we calculate the damping factor of these roots to be

$$\zeta = \frac{-\ln r}{\sqrt{\ln^2 r + \theta^2}} = \frac{-\ln(0.954)}{\sqrt{\ln^2(0.954) + (0.091)^2}} = 0.46$$

the time constant is

$$\tau = \frac{-T}{\ln r} = \frac{-0.1}{\ln(0.954)} = 2.12 \text{ s}$$

Suppose that we decide that this value of ζ is satisfactory, but that a time constant of 1.0 s is required. Then

To increase response speed ∴

$$\ln r = -\frac{T}{\tau} = -0.1$$

[To Find K_1, K_2 For this Case]

or $r = 0.905$.

for θ, we have

$$\theta^2 = \frac{\ln^2 r}{\zeta^2} - \ln^2 r = \frac{\ln^2(0.905)}{(0.46)^2} - \ln^2(0.905)$$

or θ is equal to 0.193 rad, or 11.04°. Then the desired root locations are

$$\lambda_{1,2} = 0.905\underline{/\pm 11.04^\circ} = 0.888 \pm j0.173$$

Hence the desired characteristic equation is given by

$$(z - 0.888 - j0.173)(z - 0.888 + j0.173) = z^2 - 1.776z + 0.819$$

From (27),

$$K_1 = 105[\lambda_1\lambda_2 - (\lambda_1 + \lambda_2) + 1.0]$$
$$= 105[0.819 - (1.776) + 1.0] = 4.52$$
$$K_2 = 14.67 - 5.34\lambda_1\lambda_2 - 5.17(\lambda_1 + \lambda_2) = 1.12$$

Suppose that we decide that the foregoing value of $\zeta = 0.46$ is satisfactory, but that a time constant of $\tau = 0.5$ s is required. This design is given as (H.W). The gains obtained from this design are $K_1 = 16.0$ and $K_2 = 3.26$.

Thus we see that an attempt to increase the speed of response of the system results in larger signals at the plant input. These larger signals may force the system into a nonlinear region of operation, which in some cases may be undesirable.

7.7 Ackermann's formula

We begin with the plant model

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}u(k)$$

The matrix polynomial $\lambda_c(\mathbf{A})$ is formed using the coefficients of the desired characteristic equation (17).

$$\lambda_c(\mathbf{A}) = \mathbf{A}^n + \lambda_{n-1}\mathbf{A}^{n-1} + \cdots + \lambda_1\mathbf{A} + \lambda_0\mathbf{I} \qquad (28)$$

Then Ackermann's formula for the gain matrix $\mathbf{K}$ is given by

$$\mathbf{K} = [0 \quad 0 \quad \cdots \quad 0 \quad 1][\mathbf{B} \quad \mathbf{AB} \quad \cdots \quad \mathbf{A}^{n-2}\mathbf{B} \quad \mathbf{A}^{n-1}\mathbf{B}]^{-1}\lambda_c(\mathbf{A}) \qquad (29)$$

The problem here is the inverse matrix in (29).

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We will solve the design in Example via Ackermann's formula. The plant model is given by

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}u(k) = \begin{bmatrix} 1 & 0.0952 \\ 0 & 0.905 \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} 0.00484 \\ 0.0952 \end{bmatrix}u(k)$$

and the desired characteristic equation is

$$\lambda_c(z) = z^2 - 1.776z + 0.819$$

Hence

$$\lambda_c(\mathbf{A}) = \begin{bmatrix} 1 & 0.0952 \\ 0 & 0.905 \end{bmatrix}^2 - 1.776\begin{bmatrix} 1 & 0.0952 \\ 0 & 0.905 \end{bmatrix} + 0.819\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= A^2 + \lambda_1 A + \lambda_0 I$$

or

$$\lambda_c(\mathbf{A}) = \begin{bmatrix} 0.043 & 0.01228 \\ 0 & 0.03075 \end{bmatrix}$$

Also,

$$\mathbf{AB} = \begin{bmatrix} 1 & 0.0952 \\ 0 & 0.905 \end{bmatrix}\begin{bmatrix} 0.00484 \\ 0.0952 \end{bmatrix} = \begin{bmatrix} 0.0139 \\ 0.0862 \end{bmatrix}$$

Thus

$$[\mathbf{B}\ \mathbf{AB}]^{-1} = \begin{bmatrix} 0.00484 & 0.0139 \\ 0.0952 & 0.0862 \end{bmatrix}^{-1} = \begin{bmatrix} -95.13 & 15.34 \\ 105.1 & -5.342 \end{bmatrix}$$

