2.5 Properties of Z-transform

1. Linearity: $Z\{a \cdot f(kT)\} = a \cdot Z\{f(kT)\}$

2. Addition: $Z\{f_1(kT) \pm f_2(kT)\} = Z\{f_1(kT)\} \pm Z\{f_2(kT)\}$

3. Time Shift: $Z\{f(kT - nkT)\} = z^{-n} Z\{f(kT)\}$

4. Convolution: $Z\{\sum_{i=-\infty}^{+\infty} f_1(iT) f_2(kT - iT)\} = F_1(z)F_2(z)$

5. Scaling: $Z\{r^{-k} f(kT)\} = F(rz)$

* **6. Final Value Theorem:** (valid only for stable systems)

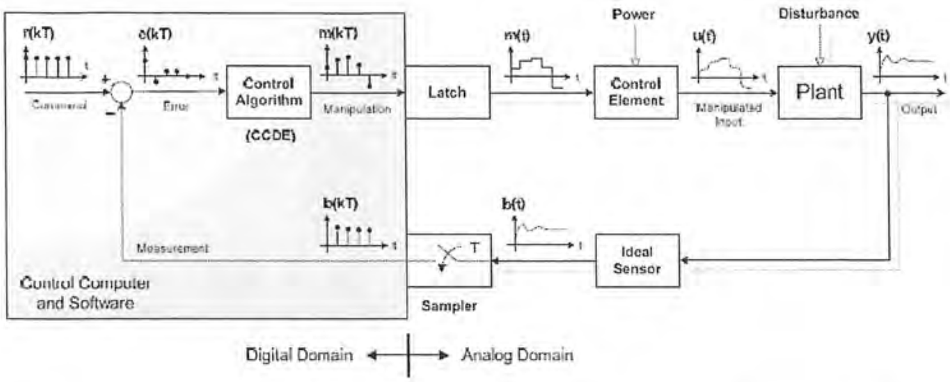
$$f(\infty) = \lim_{k\to\infty} f(kT) = \lim_{z\to 1} \frac{(z-1)}{z} F(z)$$

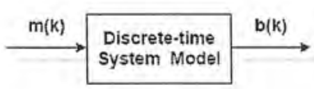
where $F(z) = Z\{f(kT)\}$

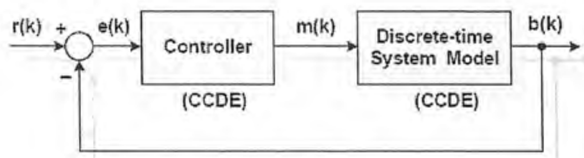
* **7. Initial Value Theorem:**

$$f(0) = \lim_{z\to\infty} F(z)$$

2.6 Discrete-time System Modeling



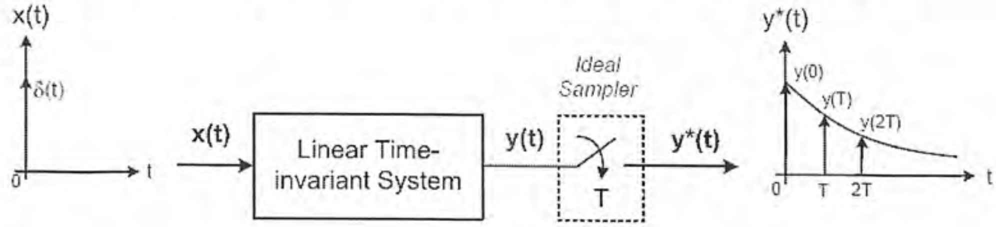




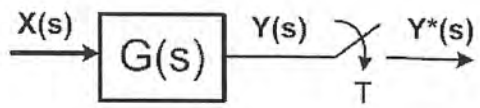
Assumptions in Digital Systems Modeling

- A/D and D/A conversions are *non-linear* operations:
  - Such converters cause a major difficulty in developing mathematical models in discrete-time domain.
- For the sake of convenience, these elements embedded inside the I/O interfaces are ignored.
  - Implies the use of *high-resolution* converters in the actual control system.
- Thus, only latch (i.e. zero-order hold) and sampler are taken into consideration.

