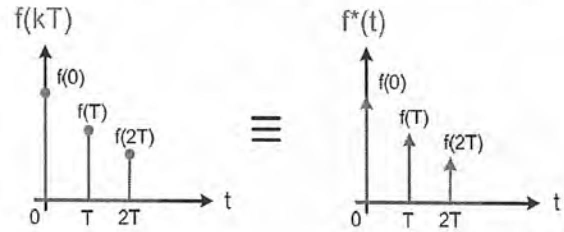
2.2 Z-transform of Sampled Signals

A discrete-time signal can be visualized as a sequence of binary numbers. As a mathematical abstraction, this sequence can be represented by a sum of impulse (Dirac-delta) functions:



$$f^*(t) = f(0)\delta(t) + f(T)\delta(t-T) + f(2T)\delta(t-2T) + \ldots = \sum_{k=0}^{\infty} f(kT)\delta(t-kT)$$

where

$$\delta(t) \triangleq \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases} \qquad \text{while} \qquad \int_{0^-}^{0^+} \delta(t)dt = 1$$

Using the transport delay property of Laplace transforms, we have

$$L\{\delta(t-T)\} = e^{-sT} \cdot 1, \quad L\{\delta(t-2T)\} = e^{-s2T} \cdot 1, \ldots$$

Therefore, $F^*(s) = L\{f^*(t)\} = f(0) + f(T)e^{-sT} + f(2T)e^{-s2T} + \ldots$

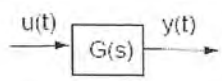
Define a new complex variable as $z \equiv e^{sT}$:

$$F(z) \triangleq f(0) + f(T)z^{-1} + f(2T)z^{-2} + \ldots = \sum_{k=0}^{\infty} f(kT)z^{-k}$$

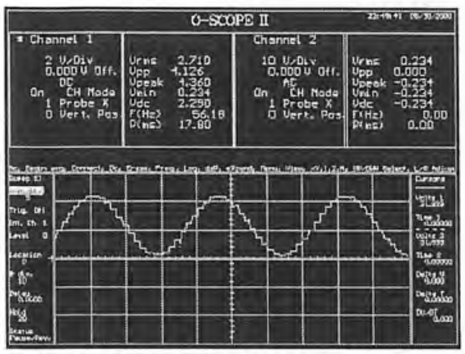
Consequently, $\boxed{F(z) = Z\{f^*(t)\} = Z\{f(kT)\}}$

where Z{f*(t)} is called the *Z-transform* of sampled time function f*(t).

The Need for Z-transforms

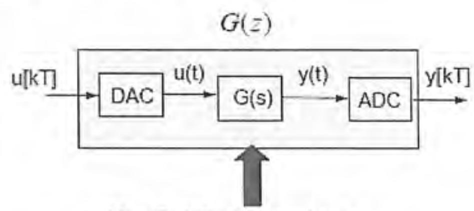


In continuous-time: You design controllers with differential equations (and implement with op-amps), with Laplace transforms, or state-space.



ADC takes time: ZOH Phenomena

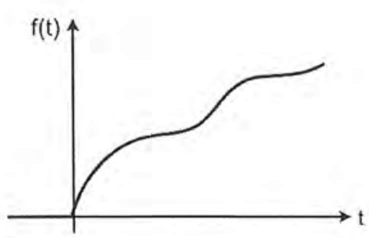
In discrete-time: You can design controllers with difference equations (and implement with code), with Z-transforms, or state-space.



Must find discrete version to account for hold phenomena

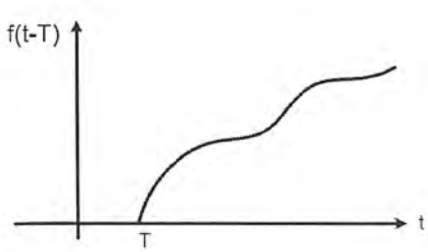
ZOH Discrete Version: $G(z) = \left(1 - z^{-1}\right) Z\left\{ L^{-1}\left[\frac{G(s)}{s} \right] \right\}$ (1)

Transport Delay Property



Recall that

$$L\{f(t)\} = F(s)$$



while

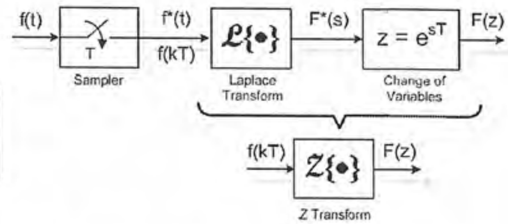
$$L\{f(t-T)\} = e^{-sT}F(s)$$

2.3 Conventions in Z-transforms

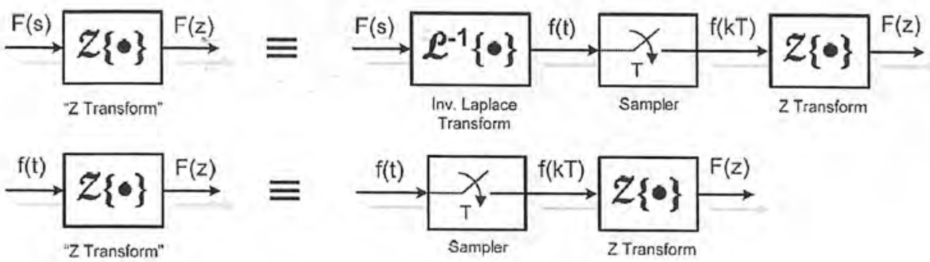
Z-transform is only defined for *sampled time functions* as

