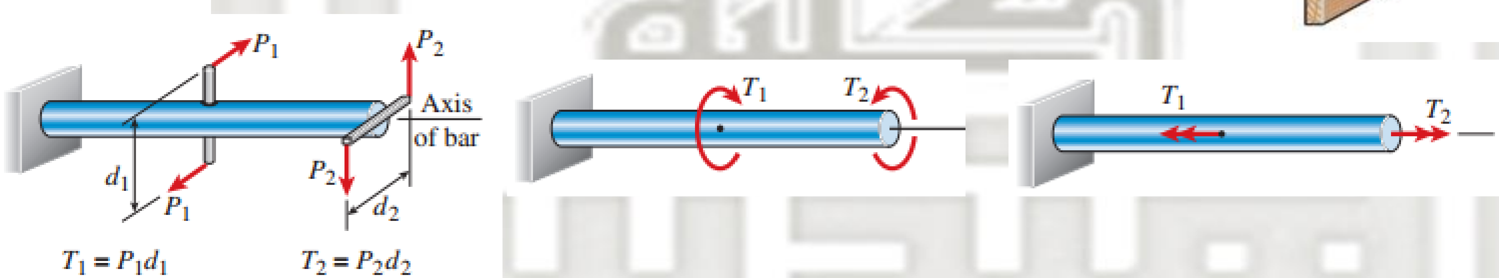
CHAPER FOUR – TORSION

Introduction:

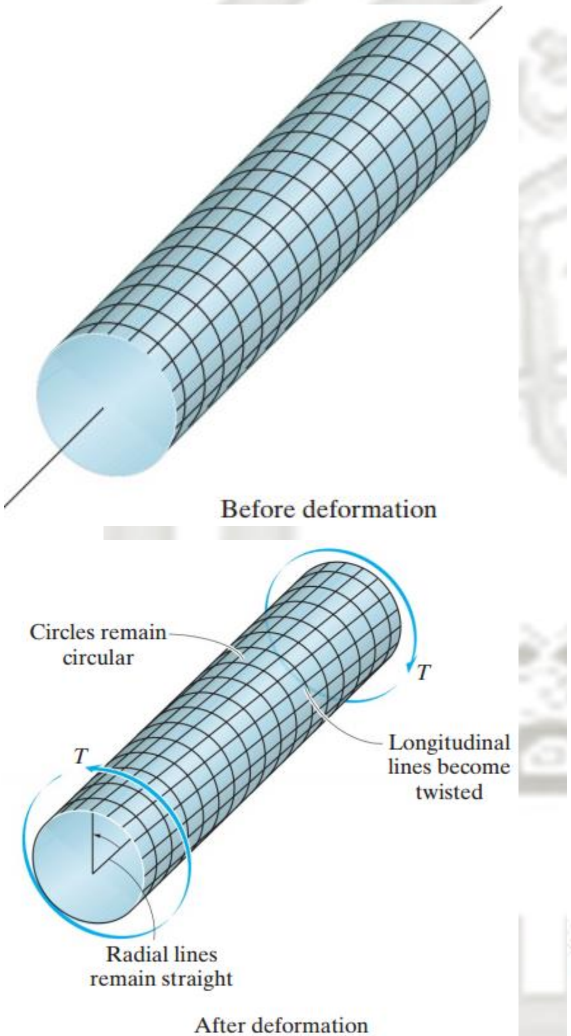
Torsion refers to the twisting of a straight bar when it is loaded by moments (or torques) that tend to produce rotation about the longitudinal axis of the bar. Its effect is of primary concern in the design of drive shafts used in vehicles and machinery, and for this reason it is important to be able to determine the stress and the deformation that occur in a shaft when it is subjected to torsional loads.

