Chapter 8

MAGNETIC FORCES, MATERIALS, AND DEVICES

Do all the good you can,
By all the means you can,
In all the ways you can,
In all the places you can,
At all the times you can,
To all the people you can,
As long as ever you can.

—JOHN WESLEY

8.1 INTRODUCTION

Having considered the basic laws and techniques commonly used in calculating magnetic field **B** due to current-carrying elements, we are prepared to study the force a magnetic field exerts on charged particles, current elements, and loops. Such a study is important to problems on electrical devices such as ammeters, voltmeters, galvanometers, cyclotrons, plasmas, motors, and magnetohydrodynamic generators. The precise definition of the magnetic field, deliberately sidestepped in the previous chapter, will be given here. The concepts of magnetic moments and dipole will also be considered.

Furthermore, we will consider magnetic fields in material media, as opposed to the magnetic fields in vacuum or free space examined in the previous chapter. The results of the preceding chapter need only some modification to account for the presence of materials in a magnetic field. Further discussions will cover inductors, inductances, magnetic energy, and magnetic circuits.

8.2 FORCES DUE TO MAGNETIC FIELDS

There are at least three ways in which force due to magnetic fields can be experienced. The force can be (a) due to a moving charged particle in a **B** field, (b) on a current element in an external **B** field, or (c) between two current elements.

A. Force on a Charged Particle

According to our discussion in Chapter 4, the electric force $\mathbf{F}_e$ on a stationary or moving electric charge Q in an electric field is given by Coulomb's experimental law and is related to the electric field intensity $\mathbf{E}$ as

$$\mathbf{F}_e = Q\mathbf{E} \tag{8.1}$$

This shows that if Q is positive, $\mathbf{F}_e$ and $\mathbf{E}$ have the same direction.

A magnetic field can exert force only on a moving charge. From experiments, it is found that the magnetic force $\mathbf{F}_m$ experienced by a charge Q moving with a velocity $\mathbf{u}$ in a magnetic field $\mathbf{B}$ is

$$\mathbf{F}_m = Q\mathbf{u} \times \mathbf{B} \tag{8.2}$$

This clearly shows that $\mathbf{F}_m$ is perpendicular to both $\mathbf{u}$ and $\mathbf{B}$.

From eqs. (8.1) and (8.2), a comparison between the electric force $\mathbf{F}_e$ and the magnetic force $\mathbf{F}_m$ can be made. $\mathbf{F}_e$ is independent of the velocity of the charge and can perform work on the charge and change its kinetic energy. Unlike $\mathbf{F}_e$, $\mathbf{F}_m$ depends on the charge velocity and is normal to it. $\mathbf{F}_m$ cannot perform work because it is at right angles to the direction of motion of the charge ($\mathbf{F}_m \cdot d\mathbf{l} = 0$); it does not cause an increase in kinetic energy of the charge. The magnitude of $\mathbf{F}_m$ is generally small compared to $\mathbf{F}_e$ except at high velocities.

For a moving charge Q in the presence of both electric and magnetic fields, the total force on the charge is given by

$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m$$

or

$$\boxed{\mathbf{F} = Q(\mathbf{E} + \mathbf{u} \times \mathbf{B})} \tag{8.3}$$

This is known as the *Lorentz force equation.*[1] It relates mechanical force to electrical force. If the mass of the charged particle moving in $\mathbf{E}$ and $\mathbf{B}$ fields is m, by Newton's second law of motion.

$$\mathbf{F} = m\frac{d\mathbf{u}}{dt} = Q\,(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \tag{8.4}$$

The solution to this equation is important in determining the motion of charged particles in $\mathbf{E}$ and $\mathbf{B}$ fields. We should bear in mind that in such fields, energy transfer can be only by means of the electric field. A summary on the force exerted on a charged particle is given in Table 8.1.

Since eq. (8.2) is closely parallel to eq. (8.1), which defines the electric field, some authors and instructors prefer to begin their discussions on magnetostatics from eq. (8.2) just as discussions on electrostatics usually begin with Coulomb's force law.

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[1]After Hendrik Lorentz (1853–1928), who first applied the equation to electric field motion.

TABLE 8.1 Force on a Charged Particle

State of Particle	E Field	B Field	Combined E and B Fields
Stationary	$Q\mathbf{E}$	—	$Q\mathbf{E}$
Moving	$Q\mathbf{E}$	$Q\mathbf{u} \times \mathbf{B}$	$Q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$

B. Force on a Current Element

To determine the force on a current element $I\,d\mathbf{l}$ of a current-carrying conductor due to the magnetic field $\mathbf{B}$, we modify eq. (8.2) using the fact that for convection current [see eq. (5.7)]:

$$\mathbf{J} = \rho_v \mathbf{u} \tag{8.5}$$

From eq. (7.5), we recall the relationship between current elements:

$$I\,d\mathbf{l} = \mathbf{K}\,dS = \mathbf{J}\,dv \tag{8.6}$$

Combining eqs. (8.5) and (8.6) yields

$$I\,d\mathbf{l} = \rho_v \mathbf{u}\,dv = dQ\,\mathbf{u}$$

Alternatively, $I\,d\mathbf{l} = \dfrac{dQ}{dt}\,d\mathbf{l} = dQ\,\dfrac{d\mathbf{l}}{dt} = dQ\,\mathbf{u}$

Hence,

$$\boxed{I\,d\mathbf{l} = dQ\,\mathbf{u}} \tag{8.7}$$

This shows that an elemental charge dQ moving with velocity $\mathbf{u}$ (thereby producing convection current element $dQ\,\mathbf{u}$) is equivalent to a conduction current element $I\,d\mathbf{l}$. Thus the force on a current element $I\,d\mathbf{l}$ in a magnetic field $\mathbf{B}$ is found from eq. (8.2) by merely replacing $Q\mathbf{u}$ by $I\,d\mathbf{l}$; that is,

$$d\mathbf{F} = I\,d\mathbf{l} \times \mathbf{B} \tag{8.8}$$

If the current I is through a closed path L or circuit, the force on the circuit is given by

$$\boxed{\mathbf{F} = \oint_L I\,d\mathbf{l} \times \mathbf{B}} \tag{8.9}$$

In using eq. (8.8) or (8.9), we should keep in mind that the magnetic field produced by the current element $I\,d\mathbf{l}$ does not exert force on the element itself just as a point charge does not exert force on itself. The $\mathbf{B}$ field that exerts force on $I\,d\mathbf{l}$ must be due to another element. In other words, the $\mathbf{B}$ field in eq. (8.8) or (8.9) is external to the current element $I\,d\mathbf{l}$. If instead of the line current element $I\,d\mathbf{l}$, we have surface current elements $\mathbf{K}\,dS$

or a volume current element $\mathbf{J}\,dv$, we simply make use of eq. (8.6) so that eq. (8.8) becomes

$$d\mathbf{F} = \mathbf{K}\,dS \times \mathbf{B} \qquad \text{or} \qquad d\mathbf{F} = \mathbf{J}\,dv \times \mathbf{B} \tag{8.8a}$$

while eq. (8.9) becomes

$$\mathbf{F} = \int_S \mathbf{K}\,dS \times \mathbf{B} \qquad \text{or} \qquad \mathbf{F} = \int_v \mathbf{J}\,dv \times \mathbf{B} \tag{8.9a}$$

From eq. (8.8)

The **magnetic field B** is defined as the force per unit current element.

Alternatively, $\mathbf{B}$ may be defined from eq. (8.2) as the vector which satisfies $\mathbf{F}_m/q = \mathbf{u} \times \mathbf{B}$ just as we defined electric field $\mathbf{E}$ as the force per unit charge, $\mathbf{F}_e/q$. Both of these definitions of $\mathbf{B}$ show that $\mathbf{B}$ describes the force properties of a magnetic field.

C. Force between Two Current Elements

Let us now consider the force between two elements $I_1\,d\mathbf{l}_1$ and $I_2\,d\mathbf{l}_2$. According to Biot–Savart's law, both current elements produce magnetic fields. So we may find the force $d(d\mathbf{F}_1)$ on element $I_1\,d\mathbf{l}_1$ due to the field $d\mathbf{B}_2$ produced by element $I_2\,d\mathbf{l}_2$ as shown in Figure 8.1. From eq. (8.8),

$$d(d\mathbf{F}_1) = I_1\,d\mathbf{l}_1 \times d\mathbf{B}_2 \tag{8.10}$$

But from Biot–Savart's law,

$$d\mathbf{B}_2 = \frac{\mu_o I_2\,d\mathbf{l}_2 \times \mathbf{a}_{R_{21}}}{4\pi R_{21}^2} \tag{8.11}$$

Hence,

$$d(d\mathbf{F}_1) = \frac{\mu_o I_1\,d\mathbf{l}_1 \times (I_2\,d\mathbf{l}_2 \times \mathbf{a}_{R_{21}})}{4\pi R_{21}^2} \tag{8.12}$$

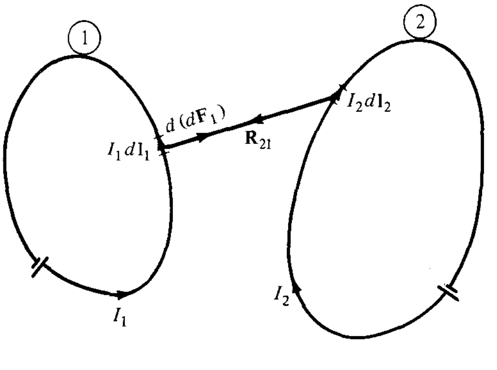


Figure 8.1 Force between two current loops.

