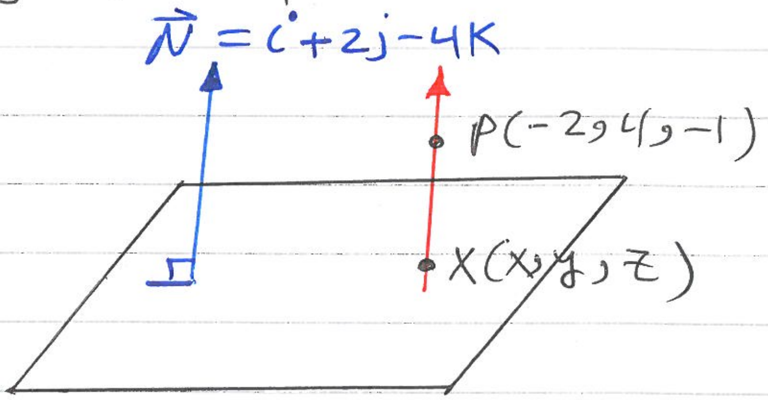
Lines and planes in Space:

EX1: (a) Find the Parametric eq. of the line passing through the point $P(-2, 4, -1)$ which is perpendicular to the plane $x+2y-4z=4$



$\vec{PX} \parallel \vec{N}$ ((perpendicular tho the Plane))

$$\frac{x-x_1}{v_1} = \frac{y-y_1}{v_2} = \frac{z-z_1}{v_3}$$

$$\frac{x-x_1}{n_1} = \frac{y-y_1}{n_2} = \frac{z-z_1}{n_3}$$

$$\frac{x+2}{1} = \frac{y-4}{2} = \frac{z+1}{-4}$$

$x+2 = c \longrightarrow x = -2 + c$

$y-4 = 2c \longrightarrow y = 4 + 2c$

$z+1 = -4c \longrightarrow z = -1 - 4c$

(b) If L is the line found in (a), Find the point of intersection of L with the plane $x + y - z = 14$

(1) $x = -2 + c$
$y = 4 + 2c$
$z = -1 - 4c$

(2) $x + y - z = 14$

The point of intersection satisfies both of the eqs.

(1) $\longrightarrow$ (2)

$$(-2 + c) + (4 + 2c) - (-1 - 4c) = 14$$

$$-2 + c + 4 + 2c + 1 + 4c = 14$$

$$7c = 11 \longrightarrow c = 11/7$$

$$x = -2 + \frac{11}{7} \longrightarrow x = -\frac{3}{7}$$

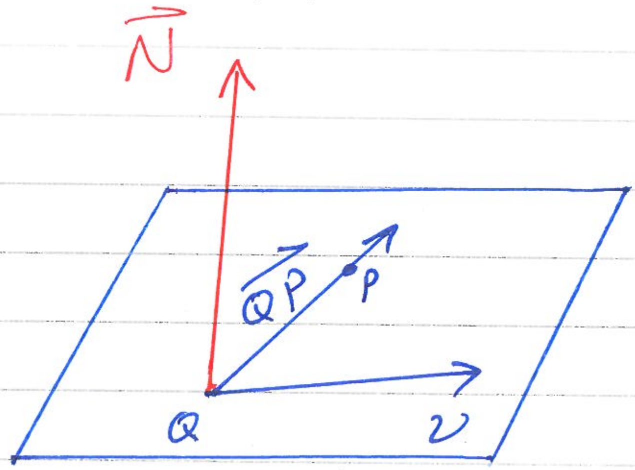
$$y = 4 + 2\frac{11}{7} \longrightarrow y = \frac{40}{7}$$

$$z = -1 - 4\frac{11}{7} \longrightarrow z = \frac{-51}{7}$$

the intersection point $\left(-\frac{3}{7}, \frac{40}{7}, \frac{-51}{7}\right)$

EX2: A line L has a parametric eqs $x = 1 + 2c$, $y = 2 - c$, $z = 3 + c$, and has a point P has coordinades $(-3, 1, -2)$.

