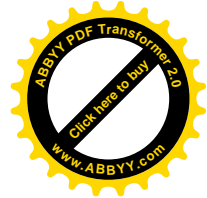


Figure P3-41



- 3-41 For the insulated corner section shown in Figure P3-41, derive an expression for the nodal equation of node (m, n) under steady-state conditions.
- 3-42 Derive the equation in Table 3-2f.
- 3-43 Derive an expression for the equation of a boundary node subjected to a constant heat flux from the environment. Use the nomenclature of Figure 3-7.
- 3-44 Set up the nodal equations for a modification of Example 3-7 in which the left half of the wire is insulated and the right half is exposed to a convection environment with $h = 200 \text{ W/m}^2 \cdot \text{°C}$ and $T = 20\text{°C}$.
- 3-45 In a proposed solar-energy application, the solar flux is concentrated on a 5-cm-OD stainless-steel tube [$k = 16 \text{ W/m} \cdot \text{°C}$] 2 m long. The energy flux on the tube surface is $20,000 \text{ W/m}^2$, and the tube wall thickness is 2 mm. Boiling water flows inside the tube with a convection coefficient of $5000 \text{ W/m}^2 \cdot \text{°C}$ and a temperature of 250°C .



Both ends of the tube are mounted in an appropriate supporting bracket, which maintains them at 100°C. For thermal-stress considerations the temperature gradient near the supports is important. Assuming a one-dimensional system, set up a numerical solution to obtain the temperature gradient near the supports.

- 3-46** An aluminum rod 2.5 cm in diameter and 15 cm long protrudes from a wall maintained at 300°C. The environment temperature is 38°C. The heat-transfer coefficient is 17 W/m² · °C. Using a numerical technique in accordance with the result of Problem 3-39, obtain values for the temperature along the rod. Subsequently obtain the heat flow from the wall at $x = 0$. *Hint:* The boundary condition at the end of the rod may be expressed by

$$T_m \left[\frac{h \Delta x}{k} + \frac{hP(\Delta x)^2}{2kA} + 1 \right] - T_\infty \left[\frac{h \Delta x}{k} + \frac{hP(\Delta x)^2}{2kA} \right] - T_{m-1} = 0$$

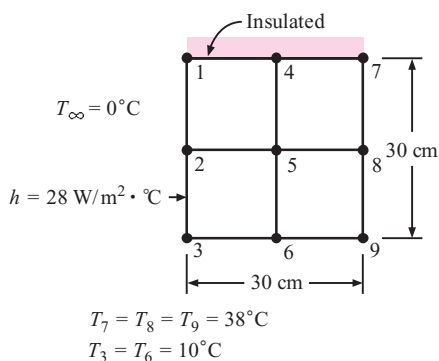
where m denotes the node at the tip of the fin. The heat flow at the base is

$$q_{x=0} = \frac{-kA}{\Delta x} (T_{m+1} - T_m)$$

where T_m is the base temperature and T_{m+1} is the temperature of the first increment.

- 3-47** Repeat Problem 3-46, using a linear variation of heat-transfer coefficient between base temperature and the tip of the fin. Assume $h = 28$ W/m² · °C at the base and $h = 11$ W/m² · °C at the tip.
- 3-48** For the wall in Problem 3-6 a material with $k = 1.4$ W/m · °C is used. The inner and outer wall temperatures are 650 and 150°C, respectively. Using a numerical technique, calculate the heat flow through the wall.
- 3-49** Repeat Problem 3-48, assuming that the outer wall is exposed to an environment at 38°C and that the convection heat-transfer coefficient is 17 W/m² · °C. Assume that the inner surface temperature is maintained at 650°C.
- 3-50** Repeat Problem 3-4, using the numerical technique.
- 3-51** In the section illustrated in Figure P3-51 the surface 1-4-7 is insulated. The convection heat transfer coefficient at surface 1-2-3 is 28 W/m² · °C. The thermal conductivity of the solid material is 5.2 W/m · °C. Using the numerical technique, compute the temperatures at nodes 1, 2, 4, and 5.

Figure P3-51



- 3-52** A glass plate 3 by 12 by 12 in [$k = 0.7$ W/m · °C] is oriented with the 12 by 12 face in a vertical position. One face loses heat by convection to the surroundings at 70°F. The other vertical face is placed in contact with a constant-temperature block at 400°F.

