Chapter 3

3.1 Stability

- A very important property of a dynamic system is its stability:
  - Internal stability
  - External stability
- *Internal stability* is concerned with the responses of all internal variables ("states").
- *External stability* studies the input-output behaviour of a dynamic system.

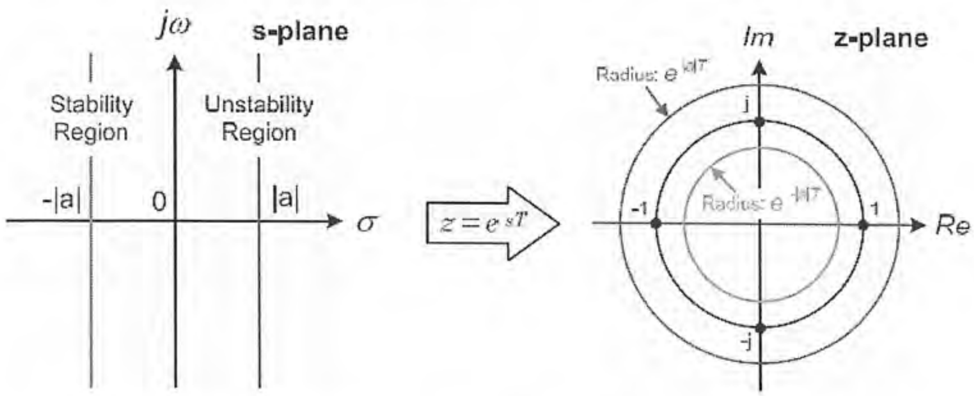
External Stability

- Most common definition of appropriate response for external stability is that for every *bounded input* (BI), the dynamic system should have a *bounded output* (BO).
- If this condition is satisfied, the system is said to be BIBO stable.

Stability for Continuous-time Systems

- A system is *asymptotically* stable if it returns to its initial equilibrium state after the application of an *impulse*.
- Stability is a property of the system and it is NOT dependent on a specific input or initial conditions.
- A *continuous-time system* (CTS) is stable if and only if the real parts of its poles are all strictly negative.
  - Poles must be located at left-hand side of the s-plane!

3.2 Mapping between s- and z-plane



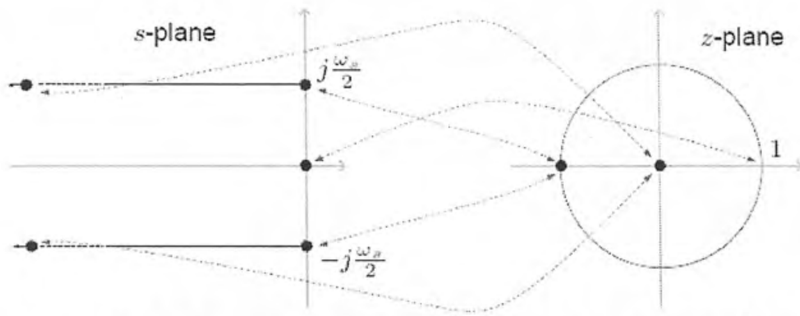
Let $s = \sigma + j\omega$ where $j \triangleq \sqrt{-1}$

Since $z \triangleq e^{sT}$

$$z = e^{\sigma T + j\omega T} = e^{\sigma T} \cdot e^{j\omega T}$$

$$z = \underbrace{e^{\sigma T}}_{Radius} \underbrace{(\cos\omega T + j\sin\omega T)}_{Depicts\ \ unit\ circle}$$

$$z = e^{sT} \qquad\qquad s = \frac{1}{T}\log z$$



Correspondence between the primary strip in the s-plane and the unity circle in the z-plane.

