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CHAPTER

4 Unsteady-State Conduction

4-1 | INTRODUCTION

If a solid body is suddenly subjected to a change in environment, some time must elapse before an equilibrium temperature condition will prevail in the body. We refer to the equilibrium condition as the steady state and calculate the temperature distribution and heat transfer by methods described in Chapters 2 and 3. In the transient heating or cooling process that takes place in the interim period before equilibrium is established, the analysis must be modified to take into account the change in internal energy of the body with time, and the boundary conditions must be adjusted to match the physical situation that is apparent in the unsteady-state heat-transfer problem. Unsteady-state heat-transfer analysis is obviously of significant practical interest because of the large number of heating and cooling processes that must be calculated in industrial applications.

To analyze a transient heat-transfer problem, we could proceed by solving the general heat-conduction equation by the separation-of-variables method, similar to the analytical treatment used for the two-dimensional steady-state problem discussed in Section 3-2. We give one illustration of this method of solution for a case of simple geometry and then refer the reader to the references for analysis of more complicated cases. Consider the infinite plate of thickness $2L$ shown in Figure 4-1. Initially the plate is at a uniform temperature T_i, and at time zero the surfaces are suddenly lowered to $T = T_1$. The differential equation is

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau} \qquad \textbf{[4-1]}$$

The equation may be arranged in a more convenient form by introduction of the variable $\theta = T - T_1$. Then

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial \tau} \qquad \textbf{[4-2]}$$

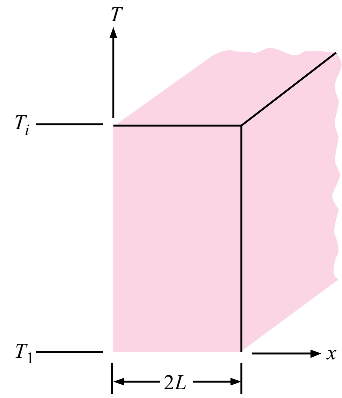
with the initial and boundary conditions

$$\theta = \theta_i = T_i - T_1 \quad \text{at } \tau = 0,\ 0 \le x \le 2L \qquad \textbf{[}\boldsymbol{a}\textbf{]}$$

$$\theta = 0 \quad \text{at } x = 0,\ \tau > 0 \qquad \textbf{[}\boldsymbol{b}\textbf{]}$$

$$\theta = 0 \quad \text{at } x = 2L,\ \tau > 0 \qquad \textbf{[}\boldsymbol{c}\textbf{]}$$

Figure 4-1 | Infinite plate subjected to sudden cooling of surfaces.



Assuming a product solution $\theta(x, \tau) = X(x)\mathcal{H}(\tau)$ produces the two ordinary differential equations

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0$$

$$\frac{d\mathcal{H}}{d\tau} + \alpha\lambda^2\mathcal{H} = 0$$

where λ^2 is the separation constant. In order to satisfy the boundary conditions it is necessary that $\lambda^2 > 0$ so that the form of the solution becomes

$$\theta = (C_1 \cos \lambda x + C_2 \sin \lambda x)e^{-\lambda^2 \alpha\tau}$$

From boundary condition (b), $C_1 = 0$ for $\tau > 0$. Because C_2 cannot also be zero, we find from boundary condition (c) that $\sin 2L\lambda = 0$, or

$$\lambda = \frac{n\pi}{2L} \qquad n = 1, 2, 3, \ldots$$

The final series form of the solution is therefore

$$\theta = \sum_{n=1}^{\infty} C_n e^{-[n\pi/2L]^2 \alpha\tau} \sin \frac{n\pi x}{2L}$$

This equation may be recognized as a Fourier sine expansion with the constants C_n determined from the initial condition *(a)* and the following equation:

$$C_n = \frac{1}{L}\int_0^{2L} \theta_i \sin \frac{n\pi x}{2L}\, dx = \frac{4}{n\pi}\theta_i \qquad n = 1, 3, 5, \ldots$$

The final series solution is therefore

$$\frac{\theta}{\theta_i} = \frac{T - T_1}{T_i - T_1} = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} e^{-[n\pi/2L]^2 \alpha\tau} \sin \frac{n\pi x}{2L} \qquad n = 1, 3, 5 \ldots \tag{4-3}$$

We note, of course, that at time zero ($\tau = 0$) the series on the right side of Equation (4-3) must converge to unity for all values of x.

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In Section 4-4, this solution will be presented in graphical form for calculation purposes. For now, our purpose has been to show how the unsteady-heat-conduction equation can be solved, for at least one case, with the separation-of-variables method. Further information on analytical methods in unsteady-state problems is given in the references.

4-2 | LUMPED-HEAT-CAPACITY SYSTEM

We continue our discussion of transient heat conduction by analyzing systems that may be considered uniform in temperature. This type of analysis is called the *lumped-heat-capacity* method. Such systems are obviously idealized because a temperature gradient must exist in a material if heat is to be conducted into or out of the material. In general, the smaller the physical size of the body, the more realistic the assumption of a uniform temperature throughout; in the limit a differential volume could be employed as in the derivation of the general heat-conduction equation.

If a hot steel ball were immersed in a cool pan of water, the lumped-heat-capacity method of analysis might be used if we could justify an assumption of uniform ball temperature during the cooling process. Clearly, the temperature distribution in the ball would depend on the thermal conductivity of the ball material and the heat-transfer conditions from the surface of the ball to the surrounding fluid (i.e., the surface-convection heat-transfer coefficient). We should obtain a reasonably uniform temperature distribution in the ball if the resistance to heat transfer by conduction were small compared with the convection resistance at the surface, so that the major temperature gradient would occur through the fluid layer at the surface. The lumped-heat-capacity analysis, then, is one that assumes that the internal resistance of the body is negligible in comparison with the external resistance.

The convection heat loss from the body is evidenced as a decrease in the internal energy of the body, as shown in Figure 4-2. Thus,

$$q = hA(T - T_\infty) = -c\rho V \frac{dT}{d\tau} \qquad [4\text{-}4]$$

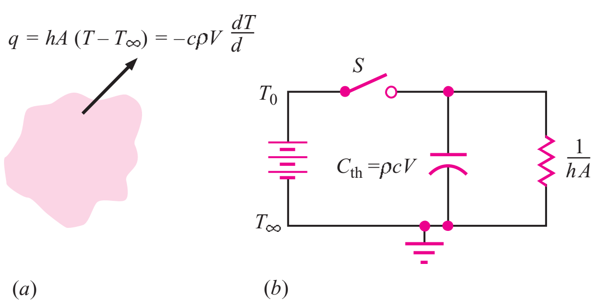
where A is the surface area for convection and V is the volume. The initial condition is written

$$T = T_0 \qquad \text{at } \tau = 0$$

so that the solution to Equation (4-4) is

$$\frac{T - T_\infty}{T_0 - T_\infty} = e^{-[hA/\rho c V]\tau} \qquad [4\text{-}5]$$

Figure 4-2 | Nomenclature for single-lump heat-capacity analysis.



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where T_∞ is the temperature of the convection environment. The thermal network for the single-capacity system is shown in Figure 4-2*b*. In this network we notice that the thermal capacity of the system is "charged" initially at the potential T_0 by closing the switch S. Then, when the switch is opened, the energy stored in the thermal capacitance is dissipated through the resistance $1/hA$. The analogy between this thermal system and an electric system is apparent, and we could easily construct an electric system that would behave exactly like the thermal system as long as we made the ratio

$$\frac{hA}{\rho c V} = \frac{1}{R_{\text{th}} C_{\text{th}}} \qquad R_{\text{th}} = \frac{1}{hA} \qquad C_{\text{th}} = \rho c V$$

equal to $1/R_e C_e$, where R_e and C_e are the electric resistance and capacitance, respectively. In the thermal system we store energy, while in the electric system we store electric charge. The flow of energy in the thermal system is called heat, and the flow of charge is called electric current. The quantity $c\rho V/hA$ is called the *time constant* of the system because it has the dimensions of time. When

$$\tau = \frac{c\rho V}{hA}$$

it is noted that the temperature difference $T - T_\infty$ has a value of 36.8 percent of the initial difference $T_0 - T_\infty$.

The reader should note that the lumped-capacity formulation assumes essentially uniform temperature throughout the solid at any instant of time so that the change in internal energy can be represented by $\rho c V dT/d\tau$. It does *not* require that the convection boundary condition have a constant value of h. In fact, variable values of h coupled with radiation boundary conditions are quite common. The specification of "time constant" in terms of the 36.8 percent value stated above implies a constant boundary condition.

For variable convection or radiation boundary conditions, numerical methods (see Section 4-6) are used to advantage to predict lumped capacity behavior. A rather general setup of a lumped-capacity solution using numerical methods and Microsoft Excel is given in Section D-6 of the Appendix. In some cases, multiple lumped-capacity formulations can be useful. An example involving the combined convection-radiation cooling of a box of electronic components is also given in this same section of the Appendix.

Applicability of Lumped-Capacity Analysis

We have already noted that the lumped-capacity type of analysis assumes a uniform temperature distribution throughout the solid body and that the assumption is equivalent to saying that the surface-convection resistance is large compared with the internal-conduction resistance. Such an analysis may be expected to yield reasonable estimates within about 5 percent when the following condition is met:

$$\frac{h(V/A)}{k} < 0.1 \qquad \textbf{[4-6]}$$

where k is the thermal conductivity of the solid. In sections that follow, we examine those situations for which this condition does *not* apply. We shall see that the lumped-capacity analysis has a direct relationship to the numerical methods discussed in Section 4-7. If one considers the ratio $V/A = s$ as a characteristic dimension of the solid, the dimensionless group in Equation (4-6) is called the *Biot number:*

$$\frac{hs}{k} = \text{Biot number} = \text{Bi}$$

The reader should recognize that there are many practical cases where the lumped-capacity method may yield good results. In Table 4-1 we give some examples that illustrate the relative validity of such cases.

Table 4-1 | Examples of lumped-capacity systems.

Physical situation	k, W/m · °C	Approximate value of h, W/m^2 · °C	$\frac{h(V/A)}{k}$
1. 3.0-cm steel cube cooling in room air	40	7.0	8.75×10^{-4}
2. 5.0-cm glass cylinder cooled by a 50-m/s airstream	0.8	180	2.81
3. Same as situation 2 but a copper cylinder	380	180	0.006
4. 3.0-cm hot copper cube submerged in water such that boiling occurs	380	10,000	0.132

We may point out that uncertainties in the knowledge of the convection coefficient of ±25 percent are quite common, so that the condition $\mathrm{Bi} = h(V/A)/k < 0.1$ should allow for some leeway in application.

Do not dismiss lumped-capacity analysis because of its simplicity. Because of uncertainties in the convection coefficient, it may not be necessary to use more elaborate analysis techniques.

Steel Ball Cooling in Air — EXAMPLE 4-1

A steel ball [$c = 0.46$ kJ/kg · °C, $k = 35$ W/m · °C] 5.0 cm in diameter and initially at a uniform temperature of 450°C is suddenly placed in a controlled environment in which the temperature is maintained at 100°C. The convection heat-transfer coefficient is 10 W/m^2 · °C. Calculate the time required for the ball to attain a temperature of 150°C.

Solution

We anticipate that the lumped-capacity method will apply because of the low value of h and high value of k. We can check by using Equation (4-6):

$$\frac{h(V/A)}{k} = \frac{(10)[(4/3)\pi(0.025)^3]}{4\pi(0.025)^2(35)} = 0.0023 < 0.1$$

so we may use Equation (4-5). We have

$$\begin{aligned}
T &= 150°\mathrm{C} \qquad \rho = 7800\ \mathrm{kg/m^3} \quad [486\ \mathrm{lb}_m/\mathrm{ft}^3] \\
T_\infty &= 100°\mathrm{C} \qquad h = 10\ \mathrm{W/m^2 \cdot °C} \quad [1.76\mathrm{Btu/h \cdot ft^2 \cdot °F}] \\
T_0 &= 450°\mathrm{C} \qquad c = 460\ \mathrm{J/kg \cdot °C} \quad [0.11\ \mathrm{Btu/lb}_m \cdot °\mathrm{F}]
\end{aligned}$$

$$\frac{hA}{\rho c V} = \frac{(10)4\pi(0.025)^2}{(7800)(460)(4\pi/3)(0.025)^3} = 3.344 \times 10^{-4}\ \mathrm{s}^{-1}$$

$$\frac{T - T_\infty}{T_0 - T_\infty} = e^{-[hA/\rho c V]\tau}$$

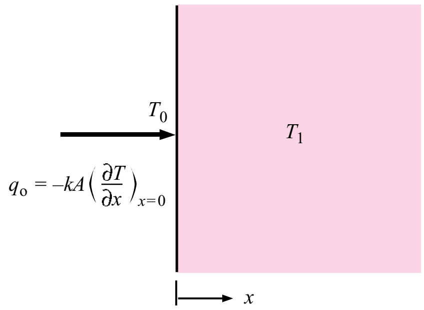
$$\frac{150 - 100}{450 - 100} = e^{-3.344 \times 10^{-4}\tau}$$

$$\tau = 5819\ \mathrm{s} = 1.62\ \mathrm{h}$$

4-3 | TRANSIENT HEAT FLOW IN A SEMI-INFINITE SOLID

Consider the semi-infinite solid shown in Figure 4-3 maintained at some initial temperature T_i. The surface temperature is suddenly lowered and maintained at a temperature T_0, and we

Figure 4-3 | Nomenclature for transient heat flow in a semi-infinite solid.



seek an expression for the temperature distribution in the solid as a function of time. This temperature distribution may subsequently be used to calculate heat flow at any x position in the solid as a function of time. For constant properties, the differential equation for the temperature distribution $T(x, \tau)$ is

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau} \qquad \textbf{[4-7]}$$

The boundary and initial conditions are

$$T(x, 0) = T_i$$
$$T(0, \tau) = T_0 \qquad \text{for } \tau > 0$$

This is a problem that may be solved by the Laplace-transform technique. The solution is given in Reference 1 as

$$\frac{T(x, \tau) - T_0}{T_i - T_0} = \text{erf}\,\frac{x}{2\sqrt{\alpha\tau}} \qquad \textbf{[4-8]}$$

where the Gauss error function is defined as

$$\text{erf}\,\frac{x}{2\sqrt{\alpha\tau}} = \frac{2}{\sqrt{\pi}}\int^{x/2\sqrt{\alpha\tau}} e^{-\eta^2}\, d\eta \qquad \textbf{[4-9]}$$

It will be noted that in this definition η is a dummy variable and the integral is a function of its upper limit. When the definition of the error function is inserted in Equation (4-8), the expression for the temperature distribution becomes

$$\frac{T(x, \tau) - T_0}{T_i - T_0} = \frac{2}{\sqrt{\pi}}\int^{x/2\sqrt{\alpha\tau}} e^{-\eta^2}\, d\eta \qquad \textbf{[4-10]}$$

The heat flow at any x position may be obtained from

$$q_x = -kA\frac{\partial T}{\partial x}$$

Performing the partial differentiation of Equation (4-10) gives

$$\begin{aligned}\frac{\partial T}{\partial x} &= (T_i - T_0)\frac{2}{\sqrt{\pi}}e^{-x^2/4\alpha\tau}\frac{\partial}{\partial x}\left(\frac{x}{2\sqrt{\alpha\tau}}\right) \\ &= \frac{T_i - T_0}{\sqrt{\pi\alpha\tau}}e^{-x^2/4\alpha\tau}\end{aligned} \qquad \textbf{[4-11]}$$

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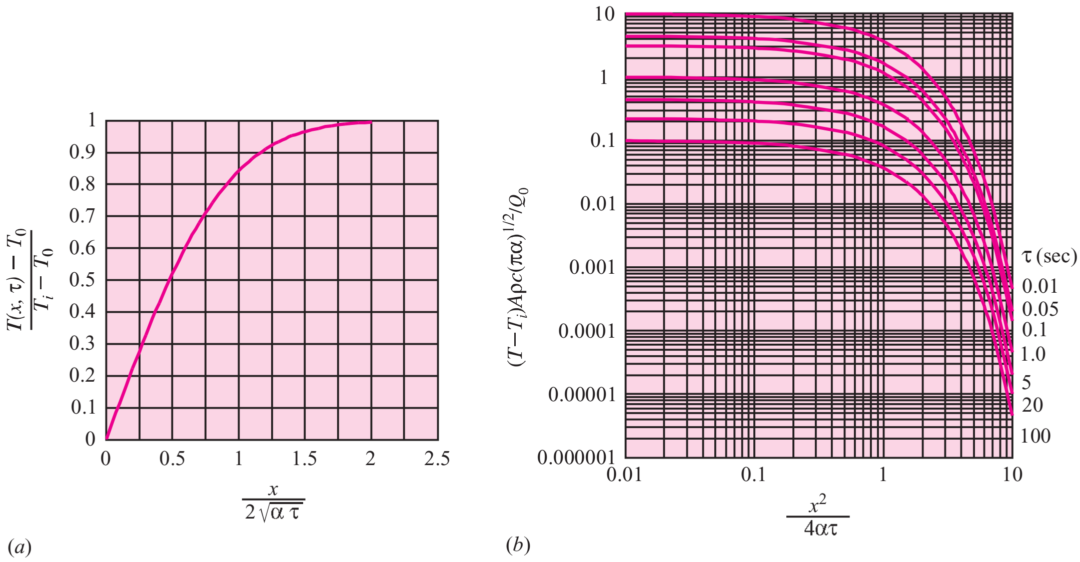
Figure 4-4 | Response of semi-infinite solid to (*a*) sudden change in surface temperature and (*b*) instantaneous surface pulse of Q_0/A J/m^2.



At the surface $(x=0)$ the heat flow is

$$q_0 = \frac{kA(T_0 - T_i)}{\sqrt{\pi\alpha\tau}} \qquad \textbf{[4-12]}$$

The surface heat flux is determined by evaluating the temperature gradient at $x=0$ from Equation (4-11). A plot of the temperature distribution for the semi-infinite solid is given in Figure 4-4. Values of the error function are tabulated in Reference 3, and an abbreviated tabulation is given in Appendix A.

Constant Heat Flux on Semi-Infinite Solid

For the same uniform initial temperature distribution, we could suddenly expose the surface to a constant surface heat flux q_0/A. The initial and boundary conditions on Equation (4-7) would then become

$$T(x,0) = T_i$$

$$\frac{q_0}{A} = -k\left.\frac{\partial T}{\partial x}\right]_{x=0} \qquad \text{for } \tau > 0$$

The solution for this case is

$$T - T_i = \frac{2q_0\sqrt{\alpha\tau/\pi}}{kA}\exp\left(\frac{-x^2}{4\alpha\tau}\right) - \frac{q_0 x}{kA}\left(1 - \operatorname{erf}\frac{x}{2\sqrt{\alpha\tau}}\right) \qquad \textbf{[4-13}\boldsymbol{a}\textbf{]}$$

Energy Pulse at Surface

Equation (4-13*a*) presents the temperature response that results from a surface heat flux that remains constant with time. A related boundary condition is that of a short, instantaneous pulse of energy at the surface having a magnitude of Q_0/A. The resulting temperature response is given by

$$T - T_i = [Q_0/A\rho c(\pi\alpha\tau)^{1/2}]\exp(-x^2/4\alpha\tau) \qquad \textbf{[4-13}\boldsymbol{b}\textbf{]}$$

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In contrast to the constant-heat-flux case where the temperature increases indefinitely for all x and times, the temperature response to the instantaneous surface pulse will die out with time, or

$$T - T_i \to 0 \text{ for all } x \text{ as } \tau \to \infty$$

This rapid exponential decay behavior is illustrated in Figure 4-4*b*.

EXAMPLE 4-2 Semi-Infinite Solid with Sudden Change in Surface Conditions

A large block of steel [$k = 45$ W/m · °C, $\alpha = 1.4 \times 10^{-5}$ m^2/s] is initially at a uniform temperature of 35°C. The surface is exposed to a heat flux (*a*) by suddenly raising the surface temperature to 250°C and (*b*) through a constant surface heat flux of 3.2×10^5 W/m^2. Calculate the temperature at a depth of 2.5 cm after a time of 0.5 min for both these cases.

Solution

We can make use of the solutions for the semi-infinite solid given as Equations (4-8) and (4-13*a*). For case *a*,

$$\frac{x}{2\sqrt{\alpha\tau}} = \frac{0.025}{(2)[(1.4 \times 10^{-5})(30)]^{1/2}} = 0.61$$

The error function is determined from Appendix A as

$$\operatorname{erf} \frac{x}{2\sqrt{\alpha\tau}} = \operatorname{erf} 0.61 = 0.61164$$

We have $T_i = 35$°C and $T_0 = 250$°C, so the temperature at $x = 2.5$ cm is determined from Equation (4-8) as

$$\begin{aligned} T(x, \tau) &= T_0 + (T_i - T_0) \operatorname{erf} \frac{x}{2\sqrt{\alpha\tau}} \\ &= 250 + (35 - 250)(0.61164) = 118.5^\circ\text{C} \end{aligned}$$

For the constant-heat-flux case *b*, we make use of Equation (4-13*a*). Since q_0/A is given as 3.2×10^5 W/m^2, we can insert the numerical values to give

$$\begin{aligned} T(x, \tau) &= 35 + \frac{(2)(3.2 \times 10^5)[(1.4 \times 10^{-5})(30)/\pi]^{1/2}}{45} e^{-(0.61)^2} \\ &\quad - \frac{(0.025)(3.2 \times 10^5)}{45}(1 - 0.61164) \\ &= 79.3^\circ\text{C} \qquad x = 2.5 \text{ cm}, \tau = 30 \text{ s} \end{aligned}$$

For the constant-heat-flux case the *surface* temperature after 30 s would be evaluated with $x = 0$ in Equation (4-13*a*). Thus,

$$T(x = 0) = 35 + \frac{(2)(3.2 \times 10^5)[(1.4 \times 10^{-5})(30)/\pi]^{1/2}}{45} = 199.4^\circ\text{C}$$

EXAMPLE 4-3 Pulsed Energy at Surface of Semi-Infinite Solid

An instantaneous laser pulse of 10 MJ/m^2 is imposed on a slab of stainless steel having properties of $\rho = 7800$ kg/m^3, $c = 460$ J/kg · °C, and $\alpha = 0.44 \times 10^{-5}$ m^2/s. The slab is initially at a uniform temperature of 40 °C. Estimate the temperature at the surface and at a depth of 2.0 mm after a time of 2 s.

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Solution

This problem is a direct application of Equation (4-13*b*). We have $Q_0/A = 10^7\ \text{J/m}^2$ and at $x = 0$

$$T_0 - T_i = Q_0/A\rho c(\pi\alpha\tau)^{1.2}$$
$$= 10^7/(7800)(460)[\pi(0.44 \times 10^{-5})(2)]^{0.5} = 530°\text{C}$$

and

$$T_0 = 40 + 530 = 570°\text{C}$$

At $x = 2.0\ \text{mm} = 0.002\ \text{m}$,

$$T - T_i = (530)\exp[-(0.002)^2/(4)(0.44 \times 10^{-5})(2)] = 473°\text{C}$$

and

$$T = 40 + 473 = 513°\text{C}$$

Heat Removal from Semi-Infinite Solid

EXAMPLE 4-4

A large slab of aluminum at a uniform temperature of 200°C suddenly has its surface temperature lowered to 70°C. What is the total heat removed from the slab per unit surface area when the temperature at a depth 4.0 cm has dropped to 120°C?