Sampled Impulse Response



In Laplace domain:



$$X(s) = L\{\delta(t)\} = 1$$

where G(s) is the transfer function of LTI system. Hence,

$$Y(s) = G(s)X(s) = G(s) \cdot 1 = G(s)$$

Note that $y(t) = g(t) = L^{-1}\{G(s)\}$

$$y^*(t) = g(0) \cdot \delta(t) + g(T) \cdot \delta(t-T) + g(2T) \cdot \delta(t-2T) + \ldots + g(kT) \cdot \delta(t-kT) + \ldots$$

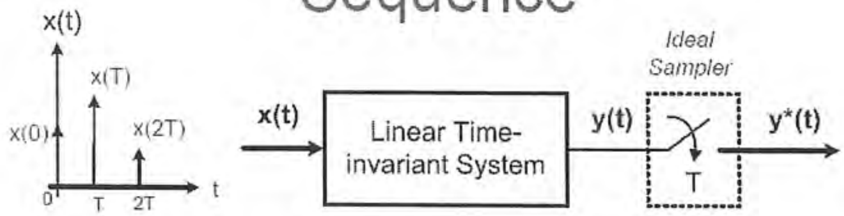
Similarly, $Y^*(s) = L\{y^*(t)\} = g(0) + g(T)e^{-sT} + g(2T)e^{-s2T} + \ldots$

Z-transform of the sampled impulse response y*(t) becomes

$$Y(z) = g(0) + g(T)z^{-1} + g(2T)z^{-2} + \ldots = \sum_{k=0}^{\infty} g(kT)z^{-k} \triangleq G(z)$$

where G(z) is called the *discrete-time transfer function* of the system.

Sampled Response to an Impulse Sequence



Superposition principle allows us to determine the overall response as a summation of individual impulse responses. For instance, when x(0)δ(t) is applied alone, the corresponding output becomes

$$y_0^*(t) = g(0)x(0)\delta(t) + g(T)x(0)\delta(t-T) + \dots$$

$$\therefore \quad Y_0(z) = G(z)\cdot x(0)$$

Likewise, the response to x(T)δ(t-T) is $Y_1(s) = G(s)\underbrace{[x(T)\cdot e^{-sT}\cdot 1]}_{delayed\ impulse}$

$$y_1(t) = L^{-1}\{x(T)e^{-sT}G(s)\} = x(T)g(t-T)$$

where g(t) = L^{-1}\{G(s)\}. Therefore, the sampled time-function is

$$y_1^*(t) = \overbrace{g(-T)}^{0\ by\ definition} \cdot x(T)\cdot\delta(t) + g(0)x(T)\delta(t-T) + g(T)x(T)\delta(t-2T) + \dots$$

$$Y_1(z) = 0z^{-0} + g(0)x(T)z^{-1} + \dots = x(T)\underbrace{\sum_{k=0}^{\infty} g(kT)z^{-k}}_{G(z)}z^{-1}$$

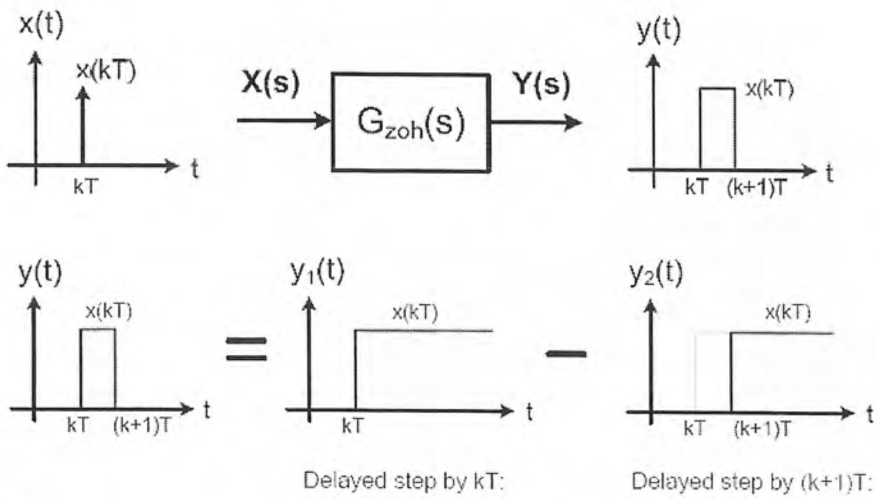
$$\therefore Y_1(z) = G(z)\cdot x(T)\cdot z^{-1}$$

Similarly, $Y_2(z) = G(z)\cdot x(T)\cdot z^{-2}$ and so on...

$$Y(z) = Y_0(z) + Y_1(z) + \ldots$$
$$= G(z)\underbrace{\left[x(0) + x(T)z^{-1} + x(2T)z^{-2} + \ldots\right]}_{Z\{x(kT)\}=X(z)}$$

$$Y(z) = G(z)\cdot X(z)$$
$$\therefore G(z) = \frac{Y(z)}{X(z)}$$

Transfer Function of ZOH / Latch



$$\frac{x(kT)}{s}e^{-skT}(1-e^{-sT}) = \frac{x(kT)}{s}e^{-skT} - \frac{x(kT)}{s}e^{-s(k+1)T}$$

