# Example

Consider the system

where

$$A \searrow \mathbf{G} = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix}, \quad B \searrow \mathbf{H} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Determine a suitable state feedback gain matrix $\mathbf{K}$ such that the system will have the closed-loop poles at

$$z = 0.5 + j0.5, \qquad z = 0.5 - j0.5$$

**First method to find (k1) and (k2)**

$$|z\mathbf{I} - \mathbf{G} + \mathbf{HK}| = \left| \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [k_1 \quad k_2] \right|$$

$$= \begin{vmatrix} z & -1 \\ 0.16 + k_1 & z + 1 + k_2 \end{vmatrix}$$

$$= z^2 + (1 + k_2)z + 0.16 + k_1 = 0$$

Now we equate this characteristic equation with the desired characteristic equation

$$(z - 0.5 - j0.5)(z - 0.5 + j0.5) = 0$$

so that

$$z^2 + (1 + k_2)z + 0.16 + k_1 = z^2 - z + 0.5$$

By comparing the coefficients of equal powers of $z$, we obtain

$$1 + k_2 = -1, \qquad 0.16 + k_1 = 0.5$$

from which we get

$$k_1 = 0.34, \qquad k_2 = -2$$

Thus, the desired state feedback gain matrix $\mathbf{K}$ is given by

$$\mathbf{K} = [k_1 \quad k_2] = [0.34 \quad -2]$$

**Second method to find (k1) and (k2)**

Note that → To apply the procedure of p 259

$$|z\mathbf{I} - \mathbf{G}| = \begin{vmatrix} z & -1 \\ 0.16 & z+1 \end{vmatrix} = z^2 + z + 0.16$$

Hence,

$$a_1 = 1, \qquad a_2 = 0.16$$

Determine a suitable state feedback gain matrix **K** such that the system will have the closed-loop poles at

$$z = 0.5 + j0.5, \qquad z = 0.5 - j0.5$$

Let us first examine the rank of the controllability matrix. The rank of

$$\underset{B}{[\mathbf{H}} \vdots \underset{AB}{\mathbf{GH}]} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

is 2. Thus, the system is completely state controllable, and therefore arbitrary pole placement is possible. The characteristic equation for the desired system is

$$|z\mathbf{I} - \mathbf{G} + \mathbf{HK}| = (z - 0.5 - j0.5)(z - 0.5 + j0.5) = z^2 - z + 0.5 = 0$$

Hence,

$$\alpha_1 = -1, \qquad \alpha_2 = 0.5$$

Hence,

$$\mathbf{K} = [\alpha_2 - a_2 \vdots \alpha_1 - a_1] = [0.5 - 0.16 \vdots -1 - 1]$$
$$= [0.34 \quad -2]$$

**Third method to find (k1) and (k2)**

Referring to Ackermann's formula given by Equation **(29)**, we have

$$\mathbf{K} = [0 \quad 1][\mathbf{H} \vdots \mathbf{GH}]^{-1}\underset{\lambda_c(A)}{\phi(\mathbf{G})}$$

$\because \underset{\text{desired}}{\lambda_c(z)} = z^2 - z + 0.5$

$\hookrightarrow = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix}^2 - \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix} + 0.5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ I

Thus,

$$\mathbf{K} = [0 \quad 1]\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 0.34 & -2 \\ 0.32 & 2.34 \end{bmatrix}$$
$$= [0.34 \quad -2]$$

or using MATLAB

# Example

This example the same as p269, but Find < P.T.F after design, write MATLAB program For T.R

Consider the system defined by

where

$$\mathbf{G} = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0]$$

Design a control system such that the desired closed-loop poles of the characteristic equation are at

$$z_1 = 0.5 + j0.5, \qquad z_2 = 0.5 - j0.5$$

Thus, the desired characteristic polynomial is given by

$$\begin{aligned} |z\mathbf{I} - \mathbf{G} + \mathbf{HK}| &= (z - 0.5 - j0.5)(z - 0.5 + j0.5) \\ &= z^2 - z + 0.5 \end{aligned}$$