ⓐ Find the eq of the plane contaning line L and point P.



$$\frac{x-1}{2} = \frac{y-2}{-1} = \frac{z-3}{1}$$

$$\vec{v} = 2i - j + k$$

Point $Q(1, 2, 3)$

$$\vec{QP} = (-3-1)i + (1-2)j + (-2-3)k$$

$$\vec{QP} = -4i - j - 5k$$

$$\vec{N} = \vec{v} \times \vec{QP} = \begin{vmatrix} i & j & k \\ 2 & -1 & 1 \\ -4 & -1 & -5 \end{vmatrix}$$

$$\vec{N} = + \begin{vmatrix} -1 & 1 \\ -1 & -5 \end{vmatrix} i - \begin{vmatrix} 2 & 1 \\ -4 & -5 \end{vmatrix} j + \begin{vmatrix} 2 & -1 \\ -4 & -1 \end{vmatrix} k$$

$$\vec{N} = 6i + 6j - 6k$$

$$n_1 = 6, \quad n_2 = 6, \quad n_3 = -6$$

$$n_1 x + n_2 y + n_3 z = D$$

$$6x + 6y - 6z = D$$

$$D = n_1 x_1 + n_2 y_1 + n_3 z_1$$

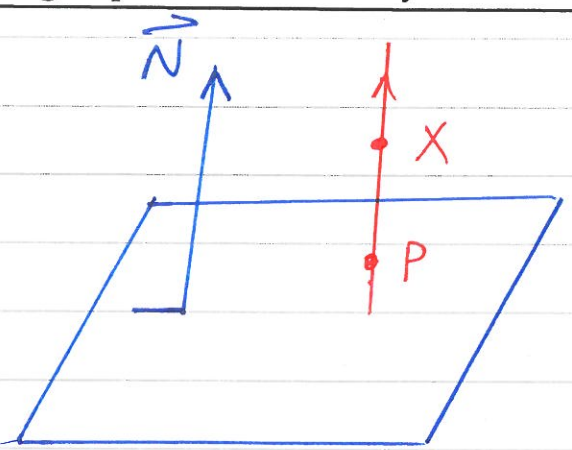
$$D = 6(-3) + 6(1) + (-6)(-2)$$

$$D = -18 + 6 + 12 = 0$$

$$6x + 6y - 6z = 0$$

or $\boxed{x + y - z = 0}$ eqn of plane

(b) Find the parametric eqs for the line through P normal to the plane in (a)

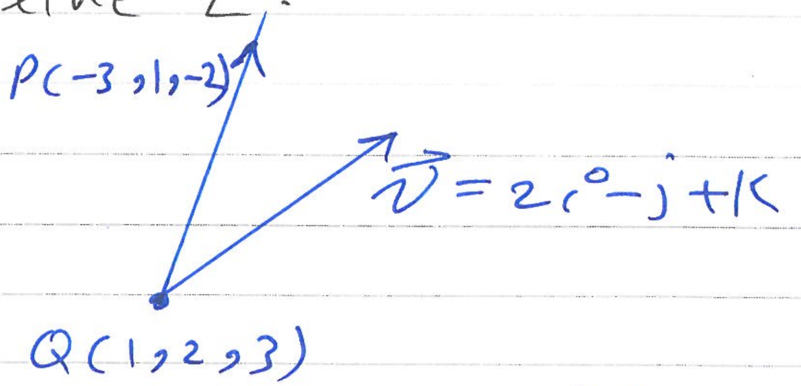


$\vec{PX} \parallel \vec{N}$

$$\frac{x+3}{6} = \frac{y-1}{6} = \frac{z+2}{-6}$$

$x = -3 + 6c$
$y = 1 + 6c$
$z = -2 - 6c$

(c) Find the distance from point P to the line L.



$$D = |\vec{QP}| \sin\theta = \frac{|\vec{PQ} \times \vec{v}|}{|\vec{v}|}$$

$$D = \frac{\sqrt{6^2+6^2+(-6)^2}}{\sqrt{2^2+(-1)^2+1^2}} = 3\sqrt{2}$$

EX3: Show that the lines

$$\frac{x-1}{-4} = \frac{y-2}{3} = \frac{z-4}{-2}$$

and

$$\frac{x-2}{-1} = \frac{y-1}{1} = \frac{z+2}{6}$$

intersect and find the equation of the plane they determine.