Input function is the delayed impulse with a magnitude of x(kT):

$$X(s) = x(kT) \cdot 1 \cdot e^{-skT}$$

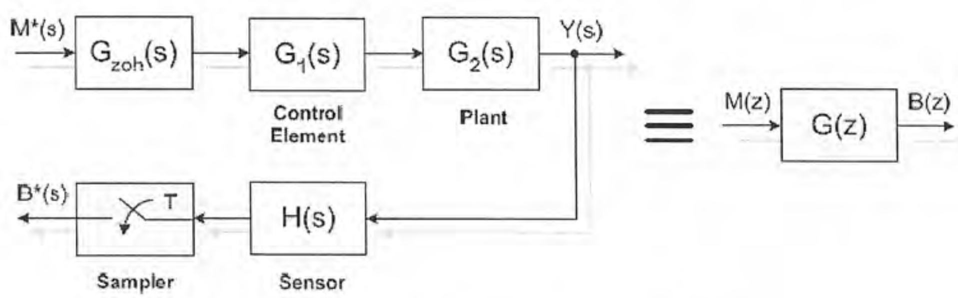
Similarly, the output function becomes

$$Y(s) = \frac{x(kT)}{s} \cdot e^{-skT}(1 - e^{-sT})$$

Since $$G_{zoh}(s) = \frac{Y(s)}{X(s)}$$

$$\boxed{G_{zoh}(s) = \frac{1 - e^{-sT}}{s}}$$

Discrete-time Model



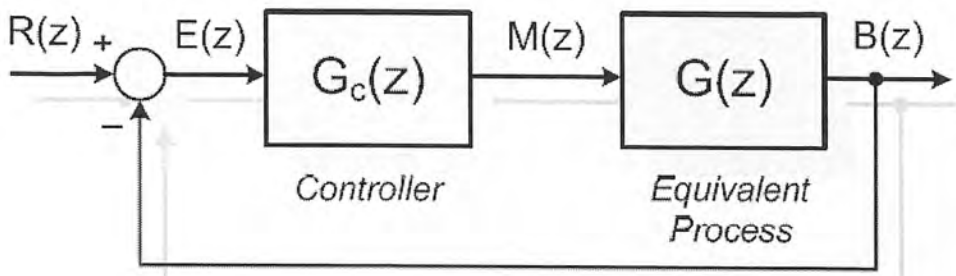
$$\frac{B(z)}{M(z)} = G(z) = Z\{G_{zoh}(s) \cdot G_1(s) \cdot G_2(s) \cdot H(s)\}$$

Since $G_{zoh}(s) = \frac{1 - e^{-sT}}{s} = \frac{1 - z^{-1}}{s}$

Using the third property of Z-transforms yields

$$G(z) = (1 - z^{-1})Z\left\{\frac{G_1(s) \cdot G_2(s) \cdot H(s)}{s}\right\}$$

Overall System Model



$$\frac{B(z)}{R(z)} = \frac{G_c(z)G(z)}{1 + G_c(z)G(z)}$$

Operators / Complex Variables

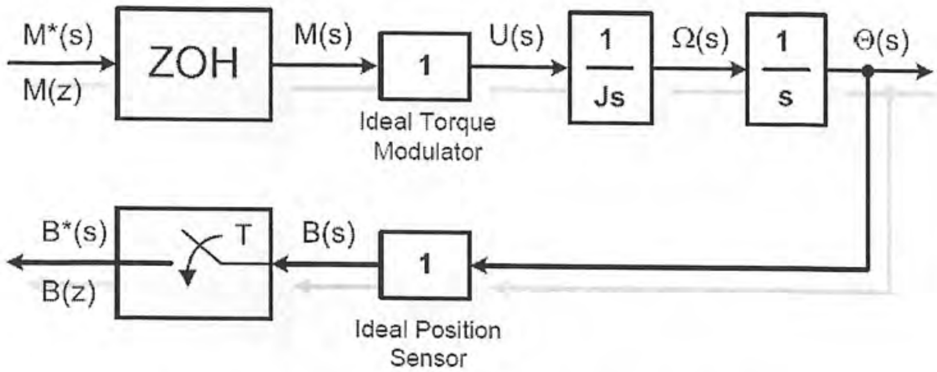
Continuous-time:

Laplace Domain	Time Domain	Physical Meaning
s	D operator	d/dt (derivative)
1/s	1/D	∫dt (integration in time)

