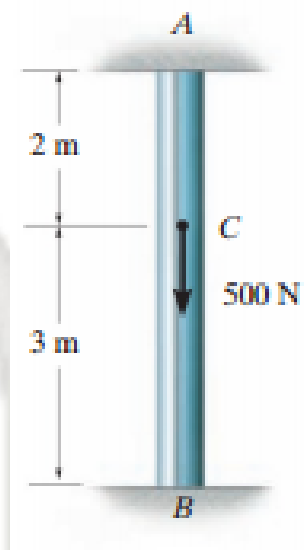
Statically indeterminate members (axially loaded only

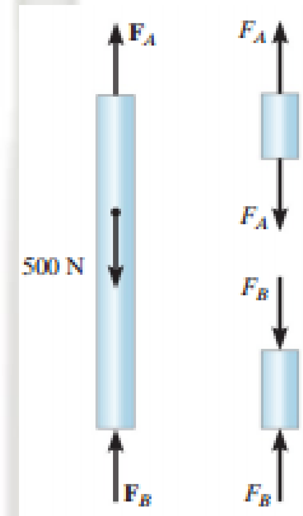
When the reactive forces or the internal resisting forces over a cross section exceed the number of independent equations of equilibrium, the structure is called **statically indeterminate**. These cases require the use of additional relations that depend on the elastic deformations in the members.

Consider the bar shown in Figure, which is fixed supported at both of its ends. From its free-body diagram, there are two unknown support reactions. Equilibrium requires:



$$\sum Fy = 0 \rightarrow F_B + F_A - 500 = 0$$

This type of problem is called **statically indeterminate,** since the equilibrium equation is not sufficient to determine both reactions on the bar. An additional equation needed for solution called **compatibility equation** that specifies the conditions for displacement. In this case, the displacement of end A of the bar with respect to end B to equal zero, since the end supports are fixed. Realizing that the internal force in segment AC is (+FA), and in segment CB it is (-FB), then the compatibility equation can be written as



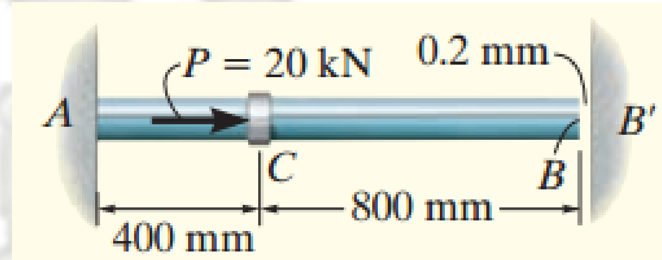
$$\frac{F_A \times 2}{AE} - \frac{F_B \times 3}{AE} = 0 \rightarrow F_A = 1.5F_B$$

$$F_A = 300N$$

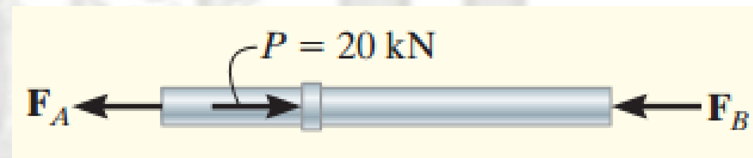
$$F_B = 200N$$

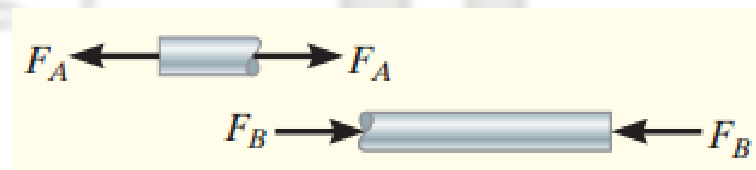
EXAMPLE 2-10

The steel rod shown in Figure, has a diameter of 10 mm. It is fixed to the wall at A, and before it is loaded, there is a gap of 0.2 mm between the wall at B and the rod. Determine the reactions on the rod if it is subjected to an axial force of P = 20 kN. (Neglect the size of the collar at C). Take Est = 200 GPa.