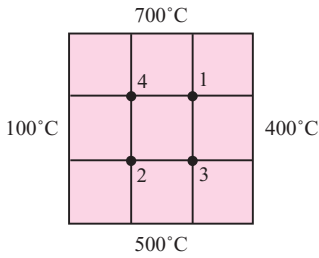


The other four faces are insulated. The convection heat-transfer coefficient varies approximately as

$$h_x = 0.22(T_s - T_\infty)^{1/4} x^{-1/4} \quad \text{Btu/h} \cdot \text{ft}^2 \cdot ^\circ\text{F}$$

where T_s and T_∞ are in degrees Fahrenheit, T_s is the local surface temperature, and x is the vertical distance from the bottom of the plate, measured in feet. Determine the convection heat loss from the plate, using an appropriate numerical analysis.

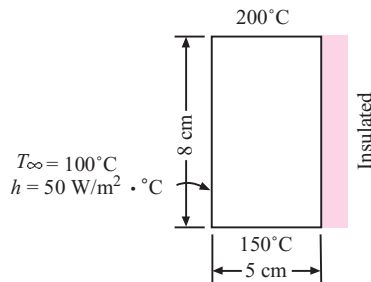
Figure P3-53



3-53 In Figure P3-53, calculate the temperatures at points 1, 2, 3, and 4 using the numerical method.

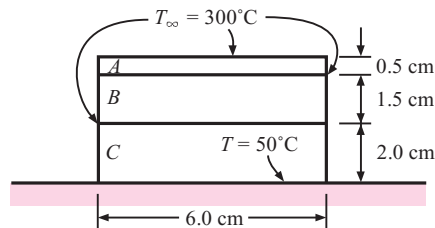
3-54 For the block shown in Figure P3-54, calculate the steady-state temperature distribution at appropriate nodal locations using the numerical method. $k = 3.2 \text{ W/m} \cdot ^\circ\text{C}$.

Figure P3-54



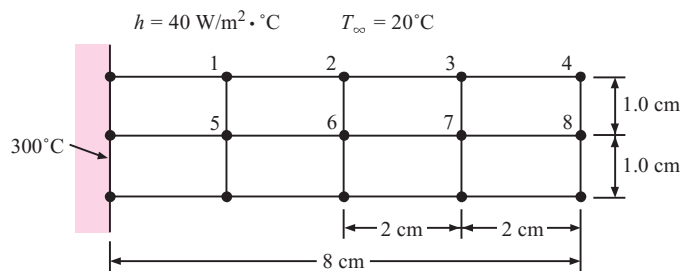
3-55 The composite strip in Figure P3-55 is exposed to the convection environment at 300°C and $h = 40 \text{ W/m}^2 \cdot ^\circ\text{C}$. The material properties are $k_A = 20 \text{ W/m} \cdot ^\circ\text{C}$, $k_B = 1.2 \text{ W/m} \cdot ^\circ\text{C}$, and $k_C = 0.5 \text{ W/m} \cdot ^\circ\text{C}$. The strip is mounted on a plate maintained at the constant temperature of 50°C . Calculate the heat transfer from the strip to plate per unit length of strip. Assume two-dimensional heat flow.

Figure P3-55



3-56 The fin shown in Figure P3-56 has a base maintained at 300°C and is exposed to the convection environment indicated. Calculate the steady-state temperatures of the nodes shown and the heat loss if $k = 1.0 \text{ W/m} \cdot ^\circ\text{C}$.

