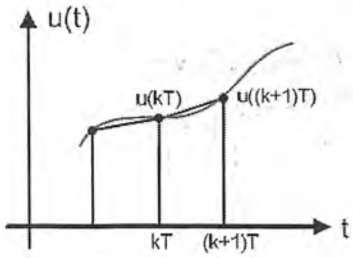
3) Trapezoidal (Tustin) Rule

Let u(t) = dx/dt. The trapezoidal integration rule leads to



$$x((k+1)T) = x(kT) + \tfrac{T}{2}[u((k+1)T) + u(kT)]$$
$$x((k+1)T) - x(kT) = \tfrac{T}{2}[u((k+1)T) + u(kT)]$$

In terms of q:

$$(q-1)x(k) = \tfrac{T}{2}(q+1)u(k)$$
$$u(k) = \frac{dx}{dt} \cong \frac{2}{T}\cdot\frac{q-1}{q+1}\cdot x(k) \Rightarrow \quad \frac{d}{dt} \cong \frac{2}{T}\cdot\frac{q-1}{q+1}$$

As d/dt corresponds to s variable while q operator is equivalent to z variable, we have

$$\boxed{s \cong \frac{2}{T}\cdot\frac{z-1}{z+1}}$$

Therefore, $G'(z) \cong G(s = \frac{2}{T}\cdot\frac{z-1}{z+1})$

Alternative Derivation for Tustin (Bilinear) Transformation

Since $z \triangleq e^{sT} = \dfrac{e^{sT/2}}{e^{-sT/2}}$ Recall that $e^{x} = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \ldots$

Then

$$z \cong \frac{1 + \frac{1}{1!}\left(\frac{sT}{2}\right) + \frac{1}{2!}\left(\frac{sT}{2}\right)^2 + \ldots}{1 + \frac{1}{1!}\left(\frac{-sT}{2}\right) + \frac{1}{2!}\left(\frac{-sT}{2}\right)^2 + \ldots}$$

If T is small, the higher order terms of the Taylor Series can be neglected:

$$z \cong \frac{1 + \frac{sT}{2}}{1 - \frac{sT}{2}}$$

Solving for s leads to

$$\boxed{s \cong \frac{2}{T}\cdot\frac{z-1}{z+1}}$$

4)Pole-Zero matching method

Consider the control system

$$G(s) = Ks\frac{(s+a1)(s+a2)(s+a3)..........(s+am)}{(s+b1)(s+b2)..............................(s+bn)}$$

where

$n \succ m$

$$G(z) = Kz\frac{(z+1)^{n-m}(z+z1)(z+z2)..........(z+zm)}{(z+p1)(z+p2)..............................(z+pn)}$$

where

$Zi = -e^{-aiT} i = 1,...m$

$Pi = -e^{-biT} i = 1,........n$

$$KzG(z)at|z = 1 = Kz2^{n-m}\frac{(1+z1)(1+z2)-----(1+zm)}{(1+p1)(1+p2)-----(1+Pn)} = G(s)at|s = 0$$

Example:

If G(s)=a/(s+a)

$$G(z) = Kz\frac{(z+1)^{1-0}}{(z+P1)}$$

Where

$P1 = -e^{-aT}$

$$G(z)a|z = 1 = Kz * 2/(1-e^{-aT}) = G(s)a|S = 0 = \frac{a}{a} = 1$$

$$Kz = \frac{1-e^{-aT}}{2}$$

$$G(z) = \frac{1-e^{-aT}}{2}\frac{(z+1)}{z-e^{-aT}}$$

Example:

$$G(s)=\frac{10(s+2)}{(s+1)(s+4)}$$

Calculate G(z) by using Z-P matching method. Let T=0.1sec

sol

$$G(z)=Kz\frac{(z+1)^{2-1}(z-e^{-2T})}{(z-e^{-T})(z-e^{-4T})}=Kz\frac{(z+1)}{(z-0.9049)(z-0.6703)}(z-0.8187)$$

$$G(z)at|Z=1=\frac{Kz*2*(0.1813)}{0.0952*0.3297}=Kz*11.552=G(s)at|S=0=5$$

$$Kz=\frac{5}{11.552}=0.432$$

$$G(z)=\frac{0.432(z+1)(z-0.8187)}{(z-0.9048)(z-0.6703)}$$

Example:

$$G(s)=\frac{30(s+10)}{(s^2+5s+100)}$$

Calculate G(z) by using Z-P matching method. Let T=0.1sec

sol

convert

$$s^2+5s+100=-2.5+9.69j$$
$$=a+bj$$
$$Z1,2=e^{aT}[\cos bT+j\sin bT]$$
$$=0.44+0.641j$$
$$or$$
$$z^2-0.883z+0.606$$

And apply the same procedures to complete solution.