This equation is essentially the law of force between two current elements and is analogous to Coulomb's law, which expresses the force between two stationary charges. From eq. (8.12), we obtain the total force $\mathbf{F}_1$ on current loop 1 due to current loop 2 shown in Figure 8.1 as

$$\mathbf{F}_1 = \frac{\mu_o I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{d\mathbf{l}_1 \times (d\mathbf{l}_2 \times \mathbf{a}_{R_{21}})}{R_{21}^2} \tag{8.13}$$

Although this equation appears complicated, we should remember that it is based on eq. (8.10). It is eq. (8.9) or (8.10) that is of fundamental importance.

The force $\mathbf{F}_2$ on loop 2 due to the magnetic field $\mathbf{B}_1$ from loop 1 is obtained from eq. (8.13) by interchanging subscripts 1 and 2. It can be shown that $\mathbf{F}_2 = -\mathbf{F}_1$; thus $\mathbf{F}_1$ and $\mathbf{F}_2$ obey Newton's third law that action and reaction are equal and opposite. It is worthwhile to mention that eq. (8.13) was experimentally established by Oersted and Ampere; Biot and Savart (Ampere's colleagues) actually based their law on it.

EXAMPLE 8.1

A charged particle of mass 2 kg and charge 3 C starts at point $(1, -2, 0)$ with velocity $4\mathbf{a}_x + 3\mathbf{a}_z$ m/s in an electric field $12\mathbf{a}_x + 10\mathbf{a}_y$ V/m. At time $t = 1$ s, determine

(a) The acceleration of the particle

(b) Its velocity

(c) Its kinetic energy

(d) Its position

Solution:

(a) This is an initial-value problem because initial values are given. According to Newton's second law of motion,

$$\mathbf{F} = m\mathbf{a} = Q\mathbf{E}$$

where $\mathbf{a}$ is the acceleration of the particle. Hence,

$$\mathbf{a} = \frac{Q\mathbf{E}}{m} = \frac{3}{2}(12\mathbf{a}_x + 10\mathbf{a}_y) = 18\mathbf{a}_x + 15\mathbf{a}_y \text{ m/s}^2$$

$$\mathbf{a} = \frac{d\mathbf{u}}{dt} = \frac{d}{dt}(u_x, u_y, u_z) = 18\mathbf{a}_x + 15\mathbf{a}_y$$

(b) Equating components gives

$$\frac{du_x}{dt} = 18 \rightarrow u_x = 18t + A \tag{8.1.1}$$

$$\frac{du_y}{dt} = 15 \rightarrow u_y = 15t + B \tag{8.1.2}$$

$$\frac{du_z}{dt} = 0 \rightarrow u_z = C \tag{8.1.3}$$

where A, B, and C are integration constants. But at $t = 0$, $\mathbf{u} = 4\mathbf{a}_x + 3\mathbf{a}_z$. Hence,

$$u_x(t = 0) = 4 \rightarrow 4 = 0 + A \quad \text{or} \quad A = 4$$

$$u_y(t = 0) = 0 \rightarrow 0 = 0 + B \quad \text{or} \quad B = 0$$

$$u_z(t = 0) = 3 \rightarrow 3 = C$$

Substituting the values of A, B, and C into eqs. (8.1.1) to (8.1.3) gives

$$\mathbf{u}(t) = (u_x, u_y, u_z) = (18t + 4, 15t, 3)$$

Hence

$$\mathbf{u}(t = 1 \text{ s}) = 22\mathbf{a}_x + 15\mathbf{a}_y + 3\mathbf{a}_z \text{ m/s}$$

(c) Kinetic energy (K.E.) $= \frac{1}{2}m\,|\mathbf{u}|^2 = \frac{1}{2}(2)(22^2 + 15^2 + 3^2)$
$= 718$ J

(d) $\mathbf{u} = \frac{d\mathbf{l}}{dt} = \frac{d}{dt}(x, y, z) = (18t + 4, 15t, 3)$

Equating components yields

$$\frac{dx}{dt} = u_x = 18t + 4 \rightarrow x = 9t^2 + 4t + A_1 \tag{8.1.4}$$

$$\frac{dy}{dt} = u_y = 15t \quad \rightarrow y = 7.5t^2 + B_1 \tag{8.1.5}$$

$$\frac{dz}{dt} = u_z = 3 \quad \rightarrow z = 3t + C_1 \tag{8.1.6}$$

At $t = 0$, $(x, y, z) = (1, -2, 0)$; hence,

$$x(t = 0) = 1 \rightarrow 1 = 0 + A_1 \quad \text{or} \quad A_1 = 1$$

$$y(t = 0) = -2 \rightarrow -2 = 0 + B_1 \quad \text{or} \quad B_1 = -2$$

$$z(t = 0) = 0 \rightarrow 0 = 0 + C_1 \quad \text{or} \quad C_1 = 0$$

Substituting the values of A_1, B_1, and C_1 into eqs. (8.1.4) to (8.1.6), we obtain

$$(x, y, z) = (9t^2 + 4t + 1, 7.5t^2 - 2, 3t) \tag{8.1.7}$$

Hence, at $t = 1$, $(x, y, z) = (14, 5.5, 3)$.
By eliminating t in eq. (8.1.7), the motion of the particle may be described in terms of x, y, and z.

PRACTICE EXERCISE 8.1

A charged particle of mass 1 kg and charge 2 C starts at the origin with zero initial velocity in a region where $\mathbf{E} = 3\mathbf{a}_z$ V/m. Find

(a) The force on the particle

(b) The time it takes to reach point $P(0, 0, 12 \text{ m})$

(c) Its velocity and acceleration at P

(d) Its K.E. at P.

Answer: (a) $6\mathbf{a}_z$ N, (b) 2 s, (c) $12\mathbf{a}_z$ m/s, $6\mathbf{a}_z$ m/s^2, (d) 72 J.

EXAMPLE 8.2

A charged particle of mass 2 kg and 1 C starts at the origin with velocity $3\mathbf{a}_y$ m/s and travels in a region of uniform magnetic field $\mathbf{B} = 10\mathbf{a}_z$ Wb/m^2. At $t = 4$ s, calculate

(a) The velocity and acceleration of the particle

(b) The magnetic force on it

(c) Its K.E. and location

(d) Find the particle's trajectory by eliminating t.

(e) Show that its K.E. remains constant.

Solution:

(a) $\mathbf{F} = m\dfrac{d\mathbf{u}}{dt} = Q\mathbf{u} \times \mathbf{B}$

$$\mathbf{a} = \frac{d\mathbf{u}}{dt} = \frac{Q}{m}\mathbf{u} \times \mathbf{B}$$

Hence

$$\frac{d}{dt}(u_x\mathbf{a}_x + u_y\mathbf{a}_y + u_z\mathbf{a}_z) = \frac{1}{2}\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ u_x & u_y & u_z \\ 0 & 0 & 10 \end{vmatrix} = 5(u_y\mathbf{a}_x - u_x\mathbf{a}_y)$$

By equating components, we get

$$\frac{du_x}{dt} = 5u_y \tag{8.2.1}$$

$$\frac{du_y}{dt} = -5u_x \tag{8.2.2}$$

$$\frac{du_z}{dt} = 0 \rightarrow u_z = C_o \tag{8.2.3}$$

We can eliminate u_x or u_y in eqs. (8.2.1) and (8.2.2) by taking second derivatives of one equation and making use of the other. Thus

$$\frac{d^2u_x}{dt^2} = 5\frac{du_y}{dt} = -25u_x$$

or

$$\frac{d^2u_x}{dt^2} + 25u_x = 0$$

which is a linear differential equation with solution (see Case 3 of Example 6.5)

$$u_x = C_1 \cos 5t + C_2 \sin 5t \tag{8.2.4}$$

From eqs. (8.2.1) and (8.2.4),

$$5u_y = \frac{du_x}{dt} = -5C_1 \sin 5t + 5C_2 \cos 5t \tag{8.2.5}$$

or

$$u_y = -C_1 \sin 5t + C_2 \cos 5t$$

We now determine constants C_o, C_1, and C_2 using the initial conditions. At $t = 0$, $\mathbf{u} = 3\mathbf{a}_y$. Hence,

$$u_x = 0 \rightarrow 0 = C_1 \cdot 1 + C_2 \cdot 0 \rightarrow C_1 = 0$$
$$u_y = 3 \rightarrow 3 = -C_1 \cdot 0 + C_2 \cdot 1 \rightarrow C_2 = 3$$
$$u_z = 0 \rightarrow 0 = C_o$$

Substituting the values of C_o, C_1, and C_2 into eqs. (8.2.3) to (8.2.5) gives

$$\mathbf{u} = (u_x, u_y, u_z) = (3 \sin 5t, 3 \cos 5t, 0) \tag{8.2.6}$$

Hence,

$$\begin{aligned}\mathbf{u}(t = 4) &= (3 \sin 20, 3 \cos 20, 0) \\ &= 2.739\mathbf{a}_x + 1.224\mathbf{a}_y \text{ m/s}\end{aligned}$$

$$\mathbf{a} = \frac{d\mathbf{u}}{dt} = (15 \cos 5t, -15 \sin 5t, 0)$$

and

$$\mathbf{a}(t = 4) = 6.101\mathbf{a}_x - 13.703\mathbf{a}_y \text{ m/s}^2$$

(b)

$$\mathbf{F} = m\mathbf{a} = 12.2\mathbf{a}_x - 27.4\mathbf{a}_y \text{ N}$$

or

$$\begin{aligned}\mathbf{F} &= Q\mathbf{u} \times \mathbf{B} = (1)(2.739\mathbf{a}_x + 1.224\mathbf{a}_y) \times 10\mathbf{a}_z \\ &= 12.2\mathbf{a}_x - 27.4\mathbf{a}_y \text{ N}\end{aligned}$$