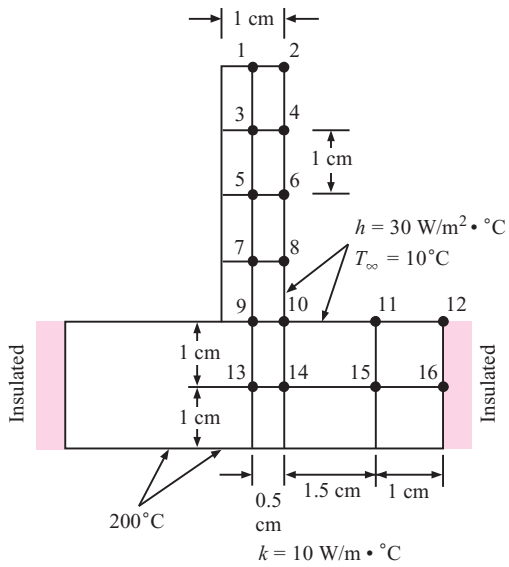
Figure P3-56





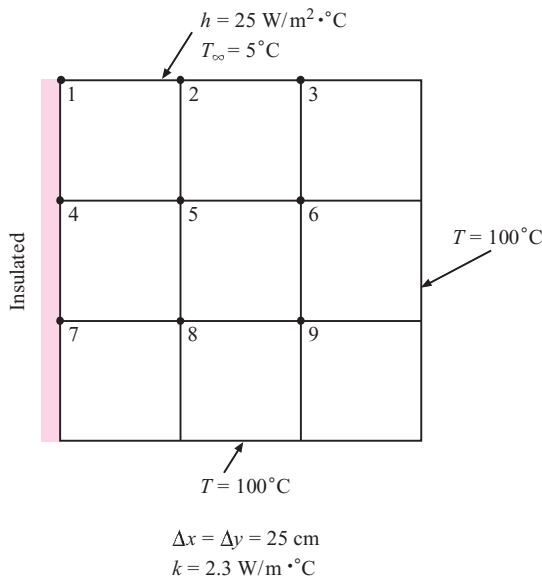
3-57 Calculate the steady-state temperatures for nodes 1 to 16 in Figure P3-57. Assume symmetry.

Figure P3-57



3-58 Calculate the steady-state temperatures for nodes 1 to 9 in Figure P3-58.

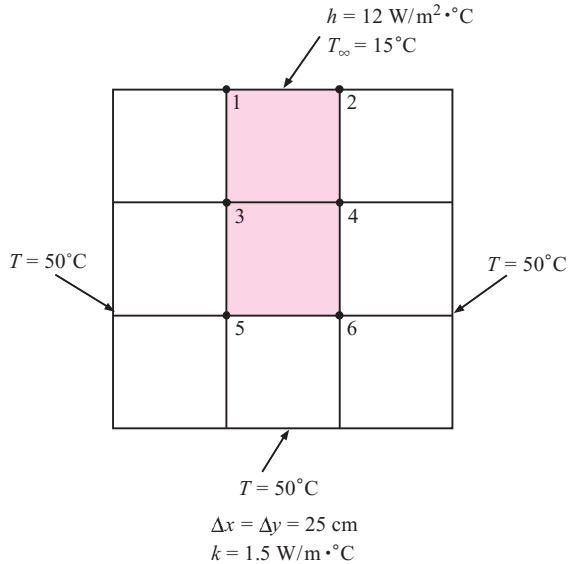
Figure P3-58





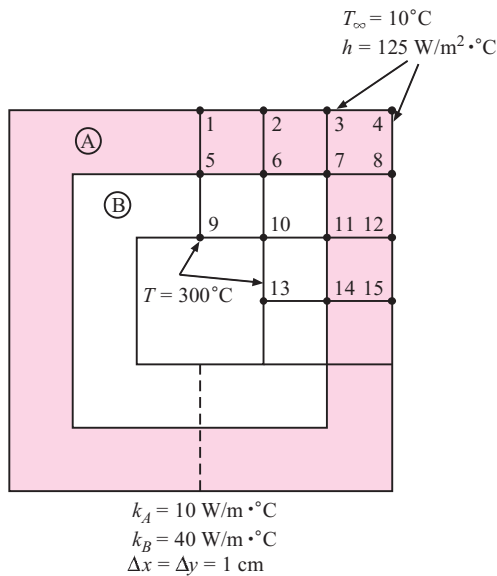
3-59 Calculate the steady-state temperatures for nodes 1 to 6 in Figure P3-59.

Figure P3-59



3-60 Calculate the temperatures for the nodes indicated in Figure P3-60. The entire outer surface is exposed to the convection environment and the entire inner surface is at a constant temperature of 300°C. Properties for materials A and B are given in the figure.