The state feedback gain matrix K can be determined as

$$\mathbf{K} = [0.34 \quad -2]$$

Using this K matrix, the state equation becomes [after design]

$$\begin{aligned} \mathbf{x}(k+1) &= (\mathbf{G} - \mathbf{HK})\mathbf{x}(k) + \mathbf{H}K_0 r(k) \\ &= \hat{\mathbf{G}}\mathbf{x}(k) + \hat{\mathbf{H}}r(k) \end{aligned}$$

where

$$\hat{\mathbf{G}} = \mathbf{G} - \mathbf{HK}, \qquad \hat{\mathbf{H}} = \mathbf{H}K_0$$

The gain constant $K_0$ can be determined in state space or can be determined in the $z$ plane using the pulse transfer function. In this example, we shall use the latter approach.

The pulse transfer function $Y(z)/R(z)$ for this system is given by

$$G(z) = \mathbf{C}(z\mathbf{I} - \hat{\mathbf{G}})^{-1}\hat{\mathbf{H}}$$

where

$$\hat{\mathbf{G}} = \mathbf{G} - \mathbf{HK} = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}[0.34 \quad -2] = \begin{bmatrix} 0 & 1 \\ -0.5 & 1 \end{bmatrix}$$

$$\hat{\mathbf{H}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} K_0 = \begin{bmatrix} 0 \\ K_0 \end{bmatrix}$$

Hence,

$$G(z) = [1 \quad 0]\begin{bmatrix} z & -1 \\ 0.5 & z-1 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ K_0 \end{bmatrix}$$

$$= \frac{K_0}{z^2 - z + 0.5}$$

Thus,

P.T.F

$$\frac{Y(z)}{R(z)} = G(z) = \frac{K_0}{z^2 - z + 0.5}$$

To determine gain constant $K_0$, we use the condition that the steady-state output $y(\infty)$ for the unit-step input is unity, or

$$\lim_{k\to\infty} y(k) = \lim_{z\to 1}(1 - z^{-1})Y(z)$$

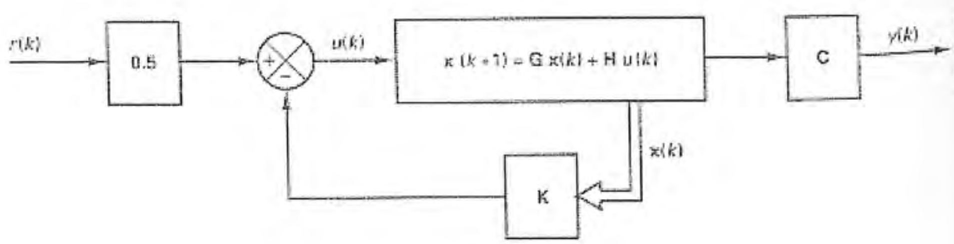
$$= \lim_{z\to 1}\frac{z-1}{z}\,\frac{K_0}{z^2 - z + 0.5}\,\frac{z}{z-1}$$

$$= 2K_0 = 1$$

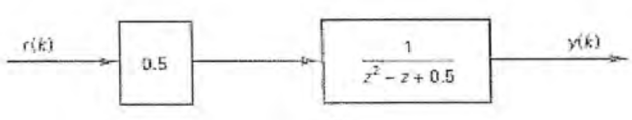
Hence, we have determined $K_0$ as

$$K_0 = 0.5$$

Figure shows block diagrams of the designed system. The unit-step response of this system can be obtained easily by use of MATLAB. MATLAB Program is a sample program for obtaining the unit-step response. Figure shows the resulting response curve.




(a)