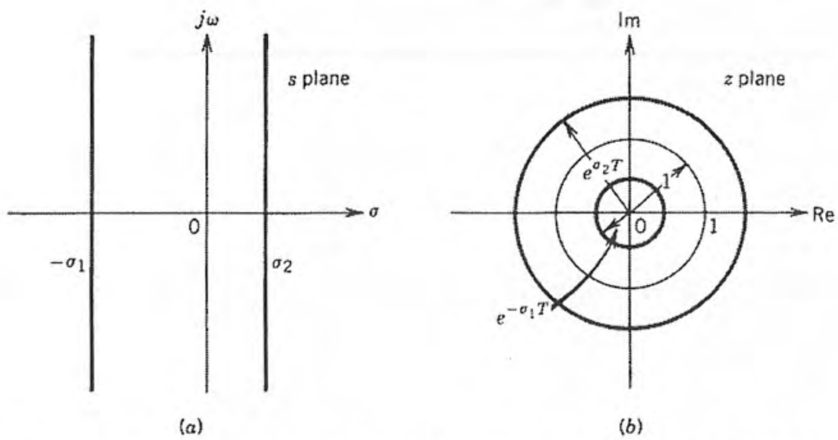
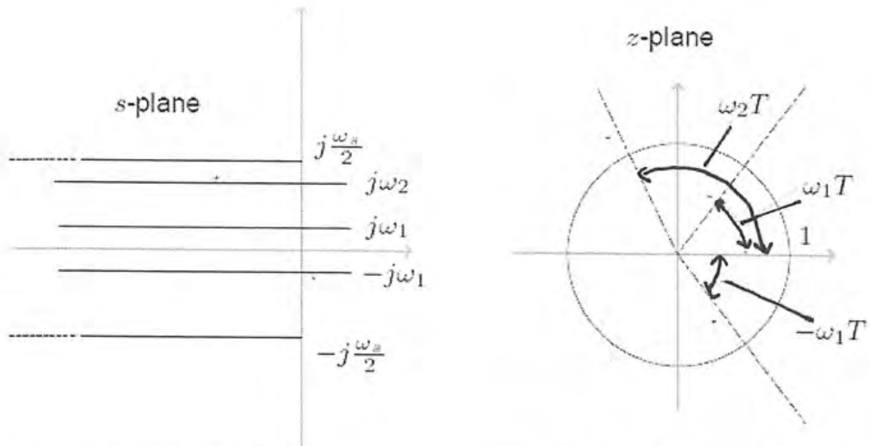


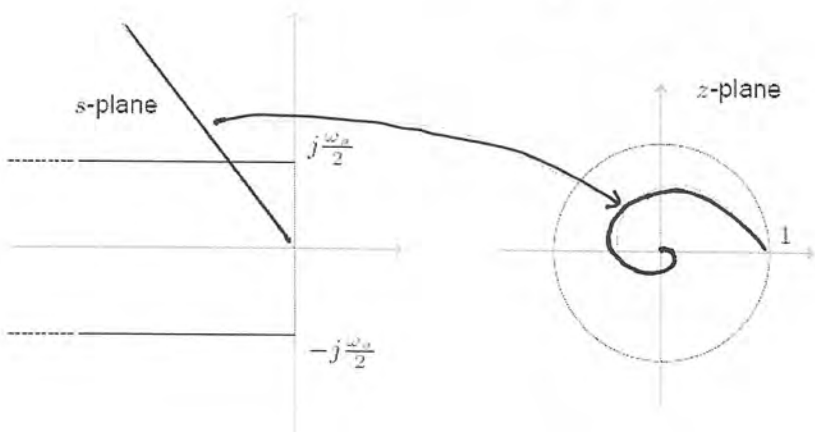
Figure (*a*) Constant attenuation lines in the s-plane; (*b*) the corresponding loci in the z-plane.

$$z = e^{sT} \qquad s = \frac{1}{T}\log z$$



Constant frequency lines: $s = \sigma + j\omega$, with ω constant, $|z| = e^{\sigma T} e^{j\omega T}$.

$$z = e^{sT} \qquad s = \frac{1}{T}\log z$$



Constant damping-ratio lines: $s = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$, with $\omega_n > 0$, $0 < \zeta < 1$ constant.
$|z| = e^{-\zeta T\omega_n}$, $\angle z = \sqrt{1-\zeta^2}\omega_n T$. The locus on the z-plane is a logarithmic spiral.