We will assume that force P is large enough to cause the rod's end B to contact the wall at B′. When this occurs, the problem becomes statically indeterminate since there are two unknowns and only one equation of equilibrium.





$$\sum F_x = 0$$

$$F_A + F_B = 20000 \rightarrow (1)$$

$$\frac{F_A \times L_{AC}}{AE} - \frac{F_B \times L_{CB}}{AE} = 0.0002$$

$$\frac{F_A \times 0.4}{AE} - \frac{F_B \times 0.8}{AE} = 0.0002$$

$$0.4F_A - 0.8F_B = 0.0002AE$$

$$0.4F_A - 0.8F_B = 0.0002 \times \pi \times 0.005^2 \times 200 \times 10^9$$

$$0.4F_A - 0.8F_B = 3141.6$$

$$F_A = 2F_B + 7854 \rightarrow (2)$$

$$2F_B + 7854 + F_B = 20000 \rightarrow F_B = 4048.67N = 4.048kN$$

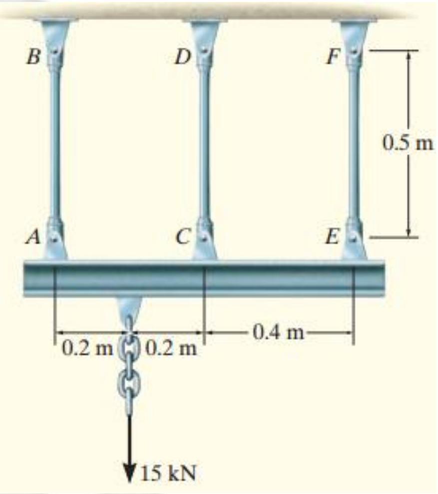
$$F_A = 2 \times 4048.67 + 7854 = 15951.33N = 15.951kN$$

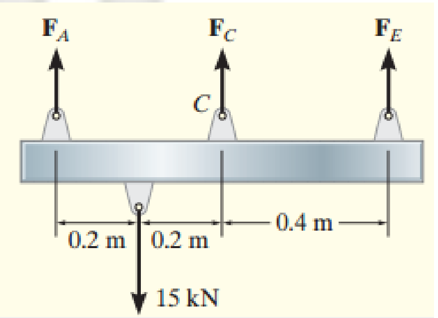
NOTE:

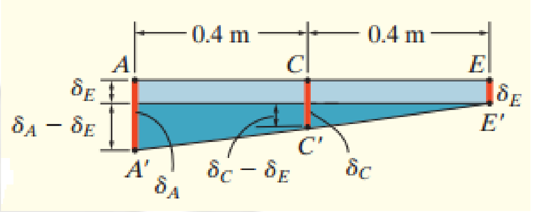
Since the answer for FB is positive, indeed end B contacts the wall at B as originally assumed. If FB were a negative quantity, the problem would be statically determinate, so that FB = 0 and FA = 20 kN.

EXAMPLE 2-11

The three steel bars shown in the Figure, are pin connected to a rigid member. If the applied load on the member is 15 kN, determine the force developed in each bar. Bars AB and EF each have a cross-sectional area of 50 mm², and bar CD has a cross-sectional area of 30 mm².