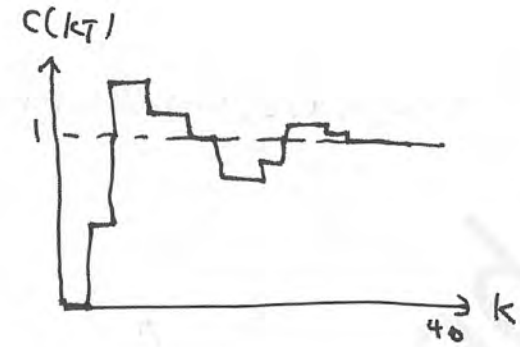


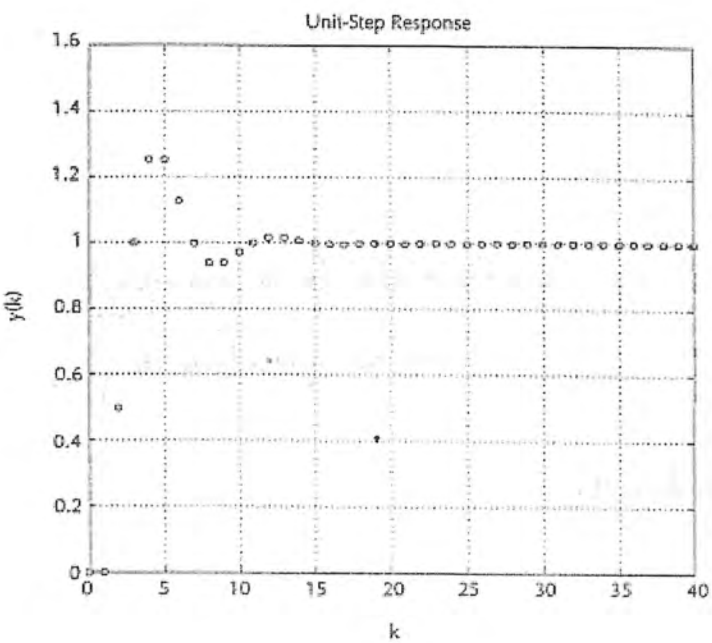

(b)

The MATLAB program is:-

```
num = [0  0   0.5];
den = [1  -1   0.5];
r = ones(1,41);
k = 0: 40;
y = filter(num,den,r);
plot(k,y,'o')
v = [0  40  0  1.6];
axis(v);
grid
title('Unit-Step Response')
xlabel('k')
ylabel('y(k)')
```

>> dstep(n, d) →



## Example

Design F/B, variable state-space will exhibit deadbeat response.

Consider the system

and

$$G = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.5 & -0.2 & 1.1 \end{bmatrix}, \quad H = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = [0 \ \ 1 \ \ 0]$$

B

Then

$$|zI - G + HK| = \begin{vmatrix} z & -1 & 0 \\ 0 & z & -1 \\ k_1 + 0.5 & k_2 + 0.2 & z + k_3 - 1.1 \end{vmatrix}$$
$$= z^3 + (k_3 - 1.1)z^2 + (k_2 + 0.2)z + k_1 + 0.5 = 0$$

By equating this characteristic equation with the desired characteristic equation (for deadbeat response),

$$z^3 = 0$$

we obtain

$$K = [k_1 \ \ k_2 \ \ k_3] = [-0.5 \ \ -0.2 \ \ 1.1]$$

→ Since the G is Controllable Canonical Form

We can use procedure of p259

or $X(k) = 0$ , $k = 3, 4, 5, \ldots$

clearly, the response is deadbeat.

## Example

Consider the system defined by

$$G(z) = \frac{z - 1}{z^2 + z + 0.16}$$

Referring to Section 7.2 obtain state-space representations for this system in the following two different forms:

1. Controllable canonical form
2. Observable canonical form

**Solution**

1. *Controllable canonical form.*
we obtain

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [1 \quad 1] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

2. *Observable canonical form.*
we obtain

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & -0.16 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [0 \quad 1] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

# Example

Consider the following pulse-transfer-function system:

$$\frac{Y(z)}{U(z)} = \frac{z^{-1}(1 + 0.8z^{-1})}{1 + 1.3z^{-1} + 0.4z^{-2}}$$

A state-space representation for this system can be given by

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.4 & -1.3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \qquad (a)$$

$$y(k) = [0.8 \quad 1] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \qquad (b)$$

A different state-space representation for the same system can be given by

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & -0.4 \\ 1 & -1.3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.8 \\ 1 \end{bmatrix} u(k) \qquad (c)$$

$$y(k) = [0 \quad 1] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \qquad (d)$$

Show that the state-space representation defined by Equations (a) and (b) gives a system which is state controllable but not observable. Show, on the other hand, that the state-space representation defined by Equations (c) and (d) gives a system that is not completely state controllable but is observable. Explain what causes the apparent difference in the controllability and observability of the same system.