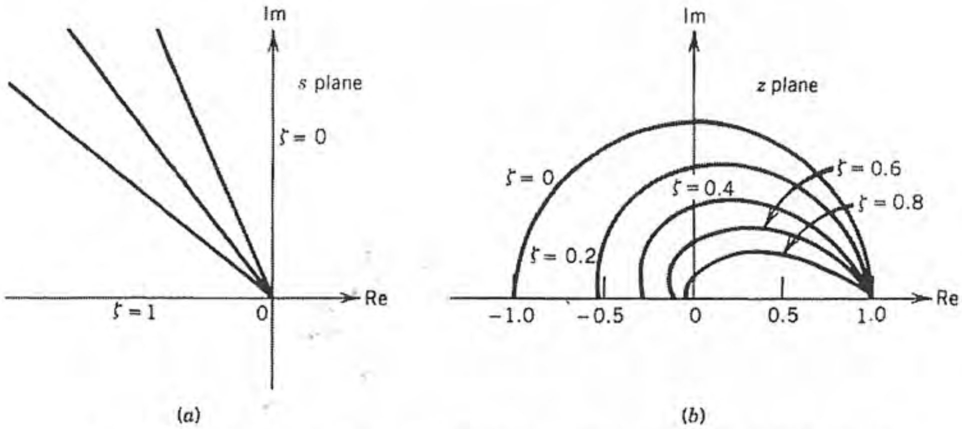
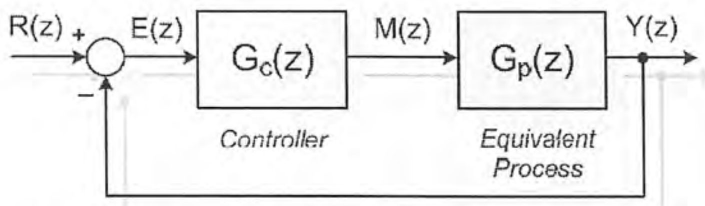


Figure (a) Constant ζ loci in the s-plane; (b) constant ζ loci in the z-plane.

Stability of Control Systems

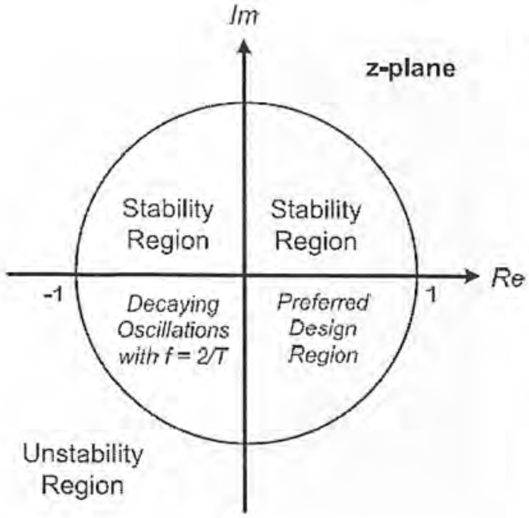


Consider the transfer function of the closed-loop control system shown:

$$\frac{Y(s)}{R(s)} = \frac{G_c(z)G_p(z)}{1+G_c(z)G_p(z)} = \frac{B(z)}{A(z)} \quad \text{where} \quad \begin{aligned} A(z) &= num\{1+G_c(z)G_p(z)\} \\ A(z) &= a_N z^N + \ldots + a_1 z + a_0 \end{aligned}$$

Hence, $$A(z) = \sum_{n=0}^{N} a_n z^n = \underbrace{\left[\prod_{i=1}^{N_r}(z+r_i)\right]}_{real\ poles}\underbrace{\left[\prod_{i=1}^{N_c}(z+p_i+jc_i)(z+p_i-jc_i)\right]}_{complex\ conjugate\ poles}$$

where $N = N_r + 2N_c$ is called the *system order.*



- For a BIBO stable system, all poles [i.e. roots of characteristic polynomial A(z)] must lie inside the unit circle.
- Systems with (dominant) poles residing inside left-hand semi-circular region exhibit forced oscillations at half the sampling frequency.
  - Very undesirable feature!