Solution

We first find the time required to attain the 120°C temperature and then integrate Equation (4-12) to find the total heat removed during this time interval. For aluminum,

$$\alpha = 8.4 \times 10^{-5}\ \text{m}^2/\text{s} \qquad k = 215\ \text{W/m}\cdot°\text{C}\ [124\ \text{Btu/h}\cdot\text{ft}\cdot°\text{F}]$$

We also have

$$T_i = 200°\text{C} \qquad T_0 = 70°\text{C} \qquad T(x,\tau) = 120°\text{C}$$

Using Equation (4-8) gives

$$\frac{120-70}{200-70} = \text{erf}\,\frac{x}{2\sqrt{\alpha\tau}} = 0.3847$$

From Figure 4-4 or Appendix A,

$$\frac{x}{2\sqrt{\alpha\tau}} = 0.3553$$

and

$$\tau = \frac{(0.04)^2}{(4)(0.3553)^2(8.4 \times 10^{-5})} = 37.72\ \text{s}$$

The total heat removed at the surface is obtained by integrating Equation (4-12):

$$\frac{Q_0}{A} = \int_0^\tau \frac{q_0}{A}\,d\tau = \int_0^\tau \frac{k(T_0 - T_i)}{\sqrt{\pi\alpha\tau}}\,d\tau = 2k(T_0 - T_i)\sqrt{\frac{\tau}{\pi\alpha}}$$
$$= (2)(215)(70-200)\left[\frac{37.72}{\pi(8.4 \times 10^{-5})}\right]^{1/2} = -21.13 \times 10^6\ \text{J/m}^2 \qquad [-1861\ \text{Btu/ft}^2]$$

4-4 | CONVECTION BOUNDARY CONDITIONS

In most practical situations the transient heat-conduction problem is connected with a convection boundary condition at the surface of the solid. Naturally, the boundary conditions for the differential equation must be modified to take into account this convection heat

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transfer at the surface. For the semi-infinite-solid problem, the convection boundary condition would be expressed by

Heat convected into surface = heat conducted into surface

or

$$hA(T_\infty - T)_{x=0} = -kA \left.\frac{\partial T}{\partial x}\right]_{x=0} \qquad \textbf{[4-14]}$$

The solution for this problem is rather involved and is worked out in detail by Schneider [1]. The result is

$$\frac{T - T_i}{T_\infty - T_i} = 1 - \operatorname{erf} X - \left[\exp\left(\frac{hx}{k} + \frac{h^2\alpha\tau}{k^2}\right)\right] \times \left[1 - \operatorname{erf}\left(X + \frac{h\sqrt{\alpha\tau}}{k}\right)\right] \qquad \textbf{[4-15]}$$

where

$X = x/(2\sqrt{\alpha\tau})$
$T_i =$ initial temperature of solid
$T_\infty =$ environment temperature

This solution is presented in graphical form in Figure 4-5.

Solutions have been worked out for other geometries. The most important cases are those dealing with (1) plates whose thickness is small in relation to the other dimensions, (2) cylinders where the diameter is small compared to the length, and (3) spheres. Results of analyses for these geometries have been presented in graphical form by Heisler [2], and nomenclature for the three cases is illustrated in Figure 4-6. In all cases the convection environment temperature is designated as T_∞ and the center temperature for $x = 0$ or $r = 0$ is T_0. At time zero, each solid is assumed to have a uniform initial temperature T_i. Temperatures in the solids are given in Figures 4-7 to 4-13 as functions of time and spatial position. In these charts we note the definitions

$$\theta = T(x, \tau) - T_\infty \qquad \text{or} \qquad T(r, \tau) - T_\infty$$
$$\theta_i = T_i - T_\infty$$
$$\theta_0 = T_0 - T_\infty$$

If a centerline temperature is desired, only one chart is required to obtain a value for θ_0 and then T_0. To determine an off-center temperature, two charts are required to calculate the product

$$\frac{\theta}{\theta_i} = \frac{\theta_0}{\theta_i}\frac{\theta}{\theta_0}$$

For example, Figures 4-7 and 4-10 would be employed to calculate an off-center temperature for an infinite plate.

The heat losses for the infinite plate, infinite cylinder, and sphere are given in Figures 4-14 to 4-16, where Q_0 represents the initial internal energy content of the body in reference to the environment temperature

$$Q_0 = \rho c V(T_i - T_\infty) = \rho c V\theta_i \qquad \textbf{[4-16]}$$

In these figures Q is the actual heat lost by the body in time τ.

Figure 4-5 | Temperature distribution in the semi-infinite solid with convection boundary condition.

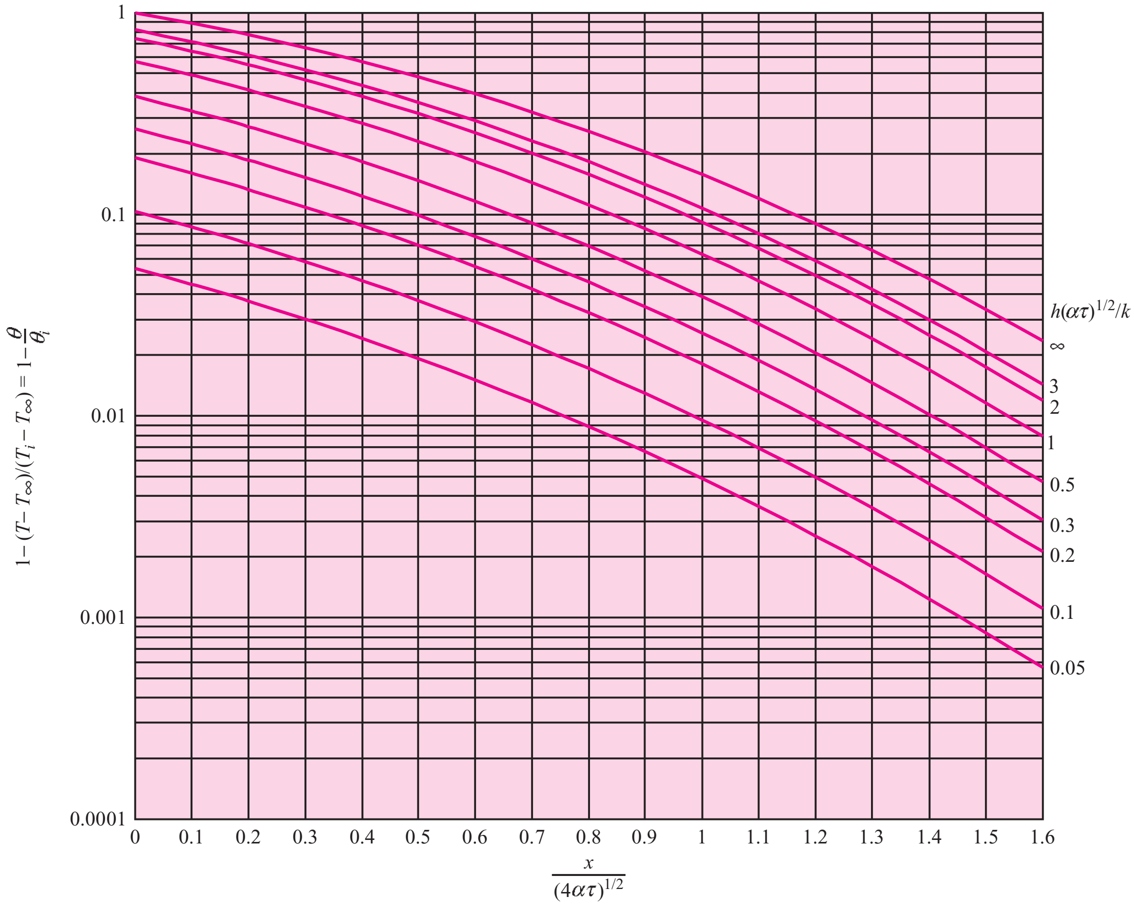


Figure 4-6 | Nomenclature for one-dimensional solids suddenly subjected to convection environment at T_∞: (*a*) infinite plate of thickness $2L$; (*b*) infinite cylinder of radius r_0; (*c*) sphere of radius r_0.

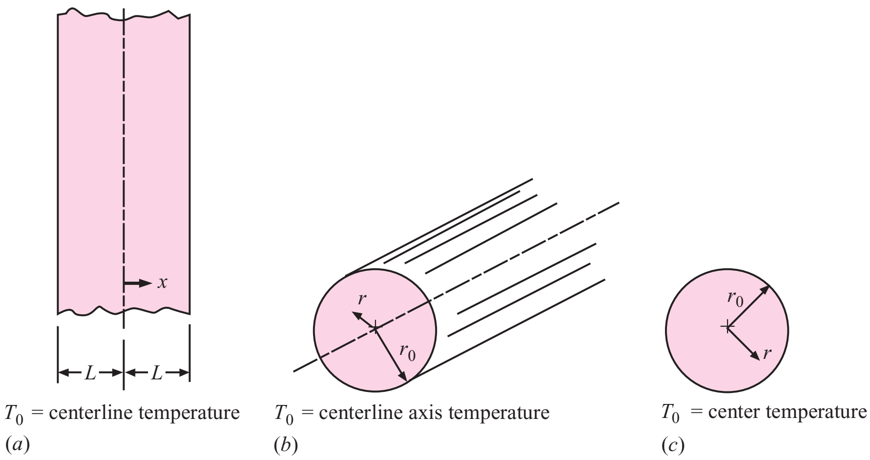
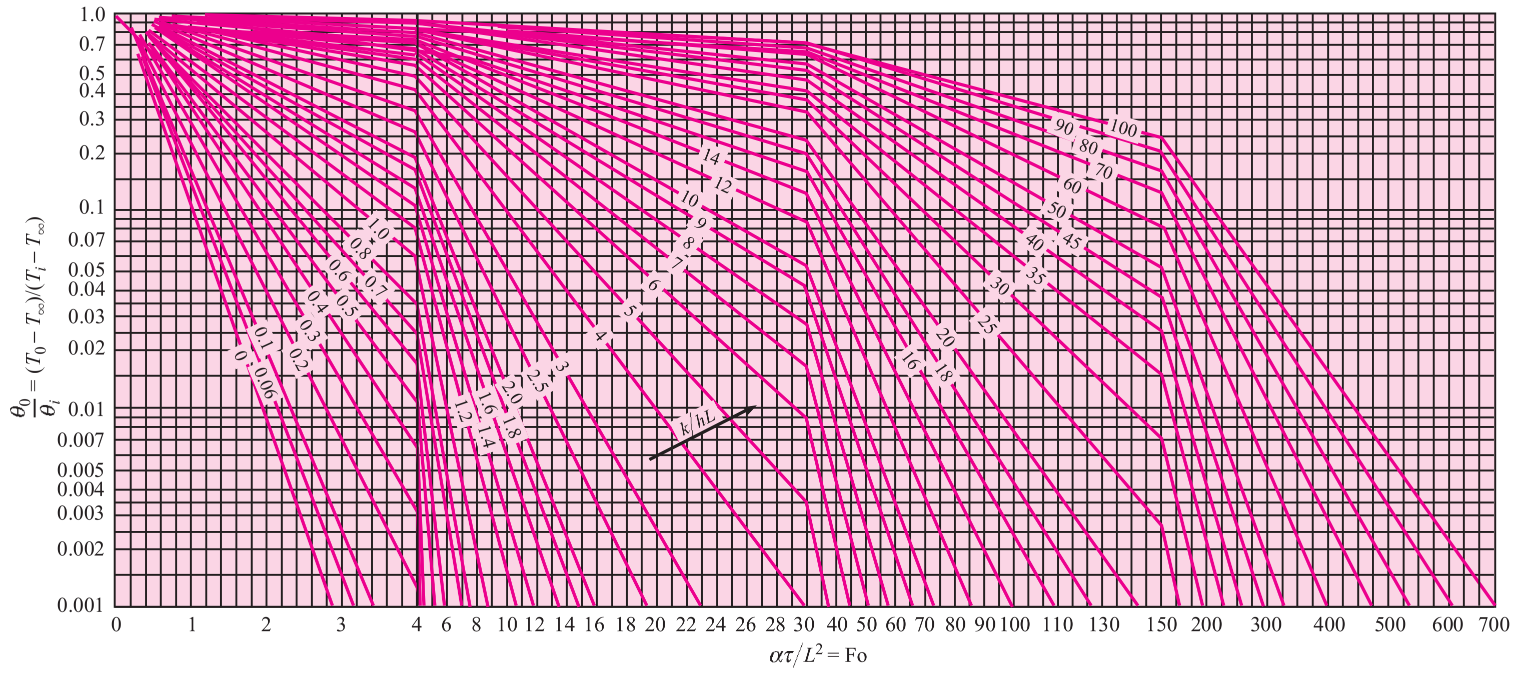


Figure 4-7 | Midplane temperature for an infinite plate of thickness $2L$: (a) full scale.



(a)

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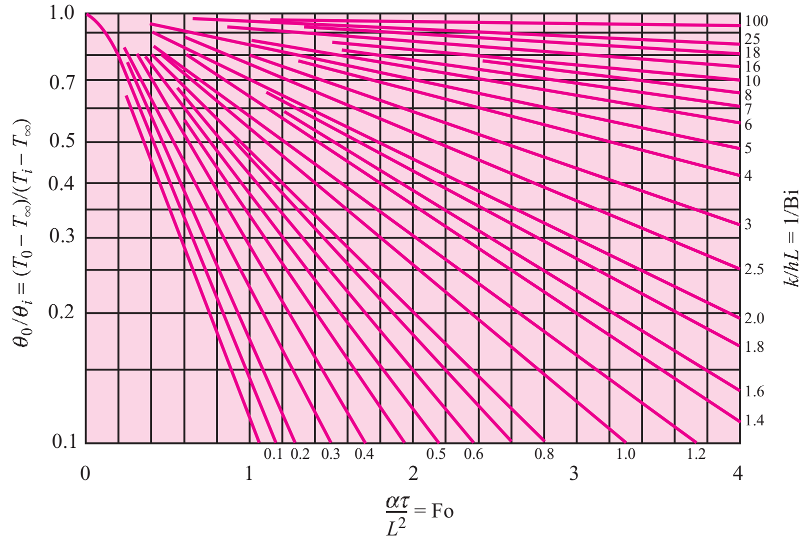
Figure 4-7 | (*Continued*). (*b*) expanded scale for $0 < \text{Fo} < 4$, from Reference 2.



(*b*)

If one considers the solid as behaving as a lumped capacity during the cooling or heating process, that is, small internal resistance compared to surface resistance, the exponential cooling curve of Figure 4-5 may be replotted in expanded form, as shown in Figure 4-13 using the Biot-Fourier product as the abscissa. We note that the following parameters apply for the bodies considered in the Heisler charts.

$$(A/V)_{\text{inf plate}} = 1/L$$
$$(A/V)_{\text{inf cylinder}} = 2/r_0$$
$$(A/V)_{\text{sphere}} = 3/r_0$$

Obviously, there are many other practical heating and cooling problems of interest. The solutions for a large number of cases are presented in graphical form by Schneider [7], and readers interested in such calculations will find this reference to be of great utility.

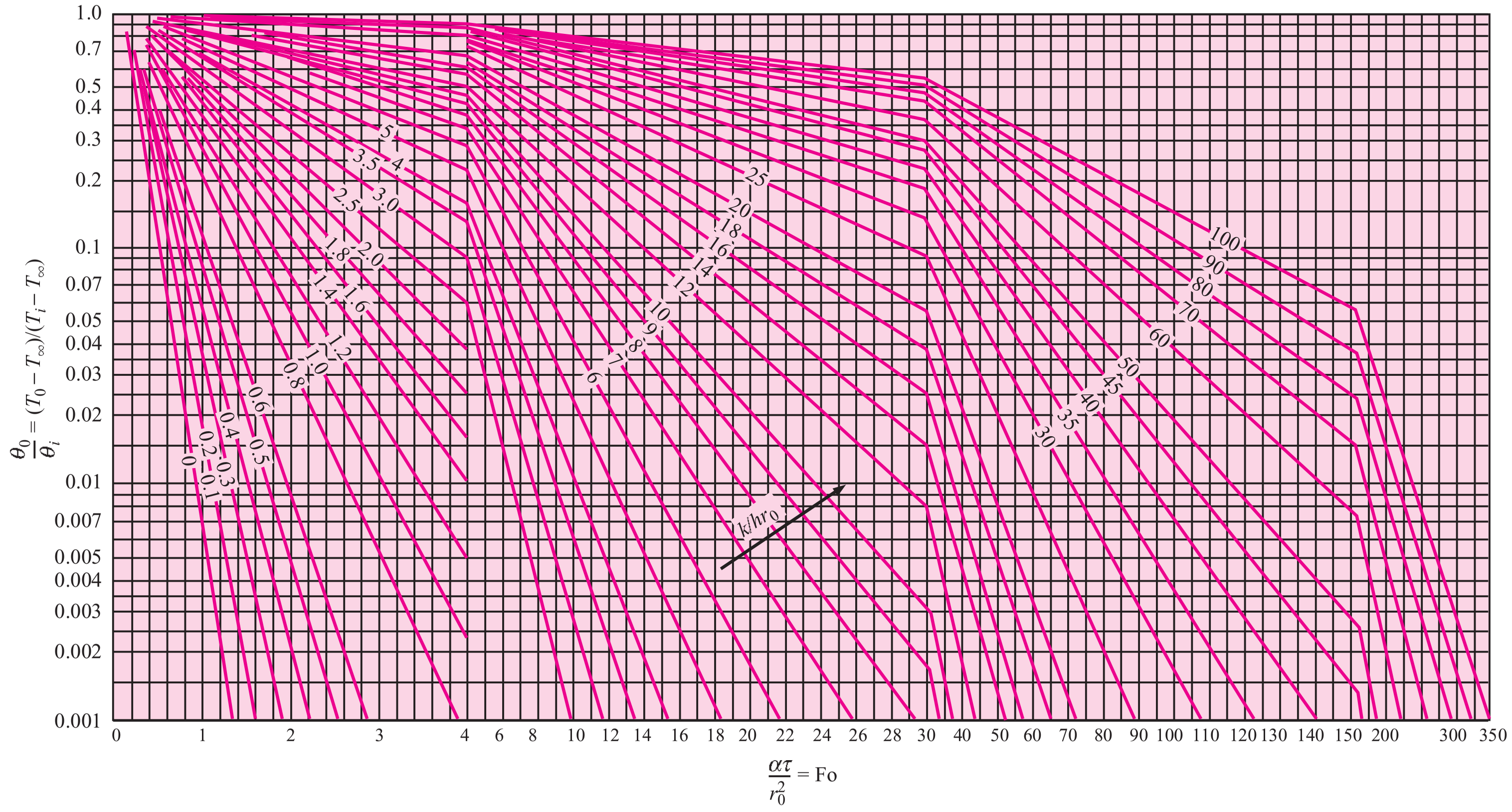
The Biot and Fourier Numbers

A quick inspection of Figures 4-5 to 4-16 indicates that the dimensionless temperature profiles and heat flows may all be expressed in terms of two dimensionless parameters called the Biot and Fourier numbers:

$$\text{Biot number} = \text{Bi} = \frac{hs}{k}$$
$$\text{Fourier number} = \text{Fo} = \frac{\alpha\tau}{s^2} = \frac{k\tau}{\rho c s^2}$$

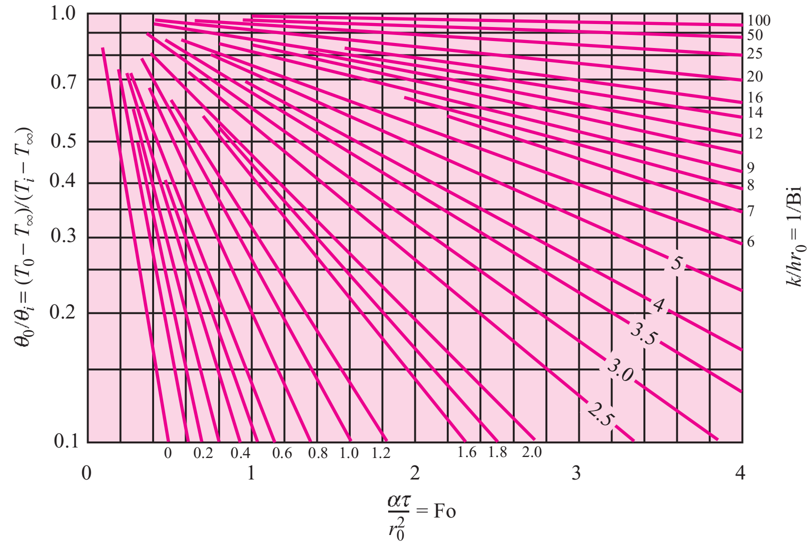
In these parameters s designates a characteristic dimension of the body; for the plate it is the half-thickness, whereas for the cylinder and sphere it is the radius. The Biot number compares the relative magnitudes of surface-convection and internal-conduction resistances

Figure 4-8 | Axis temperature for an infinite cylinder of radius r_0: (a) full scale.



(a)

Figure 4-8 | (*Continued*). (*b*) expanded scale for $0 < \text{Fo} < 4$, from Reference 2.



(*b*)

to heat transfer. The Fourier modulus compares a characteristic body dimension with an approximate temperature-wave penetration depth for a given time τ.

