Chapter 6

6.1 Digital Root-locus

1-The effect of system gain on the absolute and relative stability of the closed loop system should be investigated in addition to the transient response characteristics. Root locus method is very useful in this regard.

2- The root locus method for continuous time systems can be extended to discrete time systems without much modification since the characteristic equation of a discrete control system is of the same form as that of a continuous time control system.

In many LTI discrete time control systems, the characteristics equation may have either of the following two forms.

$$\left.\begin{array}{l} 1+G(z)H(z) = \\ \text{or} \\ 1+GH(z) \end{array}\right\} \text{-----} (1)$$

To combine both, let us define the characteristics equation as:

$$1 + L(z) = 0$$

where, $\underline{L(z) = G(z)H(z)}$ or $\underline{L(z) = GH(z)}$. $L(z)$ is popularly known as the loop pulse transfer function. From equation (1), we can write

$$L(z) = -1$$

Since $L(z)$ is a complex quantity it can be split into two equations by equating angles and magnitudes of two sides. This gives us the angle and magnitude criteria as

Angle Criterion: $\angle L(z) = \pm 180^0(2k+1), \quad k = 0, 1, 2....$

Magnitude Criterion: $|L(z)| = 1$

The values of z that satisfy both criteria are the roots of the characteristics equation or close loop poles. Before constructing the root locus, the characteristics equation $1+L(z)=0$ should be rearranged in the following form

$$1 + K\frac{(z+z_1)(z+z_2)....(z+z_m)}{(z+p_1)(z+p_2)....(z+p_n)} = 0$$

where z_i's and p_i's are zeros and poles of open loop transfer function, (m) is the number of zeros (n) is the number of poles.

6.2 Construction Rules for Root Locus

Root locus construction rules for digital systems are same as that of continuous time systems.

- 1. The root locus is symmetric about real axis. Number of root locus branches equals the number of open loop poles.
- 2. The root locus branches start from the open loop poles at K = 0 and ends at open loop zeros at K = ∞. In absence of open loop zeros, the locus tends to ∞ when K → ∞. Number of branches that tend to ∞ is equal to difference between the number of poles and number of zeros.
- 3. A portion of the real axis will be a part of the root locus if the number of poles plus number of zeros to the right of that portion is odd.
- 4. If there are n open loop poles and m open loop zeros then n - m root locus branches tend to ∞ along the straight line asymptotes drawn from a single point s = σ which is called centroid of the loci.

$$\phi_q = \frac{180°(2q+1)}{n-m}, \quad q = 0, 1, \ldots, n-m-1$$

Angle of asymptotes

- 5. Breakaway (Break in) points or the points of multiple roots are the solution of the following equation:

$$\frac{dK}{dz} = 0$$

where K is expressed as a function of z from the characteristic equation. This is a necessary but not sufficient condition. One has to check if the solutions lie on the root locus.

6. The intersection (if any) of the root locus with the imaginary axis can be determined from the Routh array.

7. The angle of departure from a complex open loop pole is given by

$$\phi_p = 180^o + \phi$$

where ϕ is the net angle contribution of all other open loop poles and zeros to that pole.

$$\phi = \sum_i \psi_i - \sum_{j \neq p} \gamma_j$$

ψ_i's are the angles contributed by zeros and γ_j's are the angles contributed by the poles.

8. The angle of arrival at a complex zero is given by

$$\phi_z = 180^o - \phi$$

where ϕ is same as in the above rule.

9. The gain at any point z_0 on the root locus is given by

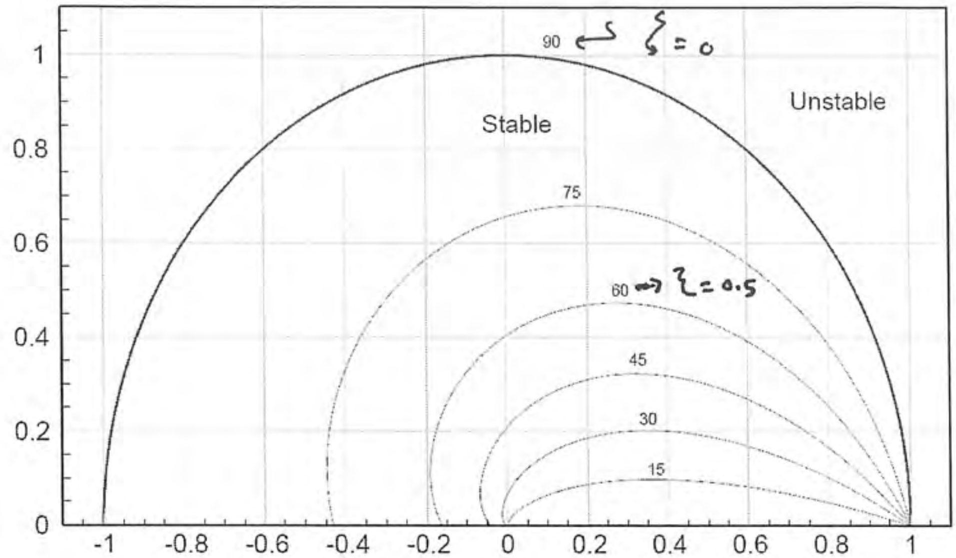
$$K = \frac{\prod_{j=1}^{n} |z_0 + p_j|}{\prod_{i=1}^{m} |z_0 + z_i|}$$

The point you want to pick in the z-plane for a good response is then:

- The dominant pole is the pole closest to s=0. In the z-plane, it's the pole closest to z=1.
- Pick 'k' so that you are stable. All poles are inside the unit circle.
- Better yet, pick 'k' so that the dominant pole has a desired damping ratio.