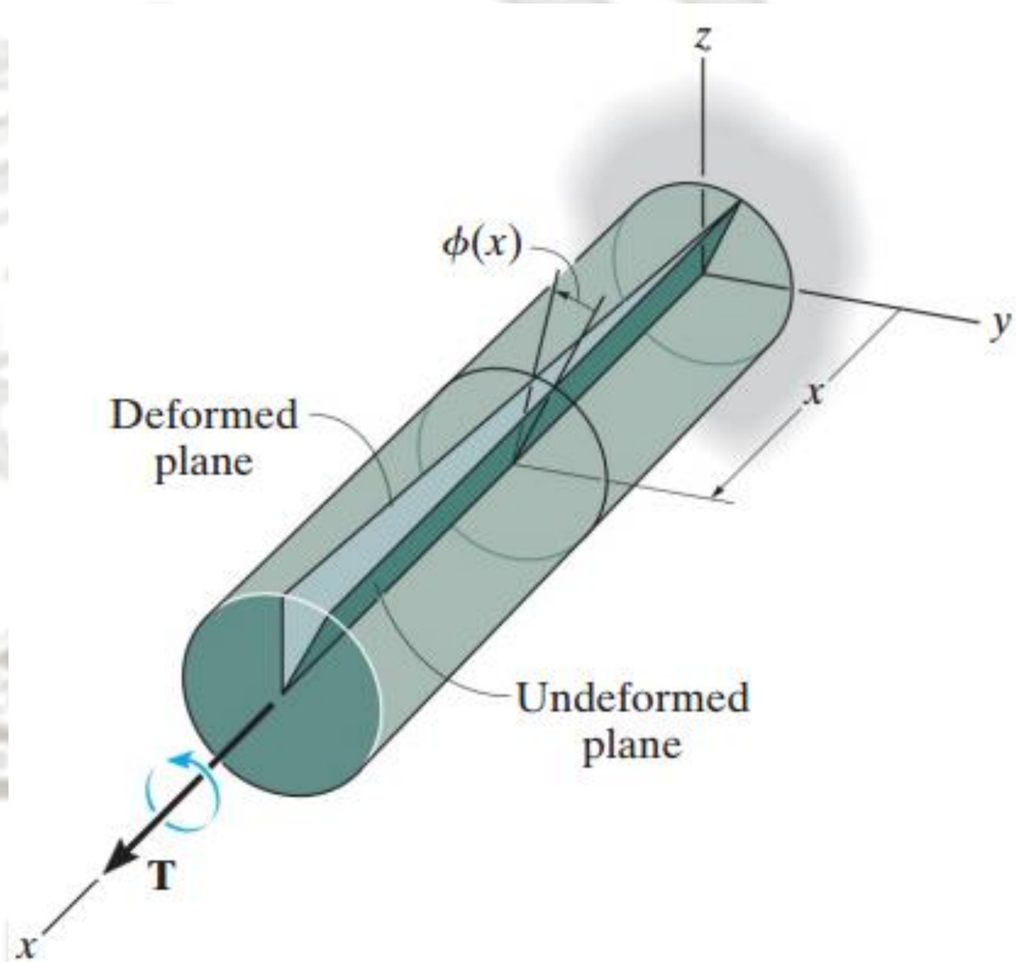
For instance, when you turn a screwdriver as shown in the figure, your hand applies a torque T to the handle and twists the rod of the screwdriver.

An idealized case of torsional loading is pictured in the figure which shows a straight bar supported at one end and loaded by two pairs of equal and opposite forces. The first pair consists of the forces $P1$ acting near the midpoint of the bar and the second pair consists of the forces $P2$ acting at the end. Each pair of forces forms a **couple** that tends to twist the bar about its longitudinal axis. The **moment of a couple** is equal to the product of one of the forces and the perpendicular distance between the lines of action of the forces; thus, the first couple has a moment $T1=P1d1$ and the second has a moment $T2=P2d2$. The SI unit for moment is the newton meter (N.m).



When the torque is applied, the longitudinal grid lines originally marked on the shaft tend to distort into a helix, that intersects the circles at equal angles. Also, all the cross sections of the shaft will remain flat and they do not warp or bulge in or out. The radial lines remain straight and rotate during this deformation. Provided the angle of twist is *small*, then the length of the shaft and its radius will remain practically unchanged.