Then the gain matrix is, from (29),

$$\mathbf{K} = [0 \quad 1]\begin{bmatrix} -95.13 & 15.34 \\ 105.1 & -5.342 \end{bmatrix}\begin{bmatrix} 0.043 & 0.01228 \\ 0 & 0.03075 \end{bmatrix}$$

$$= [4.52 \quad 1.12]$$

These results are the same as those obtained in Example [P245]

A general MATLAB program to implement the calculations in this example is given by

```
PLACE  Pole placement technique

    K = PLACE(A,B,P)  computes a state-feedback matrix K such that
    the eigenvalues of  A-B*K  are those specified in vector P.
    No eigenvalue should have a multiplicity greater than the
    number of inputs.
```

Example

Assign the eigenvalues $\{0.3 \pm j0.2\}$ to the pair

$$A = \begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The pole placement command is **place**. The following example illustrates the use of the command.

```
>> A = [0, 1; 3, 4];
>> B = [0; 1];
>> poles = [0.3 + j .2, 0.3- j 0.2];
>> K = place(A, B, poles)

K = 3.1300    3.4000
```

Example

Consider the plant given by

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2} \quad \rightarrow A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

which represents a pure inertia. With two poles at the origin, this plant is unstable. Suppose we wish to design a control law for this plant so the poles have

$$\begin{aligned} \zeta &= 0.707 \\ \omega_n &= 10 \text{ rad/sec} \end{aligned}$$

which corresponds to poles at $s = -7.07 \pm j7.07$.

Pick a sample period of $T = 0.1$ second. or using program p 250 or p252 method

Let · G(z) and after that the matrices(A) and (B) are

A=Φ and B=Γ

$$A = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} T^2/2 \\ T \end{bmatrix}$$

$$\underset{\uparrow\,A}{\Phi} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}, \underset{\uparrow\,B}{\Gamma} = \begin{bmatrix} 0.005 \\ 0.1 \end{bmatrix}$$

The characterstic equation of the controlled system is given by the determinant

$$|z\mathbf{I} - \Phi + \Gamma\mathbf{K}| = 0$$

and after some algebra we obtain the symbolic characterstic equation as

$$z^2 + (0.005K_1 + 0.1K_2 - 2)z + (0.005K_1 - 0.1K_2 + 1) = 0$$

The desired pole locations in the s plane can be mapped into the z plane using $z = e^{sT}$, and we obtain

$$z = 0.3749 \pm j0.3203$$

thus the numerical form of the desired characteristic polynomial is

$$\alpha_c(z) = z^2 - 0.7499z + 0.2432$$

Equating coefficients of like powers of z of both characteristic polynomials gives the two simultaneous equations

$$\begin{aligned} 0.005K_1 + 0.1K_2 &= 1.2501 \\ 0.005K_1 - 0.1K_2 &= -0.7568 \end{aligned}$$

from which we obtain

$$\begin{aligned} K_1 &= 49.33 \\ K_2 &= 10.03 \end{aligned}$$

Example

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 + 1}$$

with differential equation $\ddot{y} + y = u$. Using state vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$$

we have plant matrices

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0.$$

Specifications. Increase the damping to critical, keeping the same natural frequency of 1 rad/sec. Thus the desired pole locations will be

$$s = -1, -1$$

Hence $T = 1$.

Write a MATLAB program for pole-placement or using P252

Plant Discretization. Using this T, the continuous plant can be defined, then discretized using MATLAB:

```
>> A = [0 1;-1 0];
>> B = [0;1];
>> C = [1 0];
>> D = 0;
>>
>> plant_c = ss(A,B,C,D);
>> T = 1;
>> plant_d = c2d(plant_c,T)

a =
              x1        x2
   x1     0.5403    0.8415
   x2    -0.8415    0.5403

b =
            u1
   x1   0.4597
   x2   0.8415

c =
        x1  x2
   y1    1   0

d =
        u1
   y1    0

Sampling time: 1
Discrete-time model.
```

Note that the state-space matrices can be extracted from LTI model plant_d by:

A

```
>> Phi = plant_d.A

Phi =

    0.5403    0.8415
   -0.8415    0.5403
```

Control Law. The desired z-plane pole locations are found using the mapping $z = e^{sT}$.

```
>>control_s_poles = [-1 -1]';
>>control_z_poles = exp(control_s_poles*T)

control_z_poles = 0.3679
                  0.3679
```

It is a numerical anomaly of MATLAB function place that poles of multiplicity greater than the number of inputs are not allowed. One pole can therefore be offset slightly, and the gain vector K is

```
>>control_z_poles(1) = control_z_poles(1)+0.0001;
>>K = place(Phi,Gamma,control_z_poles)
place: ndigits= 19

K = -0.5655    0.7186
```

B

To change one root to $0.3679 + \underbrace{0.0001}_{\approx 0} = \underbrace{0.3680}_{\text{differ from } 0.3679}$

Example

Design a control law for the satellite attitude-control system described by P(248). Pick the z-plane roots of the closed-loop characteristic equation so that the equivalent s-plane roots have a damping ratio of $\zeta = 0.5$ and real part of $s = -1.8$ rad/sec (i.e., $s = -1.8 \pm j3.12$ rad/sec). Use a sample period of $T = 0.1$ sec.