$$\sum F_y = 0 \rightarrow F_A + F_C + F_E = 15 \rightarrow (1)$$

$$\sum M_C = 0 \rightarrow F_A \times 0.4 = 15 \times 0.2 + F_E \times 0.4$$

$$0.4F_A = 3 + 0.4F_E \rightarrow (2)$$

$$\frac{\delta_A - \delta_E}{0.8} = \frac{\delta_C - \delta_E}{0.4}$$

$$\delta_C = \frac{1}{2}\delta_A + \frac{1}{2}\delta_E$$

$$\frac{F_C L}{30E_{st}} = \frac{F_A L}{2 \times 50 \times E_{st}} + \frac{F_E L}{2 \times 50 \times E_{st}}$$

$$F_C = 0.3F_A + 0.3F_E \rightarrow (3)$$

Solving Eqs. 1–3 simultaneously yields:

$$F_A = 15 - F_C - F_E$$

$$F_C = 0.3(15 - F_C - F_E) + 0.3F_E$$

$$F_C = 4.5 - 0.3F_C \rightarrow F_C = 3.46kN$$

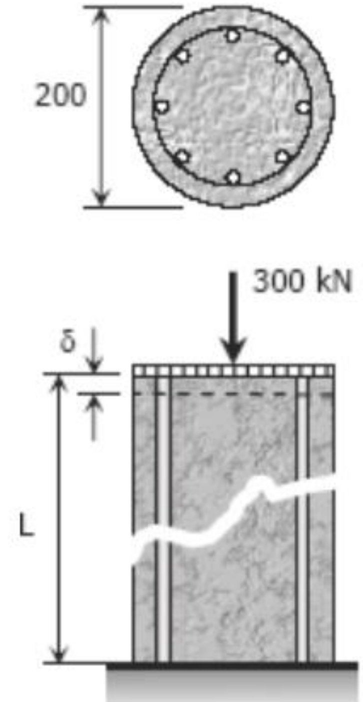
$$F_A = 15 - 3.46 - F_E = 11.538 - F_E$$

$$0.4(11.538 - F_E) = 3 + 0.4F_E \rightarrow F_E = 2.02kN$$

$$F_A = 15 - 3.46 - 2.02 = 9.52kN$$

EXAMPLE 2-12

A reinforced concrete column 200 mm in diameter is designed to carry an axial compressive load of 300 kN. Determine the required area of the reinforcing steel if the allowable stresses are 6 MPa and 120 MPa for the concrete and steel, respectively. Use Eco = 14 GPa and Est = 200 GPa.



$$\delta_{con} = \delta_{st}$$

$$\left(\frac{PL}{AE}\right)_{con} = \left(\frac{PL}{AE}\right)_{st}$$

$$\left(\frac{\sigma L}{E}\right)_{con} = \left(\frac{\sigma L}{E}\right)_{st}$$

$$\left(\frac{\sigma L}{14000}\right)_{con} = \left(\frac{\sigma L}{200000}\right)_{st} \rightarrow 100\sigma_{con} = 7\sigma_{st}$$