(c) K.E. $= 1/2m\,|\mathbf{u}|^2 = 1/2(2)\,(2.739^2 + 1.224^2) = 9$ J

$$u_x = \frac{dx}{dt} = 3 \sin 5t \rightarrow x = -\frac{3}{5}\cos 5t + b_1 \qquad (8.2.7)$$

$$u_y = \frac{dy}{dt} = 3 \cos 5t \rightarrow y = \frac{3}{5}\sin 5t + b_2 \qquad (8.2.8)$$

$$u_z = \frac{dz}{dt} = 0 \rightarrow z = b_3 \qquad (8.2.9)$$

where b_1, b_2, and b_3 are integration constants. At $t = 0$, $(x, y, z) = (0, 0, 0)$ and hence,

$$x(t = 0) = 0 \rightarrow 0 = -\frac{3}{5}\cdot 1 + b_1 \rightarrow b_1 = 0.6$$

$$y(t = 0) = 0 \rightarrow 0 = \frac{3}{5}\cdot 0 + b_2 \rightarrow b_2 = 0$$

$$z(t = 0) = 0 \rightarrow 0 = b_3$$

Substituting the values of b_1, b_2, and b_3 into eqs. (8.2.7) to (8.2.9), we obtain

$$(x, y, z) = (0.6 - 0.6 \cos 5t, 0.6 \sin 5t, 0) \qquad (8.2.10)$$

At $t = 4$ s,

$$(x, y, z) = (0.3552, 0.5478, 0)$$

(d) From eq. (8.2.10), we eliminate t by noting that

$$(x - 0.6)^2 + y^2 = (0.6)^2 (\cos^2 5t + \sin^2 5t), \qquad z = 0$$

or

$$(x - 0.6)^2 + y^2 = (0.6)^2, \qquad z = 0$$

which is a circle on plane $z = 0$, centered at (0.6, 0, 0) and of radius 0.6 m. Thus the particle gyrates in an orbit about a magnetic field line.

(e) $$\text{K.E.} = \frac{1}{2} m\,|\mathbf{u}|^2 = \frac{1}{2}(2)\,(9 \cos^2 5t + 9 \sin^2 5t) = 9 \text{ J}$$

which is the same as the K.E. at $t = 0$ and $t = 4$ s. Thus the uniform magnetic field has no effect on the K.E. of the particle.

Note that the angular velocity $\omega = QB/m$ and the radius of the orbit $r = u_o/\omega$, where u_o is the initial speed. An interesting application of the idea in this example is found in a common method of focusing a beam of electrons. The method employs a uniform magnetic field directed parallel to the desired beam as shown in Figure 8.2. Each electron emerging from the electron gun follows a helical path and is back on the axis at the same focal point with other electrons. If the screen of a cathode ray tube were at this point, a single spot would appear on the screen.

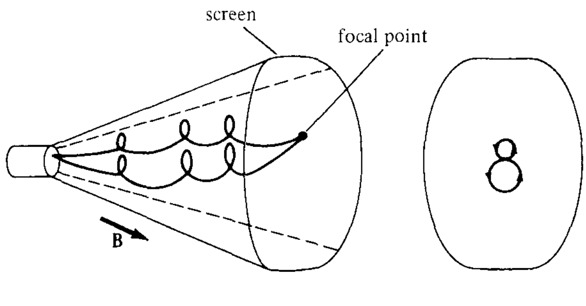


Figure 8.2 Magnetic focusing of a beam of electrons: **(a)** helical paths of electrons, **(b)** end view of paths.

PRACTICE EXERCISE 8.2

A proton of mass m is projected into a uniform field $\mathbf{B} = B_o\mathbf{a}_z$ with an initial velocity $\alpha\mathbf{a}_x + \beta\mathbf{a}_z$. (a) Find the differential equations that the position vector $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ must satisfy. (b) Show that a solution to these equations is

$$x = \frac{\alpha}{\omega}\sin\omega t, \qquad y = \frac{\alpha}{\omega}\cos\omega t, \qquad z = \beta t$$

where $\omega = eB_o/m$ and e is the charge on the proton. (c) Show that this solution describes a circular helix in space.

Answer: (a) $\frac{dx}{dt} = \alpha\cos\omega t$, $\frac{dy}{dt} = -\alpha\sin\omega t$, $\frac{dz}{dt} = \beta$, (b) and (c) Proof.

EXAMPLE 8.3

A charged particle moves with a uniform velocity $4\mathbf{a}_x$ m/s in a region where $\mathbf{E} = 20\,\mathbf{a}_y$ V/m and $\mathbf{B} = B_o\mathbf{a}_z$ Wb/m^2. Determine B_o such that the velocity of the particle remains constant.

Solution:

If the particle moves with a constant velocity, it implies that its acceleration is zero. In other words, the particle experiences no net force. Hence,

$$0 = \mathbf{F} = m\mathbf{a} = Q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

$$0 = Q(20\mathbf{a}_y + 4\mathbf{a}_x \times B_o\mathbf{a}_z)$$

or

$$-20\mathbf{a}_y = -4B_o\mathbf{a}_y$$

Thus $B_o = 5$.

This example illustrates an important principle employed in a velocity filter shown in Figure 8.3. In this application, $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{u}$ are mutually perpendicular so that $Q\mathbf{u} \times \mathbf{B}$ is

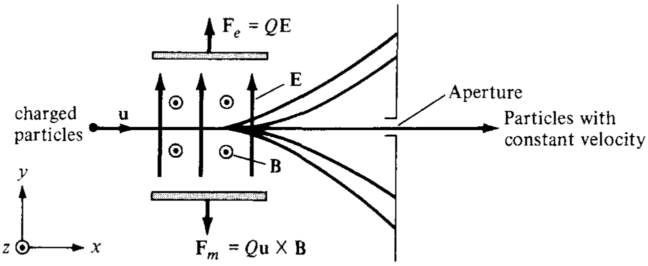


Figure 8.3 A velocity filter for charged particles.

directed opposite to $Q\mathbf{E}$, regardless of the sign of the charge. When the magnitudes of the two vectors are equal,

$$QuB = QE$$

or

$$u = \frac{E}{B}$$

This is the required (critical) speed to balance out the two parts of the Lorentz force. Particles with this speed are undeflected by the fields; they are "filtered" through the aperture. Particles with other speeds are deflected down or up, depending on whether their speeds are greater or less than this critical speed.

PRACTICE EXERCISE 8.3

Uniform **E** and **B** fields are oriented at right angles to each other. An electron moves with a speed of 8×10^6 m/s at right angles to both fields and passes undeflected through the field.

(a) If the magnitude of **B** is 0.5 mWb/m^2, find the value of **E**.

(b) Will this filter work for positive and negative charges and any value of mass?

Answer: (a) 4 kV/m, (b) Yes.

EXAMPLE 8.4

A rectangular loop carrying current I_2 is placed parallel to an infinitely long filamentary wire carrying current I_1 as shown in Figure 8.4(a). Show that the force experienced by the loop is given by

$$\mathbf{F} = -\frac{\mu_o I_1 I_2 b}{2\pi}\left[\frac{1}{\rho_o} - \frac{1}{\rho_o + a}\right]\mathbf{a}_\rho \text{ N}$$

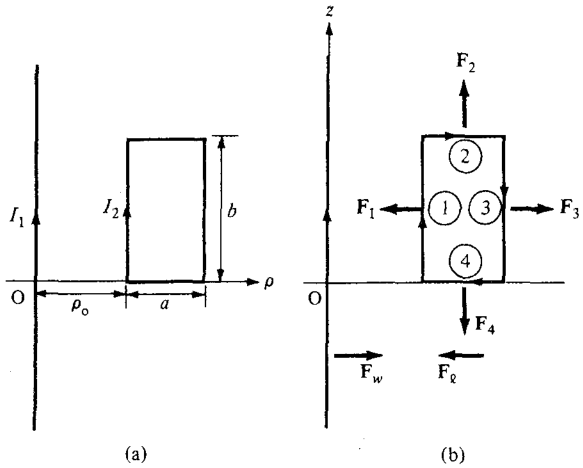


Figure 8.4 For Example 8.4: **(a)** rectangular loop inside the field produced by an infinitely long wire, **(b)** forces acting on the loop and wire.

