In previous two sections we observed that the system matrix A in observable canonical form is transpose of the system matrix in controllable canonical form. Similarly, control matrix B in observable canonical form is transpose of output matrix C in controllable canonical form. So also output matrix C in observable canonical form is transpose of control matrix B in controllable canonical form.

Example: Consider the following discrete transfer function.

$\beta_0 = 0$

$$G(z) = \frac{0.17z + 0.04}{z^2 - 1.1z + 0.24}$$ ← here $n \neq m$ (2, 1)

Find out the state variable model in 2 different canonical forms.

Solution:

The state variable model in controllable canonical form can directly be derived from the transfer function, where the A, B, C and D matrices are as follows:

$$A = \begin{bmatrix} 0 & 1 \\ -0.24 & 1.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.04 & 0.17 \end{bmatrix}, \quad D = \overset{\beta_0}{0}$$

The matrices in state model corresponding to observable canonical form are obtained as,

$$A = \begin{bmatrix} 0 & -0.24 \\ 1 & 1.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.04 \\ 0.17 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = 0$$

Example

Given the following function, Find state equations:

$$G(z) = \frac{Y(z)}{U(z)} = \frac{z^2 + 2z + 1}{z^3 + 2z^2 + z + \frac{1}{2}}$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{2} & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [1 \quad 2 \quad 1] \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

Example

Given the following transfer function, we wish to derive the state equations.

$$G(z) = \frac{Y(z)}{U(z)} = \frac{b_2 z^2 + b_1 z + b_0}{z^2 + a_1 z + a_0}$$

In this case, the order of the numerator is equal to that of the denominator. First we express the transfer function as

$$\frac{Y(z)}{U(z)} = \frac{b_2 + b_1 z^{-1} + b_0 z^{-2}}{1 + a_1 z^{-1} + a_0 z^{-2}} \frac{E(z)}{E(z)}$$

The same as normal ($n \neq m$) controllable form

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$Y(z) = b_0 X_1(z) + b_1 X_2(z) + b_2 E(z)$$

But

$$E(z) = U(z) - a_0 X_1(z) - a_1 X_2(z)$$

From these equations we obtain

$$Y(z) = b_2 U(z) + (b_0 - b_2 a_0) X_1(z) + (b_1 - b_2 a_1) X_2(z)$$

Hence the output equation is

but C is not the same

$$y(k) = [(b_0 - b_2 a_0) \quad (b_1 - b_2 a_1)] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \underbrace{b_2}_{D} u(k)$$

7.3 State Space Model to Transfer Function

Consider a discrete state variable model

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

Taking the Z-transform on both sides of Eqn. we get

$$\begin{aligned} zX(z) - z\boldsymbol{x}_0 &= AX(z) + BU(z) \\ Y(z) &= CX(z) + DU(z) \end{aligned}$$

where $\boldsymbol{x}_0$ is the initial state of the system.

$$\begin{aligned} &\Rightarrow \qquad (zI - A)X(z) = z\boldsymbol{x}_0 + BU(z) \\ &\text{or,} \quad X(z) = (zI - A)^{-1}z\boldsymbol{x}_0 + (zI - A)^{-1}BU(z) \end{aligned}$$

To find out the transfer function, we assume that the initial conditions are zero, i.e., $x_0 = 0$, thus

$$Y(z) = \Big(C(zI - A)^{-1}B + D\Big)U(z)$$

Therefore, the transfer function becomes

$$G(z) = \frac{Y(z)}{U(z)} = C(zI - A)^{-1}B + D \quad \text{---} \quad (*)$$

which has the same form as that of a continuous time system.