$$F(z) = Z\{f(kT)\} = \sum_{k=0}^{\infty} f(kT) z^{-k}$$

where $z \triangleq e^{sT}$.



However, the following conventions are commonly used for convenience:



Z-transform - Examples

1. Unit Impulse: $f^*(t) = 1 \cdot \delta(t) + 0 + 0 + \ldots$

$$Z\{f^*(t)\} = Y(z) = 1 \cdot z^{-0} + 0 \cdot z^{-1} + 0 \cdot z^{-2} + \ldots = 1$$

2. Unit Step: $f^*(t) = 1 \cdot \delta(t) + 1 \cdot \delta(t-T) + 1 \cdot \delta(t-2T) + \ldots$

↳ $F(s) = 1/s$ $\quad Z\{f^*(t)\} = 1 \cdot z^{-0} + 1 \cdot z^{-1} + 1 \cdot z^{-2} + \ldots = \sum_{k=0}^{\infty} z^{-k}$

Recall that $\sum_{k=0}^{\infty} a^k = \frac{1}{1-a} \quad (a < 1)$ ← Law

$$F(z) = \frac{1}{1-z^{-1}} \quad (|z^{-1}| < 1) = \frac{z}{z-1}$$

3. Unit Ramp: $\Rightarrow \frac{1}{s^2}$

$$f^*(t) = 0 \cdot \delta(t) + T \cdot \delta(t-T) + 2T \cdot \delta(t-2T) + \ldots$$

$$Z\{f^*(t)\} = 0 \cdot z^{-0} + T \cdot z^{-1} + 2T \cdot z^{-2} + \ldots = \sum_{k=0}^{\infty} (kT) z^{-k}$$

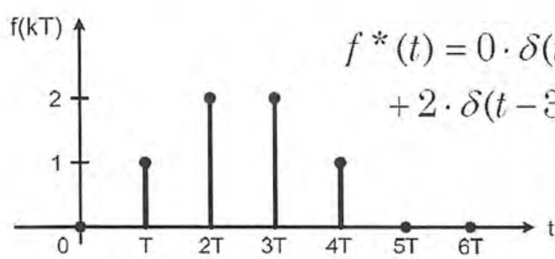
$$\boxed{F(z) = \frac{Tz^{-1}}{(1-z^{-1})^2}} = \frac{Tz}{(z-1)^2}$$

4-exponential function

$$f(s) = \frac{1}{s+a}$$

$$F(z) = \frac{1}{1 - z^{-1}e^{-aT}} = \frac{z}{z - e^{-aT}}$$

5-Arbitrary Function:



$$f^*(t) = 0 \cdot \delta(t) + 1 \cdot \delta(t-T) + 2 \cdot \delta(t-2T) + 2 \cdot \delta(t-3T) + 1 \cdot \delta(t-4T) + 0 + \ldots$$

$$Z\{f^*(t)\} = 1 \cdot z^{-1} + 2 \cdot z^{-2} + 2 \cdot z^{-3} + 1 \cdot z^{-4}$$