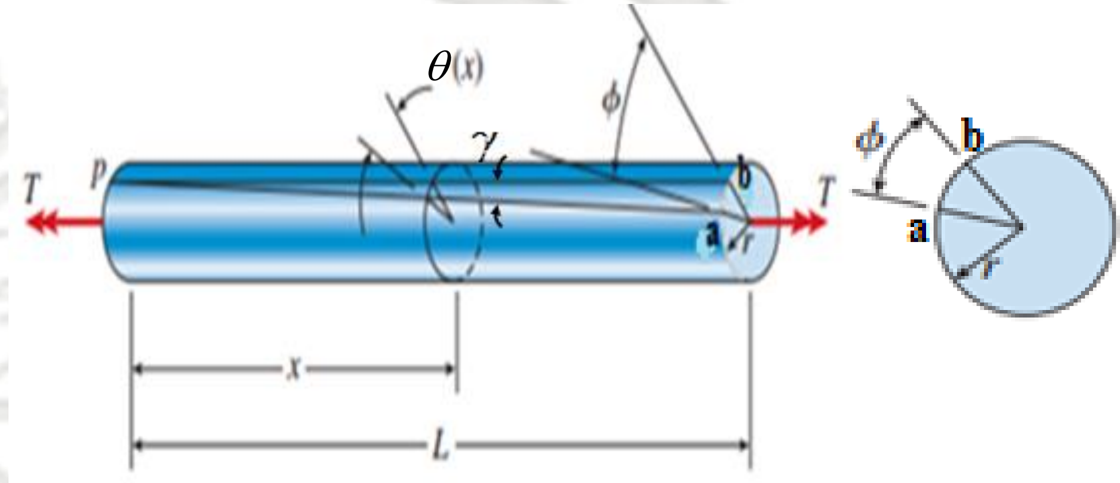


The angle of twist $\phi(x)$ increases as x increases.

Torsion Formula of Circular Sections

Assumptions:

1. A plane section before twist remains plane after twist.
2. The distribution of shear strain (γ) through the section is linear.
3. The material of the body is linear elastic.



Arc length $ab = r\phi$

And $ab = L\gamma$

$$\gamma_{\max} = \frac{r\phi}{L}$$

And $\tau_{\max} = G\gamma_{\max} = G\dfrac{r\phi}{L}$

Torque= stress x area x arm

$$dA = 2\pi\rho d\rho \quad , \quad (\rho = r)$$

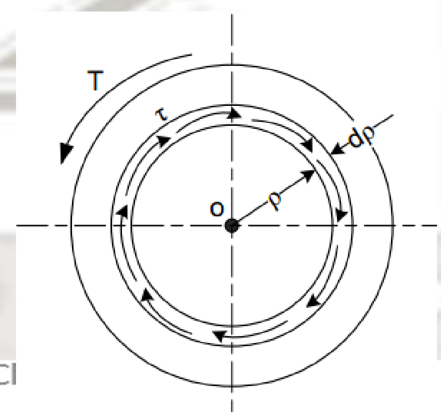
For a bar in **pure torsion**, the total angle of twist φ, equal to the rate of twist times the length of the bar that is:

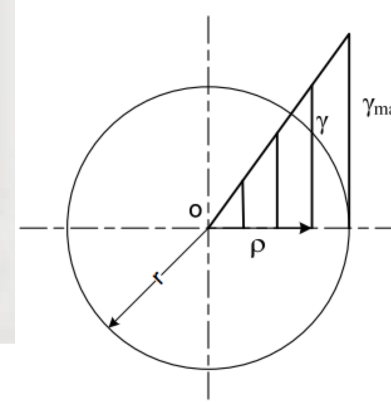
$$\phi = \theta \times L$$

$$T = \int_{area} 2\pi\rho.\tau.\rho d\rho = \int_{area} 2\pi\rho.G\rho\theta.\rho d\rho$$