A very low value of the Biot modulus means that internal-conduction resistance is negligible in comparison with surface-convection resistance. This in turn implies that the temperature will be nearly uniform throughout the solid, and its behavior may be approximated by the lumped-capacity method of analysis. It is interesting to note that the exponent of Equation (4-5) may be expressed in terms of the Biot and Fourier numbers if one takes the ratio V/A as the characteristic dimension s. Then,

$$\frac{hA}{\rho c V}\tau = \frac{h\tau}{\rho c s} = \frac{hs}{k}\frac{k\tau}{\rho c s^2} = \text{Bi Fo}$$

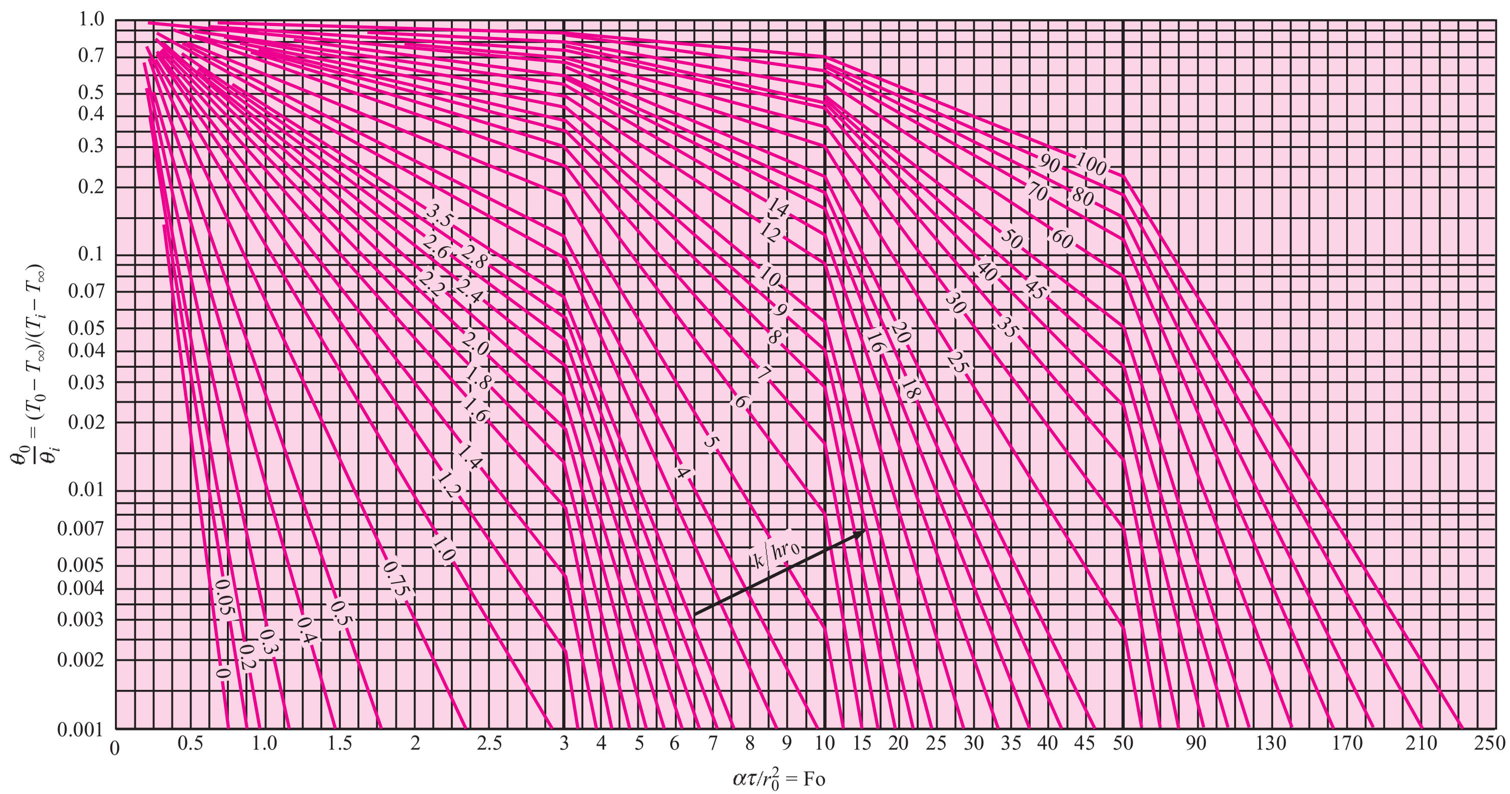
Applicability of the Heisler Charts

The calculations for the Heisler charts were performed by truncating the infinite series solutions for the problems into a few terms. This restricts the applicability of the charts to values of the Fourier number greater than 0.2.

$$\text{Fo} = \frac{\alpha\tau}{s^2} > 0.2$$

For smaller values of this parameter the reader should consult the solutions and charts given in the references at the end of the chapter. Calculations using the truncated series solutions directly are discussed in Appendix C.

Figure 4-9 | Center temperature for a sphere of radius r_0: (*a*) full scale.



(*a*)

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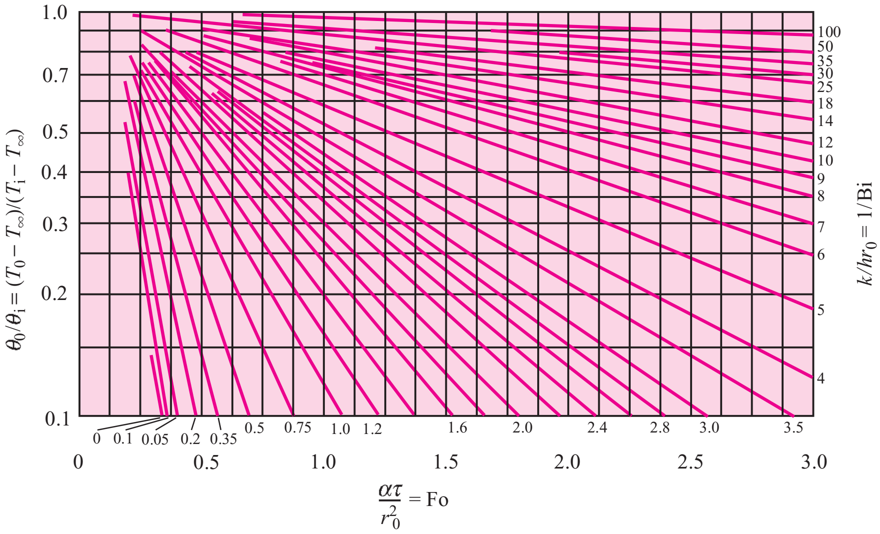
Figure 4-9 | (*Continued*). (*b*) expanded scale for $0 < \mathrm{Fo} < 3$, from Reference 2.



(*b*)

Figure 4-10 | Temperature as a function of center temperature in an infinite plate of thickness $2L$, from Reference 2.

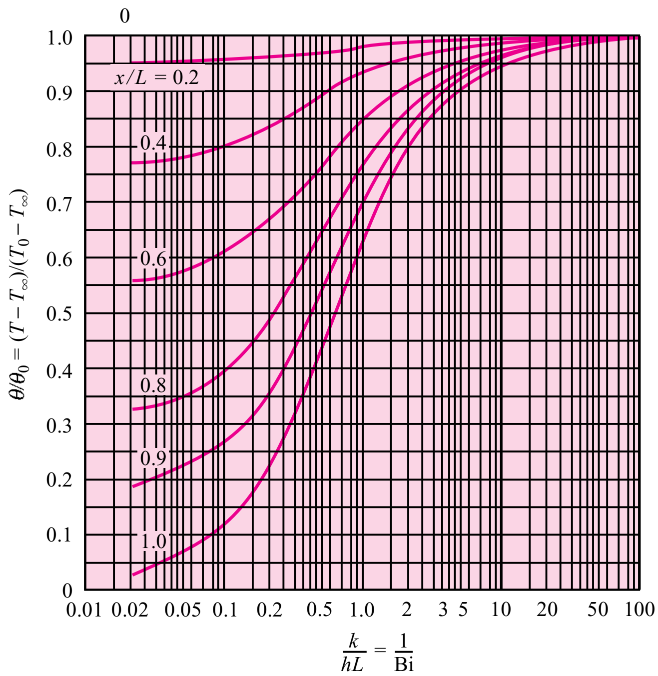


Figure 4-11 | Temperature as a function of axis temperature in an infinite cylinder of radius r_0, from Reference 2.

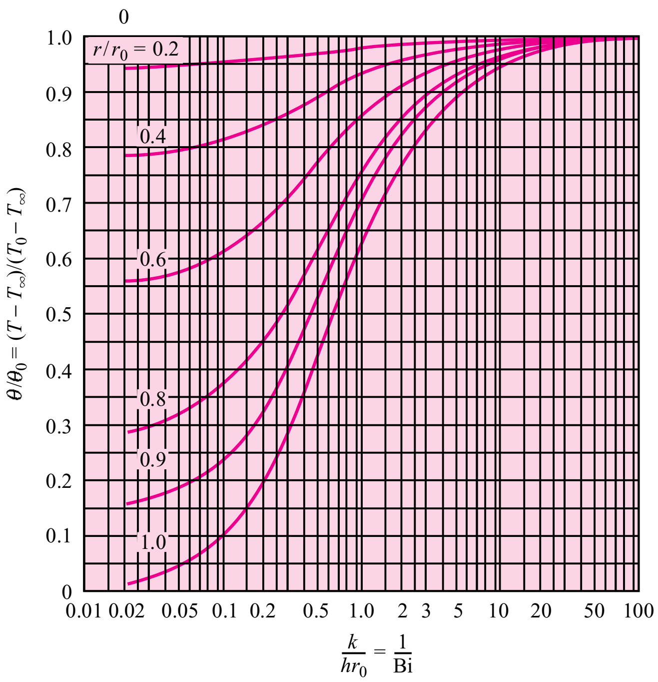


Figure 4-12 | Temperature as a function of center temperature for a sphere of radius r_0, from Reference 2.

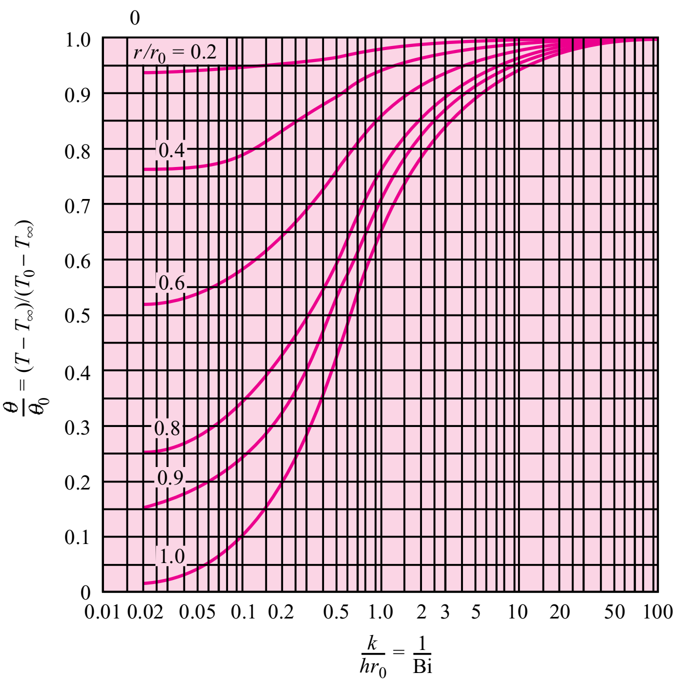


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Figure 4-13 | Temperature variation with time for solids that may be treated as lumped capacities: (*a*) $0 < \mathrm{BiFo} < 10$, (*b*) $0.1 < \mathrm{BiFo} < 1.0$, (*c*) $0 < \mathrm{BiFo} < 0.1$. Note: $(A/V)_{\text{inf plate}} = 1/L$, $(A/V)_{\text{inf cyl}} = 2/r_0$, $(A/V)_{\text{sphere}} = 3/r_0$. See Equations (4-5) and (4-6).

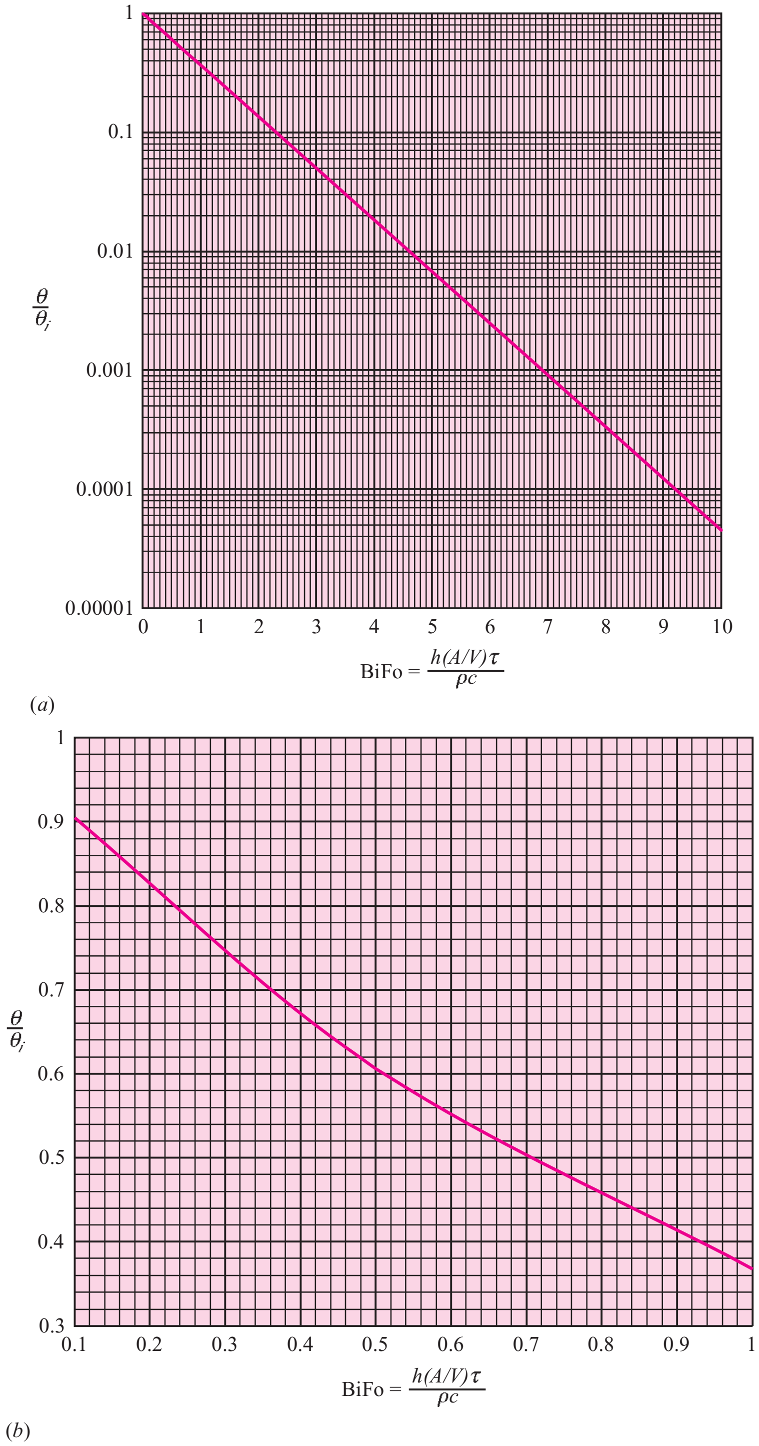
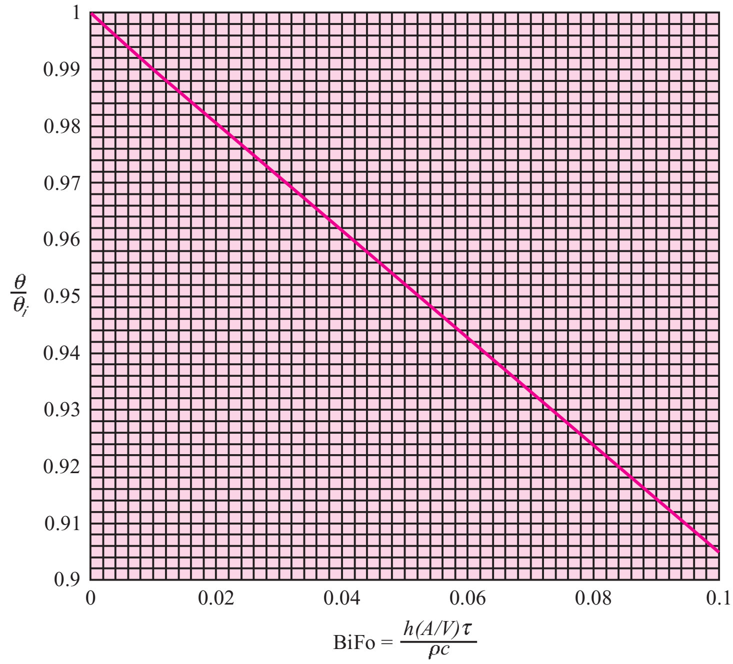


Figure 4-13 | (*Continued*).



(*c*)

Figure 4-14 | Dimensionless heat loss Q/Q_0 of an infinite plane of thickness $2L$ with time, from Reference 6.

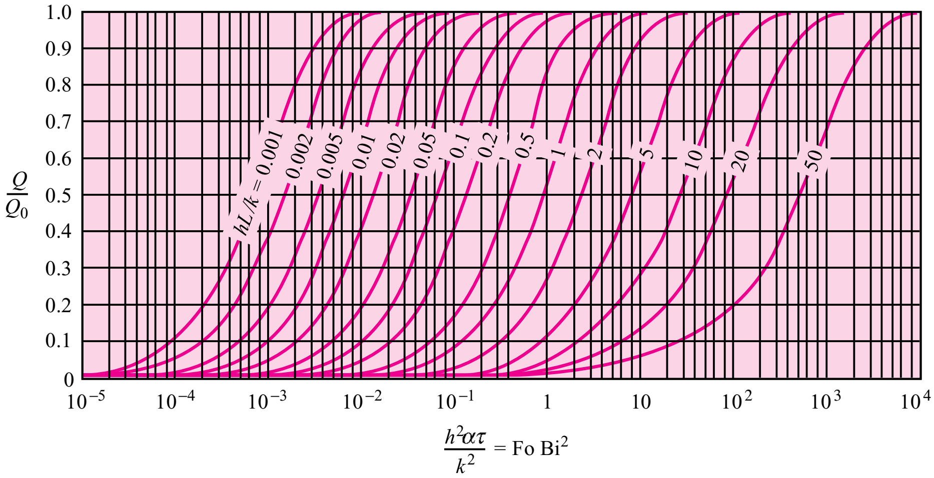


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Figure 4-15 | Dimensionlesss heat loss Q/Q_0 of an infinite cylinder of radius r_0 with time, from Reference 6.

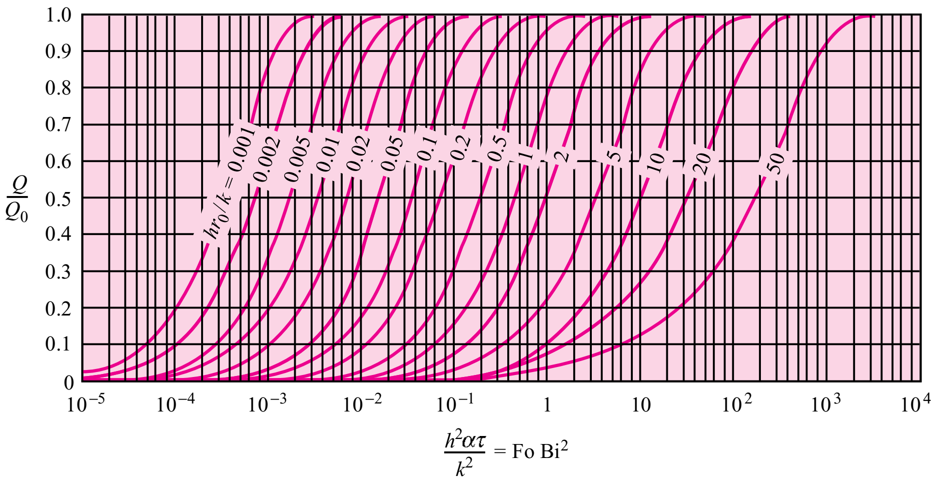
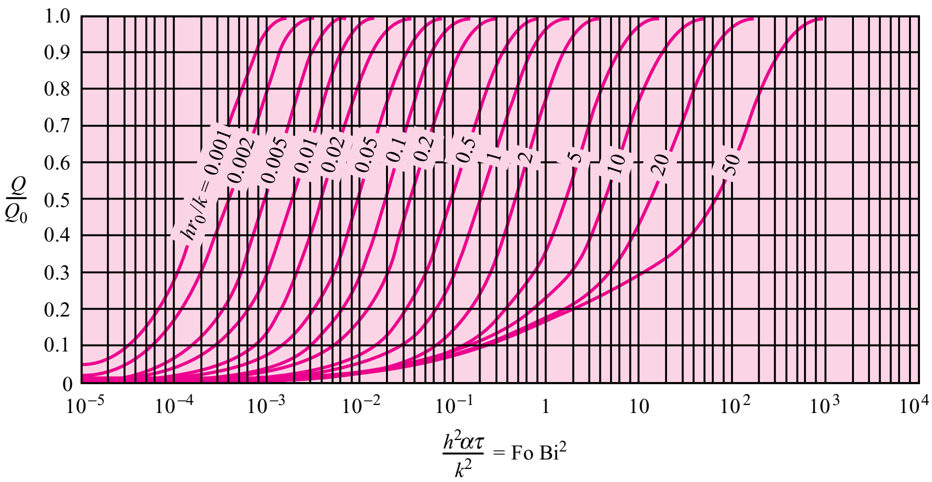


Figure 4-16 | Dimensionless heat loss Q/Q_0 of a sphere of radius r_0 with time, from Reference 6.



Sudden Exposure of Semi-Infinite Slab to Convection

EXAMPLE 4-5

The slab of Example 4-4 is suddenly exposed to a convection-surface environment of 70°C with a heat-transfer coefficient of 525 $W/m^2 \cdot °C$. Calculate the time required for the temperature to reach 120°C at the depth of 4.0 cm for this circumstance.

■ Solution

We may use either Equation (4-15) or Figure 4-5 for solution of this problem, but Figure 4-5 is easier to apply because the time appears in two terms. Even when the figure is used, an iterative procedure is required because the time appears in both of the variables $h\sqrt{\alpha\tau}/k$ and $x/(2\sqrt{\alpha\tau})$.

We seek the value of τ such that

$$\frac{T-T_i}{T_\infty - T_i} = \frac{120-200}{70-200} = 0.615 \qquad [a]$$

We therefore try values of τ and obtain readings of the temperature ratio from Figure 4-5 until agreement with Equation (*a*) is reached. The iterations are listed below. Values of k and α are obtained from Example 4-4.

τ, s	$\frac{h\sqrt{\alpha\tau}}{k}$	$\frac{x}{2\sqrt{\alpha\tau}}$	$\frac{T-T_i}{T_\infty-T_i}$ from Figure 4-5
1000	0.708	0.069	0.41
3000	1.226	0.040	0.61
4000	1.416	0.035	0.68

Consequently, the time required is approximately 3000 s.

EXAMPLE 4-6 Aluminum Plate Suddenly Exposed to Convection

A large plate of aluminum 5.0 cm thick and initially at 200°C is suddenly exposed to the convection environment of Example 4-5. Calculate the temperature at a depth of 1.25 cm from one of the faces 1 min after the plate has been exposed to the environment. How much energy has been removed per unit area from the plate in this time?

Solution

The Heisler charts of Figures 4-7 and 4-10 may be used for solution of this problem. We first calculate the center temperature of the plate, using Figure 4-7, and then use Figure 4-10 to calculate the temperature at the specified x position. From the conditions of the problem we have

$$\theta_i = T_i - T_\infty = 200 - 70 = 130 \qquad \alpha = 8.4\times 10^{-5}\ \mathrm{m^2/s} \quad [3.26\ \mathrm{ft^2/h}]$$

$$2L = 5.0\ \mathrm{cm} \qquad L = 2.5\ \mathrm{cm} \qquad \tau = 1\ \mathrm{min} = 60\ \mathrm{s}$$

$$k = 215\ \mathrm{W/m\cdot °C} \quad [124\ \mathrm{Btu/h\cdot ft\cdot °F}]$$

$$h = 525\ \mathrm{W/m^2\cdot °C} \quad [92.5\ \mathrm{Btu/h\cdot ft^2\cdot °F}]$$

$$x = 2.5 - 1.25 = 1.25\ \mathrm{cm}$$

Then

$$\frac{\alpha\tau}{L^2} = \frac{(8.4\times 10^{-5})(60)}{(0.025)^2} = 8.064 \qquad \frac{k}{hL} = \frac{215}{(525)(0.025)} = 16.38$$

$$\frac{x}{L} = \frac{1.25}{2.5} = 0.5$$

From Figure 4-7

$$\frac{\theta_0}{\theta_i} = 0.61$$

$$\theta_0 = T_0 - T_\infty = (0.61)(130) = 79.3$$

From Figure 4-10 at $x/L = 0.5$,

$$\frac{\theta}{\theta_0} = 0.98$$

and

$$\theta = T - T_\infty = (0.98)(79.3) = 77.7$$

$$T = 77.7 + 70 = 147.7°\mathrm{C}$$

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We compute the energy lost by the slab by using Figure 4-14. For this calculation we require the following properties of aluminum:

$$\rho = 2700 \text{ kg/m}^3 \qquad c = 0.9 \text{ kJ/kg} \cdot {}^\circ\text{C}$$

For Figure 4-14 we need

$$\frac{h^2\alpha\tau}{k^2} = \frac{(525)^2(8.4\times10^{-5})(60)}{(215)^2} = 0.03 \qquad \frac{hL}{k} = \frac{(525)(0.025)}{215} = 0.061$$

From Figure 4-14

$$\frac{Q}{Q_0} = 0.41$$

For unit area

$$\frac{Q_0}{A} = \frac{\rho c V\theta_i}{A} = \rho c(2L)\theta_i$$
$$= (2700)(900)(0.05)(130)$$
$$= 15.8\times10^6 \text{ J/m}^2$$

so that the heat removed per unit surface area is

$$\frac{Q}{A} = (15.8\times10^6)(0.41) = 6.48\times10^6 \text{ J/m}^2 \quad [571 \text{ Btu/ft}^2]$$

Long Cylinder Suddenly Exposed to Convection

EXAMPLE 4-7

A long aluminum cylinder 5.0 cm in diameter and initially at 200°C is suddenly exposed to a convection environment at 70°C and $h = 525$ W/m$^2 \cdot$ °C. Calculate the temperature at a radius of 1.25 cm and the heat lost per unit length 1 min after the cylinder is exposed to the environment.