Discrete-time:

z-Domain	Time Domain	Physical Meaning
z^{-1}	q^{-1}	unit delay (delay by T)

Example 3 - Motor Control



- Consider the system shown.
  a) Develop Θ(z)/M(z)
  b) Find Θ(z)/Ω(z)

Part (a)

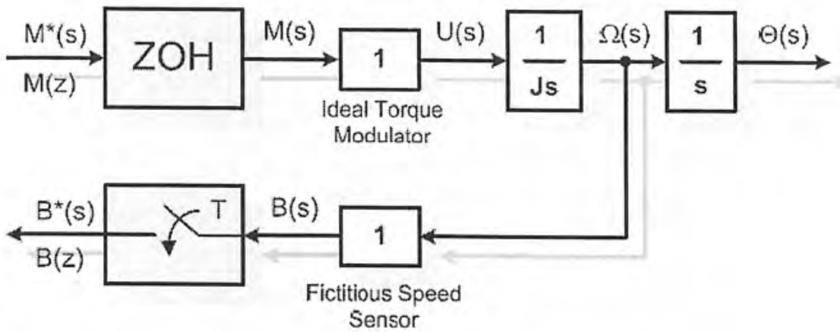
$$\frac{\Theta(z)}{M(z)} = Z\left\{\underbrace{\frac{(1-z^{-1})}{s}}_{ZOH}\cdot\underbrace{\frac{1}{J}\cdot\frac{1}{s^2}}_{Process}\right\} = \frac{1}{J}(1-z^{-1})Z\left\{\frac{1}{s^3}\right\}$$

$$\frac{\Theta(z)}{M(z)} = \frac{T^2}{2J}\cancel{(1-z^{-1})}\frac{(z^{-1}+z^{-2})}{(1-z^{-1})^{\cancel{3}\,2}}$$

$$\frac{\Theta(z)}{M(z)} = \frac{T^2}{2J}\cdot\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^2}$$

Part (b)

- To obtain Θ(z)/Ω(z), let us determine the transfer function Ω(z)/M(z) first.
- Even though the speed (Ω) is NOT actually sampled in this configuration, a *fictitious* speed sensor is added so as to obtain the desired relationship.

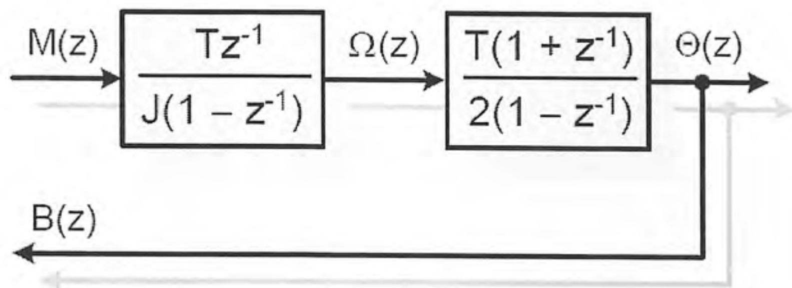


$$\frac{\Omega(z)}{M(z)} = Z\left\{\underbrace{\frac{(1-z^{-1})}{s}}_{ZOH}\cdot\underbrace{\frac{1}{J}\cdot\frac{1}{s}}_{Process}\right\} = \frac{1}{J}(1-z^{-1})Z\left\{\frac{1}{s^2}\right\}$$

$$\frac{\Theta(z)}{M(z)} = \frac{T}{J}\cancel{(1-z^{-1})}\frac{z^{-1}}{(1-z^{-1})^{\cancel{2}}}$$

$$\frac{\Omega(z)}{M(z)} = \frac{T}{J}\cdot\frac{z^{-1}}{(1-z^{-1})}$$

$$\frac{\Theta(z)}{\Omega(z)} = \frac{\frac{\Theta(z)}{M(z)}}{\frac{\Omega(z)}{M(z)}} = \frac{\frac{T^2}{2J} \cdot \frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^2}}{\frac{T}{J} \cdot \frac{z^{-1}}{(1-z^{-1})}} \Rightarrow \quad \frac{\Theta(z)}{\Omega(z)} = \frac{T}{2} \cdot \frac{(1+z^{-1})}{(1-z^{-1})}$$



A Common Mistake

- Students are oftentimes tempted to do the following:

Since Θ(s)/Ω(s) = 1/s,

$$\frac{\Theta(z)}{\Omega(z)} = Z\left\{\frac{1}{s}\right\} = \frac{1}{(1-z^{-1})}$$