The stability of the system may be determined from the locations of the closed-loop poles in the z-plane or the roots of the closed-loop characteristic equation defined by Eq.

$$P(z) = 1 + G(z) = 0$$

as follows:

1. For the system to be stable, the closed-loop poles or the roots of the characteristic equation must lie within the unit circle in the z-domain. Any closed-loop pole outside the unit circle makes the system unstable.

2. If a simple pole lies at z=1 or z=-1, then the system becomes marginally stable. Also, the system becomes marginally stable if a single pair of complex-conjugate poles lies on the unit circle in the z-domain. Any multiple closed-loop pole on the unit circle makes the system unstable.

3. Closed-loop zeros do not affect the absolute stability and therefore may be located anywhere in the z-plane. Thus, a linear time-invariant single-input–single-output discrete-time closed-loop system becomes unstable if any closed-loop poles lies outside the unit circle or any multiple closed-loop pole lies on the unit circle in the z-domain.

Example

Determine the stability of the following system:

$$\frac{Y(z)}{R(z)} = \frac{1.658\times10^{-7}z^{-1} + 6.6\times10^{-7}z^{-2} + 1.642\times10^{-7}z^{-3}}{1 - 2.98z^{-1} + 2.96z^{-2} - 0.9802z^{-3}}$$

Solution:

Rearrange the transfer function in powers of z (not z^{-1}):

$$\frac{Y(z)}{R(z)} = \frac{1.658\times10^{-7}z^{2} + 6.6\times10^{-7}z + 1.642\times10^{-7}}{z^{3} - 2.98z^{2} + 2.96z - 0.9802}$$

Solution – Roots (Cont'd)

The characteristic polynomial A(z) becomes

$$A(z) = z^{3} - 2.98z^{2} + 2.96z - 0.9802$$

Matlab function `roots` comes handy at this point:

```
roots([1 -2.98 2.96 -0.9802])
```

Hence, the roots of A(z) are found to be

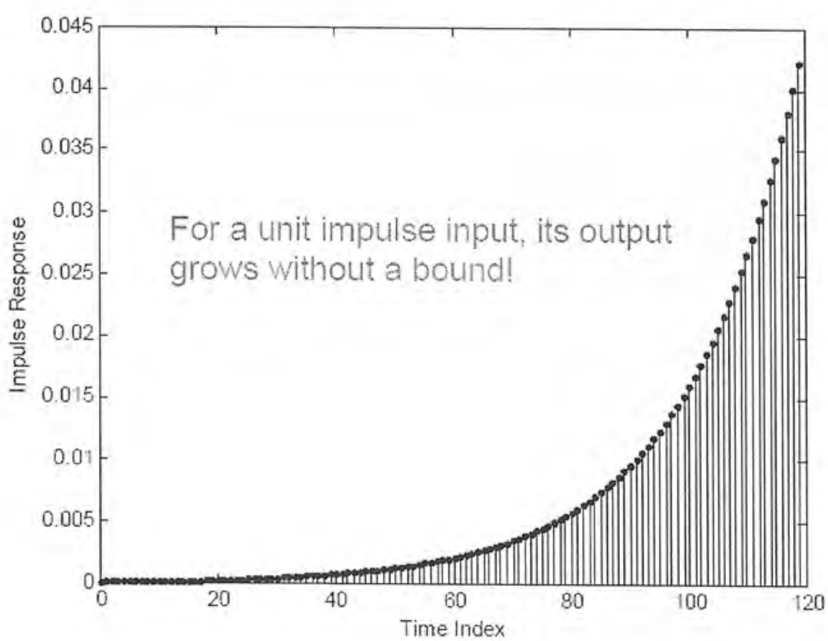
$$p_1 = 1.0525;$$
$$p_{2,3} = 0.9637 \pm j0.0499$$

Since p_1 is outside the unit circle, the system is UNSTABLE!