$$\boxed{F(z) = z^{-1}[1 + 2z^{-1}(1 + z^{-1}) + z^{-3}]}$$

Z-transform Table

$F(s)$	$f(kT)$	$F(z)$
1	$\delta(kT)$	1
$1/s$	$1(kT)$	$1/(1-z^{-1})$
$1/s^2$	kT	$Tz^{-1}/(1-z^{-1})^2$
$1/s^3$	$(kT)^2/2!$	$T^2z^{-1}(1+z^{-1})/[2(1-z^{-1})^3]$
$1/(s+a)$	e^{-akT}	$1/(1-z^{-1}e^{-aT})$
$1/(s+a)^2$	$(kT)\,e^{-akT}$	$z^{-1}Te^{-aT}/(1-z^{-1}e^{-aT})^2$
$a/[s(s+a)]$	$1-e^{-akT}$	$z^{-1}(1-e^{-aT})/[(1-z^{-1})(1-z^{-1}e^{-aT})]$
$a/[s^2(s+a)]$	$(a^{-1})(akT-1+e^{-akT})$	$z^{-1}(A+Bz^{-1})/[a(1-z^{-1})^2(1-z^{-1}e^{-aT})]$ $A \equiv aT-1+e^{-aT}$ $B \equiv 1-e^{-aT}-aTe^{-aT}$
$s/(s+a)^2$	$(1-akT)e^{-akT}$	$[1-z^{-1}e^{-aT}(1+aT)]/(1-e^{-aT}z^{-1})^2$
$a/(s^2+a^2)$	$\sin(akT)$	$z^{-1}\sin(aT)/[1-2\cos(aT)z^{-1}+z^{-2}]$
$s/(s^2+a^2)$	$\cos(akT)$	$[1-z^{-1}\cos(aT)]/[1-2\cos(aT)z^{-1}+z^{-2}]$
$(s+a)/[(s+a)^2+b^2]$	$\cos(bkT)\,e^{-akT}$	$(1-Az^{-1})/[1-2Az^{-1}+e^{-2aT}z^{-2}]$ $A \equiv e^{-aT}\cos(bT)$
$b/[(s+a)^2+b^2]$	$\sin(bkT)\,e^{-akT}$	$z^{-1}e^{-aT}\sin(bT)/[1-2Az^{-1}+e^{-2aT}z^{-2}]$ $A \equiv e^{-aT}\cos(bT)$

2.4 Miscellaneous Issues in Z-transforms

- One can break up an unknown (Laplace) function into (a summation of) familiar functions that are likely to be found in the given tables using the following methods:
  - Partial fraction expansion *or*
  - Residue method
- Similar techniques could be employed to obtain the *inverse Z-transform* of an arbitrary function of z.

2.4.1 Methods of Z-T

1- Partial Fraction Method(P.F.M)

Example: Find Z-T for

$$G(s) = \frac{1}{s(s+1)}$$

$$= \frac{A}{s} + \frac{B}{s+1}$$

$$A = \lim G(s)a|s = 0$$

$$A = 1$$

$$B = \lim G(s)a|s = -1$$

$$B = -1$$

$$G(s) = \frac{1}{s} - \frac{1}{s+1}$$

$$G(z) = \frac{z}{z-1} - \frac{z}{z-e^{-T}}$$

For repeated roots

Example: Find Z-T for

$$G(s) = \frac{1}{s^2(s+1)}$$
$$= \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1}$$
$$A = \lim G(s) at s = 0$$
$$A = 0.5$$
$$C = \lim G(s) at s = -1$$
$$C = 0.25$$
$$B = \lim \frac{1}{(2-1)!} \frac{d^{2-1}}{ds} s^2 G(s)$$
$$B = -0.25$$
$$G(z) = \frac{0.5Tz}{(z-1)^2} - \frac{0.25z}{z-1} + \frac{0.25z}{z-e^{-2T}}$$

2- Residue Method

The Law is:-

$$G(z) = \sum RES of G(s) \frac{z}{z-e^{sT}} at poles of G(s)$$

Example: Find Z-T for

$$G(s) = \frac{1}{s(s+1)}$$
$$R1 = \lim at s = 0$$
$$R1 = \lim s(\frac{1}{s(s+1)}) \frac{z}{z-e^{Ts}} = \frac{z}{z-1}$$
$$R2 = \lim_{s \to -1} (s+1) \frac{1}{s(s+1)} \frac{z}{z-e^{Ts}} = \frac{-z}{z-e^{-T}}$$

$$G(z) = R1 + R2 = \frac{z(1-e^{-T})}{(z-1)(z-e^{-T})}$$

The same result as (P.F.M)

For repeated roots

$$G(z) = \frac{1}{(q-1)!}\lim_{s\to r} \frac{d^{q-1}}{ds^{q-1}}\left[(s-r)^q G(s)\frac{z}{z-e^{sT}}\right]$$

Where q=number of repeated poles

r=value of pole

Example:

Find Z-T for

$$G(s) = \frac{1}{s^2(s+1)} \rightarrow q=2$$

$$R1 = \frac{1}{1!}\lim_{s\to 0} \frac{d}{ds}[(s-0)^2\frac{1}{s^2(s+1)}\frac{z}{(z-e^{Ts})}]$$

$$R1 = \frac{-z[-T+(z-1)]}{(z-1)^2}$$

$$R2 = \lim_{s\to -1} (s+1)[\frac{1}{s^2(s+1)}\frac{z}{z-e^{Ts}}]$$

$$R2 = \frac{z}{z-e^{Ts}}$$

$$R = R1 + R2$$

Example:

$$Z\left[\frac{10}{s(s+1)^2}\right] = \frac{10}{s(s+1)^2}\cdot\frac{z}{z-e^{Ts}}\bigg|_{s=0} + \frac{1}{(2-1)!}\lim_{s\to -1}\frac{\partial}{\partial s}\left[\frac{10}{s}\cdot\frac{z}{z-e^{Ts}}\right]$$

$$= \frac{10z}{z-1} + \frac{-z^2+ze^{-T}(1-T)}{\left(z-e^{-T}\right)^2}$$