Solution

This problem is like Example 4-6 except that Figures 4-8 and 4-11 are employed for the solution. We have

$$\theta_i = T_i - T_\infty = 200 - 70 = 130 \qquad \alpha = 8.4\times10^{-5} \text{ m}^2/\text{s}$$
$$r_0 = 2.5 \text{ cm} \qquad \tau = 1 \text{ min} = 60 \text{ s}$$
$$k = 215 \text{ W/m}\cdot{}^\circ\text{C} \qquad h = 525 \text{ W/m}^2\cdot{}^\circ\text{C} \qquad r = 1.25 \text{ cm}$$
$$\rho = 2700 \text{ kg/m}^3 \qquad c = 0.9 \text{ kJ/kg}\cdot{}^\circ\text{C}$$

We compute

$$\frac{\alpha\tau}{r_0^2} = \frac{(8.4\times10^{-5})(60)}{(0.025)^2} = 8.064 \qquad \frac{k}{hr_0} = \frac{215}{(525)(0.025)} = 16.38$$
$$\frac{r}{r_0} = \frac{1.25}{2.5} = 0.5$$

From Figure 4-8

$$\frac{\theta_0}{\theta_i} = 0.38$$

and from Figures 4-11 at $r/r_0 = 0.5$

$$\frac{\theta}{\theta_0} = 0.98$$

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so that

$$\frac{\theta}{\theta_i} = \frac{\theta_0}{\theta_i}\frac{\theta}{\theta_0} = (0.38)(0.98) = 0.372$$

and

$$\theta = T - T_\infty = (0.372)(130) = 48.4$$

$$T = 70 + 48.4 = 118.4°\text{C}$$

To compute the heat lost, we determine

$$\frac{h^2\alpha\tau}{k^2} = \frac{(525)^2(8.4 \times 10^{-5})(60)}{(215)^2} = 0.03 \qquad \frac{hr_0}{k} = \frac{(525)(0.025)}{215} = 0.061$$

Then from Figure 4-15

$$\frac{Q}{Q_0} = 0.65$$

For unit length

$$\frac{Q_0}{L} = \frac{\rho c V\theta_i}{L} = \rho c\pi r_0^2\theta_i = (2700)(900)\pi(0.025)^2(130) = 6.203 \times 10^5 \text{ J/m}$$

and the actual heat lost per unit length is

$$\frac{Q}{L} = (6.203 \times 10^5)(0.65) = 4.032 \times 10^5 \text{ J/m} \quad [116.5 \text{ Btu/ft}]$$

4-5 | MULTIDIMENSIONAL SYSTEMS

The Heisler charts discussed in Section 4-4 may be used to obtain the temperature distribution in the infinite plate of thickness $2L$, in the long cylinder, or in the sphere. When a wall whose height and depth dimensions are not large compared with the thickness or a cylinder whose length is not large compared with its diameter is encountered, additional space coordinates are necessary to specify the temperature, the charts no longer apply, and we are forced to seek another method of solution. Fortunately, it is possible to combine the solutions for the one-dimensional systems in a very straightforward way to obtain solutions for the multidimensional problems.

It is clear that the infinite rectangular bar in Figure 4-17 can be formed from two infinite plates of thickness $2L_1$ and $2L_2$, respectively. The differential equation governing this situation would be

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau} \qquad \textbf{[4-17]}$$

and to use the separation-of-variables method to effect a solution, we should assume a product solution of the form

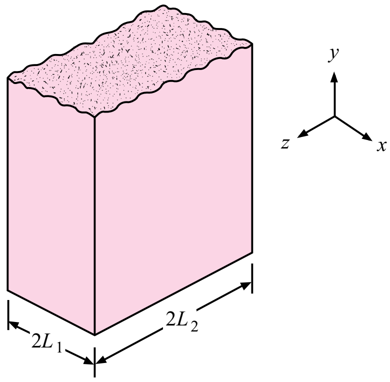
$$T(x, z, \tau) = X(x)Z(z)\Theta(\tau)$$

It can be shown that the dimensionless temperature distribution may be expressed as a product of the solutions for two plate problems of thickness $2L_1$ and $2L_2$, respectively:

$$\left(\frac{T - T_\infty}{T_i - T_\infty}\right)_{\text{bar}} = \left(\frac{T - T_\infty}{T_i - T_\infty}\right)_{2L_1\ plate}\left(\frac{T - T_\infty}{T_i - T_\infty}\right)_{2L_2\ plate} \qquad \textbf{[4-18]}$$

where T_i is the initial temperature of the bar and T_∞ is the environment temperature.

Figure 4-17 | Infinite rectangular bar.



For two infinite plates the respective differential equations would be

$$\frac{\partial^2 T_1}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T_1}{\partial \tau} \qquad \frac{\partial^2 T_2}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T_2}{\partial \tau} \tag{4-19}$$

and the product solutions assumed would be

$$T_1 = T_1(x, \tau) \qquad T_2 = T_2(z, \tau) \tag{4-20}$$

We shall now show that the product solution to Equation (4-17) can be formed from a simple product of the functions (T_1, T_2), that is,

$$T(x, z, \tau) = T_1(x, \tau)T_2(z, \tau) \tag{4-21}$$

The appropriate derivatives for substitution in Equation (4-17) are obtained from Equation (4-21) as

$$\frac{\partial^2 T}{\partial x^2} = T_2\frac{\partial^2 T_1}{\partial x^2} \qquad \frac{\partial^2 T}{\partial z^2} = T_1\frac{\partial^2 T_2}{\partial z^2}$$

$$\frac{\partial T}{\partial \tau} = T_1\frac{\partial T_2}{\partial \tau} + T_2\frac{\partial T_1}{\partial \tau}$$

Using Equations (4-19), we have

$$\frac{\partial T}{\partial \tau} = \alpha T_1\frac{\partial^2 T_2}{\partial z^2} + \alpha T_2\frac{\partial^2 T_1}{\partial x^2}$$

Substituting these relations in Equation (4-17) gives

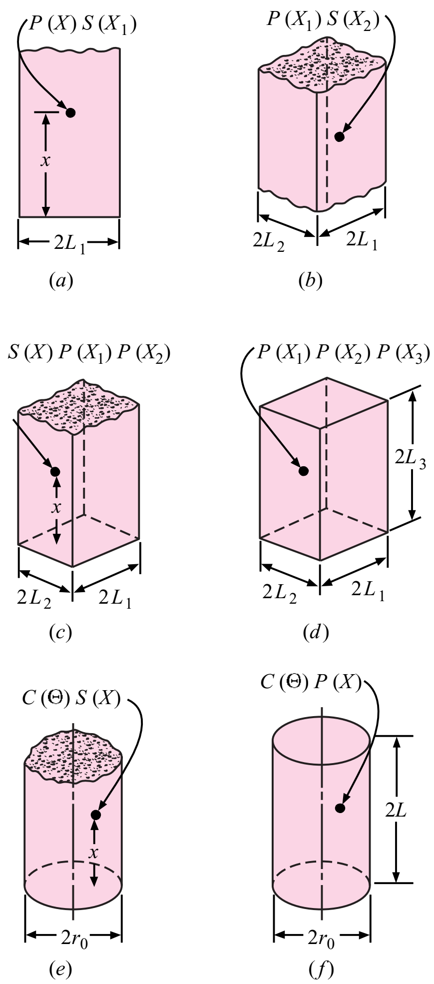
$$T_2\frac{\partial^2 T_1}{\partial x_2} + T_1\frac{\partial^2 T_2}{\partial z^2} = \frac{1}{\alpha}\left(\alpha T_1\frac{\partial^2 T_2}{\partial z^2} + \alpha T_2\frac{\partial^2 T_1}{\partial x^2}\right)$$

or the assumed product solution of Equation (4-21) does indeed satisfy the original differential equation (4-17). This means that the dimensionless temperature distribution for the infinite rectangular bar may be expressed as a product of the solutions for two plate problems of thickness $2L_1$ and $2L_2$, respectively, as indicated by Equation (4-18).

In a manner similar to that described above, the solution for a three-dimensional block may be expressed as a product of three infinite-plate solutions for plates having the thickness of the three sides of the block. Similarly, a solution for a cylinder of finite length could be expressed as a product of solutions of the infinite cylinder and an infinite plate having a thickness equal to the length of the cylinder. Combinations could also be made with the infinite-cylinder and infinite-plate solutions to obtain temperature distributions in semi-infinite bars and cylinders. Some of the combinations are summarized in Figure 4-18, where

$$C(\Theta) = \text{solution for infinite cylinder}$$
$$P(X) = \text{solution for infinite plate}$$
$$S(X) = \text{solution for semi-infinite solid}$$

Figure 4-18 | Product solutions for temperatures in multidimensional systems: (*a*) semi-infinite plate; (*b*) infinite rectangular bar; (*c*) semi-infinite rectangular bar; (*d*) rectangular parallelepiped; (*e*) semi-infinite cylinder; (*f*) short cylinder.



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The general idea is then

$$\left(\frac{\theta}{\theta_i}\right)_{\substack{\text{combined}\\ \text{solid}}} = \left(\frac{\theta}{\theta_i}\right)_{\substack{\text{intersection}\\ \text{solid 1}}} \left(\frac{\theta}{\theta_i}\right)_{\substack{\text{intersection}\\ \text{solid 2}}} \left(\frac{\theta}{\theta_i}\right)_{\substack{\text{intersection}\\ \text{solid 3}}}$$

Heat Transfer in Multidimensional Systems

Langston [16] has shown that it is possible to superimpose the heat-loss solutions for one-dimensional bodies, as shown in Figures 4-14, 4-15, and 4-16, to obtain the heat for a multidimensional body. The results of this analysis for intersection of two bodies is

$$\left(\frac{Q}{Q_0}\right)_{\text{total}} = \left(\frac{Q}{Q_0}\right)_1 + \left(\frac{Q}{Q_0}\right)_2 \left[1 - \left(\frac{Q}{Q_0}\right)_1\right] \qquad \textbf{[4-22]}$$

where the subscripts refer to the two intersecting bodies. For a multidimensional body formed by intersection of three one-dimensional systems, the heat loss is given by

$$\left(\frac{Q}{Q_0}\right)_{\text{total}} = \left(\frac{Q}{Q_0}\right)_1 + \left(\frac{Q}{Q_0}\right)_2 \left[1 - \left(\frac{Q}{Q_0}\right)_1\right] + \left(\frac{Q}{Q_0}\right)_3 \left[1 - \left(\frac{Q}{Q_0}\right)_1\right]\left[1 - \left(\frac{Q}{Q_0}\right)_2\right] \qquad \textbf{[4-23]}$$

If the heat loss is desired after a given time, the calculation is straightforward. On the other hand, if the *time to achieve a certain heat loss* is the desired quantity, a trial-and-error or iterative procedure must be employed. The following examples illustrate the use of the various charts for calculating temperatures and heat flows in multidimensional systems.

Semi-Infinite Cylinder Suddenly Exposed to Convection

EXAMPLE 4-8

A semi-infinite aluminum cylinder 5 cm in diameter is initially at a uniform temperature of 200°C. It is suddenly subjected to a convection boundary condition at 70°C with $h = 525\ \text{W/m}^2 \cdot {}^\circ\text{C}$. Calculate the temperatures at the axis and surface of the cylinder 10 cm from the end 1 min after exposure to the environment.

Solution

This problem requires a combination of solutions for the infinite cylinder and semi-infinite slab in accordance with Figure 4-18*e*. For the slab we have

$$x = 10\ \text{cm} \qquad \alpha = 8.4 \times 10^{-5}\ \text{m}^2/\text{s} \qquad k = 215\ \text{W/m} \cdot {}^\circ\text{C}$$

so that the parameters for use with Figure 4-5 are

$$\frac{h\sqrt{\alpha\tau}}{k} = \frac{(525)[(8.4 \times 10^{-5})(60)]^{1/2}}{215} = 0.173$$

$$\frac{x}{2\sqrt{\alpha\tau}} = \frac{0.1}{(2)[(8.4 \times 10^{-5})(60)]^{1/2}} = 0.704$$

From Figure 4-5

$$\left(\frac{\theta}{\theta_i}\right)_{\text{semi-infinite slab}} = 1 - 0.036 = 0.964 = S(X)$$

For the infinite cylinder we seek both the axis- and surface-temperature ratios. The parameters for use with Figure 4-8 are

$$r_0 = 2.5\ \text{cm} \qquad \frac{k}{hr_0} = 16.38 \qquad \frac{\alpha\tau}{r_0^2} = 8.064 \qquad \frac{\theta_0}{\theta_i} = 0.38$$

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This is the axis-temperature ratio. To find the surface-temperature ratio, we enter Figure 4-11, using

$$\frac{r}{r_0} = 1.0 \qquad \frac{\theta}{\theta_0} = 0.97$$

Thus

$$C(\Theta) = \left(\frac{\theta}{\theta_i}\right)_{\text{inf cyl}} = \begin{cases} 0.38 & \text{at } r = 0 \\ (0.38)(0.97) = 0.369 & \text{at } r = r_0 \end{cases}$$

Combining the solutions for the semi-infinite slab and infinite cylinder, we have

$$\begin{aligned}\left(\frac{\theta}{\theta_i}\right)_{\text{semi-infinite cylinder}} &= C(\Theta)S(X) \\ &= (0.38)(0.964) = 0.366 \qquad \text{at } r = 0 \\ &= (0.369)(0.964) = 0.356 \qquad \text{at } r = r_0\end{aligned}$$

The corresponding temperatures are

$$\begin{aligned} T &= 70 + (0.366)(200 - 70) = 117.6 \qquad \text{at } r = 0 \\ T &= 70 + (0.356)(200 - 70) = 116.3 \qquad \text{at } r = r_0 \end{aligned}$$

EXAMPLE 4-9 Finite-Length Cylinder Suddenly Exposed to Convection

A short aluminum cylinder 5.0 cm in diameter and 10.0 cm long is initially at a uniform temperature of 200°C. It is suddenly subjected to a convection environment at 70°C, and $h = 525\ \text{W/m}^2 \cdot °\text{C}$. Calculate the temperature at a radial position of 1.25 cm and a distance of 0.625 cm from one end of the cylinder 1 min after exposure to the environment.

■ Solution

To solve this problem we combine the solutions from the Heisler charts for an infinite cylinder and an infinite plate in accordance with the combination shown in Figure 4-18*f*. For the infinite-plate problem

$$L = 5 \text{ cm}$$

The x position is measured from the center of the plate so that

$$x = 5 - 0.625 = 4.375 \text{ cm} \qquad \frac{x}{L} = \frac{4.375}{5} = 0.875$$

For aluminum

$$\alpha = 8.4 \times 10^{-5}\ \text{m}^2/\text{s} \qquad k = 215\ \text{W/m} \cdot °\text{C}$$

so

$$\frac{k}{hL} = \frac{215}{(525)(0.05)} = 8.19 \qquad \frac{\alpha\tau}{L^2} = \frac{(8.4 \times 10^{-5})(60)}{(0.05)^2} = 2.016$$

From Figures 4-7 and 4-10, respectively,

$$\frac{\theta_0}{\theta_i} = 0.75 \qquad \frac{\theta}{\theta_0} = 0.95$$

so that

$$\left(\frac{\theta}{\theta_i}\right)_{\text{plate}} = (0.75)(0.95) = 0.7125$$

For the cylinder $r_0 = 2.5$ cm

$$\frac{r}{r_0} = \frac{1.25}{2.5} = 0.5 \qquad \frac{k}{hr_0} = \frac{215}{(525)(0.025)} = 16.38$$

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$$\frac{\alpha\tau}{r_0^2} = \frac{(8.4 \times 10^{-5})(60)}{(0.025)^2} = 8.064$$

and from Figures 4-8 and 4-11, respectively,

$$\frac{\theta_0}{\theta_i} = 0.38 \qquad \frac{\theta}{\theta_0} = 0.98$$

so that

$$\left(\frac{\theta}{\theta_i}\right)_{\text{cyl}} = (0.38)(0.98) = 0.3724$$

Combining the solutions for the plate and cylinder gives

$$\left(\frac{\theta}{\theta_i}\right)_{\text{short cylinder}} = (0.7125)(0.3724) = 0.265$$

Thus

$$T = T_\infty + (0.265)(T_i - T_\infty) = 70 + (0.265)(200 - 70) = 104.5^\circ\text{C}$$

Heat Loss for Finite-Length Cylinder

EXAMPLE 4-10

Calculate the heat loss for the short cylinder in Example 4-9.

Solution

We first calculate the dimensionless heat-loss ratio for the infinite plate and infinite cylinder that make up the multidimensional body. For the plate we have $L = 5$ cm $= 0.05$ m. Using the properties of aluminum from Example 4-9, we calculate

$$\frac{hL}{k} = \frac{(525)(0.05)}{215} = 0.122$$

$$\frac{h^2\alpha\tau}{k^2} = \frac{(525)^2(8.4 \times 10^{-5})(60)}{(215)^2} = 0.03$$

From Figure 4-14, for the plate, we read

$$\left(\frac{Q}{Q_0}\right)_p = 0.22$$

For the cylinder $r_0 = 2.5$ cm $= 0.025$ m, so we calculate

$$\frac{hr_0}{k} = \frac{(525)(0.025)}{215} = 0.061$$

and from Figure 4-15 we can read

$$\left(\frac{Q}{Q_0}\right)_c = 0.55$$

The two heat ratios may be inserted in Equation (4-22) to give

$$\left(\frac{Q}{Q_0}\right)_{\text{tot}} = 0.22 + (0.55)(1 - 0.22) = 0.649$$

The specific heat of aluminum is 0.896 kJ/kg · °C and the density is 2707 kg/m^3, so we calculate Q_0 as

$$\begin{aligned} Q_0 = \rho c V \theta_i &= (2707)(0.896)\pi(0.025)^2(0.1)(200 - 70) \\ &= 61.9 \text{ kJ} \end{aligned}$$

The actual heat loss in the 1-min time is thus

$$Q = (61.9 \text{ kJ})(0.649) = 40.2 \text{ kJ}$$

4-6 | TRANSIENT NUMERICAL METHOD

The charts described in Sections 4-4 and 4-5 are very useful for calculating temperatures in certain regular-shaped solids under transient heat-flow conditions. Unfortunately, many geometric shapes of practical interest do not fall into these categories; in addition, one is frequently faced with problems in which the boundary conditions vary with time. These transient boundary conditions as well as the geometric shape of the body can be such that a mathematical solution is not possible. In these cases, the problems are best handled by a numerical technique with computers. It is the setup for such calculations that we now describe. For ease in discussion we limit the analysis to two-dimensional systems. An extension to three dimensions can then be made very easily.

Consider a two-dimensional body divided into increments as shown in Figure 4-19. The subscript m denotes the x position, and the subscript n denotes the y position. Within the solid body the differential equation that governs the heat flow is

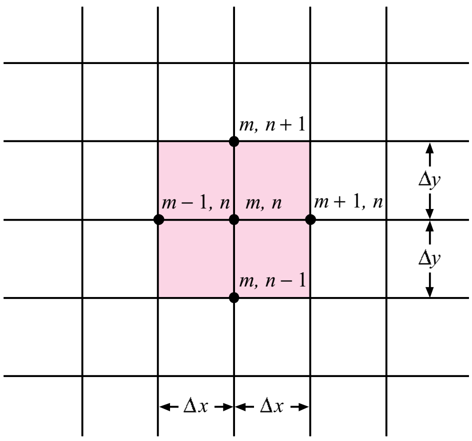
$$k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)=\rho c\,\frac{\partial T}{\partial \tau} \qquad \textbf{[4-24]}$$

assuming constant properties. We recall from Chapter 3 that the second partial derivatives may be approximated by

$$\frac{\partial^2 T}{\partial x^2}\approx\frac{1}{(\Delta x)^2}(T_{m+1,n}+T_{m-1,n}-2T_{m,n}) \qquad \textbf{[4-25]}$$

$$\frac{\partial^2 T}{\partial y^2}\approx\frac{1}{(\Delta y)^2}(T_{m,n+1}+T_{m,n-1}-2T_{m,n}) \qquad \textbf{[4-26]}$$

Figure 4-19 | Nomenclature for numerical solution of two-dimensional unsteady-state conduction problem.



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The time derivative in Equation (4-24) is approximated by

$$\frac{\partial T}{\partial \tau} \approx \frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta \tau} \qquad \textbf{[4-27]}$$

In this relation the superscripts designate the time increment. Combining the relations above gives the difference equation equivalent to Equation (4-24)

$$\frac{T_{m+1,n}^{p} + T_{m-1,n}^{p} - 2T_{m,n}^{p}}{(\Delta x)^2} + \frac{T_{m,n+1}^{p} + T_{m,n-1}^{p} - 2T_{m,n}^{p}}{(\Delta y)^2} = \frac{1}{\alpha}\frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta \tau} \qquad \textbf{[4-28]}$$

Thus, if the temperatures of the various nodes are known at any particular time, the temperatures after a time increment $\Delta\tau$ may be calculated by writing an equation like Equation (4-28) for each node and obtaining the values of $T_{m,n}^{p+1}$. The procedure may be repeated to obtain the distribution after any desired number of time increments. If the increments of space coordinates are chosen such that

$$\Delta x = \Delta y$$

the resulting equation for $T_{m,n}^{p+1}$ becomes

$$T_{m,n}^{p+1} = \frac{\alpha\,\Delta\tau}{(\Delta x)^2}\left(T_{m+1,n}^{p} + T_{m-1,n}^{p} + T_{m,n+1}^{p} + T_{m,n-1}^{p}\right) + \left[1 - \frac{4\alpha\,\Delta\tau}{(\Delta x)^2}\right]T_{m,n}^{p} \qquad \textbf{[4-29]}$$

If the time and distance increments are conveniently chosen so that

$$\frac{(\Delta x)^2}{\alpha\,\Delta\tau} = 4 \qquad \textbf{[4-30]}$$

it is seen that the temperature of node (m, n) after a time increment is simply the arithmetic average of the four surrounding nodal temperatures at the beginning of the time increment.