$$100\sigma_{con} = 7 \times 120$$

$\sigma_{con} = 8.4 MPa > 6 MPa$ **Not o.k**

$$\sigma_{con} = 6 MPa \rightarrow 100 \times 6 = 7\sigma_{st}$$

$$\sigma_{st} = 85.71 MPa < 120 MPa$$

Use $\sigma_{con} = 6 MPa$

and $\sigma_{st} = 85.71 MPa$

$$\sum F_y = 0 \rightarrow P_{con} + P_{st} = 300$$

$$\sigma_{st} A_{st} + \sigma_{con} A_{con} = 300$$

$$85.71 A_{st} + 6(\pi \times 100^2 - A_{st}) = 300 \times 1000$$

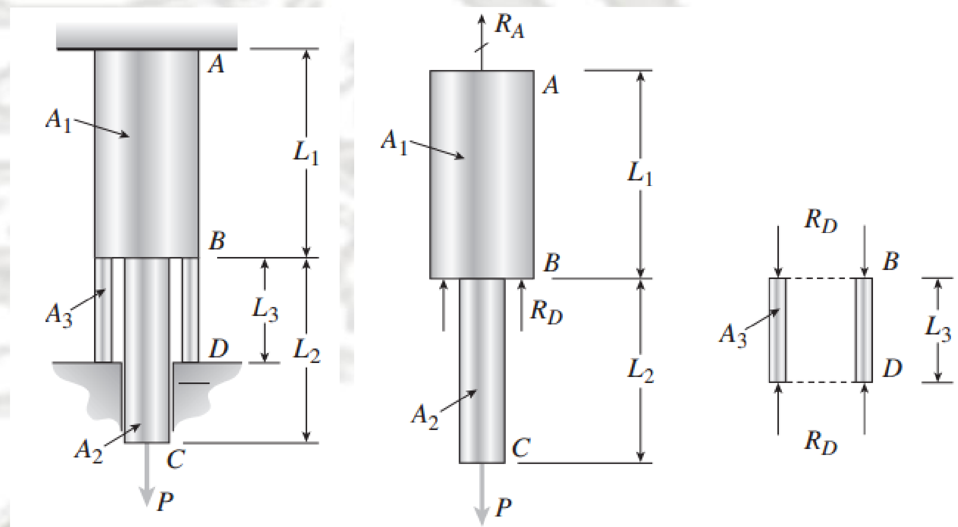
$$79.71 A_{st} + 60000\pi = 300000$$

$$A_{st} = 1398.9 mm^2$$

H.W

A circular steel bar ABC (E = 200 GPa) has cross sectional area A_1 from A to B and cross-sectional area A_2 from B to C. The bar is supported rigidly at end A and is subjected to a load P equal to 40 kN at end C. A circular steel collar BD having cross-sectional area A_3 supports the bar at B. The collar fits snugly at B and D when there is no load. Determine the elongation AC of the bar due to the load P.

(Assume L_1=$2L_3$=250 mm, L_2=225 mm, A_1=$2A_3$=960 mm^2, and A_2=300 mm^2.)



Problems Involving Temperature Changes

A change in temperature can cause a body to change its dimensions. Generally, if the temperature increases, the body will expand, whereas if the temperature decreases, it will contract. If the material is homogeneous and isotropic, it has been found from experiment that the displacement of the end of a member having a length (L) can be calculated using the formula:

$$\boxed{\delta_T = \alpha \Delta T L}$$

Where:

α = Linear coefficient of thermal expansion 1/C^{o}.

ΔT = Algebraic change in temperature of the member (T_{final}-$T_{initial}$).

L = Original length of the member

δ_T = Algebraic change in the length of the member

$$\varepsilon_{thermal} = \frac{\delta_T}{L} = \alpha \Delta T$$

$$\sigma_{thermal} = \alpha \Delta T E$$

The effect of temperature changing is only **important in statically indeterminate members**. This means, temperature change causes stresses and strains only in indeterminate elements, while determinate elements elongate and shrink freely without any stress and strain.

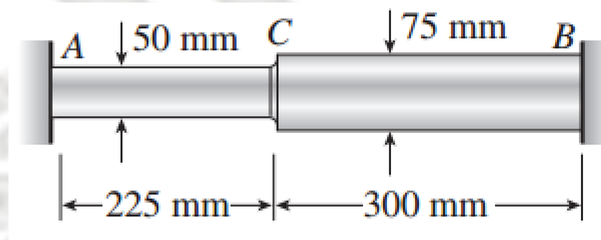
EXAMPLE 2-13

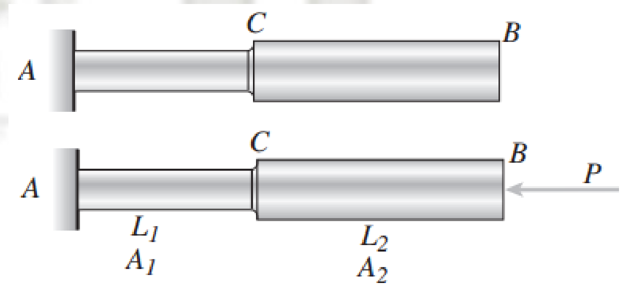
A plastic bar ACB having two different solid circular cross sections is held between rigid supports as shown in the figure. The diameters in the left and right parts are 50 mm and 75 mm, respectively. The corresponding lengths are 225 mm and 300 mm. Also, the modulus of elasticity E is 6.0 GPa, and the coefficient of thermal expansion is 100 x10^{-6}/C°. The bar is subjected to a uniform temperature increase of 30°C.Calculate the following quantities:

(a) the compressive force P in the bar.