<u>Note that :-</u> special case when there is a term (s) in num. of G(s).

Example:

$$G(s) = \frac{K(s+a)}{s(s+b)}$$

$$G(z) = \frac{Kz(z+1)^{2-1}(z-e^{-aT})}{(z-1)(z-e^{-bT})}$$

$$G(s)\,at\,S = 0 = \infty$$

$$G(z)\,at\,Z = 1 = \infty$$

problem?

sol :–

$$\lim_{S \to 0} SG(s) = \frac{Ks * a}{b}$$

$$\lim_{z \to 1} G(z)(z-1) = Kz \frac{*2}{(1-e^{-bT})}(1-e^{-aT})$$

$$Kz = \frac{Ks * a * (1-e^{-bT})}{2 * b * (1-e^{-aT})}$$

Notes on Discretization Methods

- The discretization techniques discussed here approximate the dynamics of a continuous plant *better* when the sampling intervals are kept *short*.
  - Forward- and backward difference methods are quite vulnerable to long sampling periods.
  - Despite its complexity, the Tustin transformation yields the best results in most cases.
- Since short sampling periods are essential for the successful application of these techniques, the dynamics of output interfaces (like ZOH) could be neglected under such circumstances.

- A major disadvantage of (backward) does not map to unit-circle in Z-plane. Thus we must decrease(T) to improve the approximation.

- G(s) is stable in (forward), may be give unstable G(z), hence this is an undesirable mapping.

-All stable G(s)will give stable G(z) in (bilinear or Tustin), hence ,this is why this approximation is the most commonly used.

Example

Consider the ODE representing the dynamics of a plant:

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y(t) = x(t)$$

If T = 0.01 s, obtain the discrete-time models for this system using the following techniques:

a) Forward difference
b) Backward difference
c) Tustin transformation
d) Z-transform with ZOH

Solution

Let us first develop the transfer function of this plant in s-domain:

$$(s^2+3s+2)Y(s)=X(s) \Rightarrow \quad \frac{Y(s)}{X(s)}=\frac{1}{s^2+3s+2}$$

a) Forward Difference:

$$G'(z)\cong\frac{1}{\left(\frac{z-1}{T}\right)^2+3\left(\frac{z-1}{T}\right)+2}=\frac{\frac{1}{10000}}{z^2-1.9700z+0.9702}$$

b) Backward Difference:

$$G'(z)\cong\frac{1}{\left(\frac{z-1}{Tz}\right)^2+3\left(\frac{z-1}{Tz}\right)+2}=\frac{\frac{1}{10302}z^2}{z^2-1.9705z+0.9707}$$

c) Tustin Transformation:

$$G'(z)\cong\frac{1}{\frac{4}{T^2}\left(\frac{z-1}{z-1}\right)^2+\frac{6}{T}\left(\frac{z-1}{z+1}\right)+2}=\frac{\frac{1}{40601}(z+1)^2}{z^2-1.9700z+0.9704}$$

Be careful when rounding numbers in these transfer functions. Since we are dealing with relatively small numbers, make sure to perform all calculations to the highest precision and then display *at least* 4 digits after decimal point!

d) ZOH:

$$G(s)=\left(\frac{1}{s^2+3s+2}\right)\frac{1-e^{-sT}}{s}=\left(\frac{1}{s+1}-\frac{1}{s+2}\right)\frac{1-e^{-sT}}{s}$$

$$G'(z)=Z\left\{\left(\frac{1}{s+1}-\frac{1}{s+2}\right)\frac{1-e^{-sT}}{s}\right\}=(1-z^{-1})Z\left\{\left(\frac{1}{s+1}-\frac{1}{s+2}\right)\frac{1}{s}\right\}$$

From Z-transform tables:

$$G'(z)=\frac{z^{-1}(1-e^{-T})}{1-z^{-1}e^{-T}}-\frac{\frac{1}{2}z^{-1}(1-e^{-2T})}{1-z^{-1}e^{-2T}}$$

Hence,

$$G'(z)=\frac{B_1z^{-1}+B_2z^{-2}}{1-z^{-1}(e^{-T}+e^{-2T})+z^{-2}e^{-3T}}$$

where

$$B_1\triangleq(1-e^{-T})-\tfrac{1}{2}(1-e^{-2T})$$
$$B_2\triangleq\tfrac{1}{2}e^{-T}(1-e^{-2T})-e^{-2T}(1-e^{-T})$$

In terms of numerical values, we have

$$G'(z)=\frac{(4.9503\times10^{-5})z+4.9010\times10^{-5}}{z^2-1.9702z+0.9704}$$

2.12 System Modeling with MATLAB

- Matlab provides excellent toolbox functions to model / design control systems.
- The following Matlab functions are commonly used for this purpose:
  - `tf`: to create the transfer functions of both continuous-time and discrete-time systems.
  - `c2d`: to convert continuous-time models into discrete-time ones.
  - `impulse/step`: to simulate the response of a system to impulse/step inputs.