**Solution** Consider the discrete-time control system defined by Equations (a) and (b). The rank of the controllability matrix

$$\underset{B}{[H} \vdots G\underset{B}{H]} = \begin{bmatrix} 0 & 1 \\ 1 & -1.3 \end{bmatrix}$$

is 2. Hence, the system is completely state controllable. The rank of the observability matrix

$$[C^* \vdots G^*C^*] = \begin{bmatrix} 0.8 & -0.4 \\ 1 & -0.5 \end{bmatrix}$$

is 1. Hence, the system is not observable.

Next, consider the system defined by Equations (c) and (d). The rank of the controllability matrix

$$[H \vdots GH] = \begin{bmatrix} 0.8 & -0.4 \\ 1 & -0.5 \end{bmatrix}$$

is 1. Hence, the system is not completely state controllable. The rank of the observability matrix

$$[C^* \vdots G^*C^*] = \begin{bmatrix} 0 & 1 \\ 1 & -1.3 \end{bmatrix}$$

is 2. Hence, the system is observable.

The apparent difference in the controllability and observability of the same system is caused by the fact that the original system has a pole–zero cancellation in the pulse transfer function:

$$\frac{Y(z)}{U(z)} = \frac{z + 0.8}{z^2 + 1.3z + 0.4} = \frac{z + 0.8}{(z + 0.8)(z + 0.5)}$$

## Note that:-

To prove that the P.T.F in state-space gives the same result of the general step Z-T .consider this example:-

$$G(s) = \frac{1}{s(s+2)} \quad , \text{ let } T = 1 \text{ sec.}$$

>> n =

>> d =

in Continuous state

⇓

↳ Convert to state-space, Find A, B, C, D

after that use the MATLAB program like p25.

or method p25/2 to get the discrete state-space equation.

## Example [ From P276 ]

When the sampling period is 1 sec, or $T = 1$, the discrete-time state equation and the output equation become, respectively,

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.4323 \\ 0 & 0.1353 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.2838 \\ 0.4323 \end{bmatrix} u(k)$$

and

$$y(k) = [1 \quad 0] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

The pulse-transfer-function representation of this system can be obtained from Equation ( X ), as follows: See P 231

$$\begin{aligned} F(z) &= \mathbf{C}(z\mathbf{I} - \mathbf{G})^{-1}\mathbf{H} + D \\ &= [1 \quad 0] \begin{bmatrix} z-1 & -0.4323 \\ 0 & z-0.1353 \end{bmatrix}^{-1} \begin{bmatrix} 0.2838 \\ 0.4323 \end{bmatrix} + 0 \\ &= [1 \quad 0] \begin{bmatrix} \dfrac{1}{z-1} & \dfrac{0.4323}{(z-1)(z-0.1353)} \\ 0 & \dfrac{1}{z-0.1353} \end{bmatrix} \begin{bmatrix} 0.2838 \\ 0.4323 \end{bmatrix} \\ &= \frac{0.2838z + 0.1485}{(z-1)(z-0.1353)} \\ &= \frac{0.2838z^{-1} + 0.1485z^{-2}}{(1-z^{-1})(1-0.1353z^{-1})} \end{aligned}$$

Note that the same pulse transfer function can be obtained by taking the $z$ transform of $G(s)$ when it is preceded by a sampler and zero-order hold. Assuming $T = 1$, we obtain

$$\begin{aligned} G(z) &= \mathcal{Z}\left[\frac{1-e^{-Ts}}{s} \frac{1}{s(s+2)}\right] = (1-z^{-1})\mathcal{Z}\left[\frac{1}{s^2(s+2)}\right] \\ &= (1-z^{-1})\mathcal{Z}\left[\frac{0.5}{s^2} - \frac{0.25}{s} + \frac{0.25}{s+2}\right] \\ &= (1-z^{-1})\left[\frac{0.5z^{-1}}{(1-z^{-1})^2} - \frac{0.25}{1-z^{-1}} + \frac{0.25}{1-0.1353z^{-1}}\right] \\ &= \frac{0.2838z^{-1} + 0.1485z^{-2}}{(1-z^{-1})(1-0.1353z^{-1})} \end{aligned}$$