(b) the maximum compressive stress σc.

(c) the displacement of point C.





$$\delta_T = \alpha \Delta T L = 100 \times 10^{-6} \times (30-0) \times (225+300) = 1.575mm$$

$$\delta_P = \frac{PL_1}{A_1E} + \frac{PL_2}{A_2E} = P(\frac{225}{\pi \times 25^2 \times 6000} + \frac{300}{\pi \times 37.5^2 \times 6000})$$

$$\delta_T = \delta_P$$

$$1.575 = P(\frac{225}{\pi \times 25^2 \times 6000} + \frac{300}{\pi \times 37.5^2 \times 6000})$$

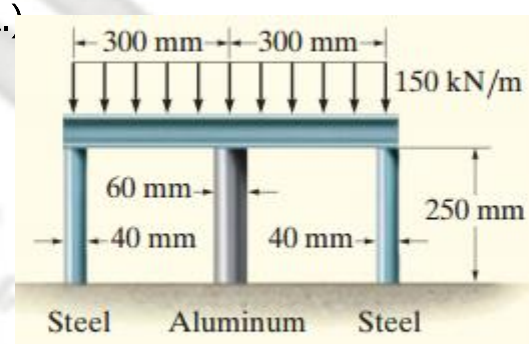
$$P = 51781N$$

$$\sigma_c = \frac{P}{A_1} = \frac{51781}{1963.5} = 26.4MPa$$

$$\delta_c = -\frac{PL_1}{A_1E} + \alpha \Delta T L_1 = -0.989 + 0.675 = -0.314mm$$

EXAMPLE 2-14

The rigid beam shown in the Figure is fixed to the top of the three posts made of steel and aluminum. The posts each have a length of 250 mm when no load is applied to the beam, and the temperature is T1=20C. Determine the force supported by each post if the bar is subjected to a uniform distributed load of 150 kN/m and the temperature is raised to T2 = 80C. (Est = 200 Gpa and Eal = 73.1 GPa.)



$$\sum F_y = 0$$

$$2F_{st} + F_{al} = 90000$$

$$\delta_{st} = -(\delta_{st})_F + (\delta_{st})_T$$

$$\delta_{al} = -(\delta_{al})_F + (\delta_{al})_T$$

$$-(\delta_{st})_F + (\delta_{st})_T = -(\delta_{al})_F + (\delta_{al})_T$$

$$-(\frac{FL}{AE})_{st} + (\alpha \Delta T L)_{st} = -(\frac{FL}{AE})_{al} + (\alpha \Delta T L)_{al}$$

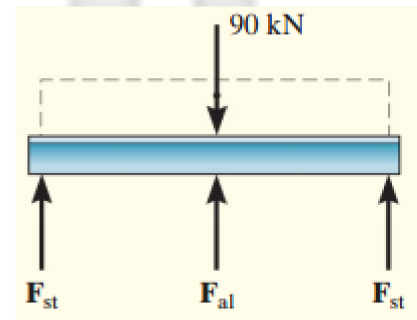
$$-\frac{F_{st} \times 0.25}{\pi \times 0.02^2 \times 200 \times 10^9} + 12 \times 10^{-6}(80-20) \times 0.25 =$$