Example

Consider the state-space model; we need to find the T.F

$$\mathbf{x}(k+1) = \begin{bmatrix} 1.35 & 0.55 \\ -0.45 & 0.35 \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}u(k)$$
$$y(k) = \begin{bmatrix} 1 & -1 \end{bmatrix}\mathbf{x}(k)$$

· Now

$$[z\mathbf{I} - \mathbf{A}] = \begin{bmatrix} z - 1.35 & -0.55 \\ 0.45 & z - 0.35 \end{bmatrix}$$

Thus

$$|z\mathbf{I} - \mathbf{A}| = z^2 - 1.7z + 0.72$$

Also,

$$\mathrm{Cof}[z\mathbf{I} - \mathbf{A}] = \begin{bmatrix} z - 0.35 & -0.45 \\ 0.55 & z - 1.35 \end{bmatrix}$$

Then

$$[z\mathbf{I} - \mathbf{A}]^{-1} = \frac{[\mathrm{Cof}[z\mathbf{I} - \mathbf{A}]]^T}{|z\mathbf{I} - \mathbf{A}|} = \frac{1}{z^2 - 1.7z + 0.72}\begin{bmatrix} z - 0.35 & 0.55 \\ -0.45 & z - 1.35 \end{bmatrix}$$

, since $D = 0$,

$$\begin{aligned} G(z) &= \mathbf{C}[z\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} \\ &= \frac{1}{z^2 - 1.7z + 0.72}\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} z - 0.35 & 0.55 \\ -0.45 & z - 1.35 \end{bmatrix}\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \\ &= \frac{1}{z^2 - 1.7z + 0.72}\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 0.5z + 0.1 \\ 0.5z - 0.9 \end{bmatrix} \\ &= \frac{1}{z^2 - 1.7z + 0.72} \end{aligned}$$

Example (State-space to polynomial) and Find diff. eqn.
Consider the state system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -0.5 & 1.5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \underbrace{2}_{D} u(k)$$

The determinant of $[Z\,I - A]$ *is* $\quad [ZI - A] = \begin{bmatrix} Z+0.5 & -1.5 \\ 1 & Z-2 \end{bmatrix}$

$$det[Z\,I - A] = Z^2 - 1.5\,Z + 0.5$$

and the adjoint of $[Z\,I - A]$ *is*

$$adj \begin{bmatrix} Z+0.5 & -1.5 \\ 1 & Z-2 \end{bmatrix} = \begin{bmatrix} Z-2 & 1.5 \\ -1 & Z+0.5 \end{bmatrix}$$

Further we compute

$$C\,adj(ZI - A)B + det[ZI - A]D = \quad = C(ZI - A)^{-1}B + D$$

بعد أن نقسم المعادلة على det

$$\begin{aligned} &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} Z-2 & 1.5 \\ -1 & Z+0.5 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} + (Z^2 - 1.5\,Z + 0.5)\,2 \\ &= (2\,Z - 6) + (2Z^2 - 3\,Z + 1) \\ &= 2\,Z^2 - Z - 5 \end{aligned}$$

So the related difference equation is given by equation : T.F

$$(Z^2 - 1.5\,Z + 0.5)\,y(k) = (2\,Z^2 - Z - 5)\,u(k) \Rightarrow \frac{y(k)}{u(k)} = \frac{2Z^2 - Z - 5}{Z^2 - 1.5\,Z + 0.5}$$

or Diff. eqn:

$$y(k+2) - 1.5\,y(k+1) + 0.5\,y(k) = 2\,u(k+2) - u(k+1) - 5\,u(k)$$

Example

Find the T.F of the following equation(let D=0)

$$\begin{aligned} H(z) &= D + C\,(zI - A)^{-1}\,B \\ &= \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} z-1 & -T \\ 0 & z-1 \end{bmatrix}^{-1} \overbrace{\begin{bmatrix} 0 \\ T/m \end{bmatrix}}^{B} \qquad A = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \\ &= \frac{1}{(z-1)^2} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} z-1 & T \\ 0 & z-1 \end{bmatrix} \begin{bmatrix} 0 \\ T/m \end{bmatrix} \\ &= \frac{T}{m} \frac{z-1}{(z-1)^2} = \boxed{\frac{T}{m} \frac{z^{-1}}{1 - z^{-1}}}. \end{aligned}$$

Generally :-

- The equation

$$\boxed{Y(z) = G(z)U(z)}$$

is the discrete counterpart of the transfer function representation $Y(s) = G(s)U(s)$ for continuous-time systems.