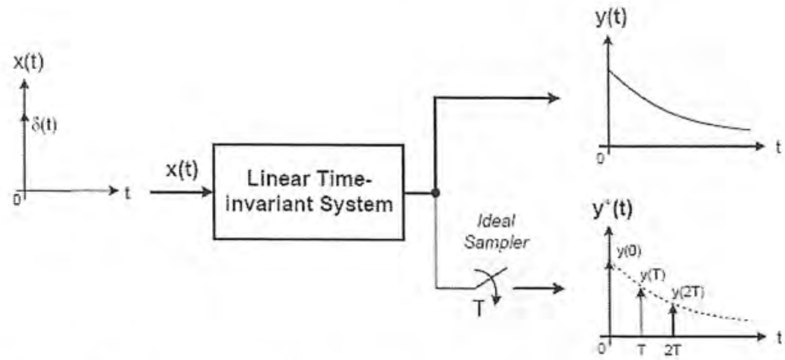
Solution – Matlab Script (Cont'd)

```
% *** Define the discrete-time TF
%     The last argument (-1) tells Matlab that
%     the sampling time is unspecified.
%
Gd = tf([1.658e-7 6.6e-7 1.642e-7],[1 -2.98 2.96 -.9802],-1);
%
% *** Calculate impulse response for 120 sampling steps...
%
[y,k] = impulse(Gd,120);
%
% *** Plot the results
%
stem(k,y,'.');
xlabel('Time Index'); ylabel('Impulse Response');
```

Solution – Response (Cont'd)



Testing Stability



- To test the stability, a unit impulse is applied to the system and its time response is observed.
- Let us assume that the impulse response is also sampled for sake of argument.

Time Response of a First Order CTS

Let the transfer function of a first order system be

$$G(s) = \frac{1}{s+a}$$

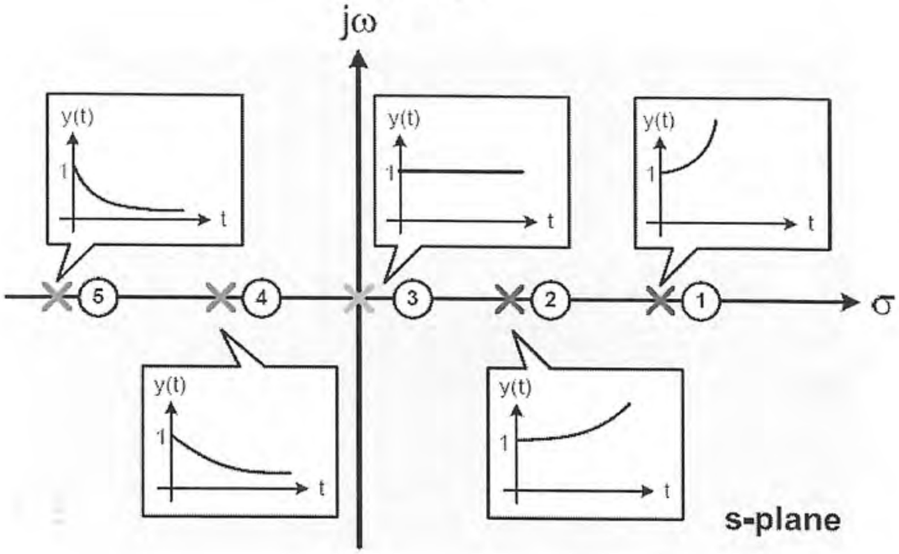
The system has one real root: $p_1 = -a$. Since X(s) = $L\{\delta(t)\}$ = 1,