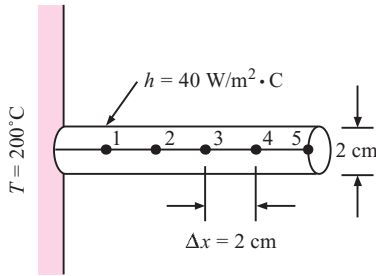
Figure P3-60





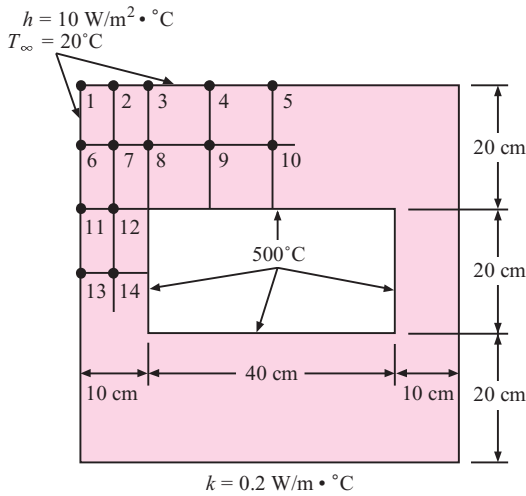
- 3-61 A rod having a diameter of 2 cm and a length of 10 cm has one end maintained at 200°C and is exposed to a convection environment at 25°C with $h = 40 \text{ W/m}^2 \cdot ^\circ\text{C}$. The rod generates heat internally at the rate of 50 MW/m^3 and the thermal conductivity is $35 \text{ W/m} \cdot ^\circ\text{C}$. Calculate the temperatures of the nodes shown in the Figure P3-61 assuming one-dimensional heat flow.

Figure P3-61



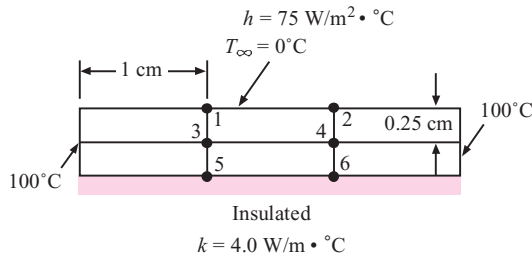
- 3-62 Calculate the steady-state temperatures of the nodes in Figure P3-62. The entire outer surface is exposed to the convection environment at 20°C and the entire inner surface is constant at 500°C. Assume $k = 0.2 \text{ W/m} \cdot ^\circ\text{C}$.

Figure P3-62



- 3-63 Calculate the steady-state temperatures for the nodes indicated in Figure P3-63.

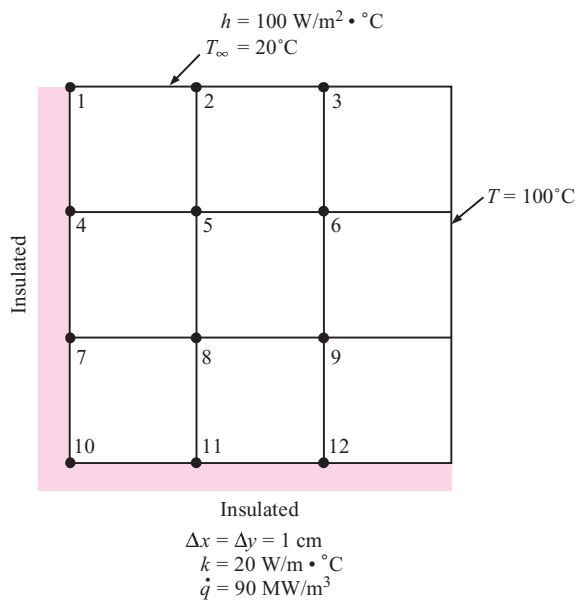
Figure P3-63





3-64 The two-dimensional solid shown in Figure P3-64 generates heat internally at the rate of 90 MW/m^3 . Using the numerical method calculate the steady-state nodal temperatures for $k = 20 \text{ W/m} \cdot ^\circ\text{C}$.

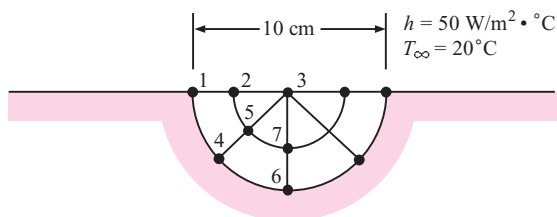
Figure P3-64



3-65 Two parallel disks having equal diameters of 30 cm are maintained at 120°C and 34°C . The disks are spaced a distance of 80 cm apart, on centers, and immersed in a conducting medium having $k = 3.4 \text{ W/m} \cdot ^\circ\text{C}$. Assuming that the disks exchange heat only on the sides facing each other, calculate the heat lost by the hotter disk, expressed in watts.