- The function $G(z)$ is the z-transform of the impulse response sequence $g[k]$ and is called the **discrete transfer function.**

- Both the discrete convolution and transfer function describe the system assuming **zero initial conditions.**

Every discrete-time, finite dimensional, linear system can be represented by state space **difference equations**, as in

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k] \\ y[k] &= Cx[k] + Du[k] . \end{aligned}$$

The relation between discrete transfer function representation and state space representation is identical to the continuous-time case,

$$\hat{G}(z) = C(zI - A)^{-1}B + D ,$$

and the same MATLAB functions can be used to define systems, e.g.,

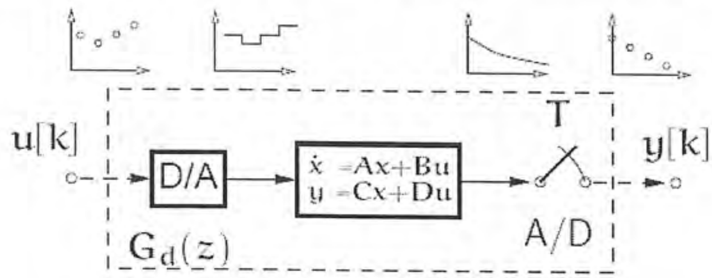
```
G1 = ss(A,B,C,D,T);
G2 = tf(Num,Den,T);
```

- Most of the state space concepts for linear continuous-time systems directly translate to discrete-time systems, described by **linear difference equations**. In this case the time variable t only takes values a set like $\{0, 1, 2, \ldots\}$.

The operation by which a continuous-time model is converted into a discrete-time one is called discretisation.

A discrete-time model is often needed, for example to simulate it with a digital computer; or to design a discrete-time controller, which is also implemented in some kind of digital computer.

The Parameter Variation Formula yields a direct method for discretisation of a continuous-time system state space model.



This discrete model gives the exact value of the variables at time $t = kT$. In MATLAB the function `[Ad,Bd] = c2d(A,B,T)` computes A_d and B_d using the above expressions.

7.4 How to find the state-space from difference equation

Example

It is desired to find a state-variable model of the system described by the difference equation

$$y(k+2) = u(k) + 1.7y(k+1) - 0.72y(k)$$

or $y(k+2) - 1.7y(k+1) + 0.72y(k) = U(k)$

Let

$$x_1(k) = y(k)$$
$$x_2(k) = x_1(k+1) = y(k+1)$$

Then

$$x_2(k+1) = y(k+2) = u(k) + 1.7x_2(k) - 0.72x_1(k)$$

or, from these equations, we write

$$x_1(k+1) = x_2(k)$$
$$x_2(k+1) = -0.72x_1(k) + 1.7x_2(k) + u(k)$$
$$y(k) = x_1(k)$$

$$\frac{Y(z)}{U(z)} = \frac{1}{z^2 - 1.7z + 0.72}$$

We may express these equations in vector-matrix form of

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 \\ -0.72 & 1.7 \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(k)$$
$$y(k) = [1 \quad 0]\mathbf{x}(k)$$

← Controllable Canonical Form

Note that :-

Equations (3) and (4) are the state equations for a linear time-invariant system and usually represent the starting point in the analysis or design of a discrete system by modern methods. However, let us first examine the connection between this approach and the z-transform method. To do this we give one method for deriving a set of discrete state-variable equations from the z-transform transfer function. [See above Ex].

Example (Diff eqn to state-space)

Consider the difference system

$$16\,y(k+3) - 20\,y(k+2) + 8\,y(k+1) - y(k) = 5\,u(k+2) - 7\,u(k+1) + 2\,u(k)$$

the state space realization is given by

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/16 & -8/16 & 20/16 \end{bmatrix} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/16 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 2 & -7 & 5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

← Controllable Form