L_1: $x = 1-4c$, $y = 2+3c$, $z = 4-2c$

L_2: $x = 2-t$, $y = 1+t$, $z = -2+6t$

setting the x, y and z equal to each other we get

$$1-4c = 2-t \quad (1)$$
$$2+3c = 1+t \quad (2)$$
$$4-2c = -2+6t \quad (3)$$

from (1) and (2) we get

$c = 0$

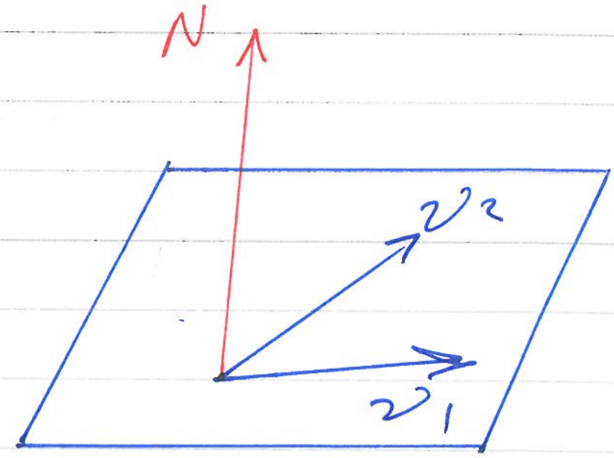
$t = 1$

putting $c=0$ and $t=1$ in eq.(3)

$$4-2(0)=-2+6(1)$$
$$4=4$$

∴ The two lines are intersect.

put $c=0$ in the first line we find intersect point $(x,y,z)=(1,2,4)$



The direction vector of the first line is:

$$v_1=-4i+3j-2K$$

The direction vector of the second line is:

$$v_2=-i+j+6K$$

we need to find a vector normal to the plane determined by the two lines

$$\vec{N} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} i & j & k \\ -4 & 3 & -2 \\ -1 & 1 & 6 \end{vmatrix}$$

$$= +\begin{vmatrix} 3 & -2 \\ 1 & 6 \end{vmatrix} i - \begin{vmatrix} -4 & -2 \\ -1 & 6 \end{vmatrix} j + \begin{vmatrix} -4 & 3 \\ -1 & 1 \end{vmatrix} k$$

$$\vec{N} = 20i + 26j - k$$

The plane determined by the two lines is:

$$n_1 x + n_2 y + n_3 z = D$$

$$20x + 26y - z = D$$

$$D = n_1 x_1 + n_2 y_1 + n_3 z_1$$

$$D = 20 * 1 + 26 * 2 - 1 * 4$$

$$D = 68$$

$$\boxed{20x + 26y - z = 68}$$

EX48: (a) Find equations for the line that passes through the point $(1,-5,6)$ and is parallel to the vector:
$v = -i + 2j - 3k$

(b) Find the point in which the required line in part (a) intersects the coordinate planes.

(a)

$$\frac{x-x_1}{v_1} = \frac{y-y_1}{v_2} = \frac{z-z_1}{v_3}$$

$$\frac{x-1}{-1} = \frac{y+5}{2} = \frac{z-6}{-3}$$

(b)

$$z=0 \longrightarrow \frac{x-1}{-1} = \frac{y+5}{2} = \frac{-6}{-3} = 2$$

$x=-1, \ y=-1$

The intersection with xy-plane:
$(-1,-1,0)$

$$y=0 \longrightarrow \frac{x-1}{-1} = \frac{z-6}{-3} = \frac{5}{2}$$

$x = -\frac{3}{2}, \ z = -\frac{3}{2}$

The intersection with xz-plane:
$(-\frac{3}{2}, 0, -\frac{3}{2})$

$x=0 \longrightarrow \frac{y+5}{2} = \frac{z-6}{-3} = \frac{-1}{-1} = 1$

$y=-3 , z=3$

The intersection with yz-plane:
$(0,-3,3)$

EX5: Determine whether the planes

$x=4y-2z \quad , \quad 8y=1+2x+4z$

are prallel, perpendicular, or neither. if neither, find the angle between them.