Solution:

Let the force on the loop be

$$\mathbf{F}_\ell = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 = I_2 \oint d\mathbf{l}_2 \times \mathbf{B}_1$$

where $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$ are, respectively, the forces exerted on sides of the loop labeled 1, 2, 3, and 4 in Figure 8.4(b). Due to the infinitely long wire

$$\mathbf{B}_1 = \frac{\mu_o I_1}{2\pi\rho_o}\,\mathbf{a}_\phi$$

Hence,

$$\begin{aligned}\mathbf{F}_1 &= I_2 \int d\mathbf{l}_2 \times \mathbf{B}_1 = I_2 \int_{z=0}^{b} dz\,\mathbf{a}_z \times \frac{\mu_o I_1}{2\pi\rho_o}\,\mathbf{a}_\phi \\ &= -\frac{\mu_o I_1 I_2 b}{2\pi\rho_o}\,\mathbf{a}_\rho \qquad \text{(attractive)}\end{aligned}$$

$\mathbf{F}_1$ is attractive because it is directed toward the long wire; that is, $\mathbf{F}_1$ is along $-\mathbf{a}_\rho$ due to the fact that loop side 1 and the long wire carry currents along the same direction. Similarly,

$$\begin{aligned}\mathbf{F}_3 &= I_2 \int d\mathbf{l}_2 \times \mathbf{B}_1 = I_2 \int_{z=b}^{0} dz\,\mathbf{a}_z \times \frac{\mu_o I_1}{2\pi(\rho_o + a)}\,\mathbf{a}_\phi \\ &= \frac{\mu_o I_1 I_2 b}{2\pi(\rho_o + a)}\,\mathbf{a}_\rho \qquad \text{(repulsive)}\end{aligned}$$

$$\begin{aligned}\mathbf{F}_2 &= I_2 \int_{\rho=\rho_o}^{\rho_o+a} d\rho\,\mathbf{a}_\rho \times \frac{\mu_o I_1\,\mathbf{a}_\phi}{2\pi\rho} \\ &= \frac{\mu_o I_1 I_2}{2\pi} \ln \frac{\rho_o + a}{\rho_o}\,\mathbf{a}_z \qquad \text{(parallel)}\end{aligned}$$

$$\mathbf{F}_4 = I_2 \int_{\rho=\rho_o+a}^{\rho_o} d\rho\, \mathbf{a}_\rho \times \frac{\mu_o I_1\, \mathbf{a}_\phi}{2\pi\rho}$$

$$= -\frac{\mu_o I_1 I_2}{2\pi} \ln \frac{\rho_o + a}{\rho_o}\, \mathbf{a}_z \qquad \text{(parallel)}$$

The total force $\mathbf{F}_\ell$ on the loop is the sum of $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$; that is,

$$\mathbf{F}_\ell = \frac{\mu_o I_1 I_2 b}{2\pi} \left[\frac{1}{\rho_o} - \frac{1}{\rho_o + a} \right] (-\mathbf{a}_\rho)$$

which is an attractive force trying to draw the loop toward the wire. The force $\mathbf{F}_w$ on the wire, by Newton's third law, is $-\mathbf{F}_\ell$; see Figure 8.4(b).

PRACTICE EXERCISE 8.4

In Example 8.4, find the force experienced by the infinitely long wire if $I_1 = 10$ A, $I_2 = 5$ A, $\rho_o = 20$ cm, $a = 10$ cm, $b = 30$ cm.

Answer: $5\mathbf{a}_\rho$ μN.

8.3 MAGNETIC TORQUE AND MOMENT

Now that we have considered the force on a current loop in a magnetic field, we can determine the torque on it. The concept of a current loop experiencing a torque in a magnetic field is of paramount importance in understanding the behavior of orbiting charged particles, d.c. motors, and generators. If the loop is placed parallel to a magnetic field, it experiences a force that tends to rotate it.

The **torque T** (or mechanical moment of force) on the loop is the vector product of the force **F** and the moment arm **r**.

That is,

$$\mathbf{T} = \mathbf{r} \times \mathbf{F} \tag{8.14}$$

and its units are Newton-meters (N · m).

Let us apply this to a rectangular loop of length ℓ and width w placed in a uniform magnetic field **B** as shown in Figure 8.5(a). From this figure, we notice that $d\mathbf{l}$ is parallel to **B** along sides 12 and 34 of the loop and no force is exerted on those sides. Thus

$$\mathbf{F} = I \int_2^3 d\mathbf{l} \times \mathbf{B} + I \int_4^1 d\mathbf{l} \times \mathbf{B}$$

$$= I \int_0^\ell dz\, \mathbf{a}_z \times \mathbf{B} + I \int_\ell^0 dz\, \mathbf{a}_z \times \mathbf{B}$$

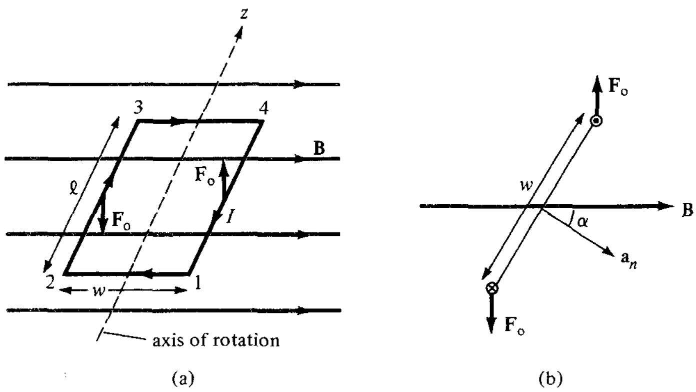


Figure 8.5 Rectangular planar loop in a uniform magnetic field.

or

$$\mathbf{F} = \mathbf{F}_o - \mathbf{F}_o = 0 \tag{8.15}$$

where $|\mathbf{F}_o| = IB\ell$ because $\mathbf{B}$ is uniform. Thus, no force is exerted on the loop as a whole. However, $\mathbf{F}_o$ and $-\mathbf{F}_o$ act at different points on the loop, thereby creating a couple. If the normal to the plane of the loop makes an angle α with $\mathbf{B}$, as shown in the cross-sectional view of Figure 8.5(b), the torque on the loop is

$$|\mathbf{T}| = |\mathbf{F}_o|\, w \sin \alpha$$

or

$$T = BI\ell w \sin \alpha \tag{8.16}$$

But $\ell w = S$, the area of the loop. Hence,

$$T = BIS \sin \alpha \tag{8.17}$$

We define the quantity

$$\boxed{\mathbf{m} = IS\mathbf{a}_n} \tag{8.18}$$

as the *magnetic dipole moment* (in A/m^2) of the loop. In eq. (8.18), $\mathbf{a}_n$ is a unit normal vector to the plane of the loop and its direction is determined by the right-hand rule: fingers in the direction of current and thumb along $\mathbf{a}_n$.

The **magnetic dipole moment** is the product of current and area of the loop; its direction is normal to the loop.

Introducing eq. (8.18) in eq. (8.17), we obtain

$$\boxed{\mathbf{T} = \mathbf{m} \times \mathbf{B}} \tag{8.19}$$

This expression is generally applicable in determining the torque on a planar loop of any arbitrary shape although it was obtained using a rectangular loop. The only limitation is that the magnetic field must be uniform. It should be noted that the torque is in the direction of the axis of rotation (the z-axis in the case of Figure 8.5a). It is directed such as to reduce α so that **m** and **B** are in the same direction. In an equilibrium position (when **m** and **B** are in the same direction), the loop is perpendicular to the magnetic field and the torque will be zero as well as the sum of the forces on the loop.

8.4 A MAGNETIC DIPOLE

A bar magnet or a small filamentary current loop is usually referred to as a *magnetic dipole.* The reason for this and what we mean by "small" will soon be evident. Let us determine the magnetic field **B** at an observation point $P(r, \theta, \phi)$ due to a circular loop carrying current I as in Figure 8.6. The magnetic vector potential at P is

$$\mathbf{A} = \frac{\mu_o I}{4\pi} \oint \frac{d\mathbf{l}}{r} \tag{8.20}$$

It can be shown that at far field ($r \gg a$, so that the loop appears small at the observation point), **A** has only ϕ-component and it is given by

$$\mathbf{A} = \frac{\mu_o I \pi a^2 \sin\theta\, \mathbf{a}_\phi}{4\pi r^2} \tag{8.21a}$$

or

$$\boxed{\mathbf{A} = \frac{\mu_o\, \mathbf{m} \times \mathbf{a}_r}{4\pi r^2}} \tag{8.21b}$$

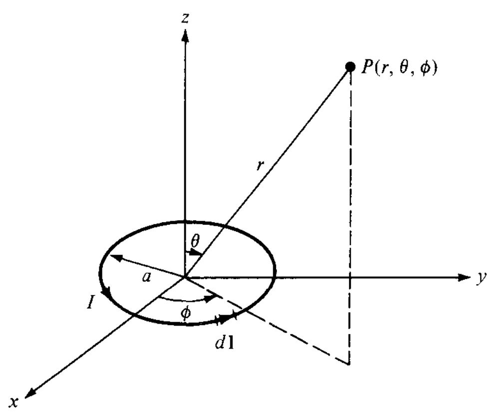


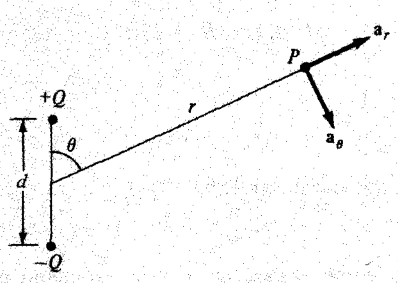
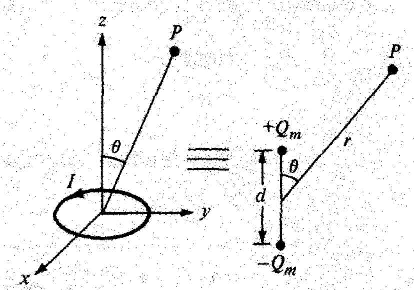
Figure 8.6 Magnetic field at P due to a current loop.

where $\mathbf{m} = I\pi a^2 \mathbf{a}_z$, the magnetic moment of the loop, and $\mathbf{a}_z \times \mathbf{a}_r = \sin\theta\, \mathbf{a}_\phi$. We determine the magnetic flux density $\mathbf{B}$ from $\mathbf{B} = \nabla \times \mathbf{A}$ as

$$\mathbf{B} = \frac{\mu_o m}{4\pi r^3}(2\cos\theta\, \mathbf{a}_r + \sin\theta\, \mathbf{a}_\theta) \tag{8.22}$$

