## 7.5 ANALYSIS OF DISCRETE-TIME LINEAR STATE-SPACE SYSTEMS

We discuss the analysis and solution of discrete-time (DT) linear time-invariant (LTI) state-variable systems. The equations derived here are very similar to the continuous-

$$\Delta(z) = |zI - A|.$$

The roots of this characteristic polynomial are the system poles. The characteristic equation is

$$\Delta(z) = |zI - A| = 0.$$

$$\boxed{\text{Steady state gain} = \mathbf{G}(1) = \mathbf{C}(\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}}$$

## Example - Analysis of DT Systems

Given DT SV system is:

$$x_{k+1} = \begin{bmatrix} 0 & -1/2 \\ 1 & 3/2 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_k$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x_k$$

a) Poles and natural modes:

Char. eqn. is

$$\det(zI - A) = 0$$

or

$$z^2 - 3/2z + 1/2 = 0$$

poles are then: $p_1 = 1.0$, $p_2 = 0.5$.

b) Natural modes: $(1)^k, (0.5)^k$

$$\Phi(z) = (zI - A)^{-1}$$

$$= \frac{\begin{bmatrix} z-3/2 & -1/2 \\ 1 & z \end{bmatrix}}{z^2 - 3/2z + 1/2}$$

c) Transfer function

$$H(z) = C\Phi(z)B + D$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{\begin{bmatrix} z-3/2 & -1/2 \\ 1 & z \end{bmatrix}}{z^2 - 3/2z + 1/2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + D$$

$$= \frac{(1+z)}{(z-1)(z-0.5)}$$

**Note that**

Here , H(z)=G(z) in our lecture.

Dc. gain $= H(1) = \infty$

$\therefore E_{ss} = \dfrac{1}{1 + \text{Dc gain}} = 0$

For step-i/p

d) Pulse response

$$h(k)=Z^{-1}(H(z)) = \left[\frac{A}{z-1} + \frac{B}{z-0.5}\right] = \frac{4}{z-1} - \frac{3}{z-0.5}$$
$$= 4(1)^k - 3(0.5)^k$$

e) Step Response of the output

$u(k)=1, \forall k \geq 0 \Leftarrow$ step input

$$U(z)=\frac{z}{z-1}$$
$$Y(z)=H(z)U(z)$$
$$y(k)=Z^{-1}(Y(z))$$
$$=Z^{-1}\frac{z(z+1)}{(z-1)^2(z-0.5)}$$
$$=6(0.5)^k - 6(1)^k + 4k(1)^{k-1}$$

f) Analytic solution for $x_k$ (Diff equ)

The solution is given by

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$$H(z)=G(z)=\frac{Y(z)}{U(z)}=\frac{(z+1)}{(z^2-1.5z+0.5)}*\frac{z^{-2}}{z^{-2}}$$
$$Y(z)[1-1.5z^{-1}+0.5z^{-2}]=U(z)[z^{-1}+z^{-2}]$$
$$Y(k)-1.5Y(k-1)+0.5Y(k-2)=U(k-1)+U(k-2)$$

Is the required difference equation.

## MATLAB statements

So the discrete equivalent state space model becomes,

$$\mathbf{x}(k+1) = \overset{\nearrow A}{\mathbf{F}}\,\mathbf{x}(k) + \overset{\nearrow B}{\mathbf{g}}\,u(k)$$
$$y(k) = \mathbf{c}\,\mathbf{x}(k) + d\,u(k)$$

MATLAB's *c2d.m* function performs the above conversion,

```
>> sysC = ss(A,B,C,D);
>> sysD = c2d(sysC,T,'zoh');
```

Zeros at: $\mathbf{c}(z\mathbf{I}-\mathbf{F})^{+}\mathbf{g} = 0$

Poles at: $|z\mathbf{I}-\mathbf{F}| = 0$

The pole equation is the same equation used to calculate the eigenvalues of the system matrix **F**. To calculate the poles and zeros using MATLAB the commands are,

```
>> poles = eig(F);
>> zeros = tzero(sysD);
```

Here F=A, g=B in our lecture.