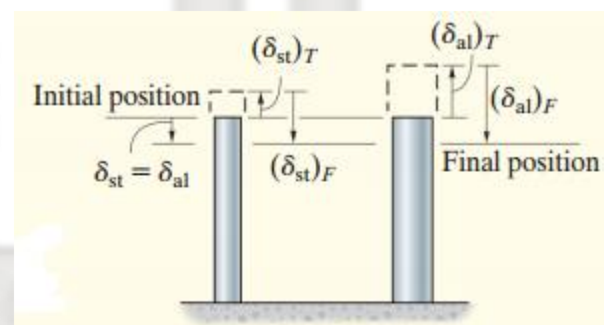
$$-\frac{F_{al} \times 0.25}{\pi \times 0.03^2 \times 73.1 \times 10^9} + 23 \times 10^{-6}(80-20) \times 0.25$$

$$F_{st} = 1.216 F_{al} - 165900$$

$$F_{st} = -16.4 kN$$

$$F_{al} = 123 kN$$

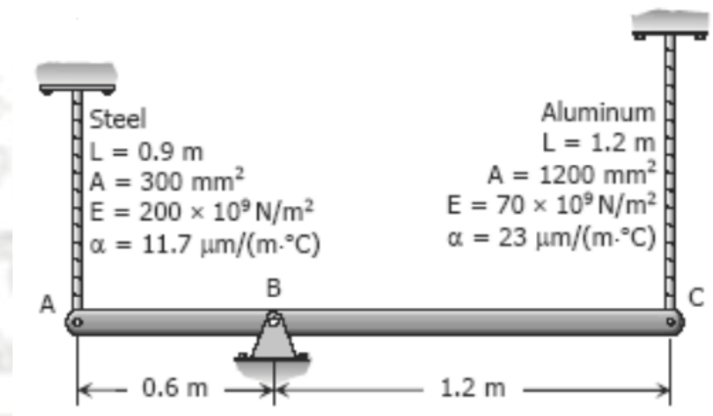




The negative value for Fst indicates that this force acts opposite to that shown in the Figure. In other words, the steel posts are in tension and the aluminum post is in compression.

EXAMPLE 2-15

The rigid bar ABC in Figure is pinned at B and attached to the two vertical rods. Initially, the bar is horizontal and the vertical rods are stress-free. Determine the stress in the aluminum rod if the temperature of the steel rod is decreased by 40°C. Neglect the weight of bar ABC.



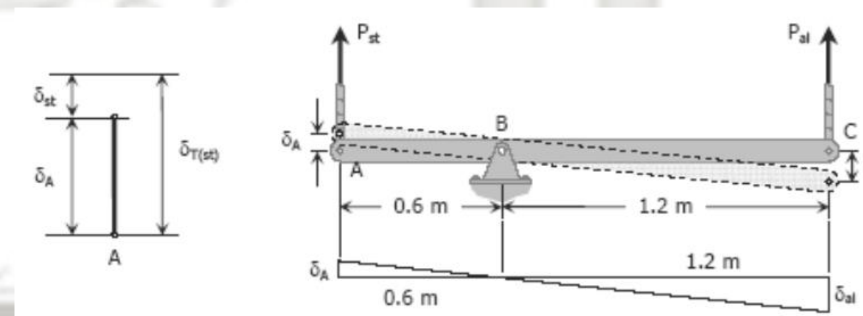
$$(\delta_{st})_T = \alpha \times \Delta T \times L = 11.7\times10^{-6}\times40\times900 = 0.4212mm$$

$$\frac{\delta_A}{0.6} = \frac{\delta_{al}}{1.2} \rightarrow \delta_A = 0.5\delta_{al}$$

$$(\delta_{st})_T - \delta_{st} = 0.5\delta_{al}$$

$$0.4212 - (\frac{PL}{AE})_{st} = 0.5(\frac{PL}{AE})_{al}$$

$$0.4212 - \frac{P_{st}\times900}{300\times200000} = 0.5\times\frac{P_{al}\times1200}{1200\times70000}$$