$$T = 2\pi G\theta \int_0^r \rho^3 d\rho = 2\pi G\theta \frac{r^4}{4}$$

$$T = \frac{\pi}{2} G\theta r^4$$





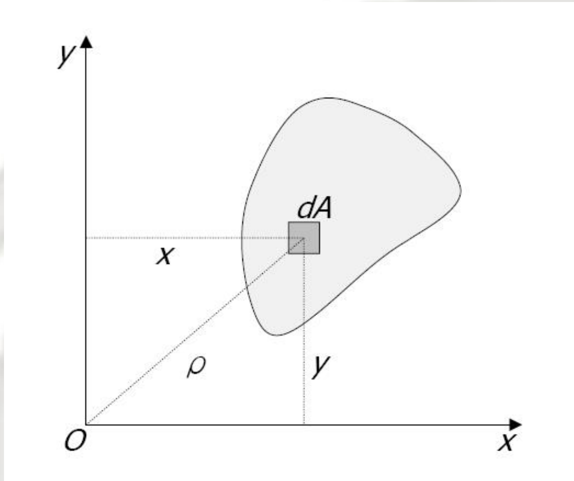
Polar Moment of Inertia

$$J = \int_{area} \rho^2 dA$$

$$J = \int_0^r 2\pi\rho.\rho^2 d\rho$$

$$J = 2\pi \int_0^r \rho^3 d\rho$$

$$J = 2\pi \frac{r^4}{4} = \frac{\pi r^4}{2}$$



$$T = JG\theta$$

$$\theta = \frac{\phi}{L}$$

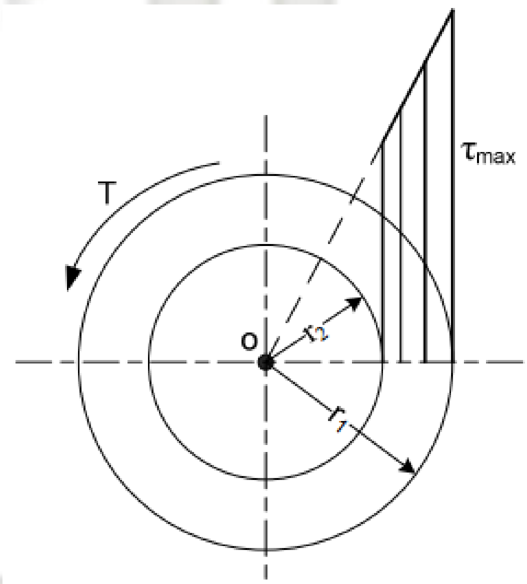
$$\phi = \theta L = \frac{TL}{GJ}$$

$$\tau_{max} = \frac{Gr\frac{TL}{GJ}}{L} = \frac{Tr}{J}$$

Hollow Circular Sections

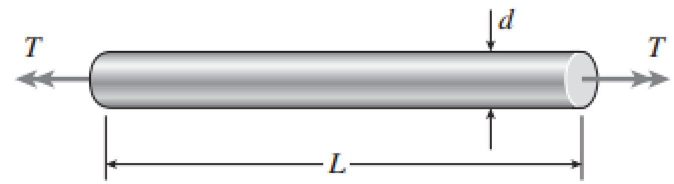
$$J = \frac{\pi}{2}(r_1^4 - r_2^4)$$

$$\tau_{max} = \frac{Tr_1}{J}$$



EXAMPLE 4-1

A plastic bar of diameter d=50 mm is to be twisted by torques T as shown in the figure until the angle of rotation between the ends of the bar is 5.0°.If the allowable shear strain in the plastic is 0.012 rad, what is the minimum permissible length of the bar?



$$d = 50mm$$

$$\phi = 5^{\circ} = 5 \times \frac{\pi}{180} = 0.08727 rad$$

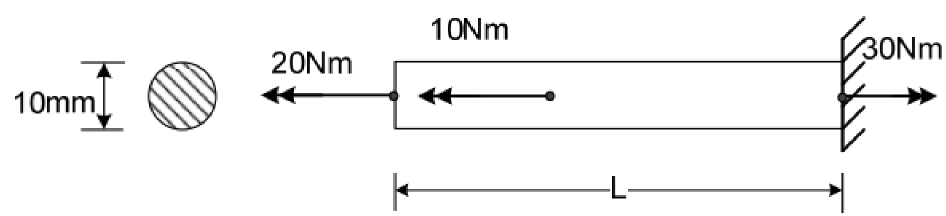
$$\gamma_{allow.} = 0.012 rad$$

$$\gamma_{\max} = \frac{r\phi}{L} = \frac{d \times \phi}{2L}$$

$$L_{\min} = \frac{d \times \phi}{2\gamma_{allow}} = \frac{50 \times 0.08727}{2 \times 0.012} = 182mm$$

EXAMPLE 4-2

Determine the maximum shearing stress occurs in the circular shaft AC.



$$J = \frac{\pi \times r^4}{2} = \frac{\pi \times 5^4}{2} = 981.7 mm^4$$

$$\tau_{\max} = \frac{Tr}{J} = \frac{30 \times 10^3 \times 5}{981.7} = 152.8 MPa$$