$x=4y-2z \longrightarrow x-4y+2z=0$

$8y=1+2x+4z \longrightarrow 2x-8y+4z=-1$

A normal vector of the first plane:

$\vec{N_1} = i-4j+2k$

A normal vector of the second plane:

$\vec{N_2} = 2i-8j+4k$
$=2(i-4j+2k)$

$\vec{N}_1 = 2\vec{N}_2 \longrightarrow$ Two planes are prallel

or

$$\vec{N}_1 \times \vec{N}_2 = \begin{vmatrix} i & j & k \\ 1 & -4 & 2 \\ 2 & -8 & 4 \end{vmatrix}$$

$$= +\begin{vmatrix} -4 & 2 \\ -8 & 4 \end{vmatrix} i - \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} j + \begin{vmatrix} 1 & -4 \\ 2 & -8 \end{vmatrix} k$$

$\vec{N}_1 \times \vec{N}_2 = 0 \longrightarrow$ Two planes are prallel

or

$$\cos\theta = \frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1||\vec{v}_2|}$$

if $\cos\theta = 1$ or $\cos\theta = -1$
the two planes are prallel

$$\cos\theta = \frac{\vec{N}_1 \cdot \vec{N}_2}{|\vec{N}_1||\vec{N}_2|} = \frac{1*2+(-4)*(-8)+2*4}{\sqrt{1^2+(-4)^2+2^2}\sqrt{2^2+(-8)^2+4^2}}$$

$$\cos\theta = \frac{42}{42} = 1 \qquad \theta = 0$$

Ex6: Two lines are said to be skew if they do not cross and they are not prallel (and hence they do not belong to the same plane). show that the lines L_1 and L_2 with parametric equations

$$x = 1 + c \,,\; y = -2 + 3c \,,\; z = 4 - c$$

and

$$x = 2t \,,\; y = 3 + t \,,\; z = -3 + 4t$$

respectivelly, are skew line.

— from paramteric eqns

$v_1 = 1i + 3j - k$

$v_2 = 2i + j + 4k$

$$\cos\theta = \frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1||\vec{v}_2|} = \frac{1}{\sqrt{231}} \neq 1 \text{ or } -1$$

so that two lines are not parallel.

To test the two lines for intersect

$$1+c=2t \quad \text{——— (1)}$$

$$-2+3c=3+t \quad \text{——— (2)}$$

$$4-c=-3+4t \quad \text{——— (3)}$$

solving (1) and (2) we get

$$c=\frac{11}{5}, \quad t=\frac{8}{5}$$

but these values do not work with our third equ

$$4-\frac{11}{5} \neq -3+4\times\frac{8}{5}$$

This shows that the two lines do not intersect.

Hence, we conclude that the two lines are skew.

Ex 7: Determine whether the lines are parallel, skew or intersect:

L_1: $x=3+c$, $y=3+3c$, $z=4-c$

L_2: $x=2-t$, $y=1-2t$, $z=6+2t$

$$\frac{x-x_1}{v_1}=\frac{y-y_1}{v_2}=\frac{z-z_1}{v_3}$$

L_1/ $$\frac{x-3}{1}=\frac{y-3}{3}=\frac{z-1}{-1}$$

$$\vec{v}_1=i+3j-k$$

L_2/ $$\frac{x-2}{-1}=\frac{y-1}{-2}=\frac{z-6}{2}$$

$$\vec{v}_2=-i-2j+2k$$

$$\vec{v}_1\times v_2=\begin{vmatrix} i & j & k \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{vmatrix}$$

$$\vec{v}_1 \times \vec{v}_2 = +\begin{vmatrix} 3 & -1 \\ -2 & 2 \end{vmatrix} i - \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} j + \begin{vmatrix} 1 & 3 \\ -1 & -2 \end{vmatrix} K$$

$\vec{v}_1 \times \vec{v}_2 = 5i - j - K \neq 0$ $\therefore$ The Two lines are not prallel.