When a one-dimensional system is involved, the equation becomes

$$T_{m}^{p+1} = \frac{\alpha\,\Delta\tau}{(\Delta x)^2}\left(T_{m+1}^{p} + T_{m-1}^{p}\right) + \left[1 - \frac{2\alpha\,\Delta\tau}{(\Delta x)^2}\right]T_{m}^{p} \qquad \textbf{[4-31]}$$

and if the time and distance increments are chosen so that

$$\frac{(\Delta x)^2}{\alpha\,\Delta\tau} = 2 \qquad \textbf{[4-32]}$$

the temperature of node m after the time increment is given as the arithmetic average of the two adjacent nodal temperatures at the beginning of the time increment.

Some general remarks concerning the use of numerical methods for solution of transient conduction problems are in order at this point. We have already noted that the selection of the value of the parameter

$$M = \frac{(\Delta x)^2}{\alpha\,\Delta\tau}$$

governs the ease with which we may proceed to effect the numerical solution; the choice of a value of 4 for a two-dimensional system or a value of 2 for a one-dimensional system makes the calculation particularly easy.

Once the distance increments and the value of M are established, the time increment is fixed, and we may not alter it without changing the value of either Δx or M, or both. Clearly, the larger the values of Δx and $\Delta\tau$, the more rapidly our solution will proceed. On the other hand, the smaller the value of these increments in the independent variables, the more accuracy will be obtained. At first glance one might assume that small distance increments could be used for greater accuracy in combination with large time increments

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to speed the solution. This is not the case, however, because the finite-difference equations limit the values of $\Delta\tau$ that may be used once Δx is chosen. Note that if $M < 2$ in Equation (4-31), the coefficient of T_m^p becomes negative, and we generate a condition that will violate the second law of thermodynamics. Suppose, for example, that the adjoining nodes are equal in temperature but less than T_m^p. After the time increment $\Delta\tau$, T_m^p may not be lower than these adjoining temperatures; otherwise heat would have to flow uphill on the temperature scale, and this is impossible. A value of $M < 2$ would produce just such an effect; so we must restrict the values of M to

$$\frac{(\Delta x)^2}{\alpha\,\Delta\tau} = \begin{cases} M \geq 2 & \text{one-dimensional systems} \\ M \geq 4 & \text{two-dimensional systems} \end{cases}$$

This restriction automatically limits our choice of $\Delta\tau$, once Δx is established.

It so happens that the above restrictions, which are imposed in a physical sense, may also be derived on mathematical grounds. It may be shown that the finite-difference solutions will not converge unless these conditions are fulfilled. The problems of stability and convergence of numerical solutions are discussed in References 7, 13, and 15 in detail.

The difference equations given above are useful for determining the internal temperature in a solid as a function of space and time. At the boundary of the solid, a convection resistance to heat flow is usually involved, so that the above relations no longer apply. In general, each convection boundary condition must be handled separately, depending on the particular geometric shape under consideration. The case of the flat wall will be considered as an example.

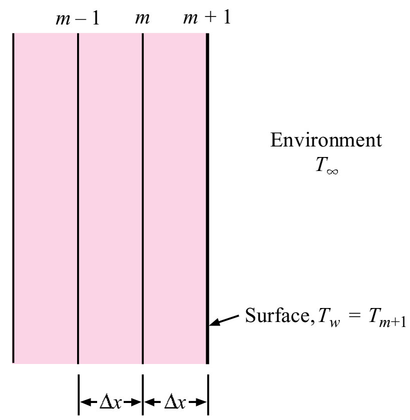
For the one-dimensional system shown in Figure 4-20 we may make an energy balance at the convection boundary such that

$$-kA\left.\frac{\partial T}{\partial x}\right]_{\text{wall}} = hA(T_w - T_\infty) \qquad \textbf{[4-33]}$$

The finite-difference approximation would be given by

$$-k\frac{\Delta y}{\Delta x}(T_{m+1} - T_m) = h\,\Delta y(T_{m+1} - T_\infty)$$

Figure 4-20 | Nomenclature for numerical solution of unsteady-state conduction problem with convection boundary condition.



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or

$$T_{m+1} = \frac{T_m + (h\,\Delta x/k)T_\infty}{1 + h\,\Delta x/k}$$

To apply this condition, we should calculate the surface temperature T_{m+1} at each time increment and then use this temperature in the nodal equations for the interior points of the solid. This is only an approximation because we have neglected the heat capacity of the element of the wall at the boundary. This approximation will work fairly well when a large number of increments in x are used because the portion of the heat capacity that is neglected is then small in comparison with the total. We may take the heat capacity into account in a general way by considering the two-dimensional wall of Figure 3-7 exposed to a convection boundary condition, which we duplicate here for convenience as Figure 4-21. We make a transient energy balance on the node (m, n) by setting the sum of the energy conducted and convected into the node equal to the increase in the internal energy of the node. Thus

$$k\,\Delta y\,\frac{T^p_{m-1,n} - T^p_{m,n}}{\Delta x} + k\,\frac{\Delta x}{2}\,\frac{T^p_{m,n+1} - T^p_{m,n}}{\Delta y} + k\,\frac{\Delta x}{2}\,\frac{T^p_{m,n-1} - T^p_{m,n}}{\Delta y}$$

$$+ h\,\Delta y(T_\infty - T^p_{m,n}) = \rho c\,\frac{\Delta x}{2}\,\Delta y\,\frac{T^{p+1}_{m,n} - T^p_{m,n}}{\Delta\tau}$$

If $\Delta x = \Delta y$, the relation for $T^{p+1}_{m,n}$ becomes

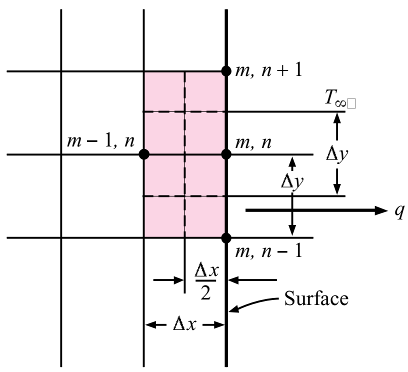
$$T^{p+1}_{m,n} = \frac{\alpha\,\Delta\tau}{(\Delta x)^2}\left\{2\frac{h\,\Delta x}{k}\,T_\infty + 2T^p_{m-1,n} + T^p_{m,n+1} + T^p_{m,n-1} + \left[\frac{(\Delta x)^2}{\alpha\,\Delta\tau} - 2\frac{h\,\Delta x}{k} - 4\right]T^p_{m,n}\right\} \qquad \textbf{[4-34]}$$

The corresponding one-dimensional relation is

$$T^{p+1}_{m} = \frac{\alpha\,\Delta\tau}{(\Delta x)^2}\left\{2\frac{h\,\Delta x}{k}T_\infty + 2T^p_{m-1} + \left[\frac{(\Delta x)^2}{\alpha\,\Delta\tau} - 2\frac{h\,\Delta x}{k} - 2\right]T^p_m\right\} \qquad \textbf{[4-35]}$$

Notice now that the selection of the parameter $(\Delta x)^2/\alpha\,\Delta\tau$ is not as simple as it is for the interior nodal points because the heat-transfer coefficient influences the choice. It is still

Figure 4-21 | Nomenclature for nodal equation with convective boundary condition.



possible to choose the value of this parameter so that the coefficient of T_m^p or $T_{m,n}^p$ will be zero. These values would then be

$$\frac{(\Delta x)^2}{\alpha\,\Delta\tau} = \begin{cases} 2\left(\dfrac{h\,\Delta x}{k}+1\right) & \text{for the one-dimensional case} \\ 2\left(\dfrac{h\,\Delta x}{k}+2\right) & \text{for the two-dimensional case} \end{cases}$$

To ensure convergence of the numerical solution, all selections of the parameter $(\Delta x)^2/\alpha\,\Delta\tau$ must be restricted according to

$$\frac{(\Delta x)^2}{\alpha\,\Delta\tau} \geq \begin{cases} 2\left(\dfrac{h\,\Delta x}{k}+1\right) & \text{for the one-dimensional case} \\ 2\left(\dfrac{h\,\Delta x}{k}+2\right) & \text{for the two-dimensional case} \end{cases}$$

Forward and Backward Differences

The equations above have been developed on the basis of a *forward-difference* technique in that the temperature of a node at a future time increment is expressed in terms of the surrounding nodal temperatures at the beginning of the time increment. The expressions are called *explicit* formulations because it is possible to write the nodal temperatures $T_{m,n}^{p+1}$ explicitly in terms of the previous nodal temperatures $T_{m,n}^p$. In this formulation, the calculation proceeds directly from one time increment to the next until the temperature distribution is calculated at the desired final state.

The difference equation may also be formulated by computing the space derivatives in terms of the temperatures at the $p+1$ time increment. Such an arrangement is called a *backward-difference* formulation because the time derivative moves backward from the times for heat conduction into the node. The equation equivalent to Equation (4-28) would then be

$$\frac{T_{m+1,n}^{p+1}+T_{m-1,n}^{p+1}-2T_{m,n}^{p+1}}{(\Delta x)^2}+\frac{T_{m,n+1}^{p+1}+T_{m,n-1}^{p+1}-2T_{m,n}^{p+1}}{(\Delta y)^2}=\frac{1}{\alpha}\frac{T_{m,n}^{p+1}-T_{m,n}^{p}}{\Delta\tau} \quad \textbf{[4-36]}$$

The equivalence to Equation (4-29) is

$$T_{m,n}^{p}=\frac{-\alpha\,\Delta\tau}{(\Delta x)^2}\left(T_{m+1,n}^{p+1}+T_{m-1,n}^{p+1}+T_{m,n+1}^{p+1}+T_{m,n-1}^{p+1}\right)+\left[1+\frac{4\alpha\,\Delta\tau}{(\Delta x)^2}\right]T_{m,n}^{p+1} \quad \textbf{[4-37]}$$

We may now note that this backward-difference formulation does not permit the explicit calculation of the T^{p+1} in terms of T^p. Rather, a whole set of equations must be written for the entire nodal system and solved simultaneously to determine the temperatures T^{p+1}. Thus we say that the backward-difference method produces an *implicit formulation* for the future temperatures in the transient analysis. The solution to the set of equations can be performed with the methods discussed in Chapter 3.

The Biot and Fourier numbers may also be defined in the following way for problems in the numerical format:

$$\mathrm{Bi}=\frac{h\,\Delta x}{k} \quad \textbf{[4-38]}$$

$$\mathrm{Fo}=\frac{\alpha\,\Delta\tau}{(\Delta x)^2} \quad \textbf{[4-39]}$$

By using this notation, Tables 4-2 and 4-3 have been constructed to summarize some typical nodal equations in both the explicit and implicit formulations. For the cases of $\Delta x=\Delta y$

displayed in Table 4-2, the most restrictive stability requirement (smallest $\Delta\tau$) is exhibited by an exterior corner node, assuming all the convection nodes have the same value of Bi.

Table 4-2 | Explicit nodal equations. (Dashed lines indicate element volume.)†

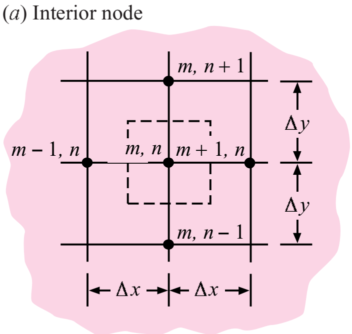
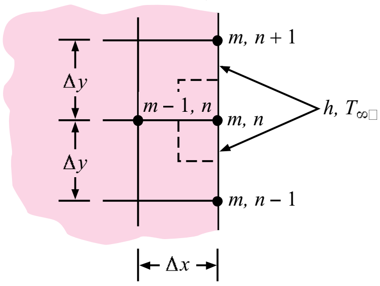
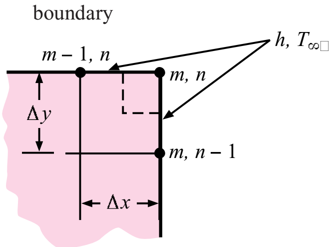
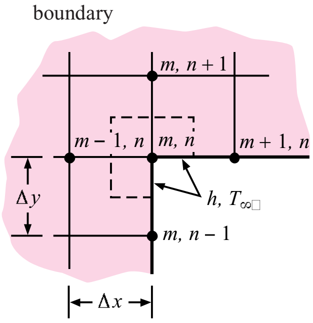
<table>
<tr><th>Physical situation</th><th>Nodal equation for $\Delta x = \Delta y$</th><th>Stability requirement</th></tr>
<tr><td>(a) Interior node
</td><td>$T^{p+1}_{m,n} = \text{Fo}\left(T^p_{m-1,n}+T^p_{m,n+1}+T^p_{m+1,n}+T^p_{m,n-1}\right) + [1-4(\text{Fo})]T^p_{m,n}$
$T^{p+1}_{m,n} = \text{Fo}\left(T^p_{m-1,n}+T^p_{m,n+1}+T^p_{m+1,n}+T^p_{m,n-1} - 4T^p_{m,n}\right)+T^p_{m,n}$</td><td>$\text{Fo} \le \frac{1}{4}$</td></tr>
<tr><td>(b) Convection boundary node
</td><td>$T^{p+1}_{m,n} = \text{Fo}\,[2T^p_{m-1,n}+T^p_{m,n+1}+T^p_{m,n-1}+2(\text{Bi})T^p_\infty] + [1-4(\text{Fo})-2(\text{Fo})(\text{Bi})]T^p_{m,n}$
$T^{p+1}_{m,n} = \text{Fo}\,[2\text{Bi}\,(T^p_\infty - T^p_{m,n}) + 2T^p_{m-1,n}+T^p_{m,n+1}+T^p_{m,n-1}-4T^p_{m,n}]+T^p_{m,n}$</td><td>$\text{Fo}(2+\text{Bi}) \le \frac{1}{2}$</td></tr>
<tr><td>(c) Exterior corner with convection boundary
</td><td>$T^{p+1}_{m,n}=2(\text{Fo})\,[T^p_{m-1,n}+T^p_{m,n-1}+2(\text{Bi})T^p_\infty]+[1-4(\text{Fo})-4(\text{Fo})(\text{Bi})]T^p_{m,n}$
$T^{p+1}_{m,n}=2\text{Fo}\,[T^p_{m-1,n}+T^p_{m,n-1}-2T^p_{m,n}+2\text{Bi}(T^p_\infty-T^p_{m,n})]+T^p_{m,n}$</td><td>$\text{Fo}(1+\text{Bi}) \le \frac{1}{4}$</td></tr>
<tr><td>(d) Interior corner with convection boundary
</td><td>$T^{p+1}_{m,n}=\frac{2}{3}(\text{Fo})\,[2T^p_{m,n+1}+2T^p_{m+1,n}+2T^p_{m-1,n}+T^p_{m,n-1}+2(\text{Bi})T^p_\infty]+[1-4(\text{Fo})-\frac{4}{3}(\text{Fo})(\text{Bi})]T^p_{m,n}$
$T^{p+1}_{m,n}=(4/3)\text{Fo}\,[T^p_{m,n+1}+T^p_{m+1,n}+T^p_{m-1,n}-3T^p_{m,n}+\text{Bi}\,(T^p_\infty-T^p_{m,n})]+T^p_{m,n}$</td><td>$\text{Fo}(3+\text{Bi}) \le \frac{3}{4}$</td></tr>
</table>

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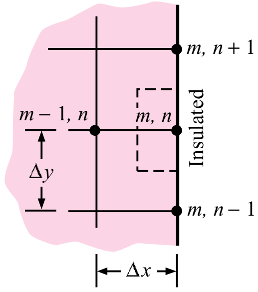
Table 4-2 | (*Continued*).

Physical situation	Nodal equation for $\Delta x = \Delta y$	Stability requirement
(*e*) Insulated boundary 	$T_{m,n}^{p+1} = \text{Fo}\,[2T_{m-1,n}^{p} + T_{m,n+1}^{p} + T_{m,n-1}^{p}] + [1 - 4(\text{Fo})]T_{m,n}^{p}$	$\text{Fo} \le \frac{1}{4}$

†Convection surfaces may be made insulated by setting $h = 0$ (Bi = 0).

Table 4-3 | Implicit nodal equations. (Dashed lines indicate volume element.)

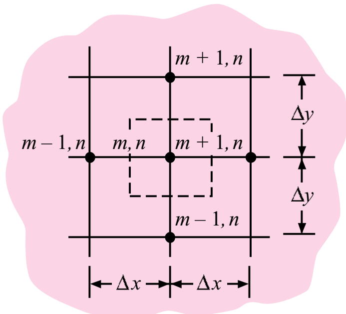
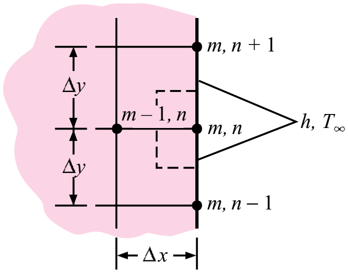
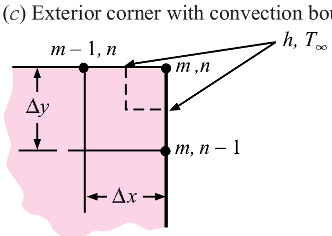
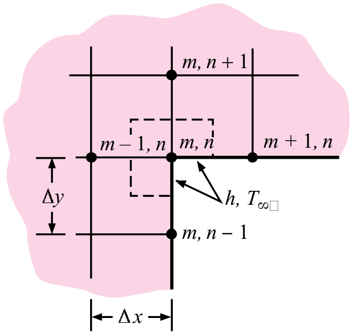
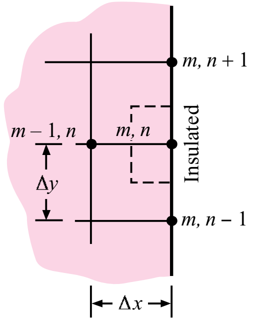
Physical situation	Nodal equation for $\Delta x = \Delta y$
(*a*) Interior node 	$[1 + 4(\text{Fo})]T_{m,n}^{p+1} - \text{Fo}\left(T_{m-1,n}^{p+1} + T_{m,n+1}^{p+1} + T_{m+1,n}^{p+1} + T_{m,n-1}^{p+1}\right) - T_{m,n}^{p} = 0$
(*b*) Convection boundary node 	$[1 + 2(\text{Fo})(2 + \text{Bi})]T_{m,n}^{p+1} - \text{Fo}\left[T_{m,n+1}^{p+1} + T_{m,n-1}^{p+1} + 2T_{m-1,n}^{p+1} + 2(\text{Bi})T_{\infty}^{p+1}\right] - T_{m,n}^{p} = 0$
(*c*) Exterior corner with convection boundary 	$[1 + 4(\text{Fo})(1 + \text{Bi})]T_{m,n}^{p+1} - 2(\text{Fo})\left[T_{m-1,n}^{p+1} + T_{m,n-1}^{p+1} + 2(\text{Bi})T_{\infty}^{p+1}\right] - T_{m,n}^{p} = 0$

Table 4-3 | (*Continued*).

Physical situation	Nodal equation for $\Delta x = \Delta y$
(*d*) Interior corner with convection boundary 	$\left[1+4(\text{Fo})\left(1+\frac{\text{Bi}}{3}\right)\right]T_{m,n}^{p+1} - \frac{2(\text{Fo})}{3}\times\left[2T_{m-1,n}^{p+1}+T_{m,n-1}^{p+1} + 2T_{m,n+1}^{p+1}+2T_{m+1,n}^{p+1}+2(\text{Bi})T_{\infty}^{p+1}\right] - T_{m,n}^{p} = 0$
(*e*) Insulated boundary 	$[1+4(\text{Fo})]T_{m,n}^{p+1} - \text{Fo}\left(2T_{m-1,n}^{p+1}+T_{m,n+1}^{p+1}+T_{m,n-1}^{p+1}\right) - T_{m,n}^{p} = 0$

The advantage of an explicit forward-difference procedure is the direct calculation of future nodal temperatures; however, the stability of this calculation is governed by the selection of the values of Δx and $\Delta\tau$. A selection of a small value of Δx automatically forces the selection of some maximum value of $\Delta\tau$. On the other hand, no such restriction is imposed on the solution of the equations that are obtained from the implicit formulation. This means that larger time increments can be selected to speed the calculation. The obvious disadvantage of the implicit method is the larger number of calculations for each time step. For problems involving a large number of nodes, however, the implicit method may result in less total computer time expended for the final solution because very small time increments may be imposed in the explicit method from stability requirements. Much larger increments in $\Delta\tau$ can be employed with the implicit method to speed the solution.

Most problems will involve only a modest number of nodes and the explicit formulation will be quite satisfactory for a solution, particularly when considered from the standpoint of the more generalized formulation presented in the following section.

For a discussion of many applications of numerical analysis to transient heat conduction problems, the reader is referred to References 4, 8, 13, 14, and 15.

It should be obvious to the reader by now that finite-difference techniques may be applied to almost any situation with just a little patience and care. Very complicated problems then become quite easy to solve with only modest computer facilities. The use of Microsoft Excel for solution of transient heat-transfer problems is discussed in Appendix D.

Finite-element methods for use in conduction heat-transfer problems are discussed in References 9 to 13. A number of software packages are available commercially.

4-7 | THERMAL RESISTANCE AND CAPACITY FORMULATION

As in Chapter 3, we can view each volume element as a node that is connected by thermal resistances to its adjoining neighbors. For steady-state conditions the net energy transfer into the node is zero, while for the unsteady-state problems of interest in this chapter the net energy transfer into the node must be evidenced as an increase in internal energy of the element. Each volume element behaves like a small "lumped capacity," and the interaction of all the elements determines the behavior of the solid during a transient process. If the internal energy of a node *i* can be expressed in terms of specific heat and temperature, then its rate of change with time is approximated by

$$\frac{\Delta E}{\Delta \tau} = \rho c \Delta V \frac{T_i^{p+1} - T_i^p}{\Delta \tau}$$

where ΔV is the volume element. If we define the thermal capacity as

$$C_i = \rho_i c_i \, \Delta V_i \tag{4-40}$$

then the general resistance-capacity formulation for the energy balance on a node is

$$q_i + \sum_j \frac{T_j^p - T_i^p}{R_{ij}} = C_i \frac{T_i^{p+1} - T_i^j}{\Delta \tau} \tag{4-41}$$

where all the terms on the left are the same as in Equation (3-31). The resistance and volume elements for a variety of geometries and boundary conditions were given in Tables 3-3 and 3-4. Physical systems where the internal energy E involves phase changes can also be accommodated in the above formulation but are beyond the scope of our discussion.

The central point is that use of the concepts of thermal resistance and capacitance enables us to write the forward-difference equation for all nodes and boundary conditions in the single compact form of Equation (4-41). The setup for a numerical solution then becomes a much more organized process that can be adapted quickly to the computational methods at hand.