$$Y(s) = G(s)X(s) = \frac{1}{s+a} \cdot 1$$

Therefore, the time response becomes

$$y(t) = L^{-1}\{Y(s)\} = L^{-1}\{G(s)\} = e^{-at}$$

Time Response vs. Root Locus of CTS



Time Response vs. Root Locus of DTS

Since y(t) = e^{-at}, the sampled response y*(t) becomes

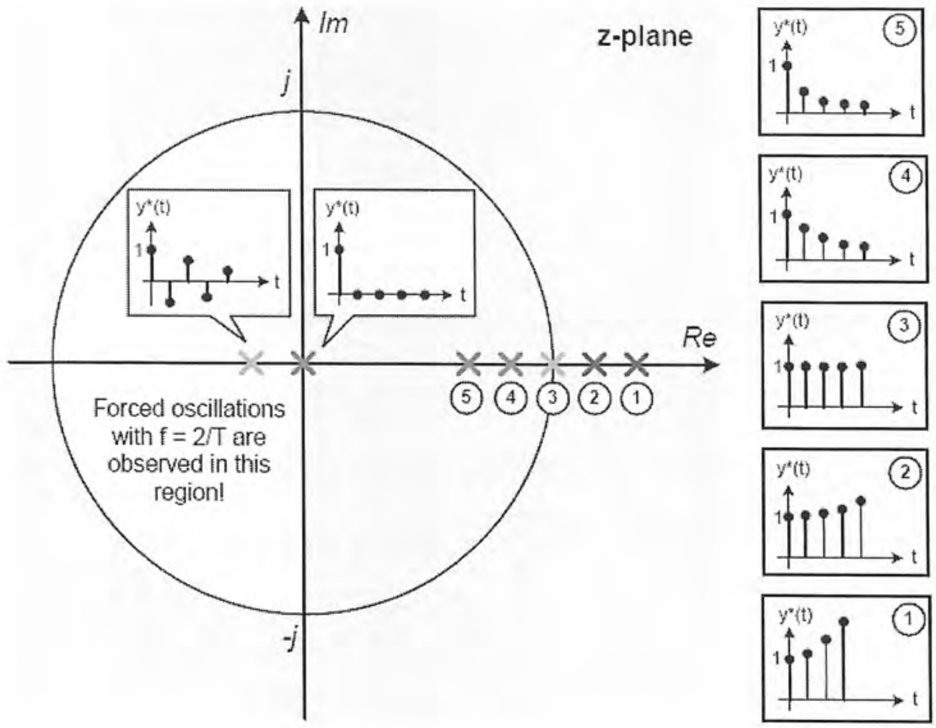
$$y^*(t) = y(kT) = e^{-akT}$$

Hence, its Z-transform leads to

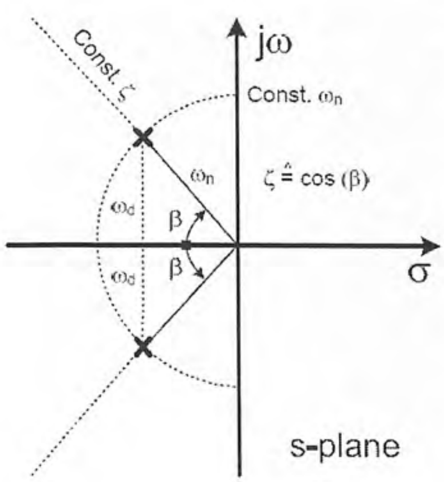
$$Y(z) = Z\{y(kT)\} = \frac{1}{1 - e^{-aT} z^{-1}} = \frac{z}{z - e^{-aT}}$$

Note that X(z) = 1. The corresponding discrete-time system has only one pole at $p_1 = e^{-aT}$.

Root Locus of DTS (Cont'd)



s-plane Analysis of Second Order CTS



The transfer function of a second-order system is expressed as

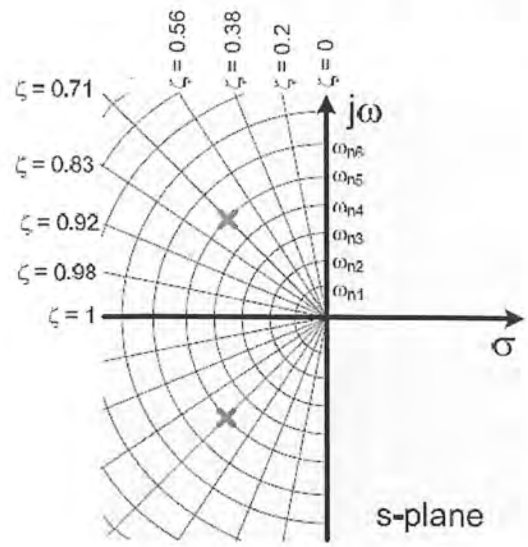
$$\frac{Y(s)}{X(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$= \frac{K\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$

where ζ is the damping ratio; ω_n is the natural frequency; ω_d is called damped (natural) frequency:

$$\omega_d \triangleq \omega_n\sqrt{1-\zeta^2}$$

s-plane Analysis (Cont'd)