It is interesting to compare eqs. (8.21) and (8.22) with similar expressions in eqs. (4.80) and (4.82) for electrical potential V and electric field intensity $\mathbf{E}$ due to an electric dipole. This comparison is done in Table 8.2, in which we notice the striking similari-

TABLE 8.2 Comparison between Electric and Magnetic Monopoles and Dipoles

Electric	Magnetic
$V = \dfrac{Q}{4\pi\varepsilon_o r}$ $\mathbf{E} = \dfrac{Q\mathbf{a}_r}{4\pi\varepsilon_o r^2}$  Monopole (point charge)	Does not exist  Monopole (point charge)
$V = \dfrac{Q\cos\theta}{4\pi\varepsilon_o r^2}$ $\mathbf{E} = \dfrac{Qd}{4\pi\varepsilon_o r^3}(2\cos\theta\, \mathbf{a}_r + \sin\theta\, \mathbf{a}_\theta)$  Dipole (two point charge)	$\mathbf{A} = \dfrac{\mu_o m \sin\theta\, \mathbf{a}_\phi}{4\pi r^2}$ $\mathbf{B} = \dfrac{\mu_o m}{4\pi r^3}(2\cos\theta\mathbf{a}_r + \sin\theta\mathbf{a}_\theta)$  Dipole (small current loop or bar magnet)

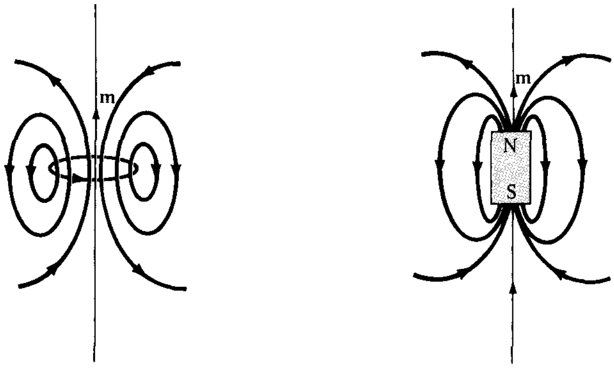


Figure 8.7 The **B** lines due to magnetic dipoles: **(a)** a small current loop with $\mathbf{m} = I\mathbf{S}$, **(b)** a bar magnet with $\mathbf{m} = Q_m\boldsymbol{\ell}$.

ties between **B** as far field due to a small current loop and **E** at far field due to an electric dipole. It is therefore reasonable to regard a small current loop as a magnetic dipole. The **B** lines due to a magnetic dipole are similar to the **E** lines due to an electric dipole. Figure 8.7(a) illustrates the **B** lines around the magnetic dipole $\mathbf{m} = I\mathbf{S}$.

A short permanent magnetic bar, shown in Figure 8.7(b), may also be regarded as a magnetic dipole. Observe that the **B** lines due to the bar are similar to those due to a small current loop in Figure 8.7(a).

Consider the bar magnet of Figure 8.8. If Q_m is an isolated magnetic charge (pole strength) and ℓ is the length of the bar, the bar has a dipole moment $Q_m\boldsymbol{\ell}$. (Notice that Q_m does exist; however, it does not exist without an associated $-Q_m$. See Table 8.2.) When the bar is in a uniform magnetic field **B**, it experiences a torque

$$\mathbf{T} = \mathbf{m} \times \mathbf{B} = Q_m\boldsymbol{\ell} \times \mathbf{B} \tag{8.23}$$

where $\boldsymbol{\ell}$ points in the direction south-to-north. The torque tends to align the bar with the external magnetic field. The force acting on the magnetic charge is given by

$$\mathbf{F} = Q_m\mathbf{B} \tag{8.24}$$

Since both a small current loop and a bar magnet produce magnetic dipoles, they are equivalent if they produce the same torque in a given **B** field; that is, when

$$T = Q_m\ell B = ISB \tag{8.25}$$

Hence,

$$Q_m\ell = IS \tag{8.26}$$

showing that they must have the same dipole moment.

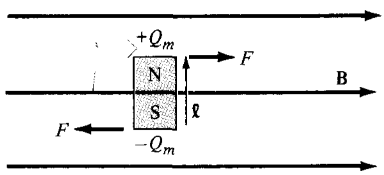


Figure 8.8 A bar magnet in an external magnetic field.

EXAMPLE 8.5

Determine the magnetic moment of an electric circuit formed by the triangular loop of Figure 8.9.

Solution:

From Problem 1.18(c), the equation of a plane is given by $Ax + By + Cz + D = 0$ where $D = -(A^2 + B^2 + C^2)$. Since points (2, 0, 0), (0, 2, 0), and (0, 0, 2) lie on the plane, these points must satisfy the equation of the plane, and the constants A, B, C, and D can be determined. Doing this gives $x + y + z = 2$ as the plane on which the loop lies. Thus we can use

$$\mathbf{m} = IS\mathbf{a}_n$$

where

$$\begin{aligned} S &= \text{loop area} = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2}(2\sqrt{2})(2\sqrt{2})\sin 60° \\ &= 4 \sin 60° \end{aligned}$$

If we define the plane surface by a function

$$f(x,y,z) = x + y + z - 2 = 0,$$

$$\mathbf{a}_n = \pm\frac{\nabla f}{|\nabla f|} = \pm\frac{(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)}{\sqrt{3}}$$

We choose the plus sign in view of the direction of the current in the loop (using the right-hand rule, **m** is directed as in Figure 8.9). Hence

$$\begin{aligned} \mathbf{m} &= 5\,(4 \sin 60°)\,\frac{(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)}{\sqrt{3}} \\ &= 10(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)\ \text{A} \cdot \text{m}^2 \end{aligned}$$

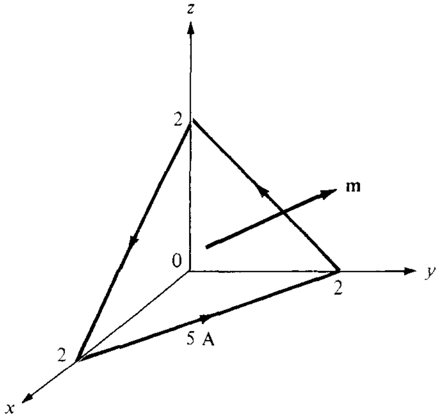


Figure 8.9 Triangular loop of Example 8.5.

PRACTICE EXERCISE 8.5

A rectangular coil of area 10 cm^2 carrying current of 50 A lies on plane $2x + 6y - 3z = 7$ such that the magnetic moment of the coil is directed away from the origin. Calculate its magnetic moment.

Answer: $(1.429\mathbf{a}_x + 4.286\mathbf{a}_y - 2.143\mathbf{a}_z) \times 10^{-2}\ \text{A} \cdot \text{m}^2$

EXAMPLE 8.6

A small current loop L_1 with magnetic moment $5\mathbf{a}_z$ A/m^2 is located at the origin while another small loop current L_2 with magnetic moment $3\mathbf{a}_y$ A · m^2 is located at $(4, -3, 10)$. Determine the torque on L_2.

Solution:

The torque $\mathbf{T}_2$ on the loop L_2 is due to the field $\mathbf{B}_1$ produced by loop L_1. Hence,

$$\mathbf{T}_2 = \mathbf{m}_2 \times \mathbf{B}_1$$

Since $\mathbf{m}_1$ for loop L_1 is along $\mathbf{a}_z$, we find $\mathbf{B}_1$ using eq. (8.22):

$$\mathbf{B}_1 = \frac{\mu_o m_1}{4\pi r^3}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta)$$

Using eq. (2.23), we transform $\mathbf{m}_2$ from Cartesian to spherical coordinates:

$$\mathbf{m}_2 = 3\mathbf{a}_y = 3\,(\sin\theta\sin\phi\,\mathbf{a}_r + \cos\theta\sin\phi\,\mathbf{a}_\theta + \cos\phi\,\mathbf{a}_\phi)$$

At $(4, -3, 10)$,

$$r = \sqrt{4^2 + (-3)^2 + 10^2} = 5\sqrt{5}$$

$$\tan\theta = \frac{\rho}{z} = \frac{5}{10} = \frac{1}{2} \rightarrow \sin\theta = \frac{1}{\sqrt{5}}, \qquad \cos\theta = \frac{2}{\sqrt{5}}$$

$$\tan\phi = \frac{y}{x} = \frac{-3}{4} \rightarrow \sin\phi = \frac{-3}{5}, \qquad \cos\phi = \frac{4}{5}$$

Hence,

$$\mathbf{B}_1 = \frac{4\pi \times 10^{-7} \times 5}{4\pi\, 625\sqrt{5}}\left(\frac{4}{\sqrt{5}}\mathbf{a}_r + \frac{1}{\sqrt{5}}\mathbf{a}_\theta\right)$$
$$= \frac{10^{-7}}{625}(4\mathbf{a}_r + \mathbf{a}_\theta)$$

$$\mathbf{m}_2 = 3\left[-\frac{3\mathbf{a}_r}{5\sqrt{5}} - \frac{6\mathbf{a}_\theta}{5\sqrt{5}} + \frac{4\mathbf{a}_\phi}{5}\right]$$

and

$$\begin{aligned}\mathbf{T} &= \frac{10^{-7}(3)}{625\,(5\sqrt{5})}(-3\mathbf{a}_r - 6\mathbf{a}_\theta + 4\sqrt{5}\mathbf{a}_\phi) \times (4\mathbf{a}_r + \mathbf{a}_\phi) \\ &= 4.293 \times 10^{-11}(-6\mathbf{a}_r + 38.78\mathbf{a}_\theta + 24\mathbf{a}_\phi) \\ &= -0.258\mathbf{a}_r + 1.665\mathbf{a}_\theta + 1.03\mathbf{a}_\phi \text{ nN} \cdot \text{m}\end{aligned}$$

PRACTICE EXERCISE 8.6

If the coil of Practice Exercise 8.5 is surrounded by a uniform field $0.6\mathbf{a}_x + 0.4\mathbf{a}_y + 0.5\mathbf{a}_z$ Wb/m^2,

(a) Find the torque on the coil.