3-66 The half-cylinder has $k = 20 \text{ W/m} \cdot ^\circ\text{C}$ and is exposed to the convection environment at 20°C . The lower surface is maintained at 300°C . Compute the temperatures for the nodes shown in Figure P3-66 and the heat loss for steady state.

Figure P3-66



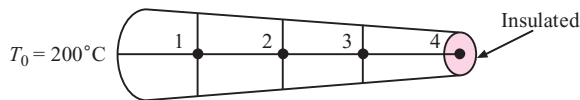
3-67 A tube has diameters of 4 mm and 5 mm and a thermal conductivity $20 \text{ W/m} \cdot ^\circ\text{C}$. Heat is generated uniformly in the tube at a rate of 500 MW/m^3 and the outside surface temperature is maintained at 100°C . The inside surface may be assumed to be insulated. Divide the tube wall into four nodes and calculate the temperature at each using the numerical method. Check with an analytical solution.

3-68 Repeat Problem 3-67 with the inside of the tube exposed to a convection condition with $h = 40 \text{ W/m}^2 \cdot ^\circ\text{C}$. Check with an analytical calculation.



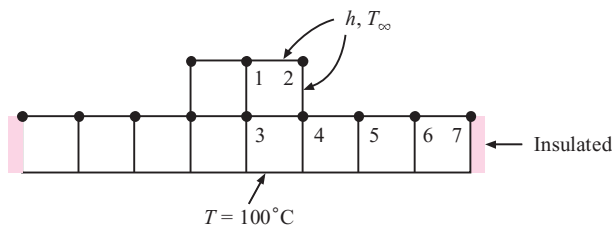
- 3-69 Rework Problem 3-57 with the surface absorbing a constant heat flux of 300 W/m^2 instead of the convection boundary condition. The bottom surface still remains at 200°C .
- 3-70 Rework Problem 3-60 with the inner surface absorbing a constant heat flux of 300 W/m^2 instead of being maintained at a constant temperature of 300°C .
- 3-71 Rework Problem 3-64 with the surface marked at a constant 100°C now absorbing a constant heat flux of 500 W/m^2 . Add nodes as necessary.
- 3-72 The tapered aluminum pin fin shown in Figure P3-72 is circular in cross section with a base diameter of 1 cm and a tip diameter of 0.5 cm. The base is maintained at 200°C and loses heat by convection to the surroundings at $T_\infty = 10^\circ\text{C}$, $h = 200 \text{ W/m}^2 \cdot ^\circ\text{C}$. The tip is insulated. Assume one-dimensional heat flow and use the finite-difference method to obtain the nodal equations for nodes 1 through 4 and the heat lost by the fin. The length of the fin is 6 cm.

Figure P3-72



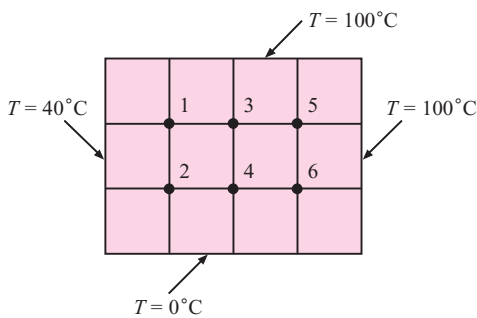
- 3-73 Write the nodal equations 1 through 7 for the symmetrical solid shown in Figure P3-73. $\Delta x = \Delta y = 1 \text{ cm}$.

Figure P3-73



- 3-74 Obtain the temperature for nodes 1 through 6 shown in Figure P3-74. $\Delta x = \Delta y = 1 \text{ cm}$.

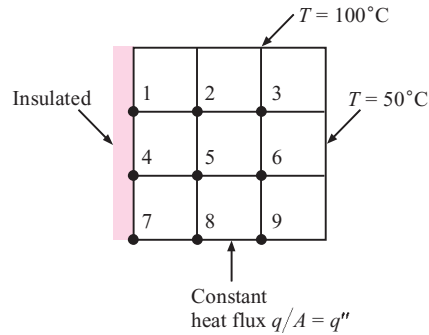
Figure P3-74





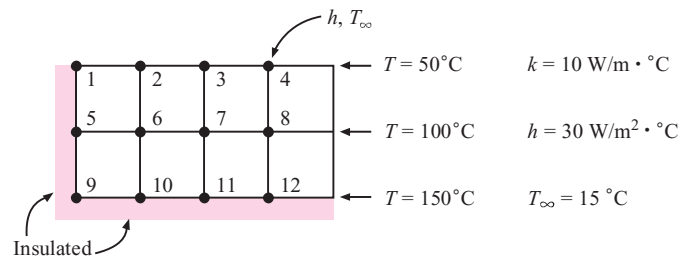
3-75 Write the nodal equations for nodes 1 through 9 shown in Figure P3-75. $\Delta x = \Delta y = 1$ cm.