# 7.6 POLE ASSIGNMENT

In this Section we present a design method known as *pole placement* or *pole assignment*. This method is similar to the root-locus design in that the closed-loop poles may be placed in desired locations. However, pole-placement design allows all closed-loop poles to be placed in desirable locations, whereas the root-locus design procedure allows only the two dominant poles to be placed. There is a cost associated with placing all closed-loop poles, however, because placing all closed-loop poles requires measurement and feedback of all the state variables of the system.

In this section a design procedure generally known as pole assignment, or pole placement, is developed. The design results in the assignment of the poles of the closed-loop transfer function (zeros of the characteristic equation) to any desired locations. There are, of course, practical implications that will be discussed as the technique is developed.

**Note that:-** sometimes this method is called full-state f/b design.

We will now develop a general procedure of pole assignment. Our $n$th-order plant is modeled by

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}u(k) \tag{6}$$

We generate the control input $u(k)$ by the relationship

$$u(k) = -\mathbf{K}\mathbf{x}(k) \tag{7}$$

where

$$\mathbf{K} = [K_1 \quad K_2 \quad \cdots \quad K_n] \tag{8}$$

Then (6) can be written as

$$\mathbf{x}(k+1) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(k) \tag{9}$$

and, this, $\mathbf{A}_f = (\mathbf{A} - \mathbf{B}\mathbf{K})$. We choose the desired pole locations

after design

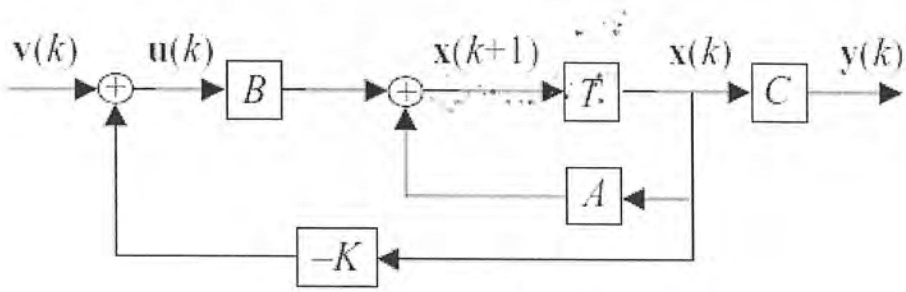
$$z = \lambda_1, \lambda_2, \ldots, \lambda_n \tag{10}$$

Then the closed-loop system characteristic polynomial is

$$\alpha_c(z) = |z\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}| = (z - \lambda_1)(z - \lambda_2)\cdots(z - \lambda_n) \tag{11}$$

In this equation there are $n$ unknowns $K_1, K_2, \ldots, K_n$, and $n$ known coefficients in the right-hand-side polynomial. We can solve for the unknown gains by equating

**Coeficients in (11).**



In the procedure above the gain matrix **K** was calculated by equating coefficients in the characteristic equation of (11). This calculation is simplified greatly if the state model of the plant is in the control canonical form (see Section 7.2), which is the form

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \vdots & \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}u(k) \qquad (12)$$

This plant has the characteristic equation

$$\alpha(z) = |z\mathbf{I} - \mathbf{A}| = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0 \qquad (13)$$

For this state model, in (11) **BK** is equal to

$$\mathbf{BK} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}[K_1 \quad K_2 \quad \cdots \quad K_n] = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & & \vdots & \\ K_1 & K_2 & \cdots & K_n \end{bmatrix} \qquad (14)$$

Hence the closed-loop system matrix is

$$\mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & & \vdots & \\ -(a_0 + K_1) & -(a_1 + K_2) & \cdots & -(a_{n-1} + K_n) \end{bmatrix} \qquad (15)$$

and the characteristic equation becomes

$$|z\mathbf{I} - \mathbf{A} + \mathbf{BK}| = z^n + (a_{n-1} + K_n)z^{n-1} + \cdots + (a_1 + K_2)z + (a_0 + K_1) = 0 \qquad (16)$$

If we write the desired characteristic equation as

$$\alpha_c(z) = z^n + \lambda_{n-1}z^{n-1} + \cdots + \lambda_1 z + \lambda_0 = 0 \qquad (17)$$

we calculate the gains by equating coefficients in (16) and (17) to yield

$$K_{i+1} = \lambda_i - a_i, \qquad i = 0, 1, \ldots, n-1 \qquad (18)$$

In general, the techniques for developing state models for plants do not result in the control canonical form (see Section 7.2). A more practical procedure for calculating the gain matrix **K** is the use of Ackermann's formula. The proof of Ackermann's formula will not be given here [in lecture], but this proof is based on transformations from a general system matrix **A** to that of the control canonical form.