(b) Show that the torque on the coil is maximum if placed on plane $2x - 8y + 4z = \sqrt{84}$. Calculate the value of the maximum torque.

Answer: (a) $0.03\mathbf{a}_x - 0.02\mathbf{a}_y - 0.02\mathbf{a}_z$ N · m, (b) 0.04387 N · m.

8.5 MAGNETIZATION IN MATERIALS

Our discussion here will parallel that on polarization of materials in an electric field. We shall assume that our atomic model is that of an electron orbiting about a positive nucleus.

We know that a given material is composed of atoms. Each atom may be regarded as consisting of electrons orbiting about a central positive nucleus; the electrons also rotate (or spin) about their own axes. Thus an internal magnetic field is produced by electrons orbiting around the nucleus as in Figure 8.10(a) or electrons spinning as in Figure 8.10(b). Both of these electronic motions produce internal magnetic fields $\mathbf{B}_i$ that are similar to the magnetic field produced by a current loop of Figure 8.11. The equivalent current loop has a magnetic moment of $\mathbf{m} = I_b S \mathbf{a}_n$, where S is the area of the loop and I_b is the bound current (bound to the atom).

Without an external **B** field applied to the material, the sum of **m**'s is zero due to random orientation as in Figure 8.12(a). When an external **B** field is applied, the magnetic

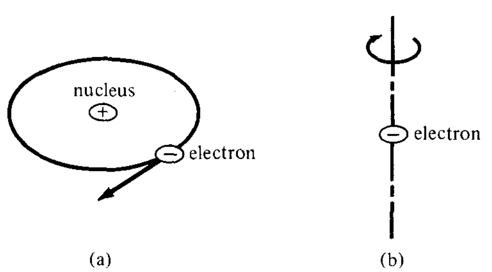


Figure 8.10 **(a)** Electron orbiting around the nucleus; **(b)** electron spin.

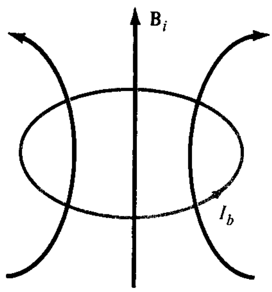


Figure 8.11 Circular current loop equivalent to electronic motion of Figure 8.10.

moments of the electrons more or less align themselves with **B** so that the net magnetic moment is not zero, as illustrated in Figure 8.12(b).

> The **magnetization M** (in amperes/meter) is the magnetic dipole moment per unit volume.

If there are N atoms in a given volume Δv and the kth atom has a magnetic moment $\mathbf{m}_k$,

$$\mathbf{M} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{N} \mathbf{m}_k}{\Delta v} \tag{8.27}$$

A medium for which **M** is not zero everywhere is said to be magnetized. For a differential volume dv', the magnetic moment is $d\mathbf{m} = \mathbf{M}\, dv'$. From eq. (8.21b), the vector magnetic potential due to $d\mathbf{m}$ is

$$d\mathbf{A} = \frac{\mu_o \mathbf{M} \times \mathbf{a}_R}{4\pi R^2}\, dv' = \frac{\mu_o \mathbf{M} \times \mathbf{R}}{4\pi R^3}\, dv'$$

According to eq. (7.46),

$$\frac{\mathbf{R}}{R^3} = \nabla' \frac{1}{R}$$

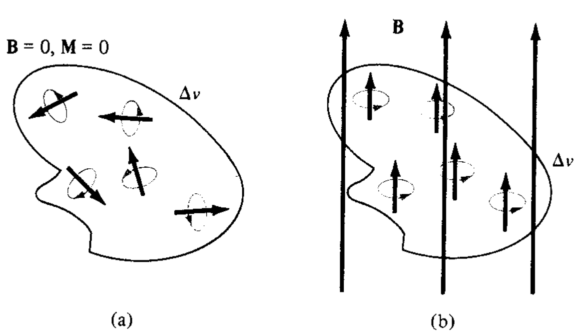


Figure 8.12 Magnetic dipole moment in a volume Δv: **(a)** before **B** is applied, **(b)** after **B** is applied.

Hence,

$$\mathbf{A} = \frac{\mu_o}{4\pi} \int \mathbf{M} \times \nabla' \frac{1}{R} \, dv' \tag{8.28}$$

Using eq. (7.48) gives

$$\mathbf{M} \times \nabla' \frac{1}{R} = \frac{1}{R} \nabla' \times \mathbf{M} - \nabla' \times \frac{\mathbf{M}}{R}$$

Substituting this into eq. (8.28) yields

$$\mathbf{A} = \frac{\mu_o}{4\pi} \int_{v'} \frac{\nabla' \times \mathbf{M}}{R} \, dv' - \frac{\mu_o}{4\pi} \int_{v'} \nabla' \times \frac{\mathbf{M}}{R} \, dv'$$

Applying the vector identity

$$\int_{v'} \nabla' \times \mathbf{F} \, dv' = -\oint_{S'} \mathbf{F} \times d\mathbf{S}$$

to the second integral, we obtain

$$\begin{aligned} \mathbf{A} &= \frac{\mu_o}{4\pi} \int_{v'} \frac{\nabla' \times \mathbf{M}}{R} \, dv' + \frac{\mu_o}{4\pi} \oint_{S'} \frac{\mathbf{M} \times \mathbf{a}_n}{R} \, dS' \\ &= \frac{\mu_o}{4\pi} \int_{v'} \frac{\mathbf{J}_b \, dv'}{R} + \frac{\mu_o}{4\pi} \oint_{S'} \frac{\mathbf{K}_b \, dS'}{R} \end{aligned} \tag{8.29}$$

Comparing eq. (8.29) with eqs. (7.42) and (7.43) (upon dropping the primes) gives

$$\boxed{\mathbf{J}_b = \nabla \times \mathbf{M}} \tag{8.30}$$

and

$$\boxed{\mathbf{K}_b = \mathbf{M} \times \mathbf{a}_n} \tag{8.31}$$

where $\mathbf{J}_b$ is the *bound volume current density* or *magnetization volume current density* (in amperes per meter square), $\mathbf{K}_b$ is the *bound surface current density* (in amperes per meter), and $\mathbf{a}_n$ is a unit vector normal to the surface. Equation (8.29) shows that the potential of a magnetic body is due to a volume current density $\mathbf{J}_b$ throughout the body and a surface current $\mathbf{K}_b$ on the surface of the body. The vector $\mathbf{M}$ is analogous to the polarization $\mathbf{P}$ in dielectrics and is sometimes called the *magnetic polarization density* of the medium. In another sense, $\mathbf{M}$ is analogous to $\mathbf{H}$ and they both have the same units. In this respect, as $\mathbf{J} = \nabla \times \mathbf{H}$, so is $\mathbf{J}_b = \nabla \times \mathbf{M}$. Also, $\mathbf{J}_b$ and $\mathbf{K}_b$ for a magnetized body are similar to ρ_{pv} and ρ_{ps} for a polarized body. As is evident in eqs. (8.29) to (8.31), $\mathbf{J}_b$ and $\mathbf{K}_b$ can be derived from $\mathbf{M}$; therefore, $\mathbf{J}_b$ and $\mathbf{K}_b$ are not commonly used.

In free space, $\mathbf{M} = 0$ and we have

$$\nabla \times \mathbf{H} = \mathbf{J}_f \qquad \text{or} \qquad \nabla \times \left(\frac{\mathbf{B}}{\mu_o}\right) = \mathbf{J}_f \tag{8.32}$$

where $\mathbf{J}_f$ is the free current volume density. In a material medium $\mathbf{M} \neq 0$, and as a result, $\mathbf{B}$ changes so that

$$\begin{aligned}\nabla \times \left(\frac{\mathbf{B}}{\mu_o}\right) &= \mathbf{J}_f + \mathbf{J}_b = \mathbf{J} \\ &= \nabla \times \mathbf{H} + \nabla \times \mathbf{M}\end{aligned}$$

or

$$\boxed{\mathbf{B} = \mu_o(\mathbf{H} + \mathbf{M})} \tag{8.33}$$

The relationship in eq. (8.33) holds for all materials whether they are linear or not. The concepts of linearity, isotropy, and homogeneity introduced in Section 5.7 for dielectric media equally apply here for magnetic media. For linear materials, $\mathbf{M}$ (in A/m) depends linearly on $\mathbf{H}$ such that

$$\boxed{\mathbf{M} = \chi_m \mathbf{H}} \tag{8.34}$$

where χ_m is a dimensionless quantity (ratio of M to H) called *magnetic susceptibility* of the medium. It is more or less a measure of how susceptible (or sensitive) the material is to a magnetic field. Substituting eq. (8.34) into eq. (8.33) yields

$$\mathbf{B} = \mu_o(1 + \chi_m)\mathbf{H} = \mu\mathbf{H} \tag{8.35}$$

or

$$\boxed{\mathbf{B} = \mu_o\mu_r\mathbf{H}} \tag{8.36}$$

where

$$\boxed{\mu_r = 1 + \chi_m = \frac{\mu}{\mu_o}} \tag{8.37}$$

The quantity $\mu = \mu_o\mu_r$ is called the *permeability* of the material and is measured in henrys/meter; the henry is the unit of inductance and will be defined a little later. The dimensionless quantity μ_r is the ratio of the permeability of a given material to that of free space and is known as the *relative permeability* of the material.