Figure P3-75



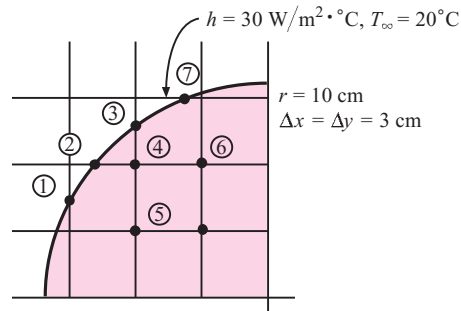
3-76 Write the nodal equation for nodes 1 through 12 shown in Figure P3-76. Express the equations in a format for Gauss-Seidel iteration.

Figure P3-76



3-77 Sometimes a square grid is desired even for a circular system. Consider the quadrant of a circle shown in Figure P3-77 with $r = 10$ cm. $\Delta x = \Delta y = 3$ cm and $k = 10 \text{ W/m}\cdot^\circ\text{C}$. Write the steady-state nodal equations for nodes 3 and 4. Make use of Tables 3-2 and 3-4.

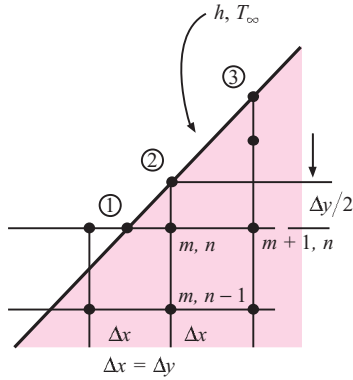
Figure P3-77





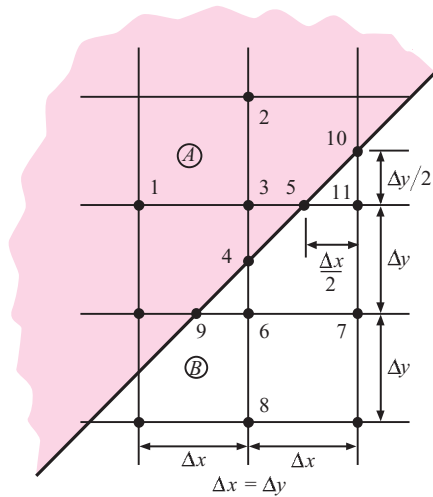
3-78 Taking Figure P3-78 as a special case of Table 3-2(*f*), write the nodal equations for nodes (*m*, *n*) and 2 for the case of $\Delta x = \Delta y$.

Figure P3-78



- 3-79 Repeat Problem 3-78 for a slanted surface that is insulated; i.e., $h = 0$.
- 3-80 If the slanted surface of Problem 3-78 is isothermal at T_∞ , what is the nodal equation for node (*m*, *n*)?
- 3-81 The slanted intersection shown in Figure P3-81 involves materials A and B. Write steady-state nodal equations for nodes 3, 4, 5, and 6 using Table 3-2(*f* and *g*) as a guide.

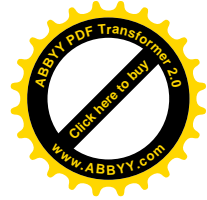
Figure P3-81



- 3-82 A horizontal plate, 25 by 50 cm, is maintained at a constant temperature of 78°C and buried in a semi-infinite medium at a depth of 5 m. The medium has an isothermal surface maintained at 15°C and a thermal conductivity of 2.8 W/m · °C. Calculate the heat lost by the plate.
- 3-83 A cube 20 cm on a side is maintained at 80°C and buried in a large medium at 10°C with a thermal conductivity of 2.3 W/m · °C. Calculate the heat lost by the cube.