To test the intersection

$3 + c = 2 - t$ (1)

$3 + 3c = 1 - 2t$ (2)

$4 - c = 6 + 2t$ (3)

from (1) and (2) we get

$c = 0$ and $t = -1$ putting in eq (3)

$4 - (0) = 6 + 2(-1)$

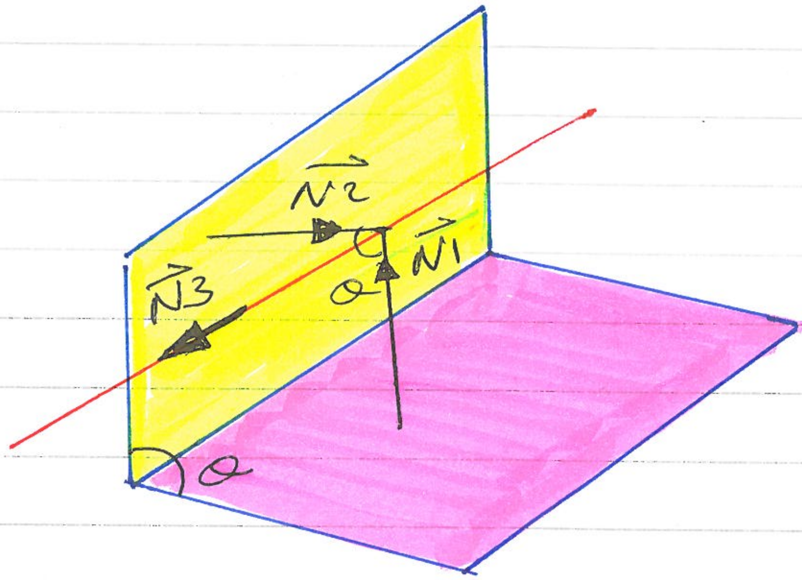
$4 = 4$

$\therefore$ The two lines are intersec

$x = 3 + 0 = 3$, $y = 3 + 3(0) = 3$, $z = 4 - 0 = 4$

The point of intersection is $(3, 3, 4)$

* Two planes are parallel if they have the same normal vector ((i.e their normal vectors are prallel)).



* If two planes are not prallel, then they intersect in a line. The angle between the two planes is the angle between their normal vectors.

Ex8: Consider the planes defined by:

$$4x - 2y + z = 2$$

and

$$2x + y - 4z = 3$$

ⓐ Find the angle between the planes.

ⓑ Find the prametric eqn for the line of intersection

ⓐ The normal vectors are

$$\vec{N_1} = 4i - 2j + K$$

$$\vec{N_2} = 2i + j - 4K$$

$$\cos\theta = \frac{\vec{N_1} \cdot \vec{N_2}}{|\vec{N_1}||\vec{N_2}|} = \frac{2}{\sqrt{21}\sqrt{21}} = \frac{2}{21}$$

$$\theta = \cos^{-1}\left(\frac{2}{21}\right) = 84.5^\circ$$

ⓑ To find the equation of the line of intersection, we need a point on the line and a direction vector. The direction vector for the line is:

$$\vec{N_3} = \vec{N_1} \times \vec{N_2} = \begin{vmatrix} i & j & K \\ 4 & -2 & 1 \\ 2 & 1 & -4 \end{vmatrix}$$

$$= + \begin{vmatrix} -2 & 1 \\ 1 & -4 \end{vmatrix} i - \begin{vmatrix} 4 & 1 \\ 2 & -4 \end{vmatrix} j + \begin{vmatrix} 4 & -2 \\ 2 & 1 \end{vmatrix} K$$

$$\vec{N_3} = 7i + 18j + 8K$$

Since the direction vector is not horizontal, The line of intersection must intersect the xy-plane (Z=0) substituting $z=0$ into the equations for the planes gives

$$4x-2y=2$$
$$2x+y=3$$

it follows that $x=1$ and $y=1$.
Thus, $(1,1,0)$ is a point on the line of intersection.

The equation of line is

$$\frac{x-x_1}{v_1}=\frac{y-y_1}{v_2}=\frac{z-z_1}{v_3}$$

$$\frac{x-1}{n_1}=\frac{y-1}{n_2}=\frac{z-0}{n_3}$$

$$\frac{x-1}{7}=\frac{y-1}{18}=\frac{z}{8}$$

The parametric eqs are:

$$x=1+7c$$
$$y=1+18c$$
$$z=8c$$