Equation (4-41) is developed by using the forward-difference concept to produce an explicit relation for each T_i^{p+1}. As in our previous discussion, we could also write the energy balance using backward differences, with the heat transfers into each *i*th node calculated in terms of the temperatures at the $p+1$ time increment. Thus,

$$q_i + \sum_j \frac{T_j^{p+1} - T_i^{p+1}}{R_{ij}} = C_i \frac{T_i^{p+1} - T_i^p}{\Delta \tau} \tag{4-42}$$

Now, as before, the set of equations produces an implicit set that must be solved simultaneously for the T_i^{p+1}, etc. The solution can be carried out by a number of methods as discussed in Chapter 3. If the solution is to be performed with a Gauss-Seidel iteration technique, then Equation (4-42) should be solved for T_i^{p+1} and expressed as

$$T_i^{p+1} = \frac{q_i + \sum_i (T_j^{p+1}/R_{ij}) + (C_i/\Delta\tau) T_i^p}{\sum_j (1/R_{ij}) + C_i/\Delta\tau} \tag{4-43}$$

It is interesting to note that in the steady-state limit of $\Delta\tau \to \infty$ this equation becomes identical with Equation (3-32), the formulation we employed for the iterative solution in Chapter 3.

The stability requirement in the explicit formulation may be examined by solving Equation (4-41) for T_i^{p+1}:

$$T_i^{p+1} = \left(q_i + \sum_j \frac{T_j^p}{R_{ij}}\right)\frac{\Delta\tau}{C_i} + \left(1 - \frac{\Delta\tau}{C_i}\sum_j \frac{1}{R_{ij}}\right)T_i^p \qquad \textbf{[4-44]}$$

The value of q_i can influence the stability, but we can choose a safe limit by observing the behavior of the equation for $q_i = 0$. Using the same type of thermodynamic argument as with Equation (4-31), we find that the coefficient of T_i^p cannot be negative. Our stability requirement is therefore

$$1 - \frac{\Delta\tau}{C_i}\sum_j \frac{1}{R_{ij}} \geq 0 \qquad \textbf{[4-45]}$$

Suppose we have a complicated numerical problem to solve with a variety of boundary conditions, perhaps nonuniform values of the space increments, etc. Once we have all the nodal resistances and capacities formulated, we then have the task of choosing the time increment $\Delta\tau$ to use for the calculation. To ensure stability we must keep $\Delta\tau$ equal to or less than a value obtained from the most restrictive nodal relation like Equation (4-45).

Solving for $\Delta\tau$ gives

$$\Delta\tau \leq \left[\frac{C_i}{\sum_j (1/R_{ij})}\right]_{\text{min}} \qquad \text{for stability} \qquad \textbf{[4-46]}$$

While Equation (4-44) is very useful in establishing the maximum allowable time increment, it may involve problems of round-off errors in computer solutions when small thermal resistances are employed. The difficulty may be alleviated by expressing T_i^{p+1} in the following form for calculation purposes:

$$T_i^{p+1} = \frac{\Delta\tau}{C_i}\left[q_i + \sum_j \frac{T_j^p - T_i^p}{R_{ij}}\right] + T_i^p \qquad \textbf{[4-47]}$$

In Table 4-2 the nodal equations for $\Delta x = \Delta y$ are listed in the formats of both equations (4-44) and (4-47). The equations listed in Table 4-2 in the form of Equation (4-47) do not include the heat-source term. If needed, the term may be added using

$$q_i = \dot{q}_i \, \Delta V_i$$

where $\dot{q}_i$ is the heat generation per unit volume and ΔV_i is the volume element shown by dashed lines in the table. For radiation input to the node,

$$q_i = q''_{i,\,\text{rad}} \times \Delta A_i$$

where $q''_{i,\text{rad}}$ is the *net radiant energy input to the node per unit area* and ΔA_i is the area of the node for radiant exchange, which may or may not be equal to the area for convection heat transfer.

We should remark that the resistance-capacity formulation is easily adapted to take into account thermal-property variations with temperature. One need only calculate the proper values of ρ, c, and k for inclusion in the C_i and R_{ij}. Depending on the nature of the problem and accuracy required, it may be necessary to calculate new values of C_i and R_{ij} for each time increment. Example 4-17 illustrates the effects of variable conductivity.

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Steady State as a Limiting Case of Transient Solution

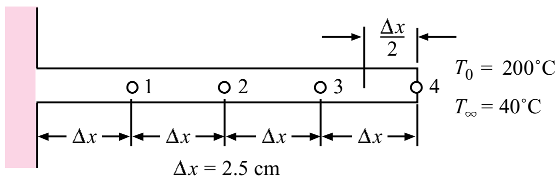
As we have seen, the steady-state numerical formulation results when the right side of Equation (4-41) is set equal to zero. It also results when the calculation of the unsteady case using either Equation (4-44) or (4-47) is carried out for a large number of time increments. While the latter method of obtaining a steady-state solution may appear rather cumbersome, it can proceed quite rapidly with a computer. We may recall that the Gauss-Seidel iteration method was employed for the solution of many steady-state numerical problems, which of course entailed many computer calculations. If variable thermal resistances resulting from either variable thermal conductivities or variations in convection boundary conditions are encountered, the steady-state limit of a transient solution may offer advantages over the direct steady-state solution counterpart. We will recall that when variable thermal resistances appear, the resulting steady-state nodal equations become nonlinear and their solution may be tedious. The transient solution for such cases merely requires that each resistance be recalculated at the end of each time increment $\Delta\tau$, or the resistances may be entered directly as variables in the nodal equations. The calculations are then carried out for a sufficiently large number of time increments until the values of the T_i^{p+1} no longer change by a significant amount. At this point, the steady-state solution is obtained as the resulting values of the T_i.

The formulation and solution of transient numerical problems using Microsoft Excel is described in Section D-5 of the Appendix, along with worked examples. An example is also given of a transient solution carried forward a sufficient length of time to achieve steady-state conditions.

EXAMPLE 4-11 Sudden Cooling of a Rod

A steel rod [$k = 50$ W/m · °C] 3 mm in diameter and 10 cm long is initially at a uniform temperature of 200°C. At time zero it is suddenly immersed in a fluid having $h = 50$ W/m^2 · °C and $T_\infty = 40$°C while one end is maintained at 200°C. Determine the temperature distribution in the rod after 100 s. The properties of steel are $\rho = 7800$ kg/m^3 and $c = 0.47$ kJ/kg · °C.

Figure Example 4-11



■ **Solution**

The selection of increments on the rod is as shown in the Figure Example 4-11. The cross-sectional area of the rod is $A = \pi(1.5)^2 = 7.069$ mm^2. The volume element for nodes 1, 2, and 3 is

$$\Delta V = A\Delta x = (7.069)(25) = 176.725 \text{ mm}^3$$

Node 4 has a ΔV of half this value, or 88.36 mm^3. We can now tabulate the various resistances and capacities for use in an explicit formulation. For nodes 1, 2, and 3 we have

$$R_{m+} = R_{m-} = \frac{\Delta x}{kA} = \frac{0.025}{(50)(7.069 \times 10^{-6})} = 70.731°\text{C/W}$$

and

$$R_\infty = \frac{1}{h(\pi d\, \Delta x)} = \frac{1}{(50)\pi(3 \times 10^{-3})(0.025)} = 84.883°\text{C/W}$$

$$C = \rho c\,\Delta V = (7800)(470)(1.7673 \times 10^{-7}) = 0.6479 \text{ J/°C}$$

For node 4 we have

$$R_{m+} = \frac{1}{hA} = 2829\text{°C/W} \qquad R_{m-} = \frac{\Delta x}{kA} = 70.731\text{°C/W}$$

$$C = \frac{\rho c\,\Delta V}{2} = 0.3240 \text{ J/°C} \qquad R_\infty = \frac{2}{h\pi d\Delta x} = 169.77\text{°C/W}$$

To determine the stability requirement we form the following table:

Node	$\sum(1/R_{ij})$	C_i	$\frac{C_i}{\sum(1/R_{ij})}$, s
1	0.04006	0.6479	16.173
2	0.04006	0.6479	16.173
3	0.04006	0.6479	16.173
4	0.02038	0.3240	15.897

Thus node 4 is the most restrictive, and we must select $\Delta\tau < 15.9$ s. Since we wish to find the temperature distribution at 100 s, let us use $\Delta\tau = 10$ s and make the calculation for 10 time increments using Equation (4-47) for the computation. We note, of course, that $q_i = 0$ because there is no heat generation. The calculations are shown in the following table.

Time increment	Node temperature			
	T_1	T_2	T_3	T_4
0	200	200	200	200
1	170.87	170.87	170.87	169.19
2	153.40	147.04	146.68	145.05
3	141.54	128.86	126.98	125.54
4	133.04	115.04	111.24	109.70
5	126.79	104.48	98.76	96.96
6	122.10	96.36	88.92	86.78
7	118.53	90.09	81.17	78.71
8	115.80	85.23	75.08	72.34
9	113.70	81.45	70.31	67.31
10	112.08	78.51	66.57	63.37

We can calculate the heat-transfer *rate* at the end of 100 s by summing the convection heat losses on the surface of the rod. Thus

$$q = \sum_i \frac{T_i - T_\infty}{R_{i\infty}}$$

and

$$q = \frac{200 - 40}{(2)(84.883)} + \frac{112.08 + 78.51 + 66.57 - (3)(40)}{84.883} + \left(\frac{1}{169.77} + \frac{1}{2829}\right)(63.37 - 40)$$
$$= 2.704 \text{ W}$$

Implicit Formulation

EXAMPLE 4-12

We can illustrate the calculation scheme for the implicit formulation by reworking Example 4-11 using only two time increments, that is, $\Delta\tau = 50$ s.

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For this problem we employ the formulation indicated by Equation (4-43), with $\Delta\tau = 50$ s. The following quantities are needed.

Node	$\dfrac{C_i}{\Delta\tau}$	$\sum_i \dfrac{1}{R_{ij}} + \dfrac{C_i}{\Delta\tau}$
1	0.01296	0.05302
2	0.01296	0.05302
3	0.01296	0.05302
4	0.00648	0.02686

We have already determined the R_{ij} in Example 4-11 and thus can insert them into Equation (4-43) to write the nodal equations for the end of the first time increment, taking all $T_i^p = 200°\text{C}$. We use the prime to designate temperatures at the end of the time increment. For node 1,

$$0.05302T_1' = \frac{200}{70.731} + \frac{T_2'}{70.731} + \frac{40}{84.833} + (0.01296)(200)$$

For node 2,

$$0.05302T_2' = \frac{T_1'}{70.731} + \frac{T_3'}{70.731} + \frac{40}{84.833} + (0.01296)(200)$$

For nodes 3 and 4,

$$0.05302T_3' = \frac{T_2'}{70.731} + \frac{T_4'}{70.731} + \frac{40}{84.833} + (0.01296)(200)$$

$$0.02686T_4' = \frac{T_3'}{70.731} + \frac{40}{2829} + \frac{40}{169.77} + (0.00648)(200)$$

These equations can then be reduced to

$$\begin{aligned}
0.05302T_1' - 0.01414T_2' \quad &= 5.8911\\
-0.01414T_1' + 0.05302T_2' - 0.01414T_3' \quad &= 3.0635\\
-0.01414T_2' + 0.05302T_3' - 0.01414T_4' &= 3.0635\\
-0.01414T_3' + 0.02686T_4' &= 1.5457
\end{aligned}$$

which have the solution

$$T_1' = 145.81°\text{C} \qquad T_2' = 130.12°\text{C}$$
$$T_3' = 125.43°\text{C} \qquad T_4' = 123.56°\text{C}$$

We can now apply the backward-difference formulation a second time using the double prime to designate the temperatures at the end of the second time increment:

$$0.05302T_1'' = \frac{200}{70.731} + \frac{T_2''}{70.731} + \frac{40}{84.833} + (0.01296)(145.81)$$

$$0.05302T_2'' = \frac{T_1''}{70.731} + \frac{T_3''}{70.731} + \frac{40}{84.833} + (0.01296)(130.12)$$

$$0.05302T_3'' = \frac{T_2''}{70.731} + \frac{T_4''}{70.731} + \frac{40}{84.833} + (0.01296)(125.43)$$

$$0.02686T_4'' = \frac{T_3''}{70.731} + \frac{40}{2829} + \frac{40}{169.77} + (0.00648)(123.56)$$

and this equation set has the solution

$$T_1'' = 123.81°\text{C} \qquad T_2'' = 97.27°\text{C}$$
$$T_3'' = 88.32°\text{C} \qquad T_4'' = 85.59°\text{C}$$

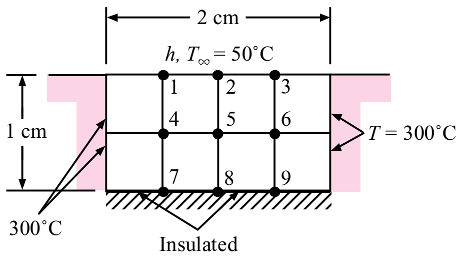
We find this calculation in substantial disagreement with the results of Example 4-11. With a larger number of time increments, better agreement would be achieved. In a problem involving a large number of nodes, the implicit formulation might involve less computer time than the explicit method, and the purpose of this example has been to show how the calculation is performed.

Cooling of a Ceramic

EXAMPLE 4-13

A 1 by 2 cm ceramic strip [$k = 3.0$ W/m · °C] is embedded in a high-thermal-conductivity material, as shown in Figure Example 4-13, so that the sides are maintained at a constant temperature of 300°C. The bottom surface of the ceramic is insulated, and the top surface is exposed to a convection environment with $h = 200$ W/m^2 · °C and $T_\infty = 50$°C. At time zero the ceramic is uniform in temperature at 300°C. Calculate the temperatures at nodes 1 to 9 after a time of 12 s. For the ceramic $\rho = 1600$ kg/m^3 and $c = 0.8$ kJ/kg · °C. Also calculate the total heat loss in this time.

Figure Example 4-13



■ **Solution**

We treat this as a two-dimensional problem with $\Delta x = \Delta y = 0.5$ cm. From symmetry $T_1 = T_3$, $T_4 = T_6$, and $T_7 = T_9$, so we have six unknown nodal temperatures. We now tabulate the various nodal resistances and capacities. For nodes 4 and 5

$$R_{m+} = R_{m-} = R_{n+} = R_{n-} = \frac{\Delta x}{kA} = \frac{0.005}{(3.0)(0.005)} = 0.3333$$

For nodes 1 and 2

$$R_{m+} = R_{m-} = \frac{\Delta x}{kA} = \frac{(0.005)(2)}{(3.0)(0.005)} = 0.6667°\text{C/W} \qquad R_{n-} = 0.3333°\text{C/W}$$

$$R_{n+} = \frac{1}{h\,\Delta x} = \frac{1}{(200)(0.005)} = 1.0°\text{C/W}$$

For nodes 7 and 8

$$R_{m+} = R_{m-} = 0.6667°\text{C/W} \qquad R_{n+} = 0.3333°\text{C/W} \qquad R_{n-} = \infty$$

For nodes 1, 2, 7, and 8 the capacities are

$$C = \frac{\rho c(\Delta x)^2}{2} = \frac{(1600)(800)(0.005)^2}{2} = 16\ \text{J/°C}$$

For nodes 4 and 5

$$C = \rho c(\Delta x)^2 = 32\ \text{J/°C}$$

The stability requirement for an explicit solution is now determined by tabulating the following quantities:

Node	$\sum \frac{1}{R_{ij}}$	C_i	$\frac{C_i}{\sum(1/R_{ij})}$, s
1	7	16	2.286
2	7	16	2.286
4	12	32	2.667
5	12	32	2.667
7	6	16	2.667
8	6	16	2.667

Thus the two convection nodes control the stability requirement, and we must choose $\Delta\tau \leq 2.286$ s. Let us choose $\Delta\tau = 2.0$ s and make the calculations for six time increments with Equation (4-47). We note once again the symmetry considerations when calculating the temperatures of nodes 2, 5, and 8, that is, $T_1 = T_3$, etc. The calculations are shown in the following table.

	Node temperature					
Time increment	T_1	T_2	T_4	T_5	T_7	T_8
0	300	300	300	300	300	300
1	268.75	268.75	300	300	300	300
2	258.98	253.13	294.14	294.14	300	300
3	252.64	245.31	289.75	287.55	297.80	297.80
4	284.73	239.48	285.81	282.38	295.19	293.96
5	246.67	235.35	282.63	277.79	292.34	290.08
6	243.32	231.97	279.87	273.95	289.71	286.32

The total heat loss during the 12-s time interval is calculated by summing the heat loss of each node relative to the initial temperature of 300°C. Thus

$$q = \sum C_i(300 - T_i)$$

where q is the heat *loss*. For this summation, since the constant-temperature boundary nodes experience no change in temperature, they can be left out. Recalling that $T_1 = T_3$, $T_4 = T_6$, and $T_7 = T_9$, we have

$$\begin{aligned}\sum C_i(300 - T_i) &= \text{nodes } (1, 2, 3, 7, 8, 9) + \text{nodes } (4, 5, 6)\\ &= 16[(6)(300) - (2)(243.2) - 231.97 - (2)(289.71)\\ &\quad - 286.32] + 32[(3)(300) - (2)(279.87) - 273.95]\\ &= 5572.3 \text{ J/m length of strip}\end{aligned}$$

The *average rate* of heat loss for the 12-s time interval is

$$\frac{q}{\Delta\tau} = \frac{5572.3}{12} = 464.4 \text{ W} \quad [1585 \text{ Btu/h}]$$

EXAMPLE 4-14 Cooling of a Steel Rod, Nonuniform h

A nickel-steel rod having a diameter of 2.0 cm is 10 cm long and initially at a uniform temperature of 200°C. It is suddenly exposed to atmospheric air at 30°C while one end of the rod is maintained at 200°C. The convection heat-transfer coefficient can be computed from

$$h = 9.0\ \Delta T^{0.175} \text{ W/m}^2 \cdot {}^\circ\text{C}$$

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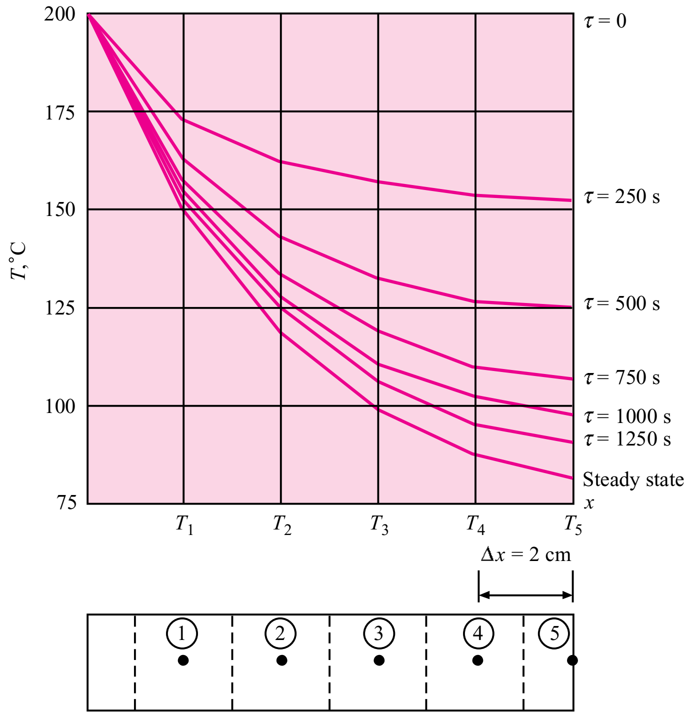
where ΔT is the temperature difference between the rod and air surroundings. The properties of nickel steel may be taken as $k = 12$ W/m · °C, $c = 0.48$ kJ/kg · °C, and $\rho = 7800$ kg/m^3. Using the numerical method, (*a*) determine the temperature distribution in the rod after 250, 500, 750, 1000, 1250 s, and for steady state; (*b*) determine the steady-state temperature distribution for a constant $h = 22.11$ W/m^2 · °C and compare with an analytical solution.

Figure Example 4-14



■ Solution

Five nodes are chosen as shown in Figure Example 4-14 with $\Delta x = 2.0$ cm. The capacitances are then

$$C_1 = C_2 = C_3 = C_4 = \frac{(7800)(480)\pi(0.02)^2(0.02)}{4} = 23.524 \text{ J/°C}$$

$$C_5 = \tfrac{1}{2}C_1 = 11.762 \text{ J/°C}$$

The resistances for nodes 1, 2, 3, and 4 are

$$\frac{1}{R_{m+}} = \frac{1}{R_{m-}} = \frac{kA}{\Delta x} = \frac{(12)\pi(0.02)^2}{(4)(0.02)} = 0.188496$$

$$\frac{1}{R_\infty} = hP\,\Delta x = (9.0)\pi(0.02)(0.02)(T-30)^{0.175} = (1.131 \times 10^{-2})(T-30)^{0.175}$$

For node 5

$$\frac{1}{R_{m-}} = 0.188496$$

$$\frac{1}{R_{m+}} = hA = 9.0\frac{\pi(0.02)^2}{4}(T-30)^{0.175} = (2.827 \times 10^{-3})(T-30)^{0.175}$$

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$$\frac{1}{R_{5\infty}} = \frac{1}{2R_{1\infty}} = (5.655 \times 10^{-3})(T - 30)^{0.175}$$

where $T_\infty = 30°C$ for all nodes. We can compute the following table for worst-case conditions of $T = 200°C$ throughout the rod. The stability requirement so established will then work for all other temperatures.

Node	$\sum(1/R_{ij})\vert_{min}$	$\dfrac{C_i}{\sum(1/R_{ij})}$, s
1	0.4048	58.11
2	0.4048	58.11
3	0.4048	58.11
4	0.4048	58.11
5	0.2093	56.197

Thus, time steps below 56 s will ensure stability. The computational procedure is complicated by the fact that the convection-resistance elements must be recalculated for each time step. Selecting $\Delta\tau = 50$ s, we have:

Node	$\Delta\tau/C_i$
1	2.1255
2	2.1255
3	2.1255
4	2.1255
5	4.251

We then use the explicit formulation of Equation (4-47) with no heat generation. The computational algorithm is thus:

1. Compute R_∞ values for the initial condition.
2. Compute temperatures at next time increment using Equation (4-47).
3. Recalculate R_∞ values based on new temperatures.
4. Repeat temperature calculations and continue until the temperature distributions are obtained at the desired times.

Results of these calculations are shown in the accompanying figure.

To determine the steady-state distribution we could carry the unsteady method forward a large number of time increments or use the steady-state method and an iterative approach. The iterative approach is required because the equations are nonlinear as a result of the variations in the convection coefficient.

We still use a resistance formulation, which is now given as Equation (3-31):

$$\sum \frac{T_j - T_i}{R_{ij}} = 0$$

The computational procedure is:

1. Calculate R_∞ values for all nodes assuming all $T_i = 200°C$.
2. Formulate nodal equations for the T_i's.
3. Solve the equations by an appropriate method.
4. Recalculate R_∞ values based on T_i values obtained in step 3.
5. Repeat the procedure until there are only small changes in T_i's.

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The results of this iteration are shown in the following table:

Iteration	T_1, °C	T_2, °C	T_3, °C	T_4, °C	T_5, °C
1	148.462	114.381	92.726	80.310	75.302
2	151.381	119.557	99.409	87.853	83.188
3	151.105	119.038	98.702	87.024	82.306
4	151.132	119.090	98.774	87.109	82.396

This steady-state temperature distribution is also plotted with the transient profiles.