- How does this compare with the heat that would be lost by a 20-cm-diameter sphere? Compare these heat transfers on a unit-volume basis.
- 3-84** A long horizontal cylinder having a diameter of 10 cm is maintained at a temperature of 100°C and centered in a 30-cm-thick slab of material for which $k = \text{W/m} \cdot ^{\circ}\text{C}$. The outside of the slab is at 20°C . Calculate the heat lost by the cylinder per unit length.
- 3-85** Work Problem 3-84 using the flux plot.
- 3-86** A horizontal plate 20 by 150 cm is buried in a large medium at a depth of 2.0 m and maintained at 50°C . The surface of the medium is at 10°C and has $k = 1.5 \text{ W/m} \cdot ^{\circ}\text{C}$. Calculate the heat lost by the plate.
- 3-87** A thin disk 10 cm in diameter is maintained at 75°C and placed on the surface of a large medium at 15°C with $k = 3 \text{ W/m} \cdot ^{\circ}\text{C}$. Calculate the heat conducted into the medium.
- 3-88** Repeat Problem 3-87 for a square 10 cm on a side. Compare the heat transfers on a per unit area basis.
- 3-89** A hot steam pipe 10 cm in diameter is maintained at 200°C and centered in a square mineral-fiber insulation 20 cm on a side. The outside surface temperature of the insulation is 35°C . Calculate the heat lost by a 20-m length of pipe if the thermal conductivity of the insulation can be taken as $50 \text{ mW/m} \cdot ^{\circ}\text{C}$.
- 3-90** A pipe having a diameter of 10 cm passes through the center of a concrete slab having a thickness of 70 cm. The surface temperature of the pipe is maintained at 100°C by condensing steam while the outer surfaces of the concrete are at 24°C . Calculate the heat lost by the pipe per meter of length.
- 3-91** Consider a circumferential fin of rectangular profile as shown in Figure 2-12. Set up nodal equations for a fin of thickness t , heat transfer coefficient h , thermal conductivity k , and heat generation rate q as a function of radial coordinate r , taking increments of Δr . Write the nodal equations for the node adjacent to the base temperature T_0 , a node in the middle of the fin, and the node at the end of the fin.
- 3-92** Set up a nodal equation for the geometry of Problem 2-123, using increments in the height of the truncated cone as the one-dimensional variable. Then work the problem with the numerical method and compare with the one-dimensional analytical solution.
- 3-93** Set up nodal equations for the geometry of Problem 2-122, using increments in an angle θ as the one-dimensional variable. Then work the problem using the numerical method and compare with the one-dimensional analytical solution.
- 3-94** A cube 30 cm on a side is buried in an infinite medium with a thermal conductivity of $1.8 \text{ W/m} \cdot ^{\circ}\text{C}$. The surface temperature of the cube is 30°C while the temperature of the medium is 10°C . Calculate the heat lost by the cube.
- 3-95** A thin horizontal disk having a diameter of 15 cm is maintained at a constant surface temperature of 87°C and buried at a depth of 20 cm in a semi-infinite medium with an adiabatic surface. The thermal conductivity of the medium is $2.7 \text{ W/m} \cdot ^{\circ}\text{C}$ and the temperature of the medium a large distance away from the disk (*not* the adiabatic the surface temperature) is 13°C . Calculate the heat lost by the disk in watts.
- 3-96** A copper rod has an internal heater that maintains its surface temperature at 50°C while it is buried vertically in a semi-infinite medium. The rod is 2 cm in diameter and 40 cm long and the isothermal surface of the medium is at 20°C . Calculate the heat lost by the rod if the thermal conductivity of the medium is $3.4 \text{ W/m} \cdot ^{\circ}\text{C}$.
- 3-97** Rework Problem 2-122, using a numerical approach with five nodes operating in increments of the radial angle θ , and compare with the analytical results of Problem 2-122.