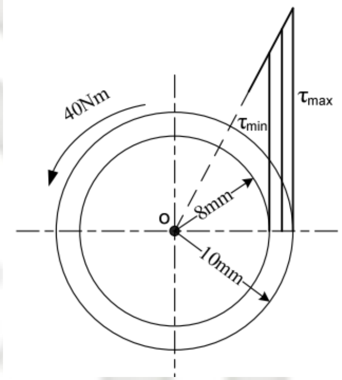
EXAMPLE 4-3

A circular hollow shaft with an outside diameter of 20mm and inside diameter of 16mm is subjected to a torque of 40Nm. Determine the torsional shear stress at the outside surface and inside surface of the shaft.

$$J=\frac{\pi}{2}(r_1^4-r_2^4)=\frac{\pi}{2}(10^4-8^4)=9274mm^4$$

$$\tau_{\max}=\tau_{out}=\frac{Tr_1}{J}=\frac{40\times10^3\times10}{9274}=43.13MPa$$

$$\tau_{\min}=\tau_{in}=\frac{Tr_2}{J}=\frac{40\times10^3\times8}{9274}=34.51MPa$$



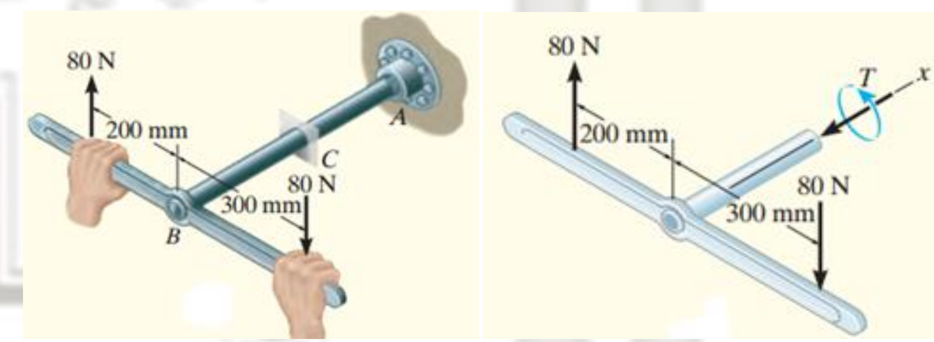
EXAMPLE 4-4

The pipe shown in the figure has an inner radius of 40 mm and an outer radius of 50 mm. If its end is tightened against the support at *A* using the torque wrench, determine the shear stress developed in the material at the inner and outer walls along the central portion of the pipe.

$$\sum M_x=0$$

$$80\times0.3+80\times0.2-T=0$$

$$T=40N.m$$



$$J=\frac{\pi}{2}(r_1^4-r_2^4)=\frac{\pi}{2}(50^4-40^4)=5796238mm^4$$

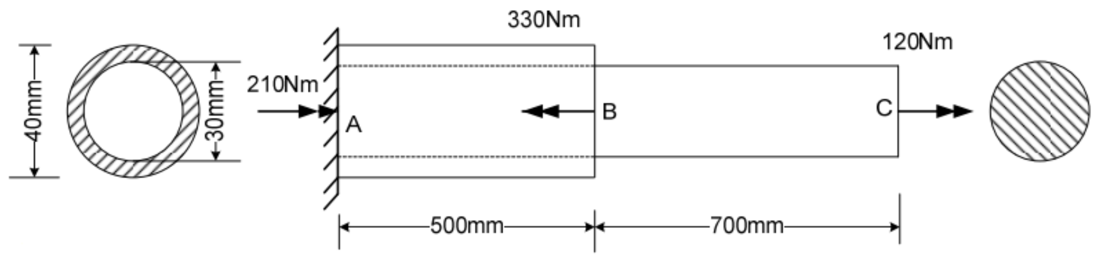
$$\tau_{out}=\frac{Tr_1}{J}=\frac{40\times10^3\times50}{5796238}=0.345MPa$$

$$\tau_{out}=\frac{Tr_1}{J}=\frac{40\times10^3\times40}{5796238}=0.276MPa$$

EXAMPLE 4-5

The steel shaft shown below is subjected to two concentrated torques at B and C. If the shear modulus of the steel material G is 80GPa, determine:

1. The angle of twist at the free end (total angle of twist).
2. The maximum shear strain in the shaft.



$$J_{AB} = \frac{\pi}{2}(r_1^4 - r_2^4) = \frac{\pi}{2}(20^4 - 15^4) = 171806 mm^4$$

$$J_{BC} = \frac{\pi}{2} r^4 = \frac{\pi}{2} \times 15^4 = 79521.5 mm^4$$

$$\phi_c = \phi_{AB} + \phi_{BC} = (\frac{TL}{GJ})_{AB} + (\frac{TL}{GJ})_{BC}$$

$$\phi_C = -\frac{210 \times 10^3 \times 500}{80 \times 10^3 \times 171806} + \frac{120 \times 10^3 \times 700}{80 \times 10^3 \times 79521.5} = 0.00556 rad.$$

$$\tau_{\max} = (\frac{Tr}{J})_{AB} = \frac{210 \times 10^3 \times 20}{171806} = 24.45 MPa$$

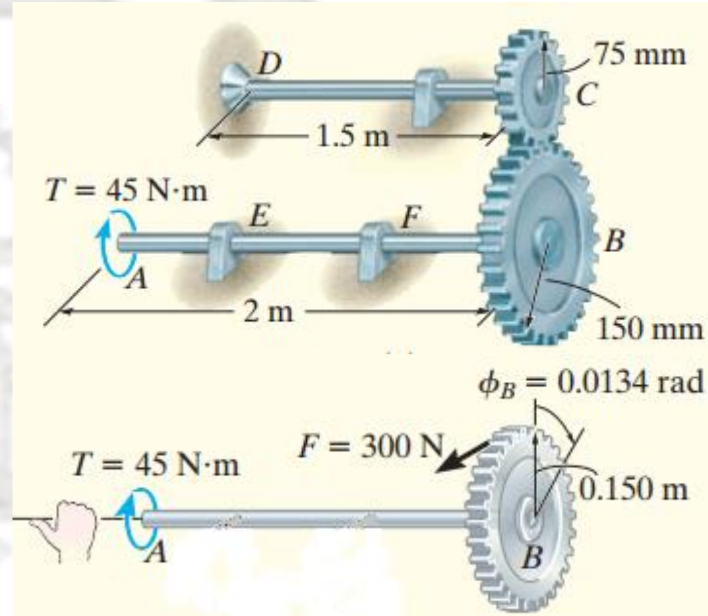
Check the shear stress at BC:

$$\tau_{\max} = (\frac{Tr}{J})_{BC} = \frac{120 \times 10^3 \times 15}{79521.5} = 22.64 MPa$$

$$\gamma_{\max} = \frac{\tau_{\max}}{G} = \frac{24.45}{80000} = 0.0003 rad.$$

EXAMPLE 4-6

The two solid steel shafts shown in figure are coupled together using the meshed gears. Determine the angle of twist of end A of shaft AB when the torque T = 45 N.m is applied. Shaft DC is fixed at D. Each shaft has a diameter of 20 mm and G = 80 GPa.



Summing moments along the axis of shaft AB yields the tangential reaction between the gears:

$$F=\frac{T_A}{L}=\frac{45}{0.15}=300N$$

$$T_D=300\times0.075=22.5N.m$$

$$J=\frac{\pi}{2}\times10^4=15708mm^4$$

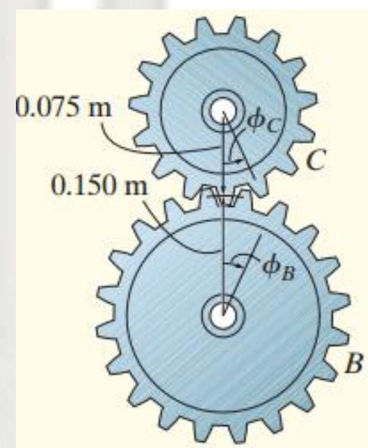
$$\phi_c=(\frac{TL}{GJ})_{DC}=\frac{22.5\times10^3\times1500}{80\times10^3\times15708}=0.0269rad.$$

$$\therefore\ \phi_B\times0.15=0.0269\times0.075\phi_{AB}$$

$$\phi_B=0.0134rad.$$

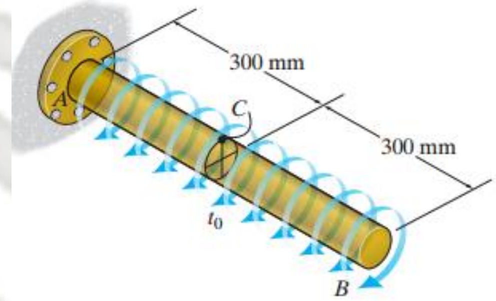
$$\phi_{AB}=(\frac{TL}{GJ})_{AB}=\frac{45\times10^3\times2000}{80\times10^3\times15708}=0.0716rad.$$

$$\therefore\ \phi_A=\phi_B+\phi_{AB}.=0.0134+0.0716=0.085rad$$



EXAMPLE 4-7

If the 40-mm-diameter rod is subjected to a uniform distributed torque of t_o =1.5 kN.m/m, determine the shear stress developed at point *C*.