The value of h for $T_i = 200$°C is 22.11 W/m^2 · °C, so the results of the first iteration correspond to a solution for a constant h of this value. The exact analytical solution is given in Equation (2-34) as

$$\frac{\theta}{\theta_0} = \frac{T - T_\infty}{T_0 - T_\infty} = \frac{\cosh m(L-x) + [h/km] \sinh m(L-x)}{\cosh mL + [h/km] \sinh mL}$$

The required quantities are

$$m = \left(\frac{hP}{kA}\right)^{1/2} = \left[\frac{(22.11)\pi(0.02)}{(12)\pi(0.01)^2}\right]^{1/2} = 19.1964$$

$$mL = (19.1964)(0.1) = 1.91964$$

$$h/km = \frac{22.22}{(12)(19.1964)} = 0.09598$$

The temperatures at the nodal points can then be calculated and compared with the numerical results in the following table. As can be seen, the agreement is excellent.

Node	x, m	$(\theta/\theta_0)_{\text{num}}$	$(\theta/\theta_0)_{\text{anal}}$	Percent deviation
1	0.02	0.6968	0.6949	0.27
2	0.04	0.4964	0.4935	0.59
3	0.06	0.3690	0.3657	0.9
4	0.08	0.2959	0.2925	1.16
5	0.1	0.2665	0.2630	1.33

We may also check the heat loss with that predicted by the analytical relation in Equation (2-34). When numerical values are inserted we obtain

$$q_{\text{anal}} = 11.874 \text{ W}$$

The heat loss for the numerical model is computed by summing the convection loss from the six nodes (including base node at 200°C). Using the temperatures for the first iteration corresponding to $h = 22.11$ W/m^2 · °C,

$$\begin{aligned} q &= (22.11)\pi(0.02)(0.02)\Big[(200-30)\left(\tfrac{1}{2}\right) + (148.462-30) \\ &\quad + (114.381-30) + (92.726-30) + (80.31-30) \\ &\quad + (75.302-30)\left(\frac{1}{2}\right)\Big] + (22.11)\pi(0.01)2(75.302-30) \\ &= 12.082 \text{ W} \end{aligned}$$

We may make a further check by calculating the energy conducted in the base. This must be the energy conducted to node 1 plus the convection lost by the base node or

$$\begin{aligned} q &= (12)\pi(0.01)^2 \frac{(200-148.462)}{0.02} + (22.11)\pi(0.02)(0.01)(200-30) \\ &= 12.076 \text{ W} \end{aligned}$$

This agrees very well with the convection calculation and both are within 1.8 percent of the analytical value.

The results of this example illustrate the power of the numerical method in solving problems that could not be solved in any other way. Furthermore, only a modest number of nodes, and thus modest computation facilities, may be required to obtain a sufficiently accurate solution. For example, the accuracy with which h will be known is typically ± 10 to 15 percent. This would overshadow any inaccuracies introduced by using relatively large nodes, as was done here.

EXAMPLE 4-15 Radiation Heating and Cooling

The ceramic wall shown in Figure Example 4-15*a* is initially uniform in temperature at 20°C and has a thickness of 3.0 cm. It is suddenly exposed to a radiation source on the right side at 1000°C. The left side is exposed to room air at 20°C with a radiation surrounding temperature of 20°C. Properties of the ceramic are $k = 3.0$ W/m · °C, $\rho = 1600$ kg/m^3, and $c = 0.8$ kJ/kg · °C. Radiation heat transfer with the surroundings at T_r may be calculated from

$$q_r = \sigma \epsilon A(T^4 - T_r{}^4) \qquad \text{W} \tag{a}$$

where $\sigma = 5.669 \times 10^{-8}$, $\epsilon = 0.8$, and T is in degrees Kelvin. The convection heat-transfer coefficient from the left side of the plate is given by

$$h = 1.92 \Delta T^{1/4} \qquad \text{W/m}^2 \cdot {}^\circ\text{C} \tag{b}$$

Convection on the right side is negligible. Determine the temperature distribution in the plate after 15, 30, 45, 60, 90, 120, and 150 s. Also determine the steady-state temperature distribution. Calculate the total heat gained by the plate for these times.

Solution

We divide the wall into five nodes as shown and must express temperatures in degrees Kelvin because of the radiation boundary condition. For node 1 the transient energy equation is

$$\sigma\epsilon\left(293^4 - T_1^{p4}\right) - 1.92\left(T_1^p - 293\right)^{5/4} + \frac{k}{\Delta x}\left(T_2^p - T_1^p\right) = \rho c \frac{\Delta x}{2} \frac{T_1^{p+1} - T_1^p}{\Delta \tau} \tag{c}$$

Similarly, for node 5

$$\sigma\epsilon\left(1273^4 - T_5^{p4}\right) + \frac{k}{\Delta x}\left(T_4^p - T_5^p\right) = \rho c \frac{\Delta x}{2} \frac{T_5^{p+1} - T_5^p}{\Delta \tau} \tag{d}$$

Equations (c) and (d) may be subsequently written

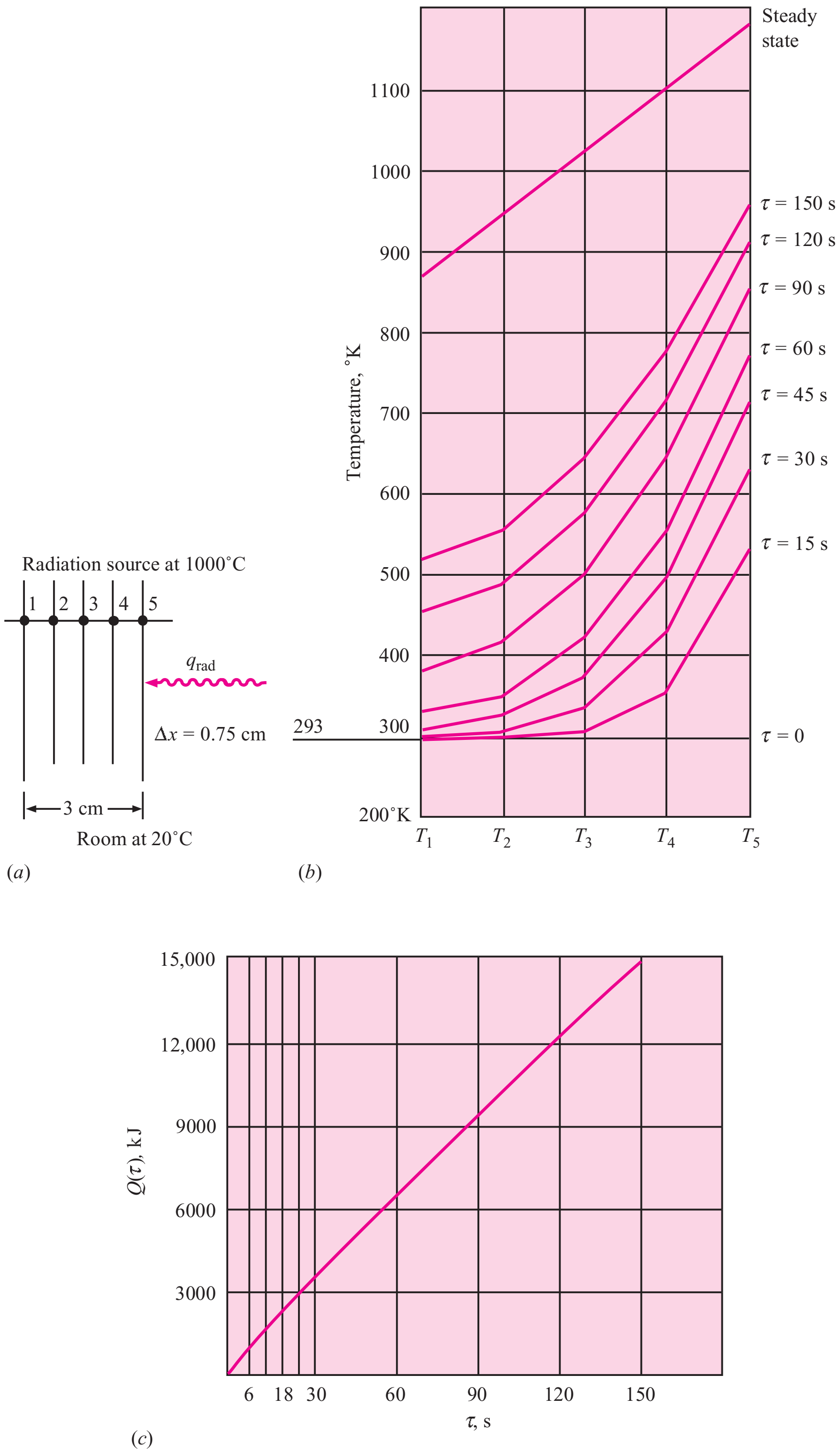
$$\begin{aligned} T_1^{p+1} &= \frac{\Delta\tau}{C_1}\left[\sigma\epsilon(293^2 + T_1^{p2})(293 + T_1^p)(293) - 1.92(T_1^p - 293)^{1/4}(293) + \frac{k}{\Delta x} T_2^p\right] \\ &\quad + \left\{1 - \frac{\Delta\tau}{C_1}\left[\sigma\epsilon(293^2 + T_1^{p2})(293 + T_1^p) - 1.92(T_1^p - 293)^{1/4} + \frac{k}{\Delta x}\right]\right\} T_1^p \end{aligned} \tag{e}$$

$$\begin{aligned} T_5^{p+1} &= \frac{\Delta\tau}{C_5}\left[\sigma\epsilon(1273^2 + T_5^{p2})(1273 + T_5^p)(1273) + \frac{k}{\Delta x} T_4^p\right] \\ &\quad + \left\{1 - \frac{\Delta\tau}{C_5}\left[\sigma\epsilon(1273^2 + T_5^{p2})(1273 + T_5^p) + \frac{k}{\Delta x}\right]\right\} T_5^p \end{aligned} \tag{f}$$

where $C_1 = C_5 = \rho c \Delta x / 2$. For the other three nodes the expressions are much simpler:

$$T_2^{p+1} = \frac{\Delta\tau}{C_2} \frac{k}{\Delta x}(T_1^p + T_3^p) + \left(1 - \frac{2k\,\Delta\tau}{C_2\,\Delta x}\right) T_2^p \tag{g}$$

Figure Example 4-15 | (*a*) Nodal system, (*b*) transient response, (*c*) heat added.



$$T_3^{p+1} = \frac{\Delta\tau}{C_3}\frac{k}{\Delta x}(T_2^p + T_4^p) + \left(1 - \frac{2k\,\Delta\tau}{C_3\,\Delta x}\right)T_3^p \qquad \textbf{[}h\textbf{]}$$

$$T_4^{p+1} = \frac{\Delta\tau}{C_4}\frac{k}{\Delta x}(T_3^p + T_5^p) + \left(1 - \frac{2k\,\Delta\tau}{C_4\,\Delta x}\right)T_4^p \qquad \textbf{[}i\textbf{]}$$

where $C_2 = C_3 = C_4 = \rho c\,\Delta x$. So, to determine the transient response, we simply choose a suitable value of $\Delta\tau$ and march through the calculations. The stability criterion is such that the coefficients of the last term in each equation cannot be negative. For Equations (g), (h), and (i) the maximum allowable time increment is

$$\Delta\tau_{\max} = \frac{C_3\,\Delta x}{2k} = \frac{(1600)(800)(0.0075)^2}{(2)(3)} = 12 \text{ s}$$

For Equation (f), the worst case is at the start when $T_5^p = 20°\text{C} = 293$ K. We have

$$C_5 = \frac{(1600)(800)(0.0075)}{2} = 4800$$

so that

$$\begin{aligned}\Delta\tau_{\max} &= \frac{4800}{(5.669\times 10^{-8})(0.8)(1273^2 + 293^2)(1273+293) + 3.0/0.0075} \\ &= 9.43 \text{ s}\end{aligned}$$

For node 1 [Equation (e)] the most restrictive condition occurs when $T_1^p = 293$. We have

$$C_1 = C_5 = 4800$$

so that

$$\begin{aligned}\Delta\tau_{\max} &= \frac{4800}{(5.669\times 10^{-8})(0.8)(293^2 + 293^2)(293+293) + 3.0/0.0075} \\ &= 11.86 \text{ s}\end{aligned}$$

So, from these calculations we see that node 5 is most restrictive and we must choose $\Delta\tau < 9.43$ s.

The calculations were performed with $\Delta\tau = 3.0$ s, and the results are shown in Figure Example 4-15b, c. Note that a straight line is obtained for the steady-state temperature distribution in the solid, which is what would be expected for a constant thermal conductivity. To compute the heat added at any instant of time we perform the sum

$$Q(\tau) = \sum C_i(T_i - 293) \qquad \textbf{[}j\textbf{]}$$

and plot the results in Figure Example 4-15c.

EXAMPLE 4-16 Transient Conduction with Heat Generation

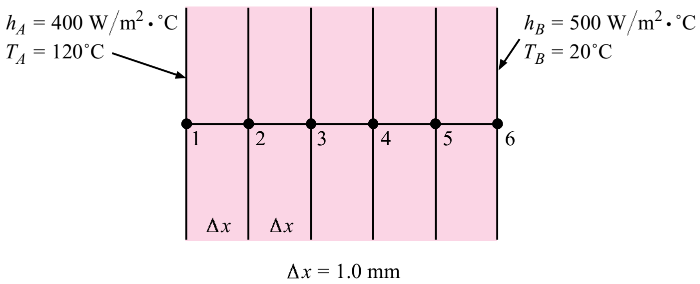
The plane wall shown has internal heat generation of 50 MW/m^3 and thermal properties of $k = 19$ W/m · °C, $\rho = 7800$ kg/m^3, and $C = 460$ J/kg · °C. It is initially at a uniform temperature of 100°C and is suddenly subjected to the heat generation and the convective boundary conditions indicated in Figure Example 4-16A. Calculate the temperature distribution after several time increments.

Solution

We use this resistance and capacity formulation and write, for unit area,

$$1/R_{12} = kA/\Delta x = (19)(1)/0.001 = 19{,}000 \text{ W/°C}$$

Figure Example 4-16a



All the conduction resistances have this value. Also,

$$1/R_{1A} = hA = (400)(1) = 400 \text{ W/°C}$$
$$1/R_{1B} = hA = (500)(1) = 500 \text{ W/°C}$$

The capacities are

$$C_1 = C_6 = \rho(\Delta x/2)c = (7800)(0.001/2)(460) = 1794 \text{ J/°C}$$
$$C_2 = C_3 = C_4 = C_5 = \rho(\Delta x)c = 3588 \text{ J/°C}$$

We next tabulate values.

Node	$\sum(1/R_{ij})$	C_i	$\dfrac{C_i}{\sum(1/R_{ij})}$
1	19,400	1794	0.092
2	38,000	3588	0.094
3	38,000	3588	0.094
4	38,000	3588	0.094
5	38,000	3588	0.094
6	19,500	1794	0.092

Any time increment $\Delta\tau$ less than 0.09 s will be satisfactory. The nodal equations are now written in the form of Equation (4-47) and the calculation marched forward on a computer.

The heat-generation terms are

$$q_i = \dot{q}\,\Delta V_i$$

so that

$$q_1 = q_6 = (50 \times 10^6)(1)(0.001/2) = 25{,}000 \text{ W}$$
$$q_2 = q_3 = q_4 = q_5 = (50 \times 106)(1)(0.001) = 50{,}000 \text{ W}$$

The computer results for several time increments of 0.09 s are shown in the following table. Because the solid stays nearly uniform in temperature at any instant of time it behaves almost like a lumped capacity. The temperature of node 3 is plotted versus time in Figure Example 4-16B to illustrate this behavior.

	Number of time increments ($\Delta\tau = 0.09$ s)			
Node	**5**	**20**	**100**	**200**
1	106.8826	123.0504	190.0725	246.3855
2	106.478	122.8867	190.9618	248.1988
3	106.1888	122.1404	190.7033	248.3325
4	105.3772	120.9763	189.3072	246.7933
5	104.4622	119.2217	186.7698	243.5786
6	102.4416	117.0056	183.0735	238.6773

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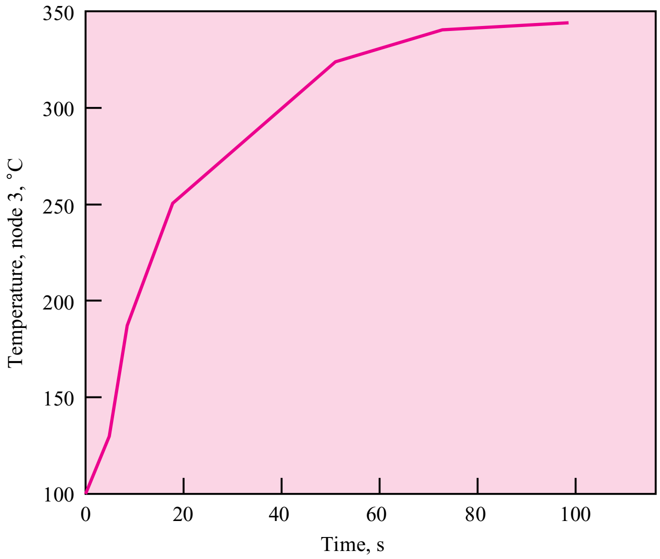
	Number of time increments ($\Delta\tau = 0.09$ s)			
Node	**500**	**800**	**1200**	**3000**
1	320.5766	340.1745	346.0174	347.2085
2	323.6071	343.5267	349.4654	350.676
3	324.2577	344.3137	350.2931	351.512
4	322.5298	342.536	348.5006	349.7165
5	318.4229	338.1934	344.0877	345.2893
6	311.9341	331.2853	337.0545	338.2306

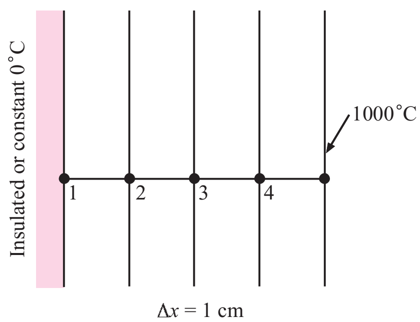
Figure Example 4-16b



EXAMPLE 4-17 Numerical Solution for Variable Conductivity

A 4.0-cm-thick slab of stainless steel (18% Cr, 8% Ni) is initially at a uniform temperature of 0°C with the left face perfectly insulated as shown in Figure Example 4-17a. The right face is suddenly raised to a constant 1000°C by an intense radiation source. Calculate the temperature distribution after (*a*) 25 s, (*b*) 50 s, (*c*) 100 s, (*d*) an interval long enough for the slab to reach a steady state, taking into account variation in thermal conductivity. Approximate the conductivity data in Appendix A with a linear relation. Repeat the calculation for the left face maintained at 0°C.

Figure Example 4-17a



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Solution

From Table A-2 we have $k = 16.3$ W/m · °C at 0°C and $k = 31$ W/m · °C at 1000°C. A linear relation for k is assumed so that

$$k = k_0(1 + \beta T)$$

where T is in degrees Celsius. Inserting the data gives

$$k = 16.3(1 + 9.02 \times 10^{-4} T) \text{ W/m} \cdot {}^\circ\text{C}$$

We also have $\rho = 7817$ kg/m^3 and $c = 460$ J/kg · °C, and use the thermal resistance-capacitance formula assuming that the resistances are evaluated at the arithmetic mean of their connecting nodal temperatures; i.e., R_{3-4} is evaluated at $(T_3 + T_4)/2$.

First, the thermal capacities are evaluated for unit area:

$$C_1 = \rho(\Delta x/2)c = (7817)(0.01/2)(460) = 17{,}980 \text{ J/m}^2 \cdot {}^\circ\text{C}$$
$$C_2 = C_3 = C_4 = \rho(\Delta x)c = (7817)(0.01)(460) = 35{,}960 \text{ J/m}^2 \cdot {}^\circ\text{C}$$

For the resistances we have the form, for unit area,

$$1/R = k/\Delta x = k_0(1 + \beta T)/\Delta x$$

Evaluating at the mean temperatures between nodes gives

$$1/R_{1-2} = (16.3)[1 + 4.51 \times 10^{-4}(T_1 + T_2)]/0.01 = 1/R_{2-1}$$
$$1/R_{2-3} = (16.3)[1 + 4.51 \times 10^{-4}(T_2 + T_3)]/0.01 = 1/R_{3-2}$$
$$1/R_{3-4} = (16.3)[1 + 4.51 \times 10^{-4}(T_3 + T_4)]/0.01 = 1/R_{4-3}$$
$$1/R_{4-1000} = (16.3)[1 + 4.51 \times 10^{-4}(T_4 + T_{1000})]/0.01 = 1/R_{1000-4}$$

The stability requirement is most severe on node 1 because it has the lowest capacity. To be on the safe side we can choose a large k of about 31 W/m · °C and calculate

$$\Delta\tau_{\text{max}} = \frac{(17{,}980)(0.01)}{31} = 5.8 \text{ s}$$

The nodal equations are now written in the form of Equation (4-47); that is to say, the equation for node 2 would be

$$T_2^{p+1} = \frac{\Delta\tau}{C_2}\Big\{1630\,[1 + 4.51 \times 10^{-4}(T_1^p + T_2^p)](T_1^p - T_2^p)$$
$$+1630\,[1 + 4.51 \times 10^{-4}(T_3^p + T_2^p)](T_3^p - T_2^p)\Big\} + T_2^p$$

A computer solution has been performed with $\Delta\tau = 5$ s and the results are shown in the tables. The steady-state solution for the insulated left face is, of course, a constant 1000°C. The steady-state distribution for the left face at 0°C corresponds to Equation (2-2) of Chapter 2. Note that, because of the nonconstant thermal conductivity, the steady-state temperature profile is not a straight line.

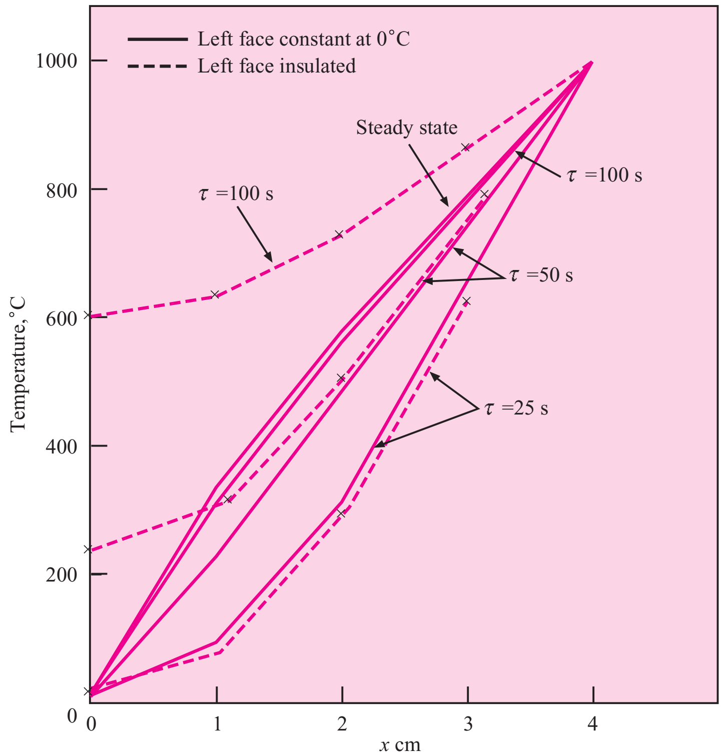
Temperatures for left face at constant 0°C, $\Delta\tau = 5$ s

Node	25 s	50 s	100 s	Steady state
1	0	0	0	0
2	94.57888	236.9619	308.2699	317.3339
3	318.7637	486.5802	565.7786	575.9138
4	653.5105	748.1359	793.7976	799.7735

Temperatures for left face insulated, $\Delta\tau = 5\ s$				
Node	**25 s**	**50 s**	**100 s**	**Steady state**
1	30.55758	232.8187	587.021	1000
2	96.67601	310.1737	623.5018	1000
3	318.7637	505.7613	721.5908	1000
4	653.5105	752.3268	855.6965	1000

These temperatures are plotted in Figure Example 4-17b.