- If the complex conjugate poles are located along the lines, the damping ratio does not change.
- If the poles are located on a particular semi-circle, the natural frequency is kept constant.

z-plane Analysis of a Second Order DTS

The transfer function of a second-order discrete-time system becomes

$$\frac{Y(z)}{X(z)} = \frac{b_1 z^{-1} + b_2 z^{-2}}{1 - 2e^{-\zeta\omega_n T}\cos(\omega_d T)z^{-1} + e^{-2\zeta\omega_n T}z^{-2}}$$

$$= \frac{}{z^2 - 2e^{-\sigma T}\cos\omega_d T\, z + e^{-2\sigma T}} \qquad \sigma = \zeta\omega_n$$

The roots of the characteristic polynomial are

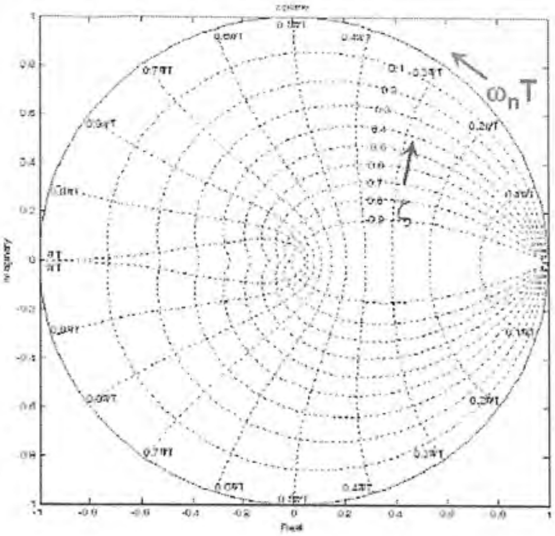
$$p_{1,2} = e^{-\zeta\omega_n T}[\cos(\omega_d T) \pm j\cdot\sin(\omega_d T)]$$

$$\uparrow$$
$$e^{-\sigma T}$$

Remark

To transform *s*-plane pole locations to the *z*-domain, the transformation $z = e^{Ts}$ where T is the sampling time, is employed.

z-plane Analysis (Cont'd)



- Mapping is slightly more complicated!
- Numbers along the radial direction shows decreasing damping ratio.
- Numbers along the perimeter of the unit circle indicate constant natural frequency.

Matlab Script to Produce "zgrid"

```
wdT = linspace(0,pi,100)'; zeta = linspace(0,.9,10)';
p1 = zeros(100,10); p2 = p1; j = sqrt(-1);
%
% Plot constant damping contours
%
for i = 1:10
  sT1 = wdT*(-zeta(i)/sqrt(1-zeta(i)^2) + j);
  sT2 = wdT*(-zeta(i)/sqrt(1-zeta(i)^2) - j);
  p1(:,i) = exp(sT1); p2(:,i) = exp(sT2);
end
close all; plot([p1 p2]); hold on
xlabel('Real'); ylabel('Imaginary'); axis('square')
%
% Plot constant (wn*T) contours
%
zeta = linspace(0,1,100)'; wnT = linspace(pi/10,pi,10)';
for i = 1:10
  sT1 = wnT(i)*(-zeta + j*sqrt(1-zeta.^2));
  sT2 = wnT(i)*(-zeta - j*sqrt(1-zeta.^2));
  p1(:,i) = exp(sT1); p2(:,i) = exp(sT2);
end
plot([p1 p2]); zgrid  % Comparison!
```