It should be borne in mind that the relationships in eqs. (8.34) to (8.37) hold only for linear and isotropic materials. If the materials are anisotropic (e.g., crystals), eq. (8.33) still holds but eqs. (8.34) to (8.37) do not apply. In this case, μ has nine terms (similar to ε in eq. 5.37) and, consequently, the fields $\mathbf{B}$, $\mathbf{H}$, and $\mathbf{M}$ are no longer parallel.

8.6 CLASSIFICATION OF MAGNETIC MATERIALS

In general, we may use the magnetic susceptibility χ_m or the relative permeability μ_r to classify materials in terms of their magnetic property or behavior. A material is said to be *nonmagnetic* if $\chi_m = 0$ (or $\mu_r = 1$); it is magnetic otherwise. Free space, air, and materials with $\chi_m = 0$ (or $\mu_r \approx 1$) are regarded as nonmagnetic.

Roughly speaking, magnetic materials may be grouped into three major classes: diamagnetic, paramagnetic, and ferromagnetic. This rough classification is depicted in Figure 8.13. A material is said to be *diamagnetic* if it has $\mu_r \lesssim 1$ (i.e., very small negative χ_m). It is *paramagnetic* if $\mu_r \gtrsim 1$ (i.e., very small positive χ_m). If $\mu_r \gg 1$ (i.e., very large positive χ_m), the material is *ferromagnetic*. Table B.3 in Appendix B presents the values μ_r for some materials. From the table, it is apparent that for most practical purposes we may assume that $\mu_r \simeq 1$ for diamagnetic and paramagnetic materials. Thus, we may regard diamagnetic and paramagnetic materials as linear and nonmagnetic. Ferromagnetic materials are always nonlinear and magnetic except when their temperatures are above curie temperature (to be explained later). The reason for this will become evident as we more closely examine each of these three types of magnetic materials.

Diamagnetism occurs in materials where the magnetic fields due to electronic motions of orbiting and spinning completely cancel each other. Thus, the permanent (or intrinsic) magnetic moment of each atom is zero and the materials are weakly affected by a magnetic field. For most diamagnetic materials (e.g., bismuth, lead, copper, silicon, diamond, sodium chloride), χ_m is of the order of -10^{-5}. In certain types of materials called *superconductors* at temperatures near absolute zero, "perfect diamagnetism" occurs: $\chi_m = -1$ or $\mu_r = 0$ and $B = 0$. Thus superconductors cannot contain magnetic fields.[2] Except for superconductors, diamagnetic materials are seldom used in practice. Although the diamagnetic effect is overshadowed by other stronger effects in some materials, all materials exhibit diamagnetism.

Materials whose atoms have nonzero permanent magnetic moment may be paramagnetic or ferromagnetic. *Paramagnetism* occurs in materials where the magnetic fields pro-

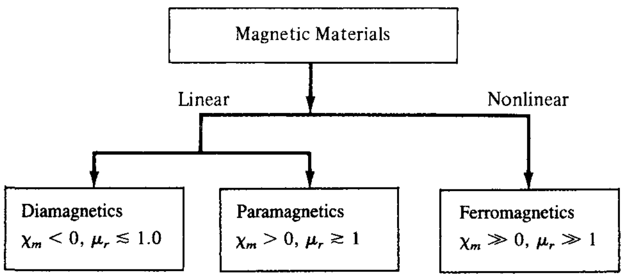


Figure 8.13 Classification of magnetic materials.

[2]An excellent treatment of superconductors is found in M. A. Plonus, *Applied Electromagnetics.* New York: McGraw-Hill, 1978, pp. 375–388. Also, the August 1989 issue of the *Proceedings of IEEE* is devoted to superconductivity.

duced by orbital and spinning electrons do not cancel completely. Unlike diamagnetism, paramagnetism is temperature dependent. For most paramagnetic materials (e.g., air, platinum, tungsten, potassium), χ_m is of the order $+10^{-5}$ to $+10^{-3}$ and is temperature dependent. Such materials find application in masers.

Ferromagnetism occurs in materials whose atoms have relatively large permanent magnetic moment. They are called ferromagnetic materials because the best known member is iron. Other members are cobalt, nickel, and their alloys. Ferromagnetic materials are very useful in practice. As distinct from diamagnetic and paramagnetic materials, ferromagnetic materials have the following properties:

1. They are capable of being magnetized very strongly by a magnetic field.
2. They retain a considerable amount of their magnetization when removed from the field.
3. They lose their ferromagnetic properties and become linear paramagnetic materials when the temperature is raised above a certain temperature known as the *curie temperature.* Thus if a permanent magnet is heated above its curie temperature (770°C for iron), it loses its magnetization completely.
4. They are nonlinear; that is, the constitutive relation $\mathbf{B} = \mu_o\mu_r\mathbf{H}$ does not hold for ferromagnetic materials because μ_r depends on $\mathbf{B}$ and cannot be represented by a single value.

Thus, the values of μ_r cited in Table B.3 for ferromagnetics are only typical. For example, for nickel $\mu_r = 50$ under some conditions and 600 under other conditions.

As mentioned in Section 5.9 for conductors, ferromagnetic materials, such as iron and steel, are used for screening (or shielding) to protect sensitive electrical devices from disturbances from strong magnetic fields. A typical example of an iron shield is shown in Figure 8.14(a) where the compass is protected. Without the iron shield, the compass gives an erroneous reading due to the effect of the external magnetic field as in Figure 8.14(b). For perfect screening, it is required that the shield have infinite permeability.

Even though $\mathbf{B} = \mu_o(\mathbf{H} + \mathbf{M})$ holds for all materials including ferromagnetics, the relationship between $\mathbf{B}$ and $\mathbf{H}$ depends on previous magnetization of a ferromagnetic

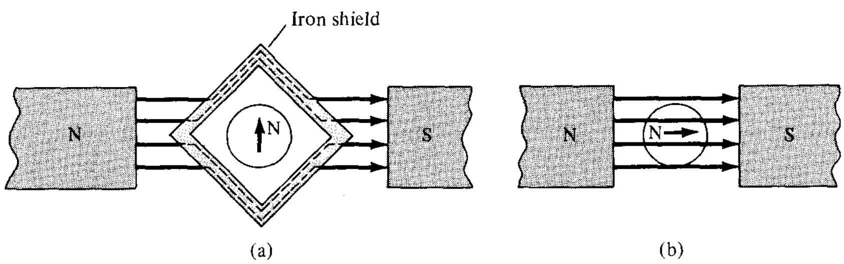


Figure 8.14 Magnetic screening: **(a)** iron shield protecting a small compass, **(b)** compass gives erroneous reading without the shield.

material—its "magnetic history." Instead of having a linear relationship between **B** and **H** (i.e., $\mathbf{B} = \mu\mathbf{H}$), it is only possible to represent the relationship by a *magnetization curve* or *B–H* curve.

A typical *B–H* curve is shown in Figure 8.15. First of all, note the nonlinear relationship between B and H. Second, at any point on the curve, μ is given by the ratio B/H and not by dB/dH, the slope of the curve.

If we assume that the ferromagnetic material whose *B–H* curve in Figure 8.15 is initially unmagnetized, as H increases (due to increase in current) from O to maximum applied field intensity H_{max}, curve OP is produced. This curve is referred to as the *virgin* or *initial magnetization curve.* After reaching saturation at P, if H is decreased, B does not follow the initial curve but lags behind H. This phenomenon of B lagging behind H is called *hysteresis* (which means "to lag" in Greek).

If H is reduced to zero, B is not reduced to zero but to B_r, which is referred to as the *permanent flux density.* The value of B_r depends on H_{max}, the maximum applied field intensity. The existence of B_r is the cause of having permanent magnets. If H increases negatively (by reversing the direction of current), B becomes zero when H becomes H_c, which is known as the *coercive field intensity.* Materials for which H_c is small are said to be magnetically hard. The value of H_c also depends on H_{max}.

Further increase in H in the negative direction to reach Q and a reverse in its direction to reach P gives a closed curve called a *hysteresis loop.* The shape of hysteresis loops varies from one material to another. Some ferrites, for example, have an almost rectangular hysteresis loop and are used in digital computers as magnetic information storage devices. The area of a hysteresis loop gives the energy loss (hysteresis loss) per unit volume during one cycle of the periodic magnetization of the ferromagnetic material. This energy loss is in the form of heat. It is therefore desirable that materials used in electric generators, motors, and transformers should have tall but narrow hysteresis loops so that hysteresis losses are minimal.

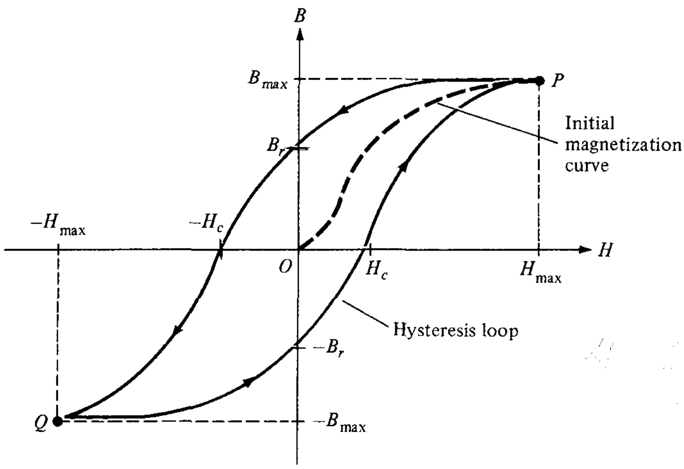


Figure 8.15 Typical magnetization (*B–H*) curve.