## Note that

The c/cs equ. Depends on the roots($\lambda_0$), ($\lambda_1$), ($\lambda_{n-1}$) and so on. This roots represents the design specifications( Mp, $\varsigma$, Wn, Ts,.....).

To introduce the pole-assignment technique, we will consider the model of a

Example (*)

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & 0.0952 \\ 0 & 0.905 \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} 0.00484 \\ 0.0952 \end{bmatrix}u(k) \quad \text{-----} (19)$$

$$y(k) = [1 \quad 0]\mathbf{x}(k)$$

We choose the control input $u(k)$ to be a linear combination of the states; that is,

$$u(k) = -K_1x_1(k) - K_2x_2(k) = -\mathbf{K}\mathbf{x}(k) \qquad (20)$$

where the gain matrix $\mathbf{K}$ is

$$\mathbf{K} = [K_1 \quad K_2]$$

Then (19) can be written as

$$\mathbf{x}(k+1) = \overset{A}{\begin{bmatrix} 1 & 0.0952 \\ 0 & 0.905 \end{bmatrix}}\mathbf{x}(k) - \overset{B}{\begin{bmatrix} 0.00484 \\ 0.0952 \end{bmatrix}}\overset{K}{[K_1x_1(k) + K_2x_2(k)]}$$

$$= \begin{bmatrix} 1 - 0.00484K_1 & 0.0952 - 0.00484K_2 \\ -0.0952K_1 & 0.905 - 0.0952K_2 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \quad \text{----} (21)$$

A-BK = Af

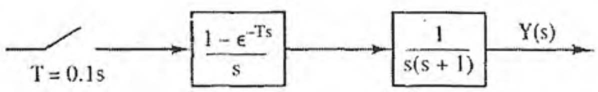



**Figure** . Servomotor system.

The matrix of (21) is the closed-loop system matrix, which we will call $A_f$. The closed-loop system equation is then

$$x(k+1) = A_f x(k) \tag{22}$$

and the characteristic equation is

$$|zI - A_f| = 0 \tag{23}$$

Evaluation of (23) yields, after some calculations, the characteristic equation

$$z^2 + (0.00484K_1 + 0.0952K_2 - 1.905)z + 0.00468K_1 - 0.0952K_2 + 0.905 = 0 \tag{24}$$

Suppose that, by some process we choose the desired characteristic-equation zero locations to be $\lambda_1$ and $\lambda_2$. Then the desired characteristic polynomial, denoted $\alpha_c(z)$, is given by

$$\alpha_c(z) = (z - \lambda_1)(z - \lambda_2) = z^2 - (\lambda_1 + \lambda_2)z + \lambda_1\lambda_2 \tag{25}$$

Equating coefficients in (24) and (25) yields the equations

$$\begin{aligned} 0.00484K_1 + 0.0952K_2 &= -(\lambda_1 + \lambda_2) + 1.905 \\ 0.00468K_1 - 0.0952K_2 &= \lambda_1\lambda_2 - 0.905 \end{aligned} \tag{26}$$

These equations are linear in $K_1$ and $K_2$, and upon solving yield

$$\begin{aligned} K_1 &= 105[\lambda_1\lambda_2 - (\lambda_1 + \lambda_2) + 1.0] \\ K_2 &= 14.67 - 5.34\lambda_1\lambda_2 - 5.17(\lambda_1 + \lambda_2) \end{aligned} \tag{27}$$

Thus we can find the gain matrix **K** that will realize *any* desired characteristic equation.

**Let in example above if k=1, and k2=0**

From (24), the characteristic equation is given by

$$z^2 - 1.9z + 0.91 = 0$$

This equation has roots at $\rightarrow z_{1,2} = 0.95 \pm j0.0866$

$$z_{1,2} = 0.954\angle \pm 0.091 \text{ rad} = r\angle \pm\theta$$