$$J = \frac{\pi}{2} r^4 = \frac{\pi}{2} \times 20^4 = 251327.4 mm^4$$

$$\tau_{\max} = \frac{Tr}{J} = \frac{1.5 \times 0.3 \times 10^6 \times 20}{251327.4} = 35.8 MPa$$

EXAMPLE 4-8

If the rod in Ex. 4-7 is subjected to a uniform distributed torque of t_o =1.5 kN.m/m, determine the rod's minimum required diameter d if the material has an allowable shear stress of τallow = 75 MPa.

$$\tau_{allow} = \frac{T_{\max} r}{J} \Rightarrow J = \frac{T_{\max} r}{\tau_{allow}}$$

$$J = \frac{1.5 \times 0.6 \times 10^6 \times r}{75} = 12000 r$$

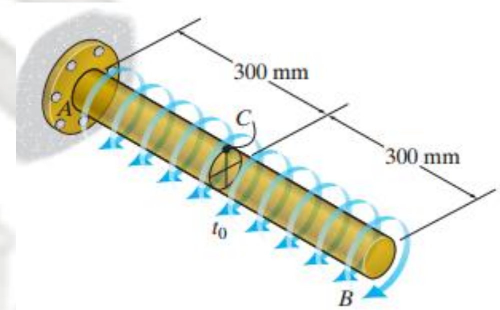
$$J = \frac{\pi}{2} r^4 \Rightarrow 12000 r = \frac{\pi}{2} r^4$$

$$r^3 = 7639.43 \Rightarrow r = 19.694 mm$$

$$\therefore d = 39.4 mm$$

EXAMPLE 4-9

If the 40-mm-diameter rod is made from a material having an allowable shear stress of τallow = 75 MPa, determine the maximum allowable intensity of the uniform distributed torque.



$$J=\frac{\pi}{2}r^4=\frac{\pi}{2}20^4=251327.4$$

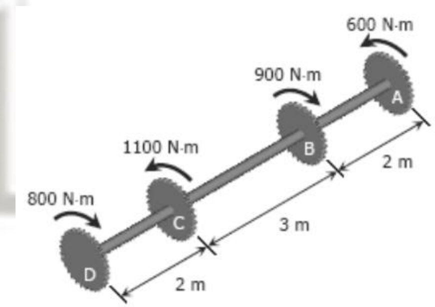
$$\tau_{allow}=\frac{T_{\max}r}{J}\Rightarrow T_{\max}=\frac{\tau_{allow}\times J}{r}=\frac{\pi}{2}r^3\tau_{allow}$$

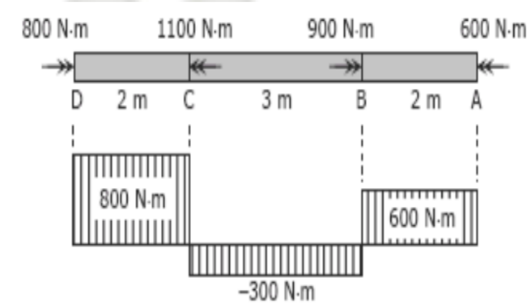
$$T_{\max}=\frac{\pi}{2}\times 20^3\times 75\times 10^{-6}=0.942kN.m$$

$$t_o=\frac{0.942}{0.6}=1.57kN.m/m$$

EXAMPLE 4-10

An aluminum shaft with a constant diameter of 50 mm is loaded by torques applied to gears attached to it as shown in figure. Using G = 28 GPa, determine the relative angle of twist of gear D relative to gear A.





$$J=\frac{\pi}{2}r^4=\frac{\pi}{2}25^4=613592.3mm^4$$

$$\phi=\sum\frac{TL}{GJ}$$

$$\phi=\frac{(800\times 2000-300\times 3000+600\times 2000)\times 10^3}{613592.3\times 28000}$$

$$\phi=0.1106$$