$z = e^{sT}$

This causes the damping lines in the s-plane to spiral in the z-plane as follows:



Root locus diagram of digital control systems

Example1

Obtain the root locus plot and the critical gain for the first-order type 1 system with loop gain:

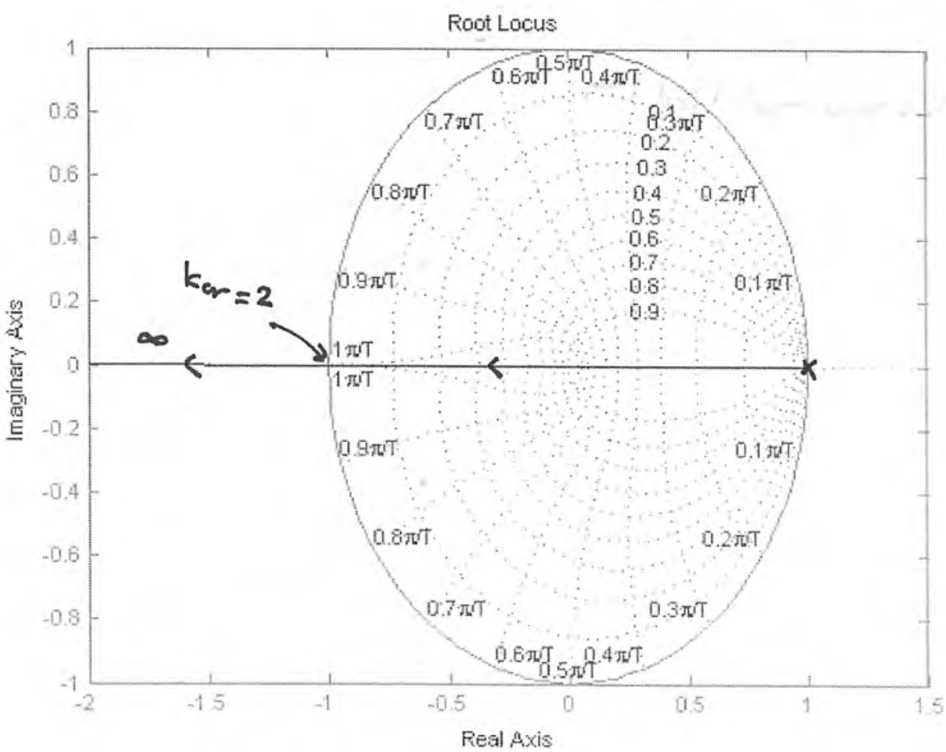
$$L(z) = \frac{1}{z-1}$$

Sol

critical gain for the system corresponds to the point (-1,0). The closed-loop characteristic equation of the system is

$z - 1 + K = 0$

Substituting z =-1 gives the critical gain Kcr = 2.



Example2

Obtain the root locus plot and the critical gain for the second-order type 1 system with loop gain:

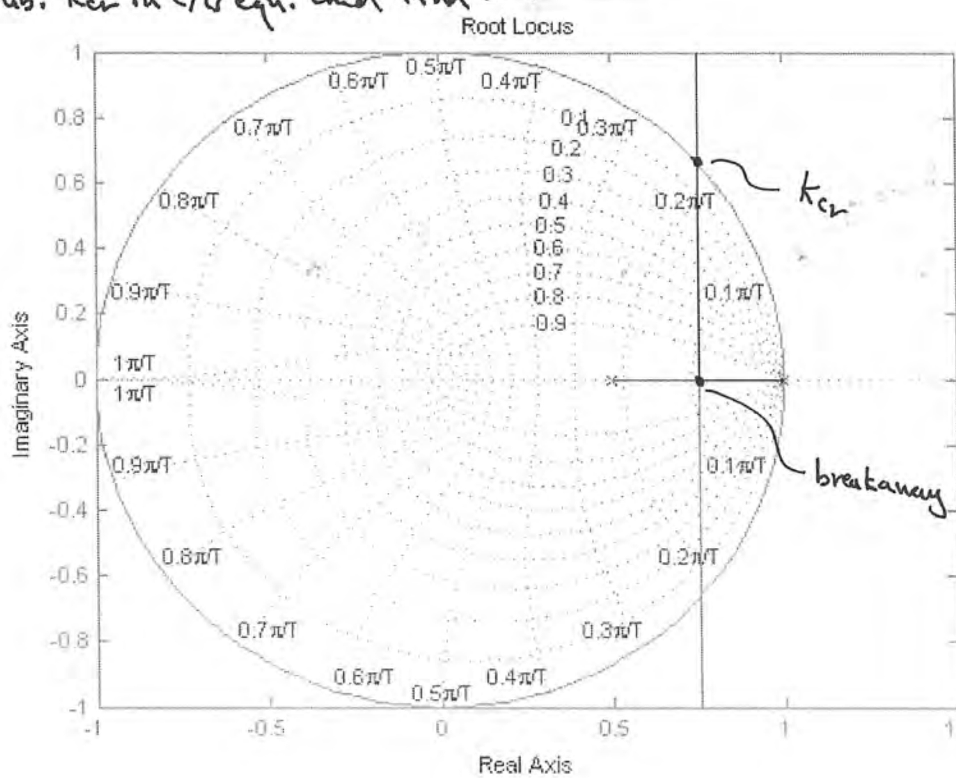
$$L(z) = \frac{1}{(z-1)(z-0.5)}$$

* Also find the C.L.poles at Kcr.

Sol

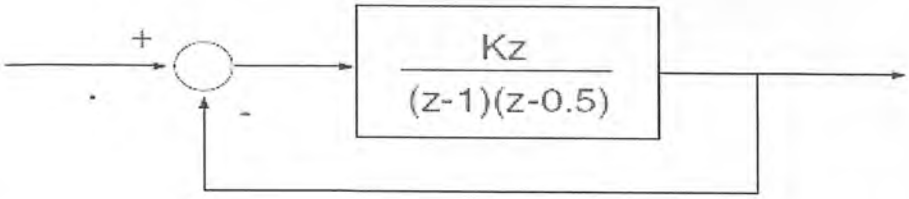
1- 2-poles are z=1 and z=0.5
2- 2-zeros are z= ∞, ∞
3- No. of asymp.=n-m=2-0=2
4- Angle of asymp.=90.
5- Use equ. Of break away and in, find breakaway=0.75.There is no break in.
6- Use jury test to find Kcr. =0.5
7- There is no intersection with j-axis
8- at this point[witk Kcr=0.5], the C.L.poles, Z12=0.75∓j0.661

Sub. Kcr in c/cs equ. and find



Example3

Repeat Example2, but with adding a zero(z).



Sol

1- 2-poles are z=1 and z=0.5
2- 2-zeros are z=0 and z= ∞
3- No. of asymp.=n-m=2-1=1
4- Angle of asymp.=180°.

5-Use equ. Of break away and in,.

$$K = -\frac{1}{G(z)} = \frac{(z-1)(z-0.5)}{z} \quad --- (*)$$

$$\frac{dK}{dz} = -\frac{(z-0.707)(z+0.707)}{z^2} = 0$$

yields the break away and break in points as $z = \pm 0.707$.

6-Use jury test to find Kcr. =3 ← there is no intersection with unit circuit So that, sub (z=-1) in equ(*)

7-The intersection with j-axis=0.7j → c/cs eqn

$z^2 - 1.5z + .5 = 0$

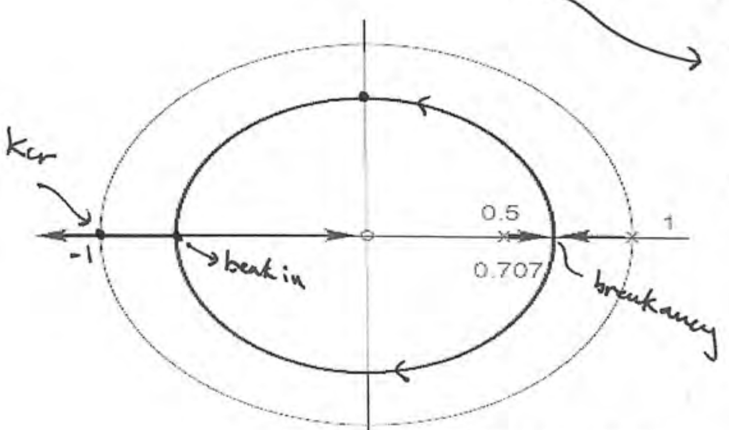
$(jw)^2 - 1.5jw + .5 = 0$

$(-w^2 + .5) - 1.5jw = 0$

Real part = 0

$w^2 = .5$

$\therefore w = \pm 0.7j$



Root Locus in z-plane

Example4 Draw the Root-locus

$$G(s) = \frac{6.4}{(s+1)(s+0.2)}.$$

Sol if T=0.1sec

The discrete time transfer function (open loop) is

$$G_d(z) = \frac{0.1183z + 0.1092}{z^2 - 1.78z + 0.7866} = 0.1183\frac{z + 0.9231}{(z - 0.9608)(z - 0.8187)}$$

1- 2-poles are z=0.9608 and z=0.8187
2- 2-zeros are z=- 0.9231 and z= ∞
3- No. of asymp.=n-m=2-1=1
4- Angle of asymp.=180°.
5- Use equ. Of break away=0.9 and in=-2.7,
6- Use jury test to find Kcr=1.9
7- Find the intersection with j-axis=1.54

Or using the MTLAB