## 7.10 How to obtain the state transition matrix $\Psi(k)$ $G^k$, state x(k) and the output y(k)

*z Transform Approach to the Solution of Discrete-Time State Equations.* We next present the solution of a discrete-time state equation by the $z$ transform method. Consider the discrete-time system described by Equation

$$\mathbf{x}(k+1) = \mathbf{G}\mathbf{x}(k) + \mathbf{H}\mathbf{u}(k) \qquad (30)$$

Taking the $z$ transform of both sides of Equation (30), we get

$$z\mathbf{X}(z) - z\mathbf{x}(0) = \mathbf{G}\mathbf{X}(z) + \mathbf{H}\mathbf{U}(z)$$

where $\mathbf{X}(z) = \mathcal{Z}[\mathbf{x}(k)]$ and $\mathbf{U}(z) = \mathcal{Z}[\mathbf{u}(k)]$. Then

$$(z\mathbf{I} - \mathbf{G})\mathbf{X}(z) = z\mathbf{x}(0) + \mathbf{H}\mathbf{U}(z)$$

Premultiplying both sides of this last equation by $(z\mathbf{I} - \mathbf{G})^{-1}$, we obtain

$$\mathbf{X}(z) = (z\mathbf{I} - \mathbf{G})^{-1} z\mathbf{x}(0) + (z\mathbf{I} - \mathbf{G})^{-1}\mathbf{H}\mathbf{U}(z) \qquad (31)$$

Taking the inverse $z$ transform of both sides of Equation (31) gives

$$\mathbf{x}(k) = \mathcal{Z}^{-1}[(z\mathbf{I} - \mathbf{G})^{-1} z]\mathbf{x}(0) + \mathcal{Z}^{-1}[(z\mathbf{I} - \mathbf{G})^{-1}\mathbf{H}\mathbf{U}(z)]$$

[illegible], we obtain

$$\Psi(k) = \mathbf{G}^k = \mathcal{Z}^{-1}[(z\mathbf{I} - \mathbf{G})^{-1} z] \qquad (32) \ **$$

State-transition matrix

and

$$\sum_{j=0}^{k-1} \mathbf{G}^{k-j-1}\mathbf{H}\mathbf{u}(j) = \mathcal{Z}^{-1}[(z\mathbf{I} - \mathbf{G})^{-1}\mathbf{H}\mathbf{U}(z)] \qquad (33)$$

where $k = 1, 2, 3, \ldots$.

# Example

Obtain the state transition matrix of the following discrete-time system:

$$\mathbf{x}(k+1) = \mathbf{G}\mathbf{x}(k) + \mathbf{H}u(k)$$

$$y(k) = \mathbf{C}\mathbf{x}(k)$$

where

$$\overset{A}{\mathbf{G}} = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix}, \quad \overset{B}{\mathbf{H}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0]$$

Then obtain the state $\mathbf{x}(k)$ and the output $y(k)$ when the input $u(k) = 1$ for $k = 0, 1, 2, \ldots$. Assume that the initial state is given by

$$\mathbf{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

↳ step input

From Equations (32) → the state transition matrix $\Psi(k)$ is

$$\Psi(k) = \mathbf{G}^k = \mathcal{Z}^{-1}[(z\mathbf{I} - \mathbf{G})^{-1} z]$$

Therefore, we first obtain $(z\mathbf{I} - \mathbf{G})^{-1}$:

$$(z\mathbf{I} - \mathbf{G})^{-1} = \begin{bmatrix} z & -1 \\ 0.16 & z+1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \dfrac{z+1}{(z+0.2)(z+0.8)} & \dfrac{1}{(z+0.2)(z+0.8)} \\ \dfrac{-0.16}{(z+0.2)(z+0.8)} & \dfrac{z}{(z+0.2)(z+0.8)} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\frac{4}{3}}{z+0.2} + \dfrac{-\frac{1}{3}}{z+0.8} & \dfrac{\frac{5}{3}}{z+0.2} + \dfrac{-\frac{5}{3}}{z+0.8} \\ \dfrac{-\frac{0.8}{3}}{z+0.2} + \dfrac{\frac{0.8}{3}}{z+0.8} & \dfrac{-\frac{1}{3}}{z+0.2} + \dfrac{\frac{4}{3}}{z+0.8} \end{bmatrix}$$