→ P248

Solution. Example showed that the discrete model for this system is

$$\Phi = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \Gamma = \begin{bmatrix} T^2/2 \\ T \end{bmatrix}.$$

Using $z = e^{sT}$ with a sample period of $T = 0.1$ sec, we find that $s = -1.8 \pm j3.12$ rad/sec translates to $z = 0.8 \pm j0.25$. The desired characteristic equation is then

$$z^2 - 1.6z + 0.70 = 0. \qquad (*)$$

and the evaluation of Eq. (11) for any control law **K** leads to

$$\left| z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} T^2/2 \\ T \end{bmatrix} [K_1 \quad K_2] \right| = 0$$

or

$$z^2 + (TK_2 - (T^2/2)K_1 - 2)z + (T^2/2)K_1 - TK_2 + 1 = 0. \qquad (**)$$

Equating coefficients in Eqs. (*) and (**) with like powers of z, we obtain two simultaneo equations in the two unknown elements of **K**

$$TK_2 + (T^2/2)K_1 - 2 = -1.6.$$
$$(T^2/2)K_1 - TK_2 + 1 = 0.70.$$

which are easily solved for the coefficients and evaluated for $T = 0.1$ sec

$$K_1 = \frac{0.10}{T^2} = 10. \qquad K_2 = \frac{0.35}{T} = 3.5.$$

Or using MATLAB

P251

Design a control law for the satellite attitude-control system as in Example . Place the z-plane closed-loop poles at $z = 0.8 \pm j0.25$.

Solution. The MATLAB statements

```
T = .1
Phi = [1 T;0 1]
Gam = [T^2/2;T]
p = [.8+i*.25;.8-i*.25]
K = acker(Phi,Gam,p)
```

A → Phi

B → Gam

acker → ackerman's Formula

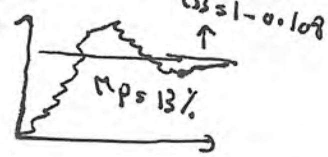
$\Rightarrow K_1 = 10.25$

$K_2 = 3.4875$

```
>> Af = A - B*K
>> [n,d] = ss2tf(Af, b, c, 0)
>> [nc, dc] = cloop(n, d, 1)
```

$$\frac{nc}{dc} = \frac{0.005z + 0.005}{z^2 - 1.6z + 0.7}$$

```
>> dstep(nc, dc)
```



For T.R

Syscl=feedback(sysd*K,1)

```
Xo = [1;0;0;0]
y = initial(sysCL,Xo)
```

$[n, d] = ss2tf[A, B \quad]$

$sysd = tf(n, d)$

7.8 How to obtain digital state-space equation from analog state-space equation .

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad (s\mathbf{I}-\mathbf{A}) = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix} \qquad (s\mathbf{I}-\mathbf{A})^{-1} = \frac{1}{(s+1)(s+2)}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$\phi(t) = \mathrm{L}^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\right] = \begin{bmatrix} 2e^{-t}-e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t} \end{bmatrix} \qquad \phi(T) = \begin{bmatrix} 2e^{-T}-e^{-2T} & e^{-T}-e^{-2T} \\ -2e^{-T}+2e^{-2T} & -e^{-T}+2e^{-2T} \end{bmatrix}$$

$$T = 1 \text{ sec.} \qquad \phi(T) = \begin{bmatrix} 0.6 & 0.2325 \\ -0.465 & -0.0972 \end{bmatrix}$$

where ϕ(T)=A in our lecture

digital

$$\theta(T) = \left[\int_0^T \phi(\tau)d\tau\right]B = \int_0^T \begin{bmatrix} e^{-\tau} - e^{-2\tau} \\ -e^{-\tau} + 2e^{-2\tau} \end{bmatrix} = \begin{bmatrix} -e^{-\tau} + e^{-2\tau}/2 \\ +e^{-\tau} - e^{-2\tau} \end{bmatrix}_0^T$$

$$\Rightarrow \begin{bmatrix} -e^{-T} + \frac{e^{-2T}}{2} - (-1 + 0.5) \\ e^{-T} - e^{-2T} - (1 - 1) \end{bmatrix}$$

$$T = 1 \text{ sec.} \qquad \theta(T) = \begin{bmatrix} 0.1998 \\ 0.2325 \end{bmatrix}$$

digital

Where Θ(T)=B in our lecture