Matlab Functions

To illustrate the use of these functions, let us revisit our previous example:

```
>> Gpc = tf([1],[1 3 2])
```

Name of T.F. — Coefficients of the numerator — Coefficients of the denominator: $1s^2 + 3s + 2$

Matlab's Output:

```
Transfer function:
      1
-------------
s^2 + 3 s + 2
```

One can define a transfer function in z-domain as well:

```
>> G = tf([0.1 0],[1 -0.9],0.01)
```

Name of T.F. — Coefficients of the numerator — Coefficients of the denominator: $1z - 0.9$ — Sampling time:T

Matlab's Output:

```
Transfer function:
 0.1 z
-------
z - 0.9

Sampling time: 0.01
```

To convert the continuous-time model into a discrete-time one:

```
>> Gpd = c2d(Gpc,0.01,'zoh')
```

Conversion method: 'zoh', 'tustin', ...

Name of T.F.   cont.-time T.F.   Sampling time: T

Matlab's Output:

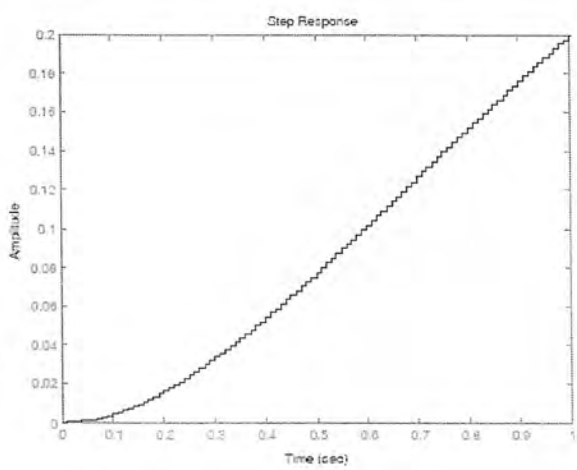
```
Transfer function:
4.95e-005 z + 4.901e-005
------------------------
 z^2 - 1.97 z + 0.9704

Sampling time: 0.01
```

To simulate the response of this system to a step-input:

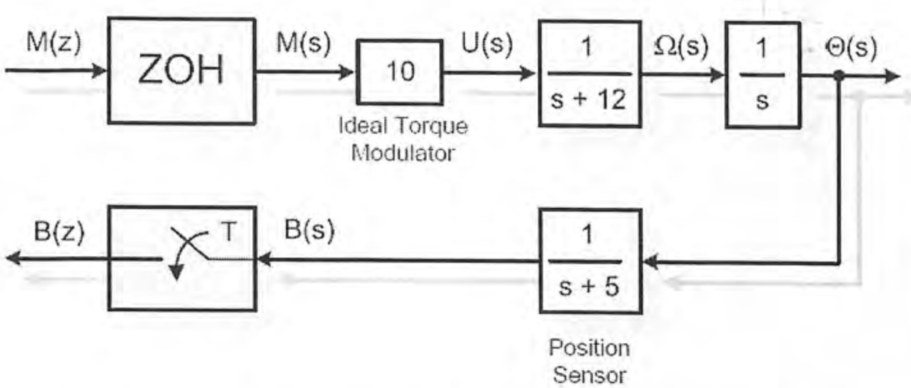
```
>> step(Gpd,1)
```

Name of T.F.   Final time

Other Matlab functions that might interest you at this point: `poly, roots, residue.`



Example



- If T = 0.01 s, obtain the transfer function B(z)/M(z) for the given system using MATLAB.

Solution

```
%
% Sample Matlab code
%
G1 = tf(10,[1 12]);
G2 = tf(1,[1 0]);
H  = tf(1,[1 5]);
G  = G1*G2*H;
Gd = c2d(G,0.01,'zoh')
```

```
>> G

Transfer function:
        10
-------------------
s^3 + 17 s^2 + 60 s
```

Output:

```
Transfer function:
1.598e-006 z^2 + 6.126e-006 z + 1.468e-006
------------------------------------------
    z^3 - 2.838 z^2 + 2.682 z - 0.8437

Sampling time: 0.01
```