The state transition matrix $\Psi(k)$ is now obtained as follows:

$$\Psi(k) = \mathbf{G}^k = \mathcal{Z}^{-1}[(z\mathbf{I} - \mathbf{G})^{-1} z]$$

$$= \mathcal{Z}^{-1}\begin{bmatrix} \dfrac{4}{3}\dfrac{z}{z+0.2} - \dfrac{1}{3}\dfrac{z}{z+0.8} & \dfrac{5}{3}\dfrac{z}{z+0.2} - \dfrac{5}{3}\dfrac{z}{z+0.8} \\ -\dfrac{0.8}{3}\dfrac{z}{z+0.2} + \dfrac{0.8}{3}\dfrac{z}{z+0.8} & -\dfrac{1}{3}\dfrac{z}{z+0.2} + \dfrac{4}{3}\dfrac{z}{z+0.8} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{3}(-0.2)^k - \frac{1}{3}(-0.8)^k & \frac{5}{3}(-0.2)^k - \frac{5}{3}(-0.8)^k \\ -\frac{0.8}{3}(-0.2)^k + \frac{0.8}{3}(-0.8)^k & -\frac{1}{3}(-0.2)^k + \frac{4}{3}(-0.8)^k \end{bmatrix}$$

Equation [illegible] gives the state transition matrix.

→ but First compute X(Z)

Next, compute $\mathbf{x}(k)$. The $z$ transform of $\mathbf{x}(k)$ is given by

$$\mathcal{Z}[\mathbf{x}(k)] = \mathbf{X}(z) = (z\mathbf{I} - \mathbf{G})^{-1} z\mathbf{x}(0) + (z\mathbf{I} - \mathbf{G})^{-1}\mathbf{H}U(z)$$

$$= (z\mathbf{I} - \mathbf{G})^{-1}[z\mathbf{x}(0) + \mathbf{H}U(z)] \longrightarrow \text{From equ (31)}$$

Since

$$U(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}$$

we obtain

$$z\mathbf{x}(0) + \mathbf{H}U(z) = \begin{bmatrix} z \\ -z \end{bmatrix} + \begin{bmatrix} \dfrac{z}{z-1} \\ \dfrac{z}{z-1} \end{bmatrix} = \begin{bmatrix} \dfrac{z^2}{z-1} \\ \dfrac{-z^2+2z}{z-1} \end{bmatrix}$$

Hence

$$\mathbf{X}(z) = (z\mathbf{I} - \mathbf{G})^{-1}[z\mathbf{x}(0) + \mathbf{H}U(z)]$$

$$= \begin{bmatrix} \dfrac{(z^2+2)z}{(z+0.2)(z+0.8)(z-1)} \\ \dfrac{(-z^2+1.84z)z}{(z+0.2)(z+0.8)(z-1)} \end{bmatrix}$$

$$
= \begin{bmatrix} \dfrac{-\frac{17}{6}z}{z+0.2} + \dfrac{\frac{22}{9}z}{z+0.8} + \dfrac{\frac{25}{18}z}{z-1} \\[2ex] \dfrac{\frac{3.4}{6}z}{z+0.2} + \dfrac{-\frac{17.6}{9}z}{z+0.8} + \dfrac{\frac{7}{18}z}{z-1} \end{bmatrix}
$$

Thus, the state vector $\mathbf{x}(k)$ is given by

$$
\mathbf{x}(k) = \mathcal{Z}^{-1}[\mathbf{X}(z)] = \begin{bmatrix} -\frac{17}{6}(-0.2)^k + \frac{22}{9}(-0.8)^k + \frac{25}{18} \\ \frac{3.4}{6}(-0.2)^k - \frac{17.6}{9}(-0.8)^k + \frac{7}{18} \end{bmatrix}
$$