Design-Oriented Problems

- 3-98** A liner of stainless steel ($k = 20 \text{ W/m} \cdot ^\circ\text{C}$), having a thickness of 3 mm, is placed on the inside surface of the solid in Problem 3-62. Assuming now that the inside surface of the stainless steel is at 500°C , calculate new values for the nodal temperatures in the low-conductivity material. Set up your nodes in the stainless steel as necessary.
- 3-99** A basement for a certain home is $4 \times 5 \text{ m}$ with a ceiling height of 3 m. The walls are concrete having a thickness of 10 cm. In the winter the convection coefficient on the inside is $10 \text{ W/m}^2 \cdot ^\circ\text{C}$ and the soil on the outside has $k = 1.7 \text{ W/m} \cdot ^\circ\text{C}$. Analyze this problem and determine an overall heat transfer coefficient U defined by $q_{\text{loss}} = UA_{\text{inside}}(T_{\text{inside}} - T_{\text{soil}})$. Determine the heat loss when $T_{\text{inside}} = 26^\circ\text{C}$ and $T_{\text{soil}} = 15^\circ\text{C}$.
- 3-100** A groundwater heat pump is a refrigeration device that rejects heat to the ground through buried pipes instead of to the local atmosphere. The heat rejection rate for such a machine at an Oklahoma location is to be 22 kW in a location where the ground temperature at depth is 17°C . The thermal conductivity of the soil at this location may be taken as $1.6 \text{ W/m} \cdot ^\circ\text{C}$. Water is to be circulated through a length of horizontal buried pipe or tube with the water entering at 29°C and leaving at 23.5°C . The convection coefficient on the inside of the pipe is sufficiently high that the inner pipe wall temperature may be assumed to be the same as the water temperature. Select an appropriate pipe/tube material, size, and length to accomplish the required cooling. You may choose standard steel pipe sizes from Table A-11. Standard tubing or plastic pipe sizes are obtained from other sources. Examine several choices before making your final selection and give reasons for that selection.
- 3-101** Professional chefs claim that gas stove burners are superior to electric burners because of the more uniform heating afforded by the gas flame and combustion products around the bottom of a cooking pan. Advocates of electric stoves note the lack of combustion products to pollute the air in the cooking area, but acknowledge that gas heat may be more uniform. Manufacturers of thick-bottomed cookware claim that their products can achieve uniformity of cooking as good as gas heat because of the “spreading” of heat through an 8-mm-thick aluminum layer on the bottom of the pan. You are asked to verify this claim. For the evaluation assume a 200-mm-diameter pan with an 8-mm-thick aluminum bottom and the interior exposed to boiling water, which produces $h = 1500 \text{ W/m}^2 \cdot ^\circ\text{C}$ at 1 atm (100°C). Observe the approximate spacing for the circular element in an electric burner and devise an appropriate numerical model to investigate the uniformity-of-heating claim. Consider such factors as contact resistance between the burner element and the bottom of the pan, and radiation transfer that might be present. Consider different heating rates (different burner element temperatures) and their effect. When the study is complete, make recommendations as to what the cookware manufacturers might prudently claim for their thick-bottomed product. Discuss uncertainties in your analysis.
- 3-102** The fin analyses of Section 2-10 assumed one-dimensional heat flows in the fins. Devise a numerical model similar to that shown in Problem 3-57 to examine the validity of this assumption. Restrict the analysis to aluminum with $k = 200 \text{ W/m} \cdot ^\circ\text{C}$. Examine several different combinations of fin thickness, fin length, and convection coefficient to determine the relative effects on temperature variation across the fin thickness. State conclusions as you think appropriate.
- 3-103** A small building 5 m wide by 7 m long by 3 m high (inside dimensions) is mounted on a flat concrete slab having a thickness of 15 cm. The walls of the building are



constructed of concrete also, with a thickness of 7 cm. The inside of the building is used for cold storage at -20°C and the outside of the building is exposed to ambient air at 30°C , with a convection coefficient of $15\text{ W/m}^2 \cdot ^{\circ}\text{C}$. The inside convection coefficient for the building is estimated at $10\text{ W/m}^2 \cdot ^{\circ}\text{C}$ and the floor slab is in contact with earth having $k = 1.8\text{ W/m} \cdot ^{\circ}\text{C}$. The earth temperature may be assumed to be 15°C . Calculate the heat gained by the building in the absence of any insulating material on the outside. Next, select two alternative insulation materials for the outside of the building from Table 2-1 and/or Table A-3. The insulation objective is to raise the outside surface temperature of the insulation to 26°C for the ambient temperature of 30°C . The refrigeration system operates in such a manner that 1 kW will produce 4000 kJ/hr of cooling, and electricity costs $\$0.085/\text{kWh}$. Economics dictates that the insulation should pay for itself in a three-year period. What is the allowable cost per unit volume of insulation to accomplish this payback objective, for the two insulating materials selected? Suppose an outside surface temperature of 24°C is chosen as the allowable value for the insulation. What would the allowable costs be for a three-year payback in this case? Make your own assumptions as to the annual hours of operation for the cooling system.

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