Figure Example 4-17b



The purpose of this example has been to show how the resistance-capacity formulation can be used to take into account property variations in a rather straightforward way. These variations may or may not be important when one considers uncertainties in boundary conditions.

4-8 | SUMMARY

In progressing through this chapter the reader will have noted analysis techniques of varying complexity, ranging from simple lumped-capacity systems to numerical computer solutions. At this point some suggestions are offered for a general approach to follow in the solution of transient heat-transfer problems.

1. First, determine if a lumped-capacity analysis can apply. If so, you may be led to a much easier calculation.

2. Check to see if an analytical solution is available with such aids as the Heisler charts and approximations.
3. If analytical solutions are very complicated, *even when already available,* move directly to numerical techniques. This is particularly true where repetitive calculations must be performed.
4. When approaching a numerical solution, recognize the large uncertainties present in convection and radiation boundary conditions. Do not insist upon a large number of nodes and computer time (and chances for error) that cannot possibly improve upon the basic uncertainty in the boundary conditions.
5. Finally, recognize that it is a rare occurrence when one has a "pure" conduction problem; there is almost always a coupling with convection and radiation. The reader should keep this in mind as we progress through subsequent chapters that treat heat convection and radiation in detail.

REVIEW QUESTIONS

1. What is meant by a lumped capacity? What are the physical assumptions necessary for a lumped-capacity unsteady-state analysis to apply?
2. What is meant by a semi-infinite solid?
3. What initial conditions are imposed on the transient solutions presented in graphical form in this chapter?
4. What boundary conditions are applied to problems in this chapter?
5. Define the error function.
6. Define the Biot and Fourier numbers.
7. Describe how one-dimensional transient solutions may be used for solution of two- and three-dimensional problems.

LIST OF WORKED EXAMPLES

4-1 Steel ball cooling in air
4-2 Semi-infinite solid with sudden change in surface conditions
4-3 Pulsed energy at surface of semi-infinite solid
4-4 Heat removal from semi-infinite solid
4-5 Sudden exposure of semi-infinite slab to convection
4-6 Aluminum plate suddenly exposed to convection
4-7 Long cylinder suddenly exposed to convection
4-8 Semi-infinite cylinder suddenly exposed to convection
4-9 Finite-length cylinder suddenly exposed to convection
4-10 Heat loss for finite-length cylinder
4-11 Sudden cooling of a rod
4-12 Implicit formulation
4-13 Cooling of a ceramic
4-14 Cooling of a steel rod, nonuniform h
4-15 Radiation heating and cooling
4-16 Transient conduction with heat generation
4-17 Numerical solution for variable conductivity

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PROBLEMS

4-1 A copper sphere initially at a uniform temperature T_0 is immersed in a fluid. Electric heaters are placed in the fluid and controlled so that the temperature of the fluid follows a periodic variation given by

$$T_\infty - T_m = A \sin \omega\tau$$

where

$T_m =$ time-average mean fluid temperature
$A =$ amplitude of temperature wave
$\omega =$ frequency

Derive an expression for the temperature of the sphere as a function of time and the heat-transfer coefficient from the fluid to the sphere. Assume that the temperatures of the sphere and fluid are uniform at any instant so that the lumped-capacity method of analysis may be used.

4-2 An infinite plate having a thickness of 2.5 cm is initially at a temperature of 150°C, and the surface temperature is suddenly lowered to 30°C. The thermal diffusivity of the material is 1.8×10^{-6} m^2/s. Calculate the center-plate temperature after 1 min by summing the first four nonzero terms of Equation (4-3). Check the answer using the Heisler charts.

4-3 What error would result from using the first four terms of Equation (4-3) to compute the temperature at $\tau = 0$ and $x = L$? (Note: temperature $= T_i$.)

4-4 A solid body at some initial temperature T_0 is suddenly placed in a room where the air temperature is T_∞ and the walls of the room are very large. The heat-transfer coefficient for the convection heat loss is h, and the surface of the solid may be assumed black. Assuming that the temperature in the solid is uniform at any instant, write the differential equation for the variation in temperature with time, considering both radiation and convection.

4-5 A 20 by 20 cm slab of copper 5 cm thick at a uniform temperature of 260°C suddenly has its surface temperature lowered to 35°C. Using the concepts of thermal resistance and capacitance and the lumped-capacity analysis, find the time at which the center temperature becomes 90°C; $\rho = 8900$ kg/m^3, $c_p = 0.38$ kJ/kg · °C, and $k = 370$ W/m · °C.

4-6 A piece of aluminum weighing 6 kg and initially at a temperature of 300°C is suddenly immersed in a fluid at 20°C. The convection heat-transfer coefficient is 58 W/m^2 · °C. Taking the aluminum as a sphere having the same weight as that given, estimate the time required to cool the aluminum to 90°C, using the lumped-capacity method of analysis.

4-7 Two identical 7.5-cm cubes of copper at 425 and 90°C are brought into contact. Assuming that the blocks exchange heat only with each other and that there is no resistance to heat flow as a result of the contact of the blocks, plot the temperature of each block as a function of time, using the lumped-capacity method of analysis. That is, assume the resistance to heat transfer is the conduction resistance of the two blocks. Assume that all surfaces are insulated except those in contact.

4-8 Repeat Problem 4-7 for a 7.5-cm copper cube at 425°C in contact with a 7.5-cm steel cube at 90°C. Sketch the thermal circuit.

4-9 An infinite plate of thickness $2L$ is suddenly exposed to a constant-temperature radiation heat source or sink of temperature T_s. The plate has a uniform initial

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temperature of T_i. The radiation heat loss from each side of the plate is given by $q = \sigma\epsilon A(T^4 - T_s^4)$, where σ and ϵ are constants and A is the surface area. Assuming that the plate behaves as a lumped capacity, that is, $k \to \infty$, derive an expression for the temperature of the plate as a function of time.

4-10 A stainless-steel rod (18% Cr, 8% Ni) 6.4 mm in diameter is initially at a uniform temperature of 25°C and is suddenly immersed in a liquid at 150°C with $h = 120$ W/m$^2 \cdot$ °C. Using the lumped-capacity method of analysis, calculate the time necessary for the rod temperature to reach 120°C.

4-11 A 5-cm-diameter copper sphere is initially at a uniform temperature of 200°C. It is suddenly exposed to an environment at 20°C having a heat-transfer coefficient $h = 28$ W/m$^2 \cdot$ °C. Using the lumped-capacity method of analysis, calculate the time necessary for the sphere temperature to reach 90°C.

4-12 A stack of common building brick 1 m high, 3 m long, and 0.5 m thick leaves an oven, where it has been heated to a uniform temperature of 300°C. The stack is allowed to cool in a room at 35°C with an air-convection coefficient of 15 W/m$^2 \cdot$ °C. The bottom surface of the brick is on an insulated stand. How much heat will have been lost when the bricks cool to room temperature? How long will it take to lose half this amount, and what will the temperature at the geometric center of the stack be at this time?

4-13 A copper sphere having a diameter of 3.0 cm is initially at a uniform temperature of 50°C. It is suddenly exposed to an airstream of 10°C with $h = 15$ W/m$^2 \cdot$ °C. How long does it take the sphere temperature to drop to 25°C?

4-14 An aluminum sphere, 5.0 cm in diameter, is initially at a uniform temperature of 50°C. It is suddenly exposed to an outer-space radiation environment at 0 K (no convection). Assuming the surface of aluminum is blackened and lumped-capacity analysis applies, calculate the time required for the temperature of the sphere to drop to −110°C.

4-15 An aluminum can having a volume of about 350 cm^3 contains beer at 1°C. Using a lumped-capacity analysis, estimate the time required for the contents to warm to 15°C when the can is placed in a room at 20°C with a convection coefficient of 15 W/m$^2 \cdot$ °C. Assume beer has the same properties as water.

4-16 A 12-mm-diameter aluminum sphere is heated to a uniform temperature of 400°C and then suddenly subjected to room air at 20°C with a convection heat-transfer coefficient of 10 W/m$^2 \cdot$ °C. Calculate the time for the center temperature of the sphere to reach 200°C.

4-17 A 4-cm-diameter copper sphere is initially at a uniform temperature of 200°C. It is suddenly exposed to a convection environment at 30°C with $h = 20$ W/m$^2 \cdot$ °C. Calculate the time necessary for the center of the sphere to reach a temperature of 80°C.

4-18 When a sine-wave temperature distribution is impressed on the surface of a semi-infinite solid, the temperature distribution in the solid is given by

$$T_{x,\tau} - T_m = A \exp\left(-x\sqrt{\frac{\pi n}{\alpha}}\right) \sin\left(2\pi n\tau - x\sqrt{\frac{\pi n}{\alpha}}\right)$$

where

$T_{x,\tau} =$ temperature at depth x and time τ after start of temperature wave at surface

$T_m =$ mean surface temperature

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$n =$ frequency of wave, cycles per unit time
$A =$ amplitude of temperature wave at surface

If a sine-wave temperature distribution is impressed on the surface of a large slab of concrete such that the temperature varies from 35 to 90°C and a complete cycle is accomplished in 15 min, find the heat flow through a plane 5 cm from the surface 2 h after the start of the initial wave.

4-19 Using the temperature distribution of Problem 4-18, show that the time lag between maximum points in the temperature wave at the surface and at a depth x is given by

$$\Delta\tau = \frac{x}{2}\sqrt{\frac{1}{\alpha\pi n}}$$

4-20 A thick concrete wall having a uniform temperature of 54°C is suddenly subjected to an airstream at 10°C. The heat-transfer coefficient is 10 W/m^2 · °C. Calculate the temperature in the concrete slab at a depth of 7 cm after 30 min.

4-21 A very large slab of copper is initially at a temperature of 300°C. The surface temperature is suddenly lowered to 35°C. What is the temperature at a depth of 7.5 cm 4 min after the surface temperature is changed?

4-22 On a hot summer day a concrete driveway may reach a temperature of 50°C. Suppose that a stream of water is directed on the driveway so that the surface temperature is suddenly lowered to 10°C. How long will it take to cool the concrete to 25°C at a depth of 5 cm from the surface?

4-23 A semi-infinite slab of copper is exposed to a constant heat flux at the surface of 0.5 MW/m^2. Assume that the slab is in a vacuum, so that there is no convection at the surface. What is the surface temperature after 5 min if the initial temperature of the slab is 20°C? What is the temperature at a distance of 15 cm from the surface after 5 min?

4-24 A semi-infinite slab of material having $k = 0.1$ W/m · °C and $\alpha = 1.1 \times 10^{-7}$ m^2/s is maintained at an initially uniform temperature of 20°C. Calculate the temperature at a depth of 5 cm after 100 s if (a) the surface temperature is suddenly raised to 150°C, (b) the surface is suddenly exposed to a convection source with $h = 40$ W/m^2 · °C and 150°C, and (c) the surface is suddenly exposed to a constant heat flux of 350 W/m^2.

4-25 A brick wall having a thickness of 10 cm is initially uniform in temperature at 25°C. One side is insulated. The other side is suddenly exposed to a convection environment with $T = 0$°C and $h = 200$ W/m^2 · °C. Using whatever method is suitable, plot the temperature of the insulated surface as a function of time. How might this calculation be applicable to building design?

4-26 A large slab of copper is initially at a uniform temperature of 90°C. Its surface temperature is suddenly lowered to 30°C. Calculate the heat-transfer rate through a plane 7.5 cm from the surface 10 s after the surface temperature is lowered.

4-27 A large slab of aluminum at a uniform temperature of 30°C is suddenly exposed to a constant surface heat flux of 15 kW/m^2. What is the temperature at a depth of 2.5 cm after 2 min?

4-28 For the slab in Problem 4-27, how long would it take for the temperature to reach 150°C at the depth of 2.5 cm?

4-29 A piece of ceramic material [$k = 0.8$ W/m · °C, $\rho = 2700$ kg/m^3, $c = 0.8$ kJ/kg · °C] is quite thick and initially at a uniform temperature of 30°C. The surface of the material is suddenly exposed to a constant heat flux of 650 W/m^2. Plot the temperature at a depth of 1 cm as a function of time.

4-30 An aluminum sphere having a diameter of 5.6 cm is initially at a uniform temperature of 355°C and is suddenly exposed to a convection environment at $T = 23$°C with a convection heat transfer coefficient of 78 W/m^2 · °C. Calculate the time for the center of the sphere to cool to a temperature of 73°C. Express the answer in seconds.

4-31 A large thick layer of ice is initially at a uniform temperature of −20°C. If the surface temperature is suddenly raised to −1°C, calculate the time required for the temperature at a depth of 1.5 cm to reach −11°C. The properties of ice are $\rho = 57$ lb$_m$/ft^3, $c_p = 0.46$ Btu/lbm · °F, $k = 1.28$ Btu/h · ft · °F, $\alpha = 0.048$ ft^2/h.

4-32 A large slab of concrete (stone 1-2-4 mix) is suddenly exposed to a constant radiant heat flux of 900 W/m^2. The slab is initially uniform in temperature at 20°C. Calculate the temperature at a depth of 10 cm in the slab after a time of 9 h.

4-33 A very thick plate of stainless steel (18% Cr, 8% Ni) at a uniform temperature of 300°C has its surface temperature suddenly lowered to 100°C. Calculate the time required for the temperature at a depth of 3 cm to attain a value of 200°C.

4-34 A large slab has properties of common building brick and is heated to a uniform temperature of 40°C. The surface is suddenly exposed to a convection environment at 2°C with $h = 25$ W/m^2 · °C. Calculate the time for the temperature to reach 20°C at a depth of 8 cm.

4-35 A large block having the properties of chrome brick at 200°C is at a uniform temperature of 30°C when it is suddenly exposed to a surface heat flux of 3×10^4 W/m^2. Calculate the temperature at a depth of 3 cm after a time of 10 min. What is the surface temperature at this time?

4-36 A slab of copper having a thickness of 3.0 cm is initially at 300°C. It is suddenly exposed to a convection environment on the top surface at 80°C while the bottom surface is insulated. In 6 min the surface temperature drops to 140°C. Calculate the value of the convection heat-transfer coefficient.

4-37 A large slab of aluminum has a thickness of 10 cm and is initially uniform in temperature at 400°C. Suddenly it is exposed to a convection environment at 90°C with $h = 1400$ W/m^2 · °C. How long does it take the centerline temperature to drop to 180°C?

4-38 A horizontal copper plate 10 cm thick is initially uniform in temperature at 250°C. The bottom surface of the plate is insulated. The top surface is suddenly exposed to a fluid stream at 80°C. After 6 min the surface temperature has dropped to 150°C. Calculate the convection heat-transfer coefficient that causes this drop.

4-39 A large slab of aluminum has a thickness of 10 cm and is initially uniform in temperature at 400°C. It is then suddenly exposed to a convection environment at 90°C with $h = 1400$ W/m^2 · °C. How long does it take the center to cool to 180°C?

4-40 A plate of stainless steel (18% Cr, 8% Ni) has a thickness of 3.0 cm and is initially uniform in temperature at 500°C. The plate is suddenly exposed to a convection environment on both sides at 40°C with $h = 150$ W/m^2 · °C. Calculate the times for the center and face temperatures to reach 120°C.

4-41 A steel cylinder 10 cm in diameter and 10 cm long is initially at 300°C. It is suddenly immersed in an oil bath that is maintained at 40°C, with $h = 280$ W/m^2 · °C. Find (*a*) the temperature at the center of the solid after 2 min and (*b*) the temperature at the center of one of the regular faces after 2 min.

4-42 Derive an expression for the heat flux per unit area at depth x and time τ when a semi-infinite solid is suddenly exposed to an instantaneous energy pulse at the surface of strength Q_0/A.

4-43 Buildings of various constructions exhibit different responses to thermal changes in climate conditions. Consider a 10-cm-thick wall of normal weight structural concrete ($c = 0.9$ kJ/kg · °C) suddenly exposed to a "blue norther" at −10°C with a convection coefficient of 65 W/m^2 · °C. The wall is initially at 15°C. Estimate the time required for the wall temperature to drop to 5°C. State the assumptions.

4-44 A semi-infinite solid of aluminum is coated with a special chemical material that reacts suddenly to ultraviolet radiation and releases energy in the amount of 1.0 MJ/m^2. If the solid is initially uniform in temperature at 20°C, calculate the temperature at a depth of 2.3 cm after 1.8 s.

4-45 A semi-infinite solid of stainless steel (18% Cr, 8% Ni) is initially at a uniform temperature of 0°C. The surface is pulsed with a laser with 10 MJ/m^2 instantaneous energy. Calculate the temperature at the surface and depth of 1 cm after a time of 3 s.

4-46 What strength pulse would be necessary to produce the same temperature effect at a depth of 1.2 cm as that experienced at a depth of 1.0 cm?

4-47 Calculate the heat flux at $x = 1$ cm and $\tau = 3$ s for the conditions of Problem 4-45.

4-48 A semi-infinite solid of aluminum is to be pulsed with a laser at the surface such that a temperature of 600°C will be attained at a depth of 2 mm, 0.2 s after the pulse. The solid is initially at 30°C. Calculate the strength of pulse required, expressed in MJ/m^2.

4-49 A slab of polycrystalline aluminum oxide is to be pulsed with a laser to produce a temperature of 900°C at a depth of 0.2 mm after a time of 0.2 s. The solid is initially at 40°C. Calculate the strength of pulse required expressed in MJ/m^2.

4-50 Repeat Problem 4-49 for window glass.

4-51 An aluminum bar has a diameter of 11 cm and is initially uniform in temperature at 300°C. If it is suddenly exposed to a convection environment at 50°C with $h = 1200$ W/m^2 · °C, how long does it take the center temperature to cool to 80°C? Also calculate the heat loss per unit length.

4-52 A fused-quartz sphere has a thermal diffusivity of 9.5×10^{-7} m^2/s, a diameter of 2.5 cm, and a thermal conductivity of 1.52 W/m · °C. The sphere is initially at a uniform temperature of 25°C and is suddenly subjected to a convection environment at 200°C. The convection heat-transfer coefficient is 110 W/m^2 · °C. Calculate the temperatures at the center and at a radius of 6.4 mm after a time of 3 min.

4-53 Lead shot may be manufactured by dropping molten-lead droplets into water. Assuming that the droplets have the properties of solid lead at 300°C, calculate the time for the center temperature to reach 120°C when the water is at 100°C with $h =$ 5000 W/m^2 · °C, $d = 1.5$ mm.

4-54 A steel sphere 10 cm in diameter is suddenly immersed in a tank of oil at 10°C. The initial temperature of the sphere is 220°C; $h = 5000$ W/m^2 · °C. How long will it take the center of the sphere to cool to 120°C?

4-55 A boy decides to place his glass marbles in an oven at 200°C. The diameter of the marbles is 15 mm. After a while he takes them from the oven and places them in room air at 20°C to cool. The convection heat-transfer coefficient is approximately 14 W/m^2 · °C. Calculate the time the boy must wait until the center temperature of the marbles reaches 50°C.

4-56 A lead sphere with $d = 1.5$ mm and initial temperature of 200°C is suddenly exposed to a convection environment at 100°C and $h = 5000$ W/m^2 · °C. Calculate the time for the center temperature to reach 120°C.

4-57 A long steel bar 5 by 10 cm is initially maintained at a uniform temperature of 250°C. It is suddenly subjected to a change such that the environment temperature is lowered to 35°C. Assuming a heat-transfer coefficient of 23 $W/m^2 \cdot °C$, use a numerical method to estimate the time required for the center temperature to reach 90°C. Check this result with a calculation using the Heisler charts.

4-58 A steel bar 2.5 cm square and 7.5 cm long is initially at a temperature of 250°C. It is immersed in a tank of oil maintained at 30°C. The heat-transfer coefficient is 570 $W/m^2 \cdot °C$. Calculate the temperature in the center of the bar after 3 min.

4-59 A cube of aluminum 10 cm on each side is initially at a temperature of 300°C and is immersed in a fluid at 100°C. The heat-transfer coefficient is 900 $W/m^2 \cdot °C$. Calculate the temperature at the center of one face after 1 min.

4-60 A short concrete cylinder 15 cm in diameter and 30 cm long is initially at 25°C. It is allowed to cool in an atmospheric environment in which the temperature is 0°C. Calculate the time required for the center temperature to reach 10°C if the heat-transfer coefficient is 17 $W/m^2 \cdot °C$.

4-61 Assume that node m in Problem 3-39 occurs along a circular rod having a diameter of 2 cm with $\Delta x = 1$ cm. The material is glass with $k = 0.8$ $W/m \cdot °C$, $\rho = 2700$ kg/m^3, $c = 0.84$ $kJ/kg \cdot °C$. The convection surrounding condition is $h = 50$ $W/m^2 \cdot °C$ and $T_\infty = 35°C$. Write the transient nodal equation for node m and determine the corresponding maximum allowable time increment, expressed in seconds.

4-62 A 4.0-cm cube of aluminum is initially at 450°C and is suddenly exposed to a convection environment at 100°C with $h = 120$ $W/m^2 \cdot °C$. How long does it take the cube to cool to 250°C?

4-63 A cube of aluminum 11 cm on each side is initially at a temperature of 400°C. It is suddenly immersed in a tank of oil maintained at 85°C. The convection coefficient is 1100 $W/m^2 \cdot °C$. Calculate the temperature at the center of one face after a time of 1 min.

4-64 An aluminum cube 5 cm on a side is initially at a uniform temperature of 100°C and is suddenly exposed to room air at 25°C. The convection heat-transfer coefficient is 20 $W/m^2 \cdot °C$. Calculate the time required for the geometric center temperature to reach 50°C.

4-65 A stainless steel cylinder (18% Cr, 8% Ni) is heated to a uniform temperature of 200°C and then allowed to cool in an environment where the air temperature is maintained constant at 30°C. The convection heat-transfer coefficient may be taken as 200 $W/m^2 \cdot °C$. The cylinder has a diameter of 10 cm and a length of 15 cm. Calculate the temperature of the geometric center of the cylinder after a time of 10 min. Also calculate the heat loss.

4-66 A cylinder having a diameter of 15 cm and a length of 30 cm is initially uniform in temperature at 300°C. It is suddenly exposed to a convection environment at 20°C with $h = 35$ $W/m^2 \cdot °C$. Properties of the solid are $k = 2.3$ $W/m \cdot °C$, $\rho = 300$ kg/m^3, and $c = 840$ $J/kg \cdot °C$. Calculate the time for (a) the center and (b) the center of one face to reach a temperature of 120°C. Also calculate the heat loss for each case.

4-67 A rectangular solid is 15 by 10 by 20 cm and has the properties of fireclay brick. It is initially uniform in temperature at 300°C and then suddenly exposed to a convection environment at 80°C and $h = 110$ $W/m^2 \cdot °C$. Calculate the time for (a) the geometric center and (b) the center of each face to reach a temperature of 190°C. Also calculate the heat loss for each of these times.

4-68 Calculate the heat loss for both cases in Problem 4-45.