Finally, the output $y(k)$ is obtained as follows:

$$
y(k) = \mathbf{C}\mathbf{x}(k) = [1 \quad 0]\begin{bmatrix} -\frac{17}{6}(-0.2)^k + \frac{22}{9}(-0.8)^k + \frac{25}{18} \\ \frac{3.4}{6}(-0.2)^k - \frac{17.6}{9}(-0.8)^k + \frac{7}{18} \end{bmatrix}
$$

$$
= -\frac{17}{6}(-0.2)^k + \frac{22}{9}(-0.8)^k + \frac{25}{18}
$$

# Example

# Write a MATLAB program:-

Consider the following example: If the continuous-time system is given by

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -25 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u \qquad (*)
$$

then, assuming the sampling period to be 0.05 sec, we obtain **G** and **H** as follows:

```
A = [0   1;-25   -4];
B = [0;1];
[G,H] = c2d(A,B,0.05)

G =

      0.9709      0.0448
     -1.1212      0.7915

H =

      0.0012
      0.0448
```

Note that the state matrix **G** and input matrix **H** of the discrete-time state-space equation

$$
\mathbf{x}(k+1) = \mathbf{G}\mathbf{x}(k) + \mathbf{H}u(k)
$$

depend on the sampling period $T$. For example, consider discretization of the continuous-time system given by Equation $(*)$ with two more different sampling periods: $T = 0.2$ sec and $T = 1$ sec. As seen in the previous and following MATLAB outputs, a set of matrices **G** and **H** differs for a different sampling period $T$.

```
A = [0  1;-25  -4];
B = [0;1];
[G,H] = c2d(A,B,0.2)

G =

     0.6401    0.1161
    -2.9017    0.1758

H =

     0.0144
     0.1161
```

```
A = [0  1;-25  -4];
B = [0;1];
[G,H] = c2d(A,B,1)

G =

    -0.0761   -0.0293
     0.7321    0.0410

H =

     0.0430
    -0.0293
```

# Example

As another example, consider the following system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 20.601 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.4905 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0.5 \end{bmatrix}$$

Assuming that the sampling period $T$ is 0.05 sec and without specifying the format, we get the following discrete-time state equation:

$$\mathbf{x}(k+1) = \mathbf{G}\mathbf{x}(k) + \mathbf{H}u(k)$$

where matrices $\mathbf{G}$ and $\mathbf{H}$ can be found in the following computer output:

```
A = [0        1  0  0;
     20.601   0  0  0;
     0        0  0  1;
     -0.4905  0  0  0];
B = [0;-1;0;0.5];
[G,H] = c2d(A,B,0.05)

G =

     1.0259    0.0504         0         0
     1.0389    1.0259         0         0
    -0.0006   -0.0000    1.0000    0.0500
    -0.0247   -0.0006         0    1.0000

H =

    -0.0013
    -0.0504
     0.0006
     0.0250
```

## H.W

**1-**design full-state f/b controller for

$$A = \begin{bmatrix} 1.5 & -0.5 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Such that the closed-loop poles in S-domain at $s12 = -2 \pm 2j$. let T= 0.2 sec.

**2-** design full-state f/b controller for

$$A = \begin{bmatrix} 0 & 0.1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0.005 \\ 0.1 \end{bmatrix}$$

If T=0.1sec and design requirements are 1-Wn=10rad/sec , 2- $\varsigma = 0.707$

**3-** design full-state f/b controller for

$$A = \begin{bmatrix} 1 & 0.0952 \\ 0 & 0.905 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0.00484 \\ 0.0952 \end{bmatrix}$$

If $Q(z) = (z-a)(z-b)$

And T=0.1sec, also calculate the c/cs Q(z) if k1=1 and k2=0

**4-** design full-state f/b controller for

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

If 1- $\varsigma = 0.8$ , 2- Ts=1 sec and let the 3rd root is ($\varsigma Wn$).

**5-** design full-state f/b controller for

$$G(z) = \frac{1}{z^2}$$

That yields closed-loop poles with $\varsigma = 0.8$ and Wn=3rad/sec. If T=0.1sec

**6-** design full-state f/b controller for

$$G(z) = \frac{1}{z^2 + 3z + 26}$$

If Mp=1% and Ts=2sec. let T=0.1sec

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