$$28080 - P_{st} = 0.4762P_{al} \rightarrow (1)$$

$$\sum M_B = 0$$

$$0.6P_{st} = 1.2P_{al} \rightarrow P_{st} = 2P_{al} \rightarrow (2)$$

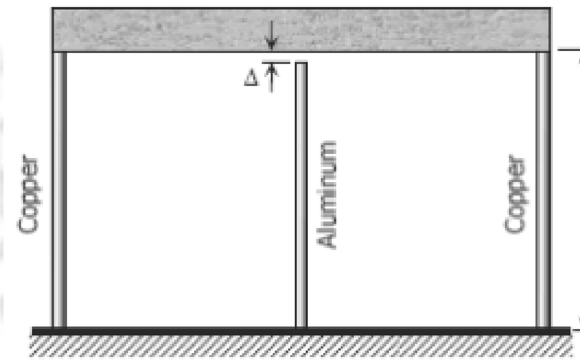
$$280800 - 2P_{al} = 0.4762P_{st}$$

$$P_{al} = 11340N$$

$$\sigma_{al} = \frac{P_{al}}{A_{al}} = \frac{11340}{1200} = 9.45MPa$$

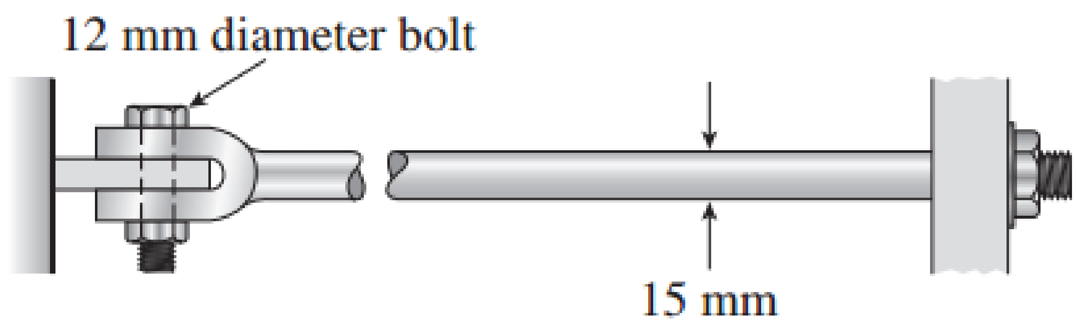
H.W

As shown in Figure, there is a gap between the aluminum bar and the rigid slab that is supported by two copper bars. At 10°C, δ = 0.18 mm. Neglecting the mass of the slab, calculate the stress in each rod when the temperature in the assembly is increased to 95°C. For each copper bar, A= 500 mm2, E = 120 GPa, and α = 16.8 μm/(m·°C). For the aluminum bar, A = 400 mm2, E = 70 GPa, and α = 23.1 μm/(m·°C).



EXAMPLE 2-16

A steel rod of diameter 15 mm is held (without any initial stresses) between rigid walls by the arrangement shown in the Figure. Calculate the temperature drop ΔT (degrees Celsius) at which the average shear stress in the 12-mm diameter bolt becomes 45 MPa. (For the steel rod, use α=12x10^{-6}/°C and E=200 GPa.)



$$\delta = \alpha \times \Delta T \times L$$

$$\frac{PL}{AE} = \alpha \times \Delta T \times L$$

$$P = AE \times \alpha \times \Delta T$$

$$V = \frac{P}{2}$$ **Double Shear**

$$V = \frac{AE \times \alpha \times \Delta T}{2}$$

$$\tau = \frac{V}{A_{bolt}} = \frac{AE \times \alpha \times \Delta T}{2A_{bolt}}$$

$$\Delta T = \frac{2 \times \tau \times A_{bolt}}{AE \times \alpha}$$

$$\Delta T = \frac{2 \times 45 \times \frac{\pi}{4} 12^2}{200 \times \frac{\pi}{4} 15^2 \times 12 \times 10^{-6}}$$

$$\Delta T = 24^{\circ} C$$