EXAMPLE 8.7 Region $0 \leq z \leq 2$ m is occupied by an infinite slab of permeable material ($\mu_r = 2.5$). If $\mathbf{B} = 10y\mathbf{a}_x - 5x\mathbf{a}_y$ mWb/m^2 within the slab, determine: (a) $\mathbf{J}$, (b) $\mathbf{J}_b$, (c) $\mathbf{M}$, (d) $\mathbf{K}_b$ on $z = 0$.

Solution:

(a) By definition,

$$\mathbf{J} = \nabla \times \mathbf{H} = \nabla \times \frac{\mathbf{B}}{\mu_o \mu_r} = \frac{1}{4\pi \times 10^{-7}(2.5)}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\mathbf{a}_z$$
$$= \frac{10^6}{\pi}(-5 - 10)10^{-3}\mathbf{a}_z = -4.775\mathbf{a}_z \text{ kA/m}^2$$

(b) $\mathbf{J}_b = \chi_m \mathbf{J} = (\mu_r - 1)\mathbf{J} = 1.5(-4.775\mathbf{a}_z) \cdot 10^3$
$= -7.163\mathbf{a}_z$ kA/m^2

(c) $\mathbf{M} = \chi_m \mathbf{H} = \chi_m \dfrac{\mathbf{B}}{\mu_o \mu_r} = \dfrac{1.5(10y\mathbf{a}_x - 5x\mathbf{a}_y) \cdot 10^{-3}}{4\pi \times 10^{-7}(2.5)}$
$= 4.775y\mathbf{a}_x - 2.387x\mathbf{a}_y$ kA/m

(d) $\mathbf{K}_b = \mathbf{M} \times \mathbf{a}_n$. Since $z = 0$ is the lower side of the slab occupying $0 \leq z \leq 2$, $\mathbf{a}_n = -\mathbf{a}_z$. Hence,

$$\mathbf{K}_b = (4.775y\mathbf{a}_x - 2.387x\mathbf{a}_y) \times (-\mathbf{a}_z)$$
$$= 2.387x\mathbf{a}_x + 4.775y\mathbf{a}_y \text{ kA/m}$$

PRACTICE EXERCISE 8.7

In a certain region ($\mu = 4.6\mu_o$),

$$\mathbf{B} = 10e^{-y}\mathbf{a}_z \text{ mWb/m}^2$$

find: (a) χ_m, (b) $\mathbf{H}$, (c) $\mathbf{M}$.

Answer: (a) 3.6, (b) $1730e^{-y}\mathbf{a}_z$ A/m, (c) $6228e^{-y}\mathbf{a}_z$ A/m.

8.7 MAGNETIC BOUNDARY CONDITIONS

We define magnetic boundary conditions as the conditions that $\mathbf{H}$ (or $\mathbf{B}$) field must satisfy at the boundary between two different media. Our derivations here are similar to those in Section 5.9. We make use of Gauss's law for magnetic fields

$$\oint \mathbf{B} \cdot d\mathbf{S} = 0 \tag{8.38}$$

and Ampere's circuit law

$$\oint \mathbf{H} \cdot d\mathbf{l} = I \tag{8.39}$$

Consider the boundary between two magnetic media 1 and 2, characterized, respectively, by μ_1 and μ_2 as in Figure 8.16. Applying eq. (8.38) to the pillbox (Gaussian surface) of Figure 8.16(a) and allowing $\Delta h \to 0$, we obtain

$$B_{1n}\,\Delta S - B_{2n}\,\Delta S = 0 \tag{8.40}$$

Thus

$$\boxed{\mathbf{B}_{1n} = \mathbf{B}_{2n}} \qquad \text{or} \qquad \mu_1\mathbf{H}_{1n} = \mu_2\mathbf{H}_{2n} \tag{8.41}$$

since $\mathbf{B} = \mu\mathbf{H}$. Equation (8.41) shows that the normal component of **B** is continuous at the boundary. It also shows that the normal component of **H** is discontinuous at the boundary; **H** undergoes some change at the interface.

Similarly, we apply eq. (8.39) to the closed path *abcda* of Figure 8.16(b) where surface current K on the boundary is assumed normal to the path. We obtain

$$\begin{aligned} K \cdot \Delta w = H_{1t} \cdot \Delta w + H_{1n} \cdot \frac{\Delta h}{2} + H_{2n} \cdot \frac{\Delta h}{2} \\ -H_{2t} \cdot \Delta w - H_{2n} \cdot \frac{\Delta h}{2} - H_{1n} \cdot \frac{\Delta h}{2} \end{aligned} \tag{8.42}$$

As $\Delta h \to 0$, eq. (8.42) leads to

$$H_{1t} - H_{2t} = K \tag{8.43}$$

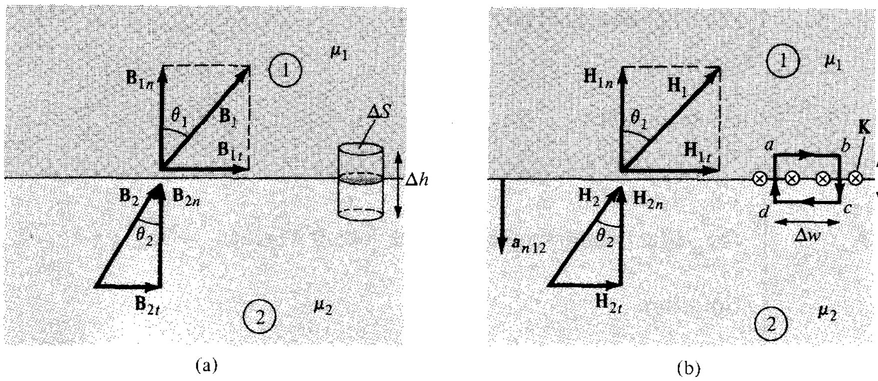


Figure 8.16 Boundary conditions between two magnetic media: **(a)** for **B**, **(b)** for **H**.

This shows that the tangential component of H is also discontinuous. Equation (8.43) may be written in terms of B as

$$\frac{B_{1t}}{\mu_1} - \frac{B_{2t}}{\mu_2} = K \tag{8.44}$$

In the general case, eq. (8.43) becomes

$$\boxed{(\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_{n12} = \mathbf{K}} \tag{8.45}$$

where $\mathbf{a}_{n12}$ is a unit vector normal to the interface and is directed from medium 1 to medium 2. If the boundary is free of current or the media are not conductors (for K is free current density), $K = 0$ and eq. (8.43) becomes

$$\boxed{\mathbf{H}_{1t} = \mathbf{H}_{2t}} \quad \text{or} \quad \frac{\mathbf{B}_{1t}}{\mu_1} = \frac{\mathbf{B}_{2t}}{\mu_2} \tag{8.46}$$

Thus the tangential component of **H** is continuous while that of **B** is discontinuous at the boundary.

If the fields make an angle θ with the normal to the interface, eq. (8.41) results in

$$B_1 \cos\theta_1 = B_{1n} = B_{2n} = B_2 \cos\theta_2 \tag{8.47}$$

while eq. (8.46) produces

$$\frac{B_1}{\mu_1} \sin\theta_1 = H_{1t} = H_{2t} = \frac{B_2}{\mu_2} \sin\theta_2 \tag{8.48}$$

Dividing eq. (8.48) by eq. (8.47) gives

$$\boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2}} \tag{8.49}$$

which is [similar to eq. (5.65)] the law of refraction for magnetic flux lines at a boundary with no surface current.

EXAMPLE 8.8

Given that $\mathbf{H}_1 = -2\mathbf{a}_x + 6\mathbf{a}_y + 4\mathbf{a}_z$ A/m in region $y - x - 2 \leq 0$ where $\mu_1 = 5\mu_o$, calculate

(a) $\mathbf{M}_1$ and $\mathbf{B}_1$

(b) $\mathbf{H}_2$ and $\mathbf{B}_2$ in region $y - x - 2 \geq 0$ where $\mu_2 = 2\mu_o$

Solution:

Since $y - x - 2 = 0$ is a plane, $y - x \leq 2$ or $y \leq x + 2$ is region 1 in Figure 8.17. A point in this region may be used to confirm this. For example, the origin (0, 0) is in this

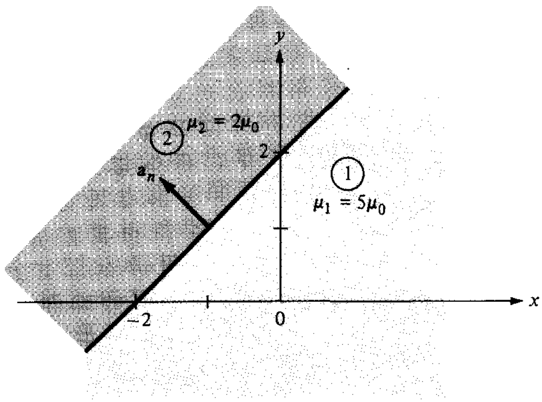


Figure 8.17 For Example 8.8.

region since $0 - 0 - 2 < 0$. If we let the surface of the plane be described by $f(x, y) = y - x - 2$, a unit vector normal to the plane is given by

$$\mathbf{a}_n = \frac{\nabla f}{|\nabla f|} = \frac{\mathbf{a}_y - \mathbf{a}_x}{\sqrt{2}}$$

(a)

$$\begin{aligned}\mathbf{M}_1 &= \chi_{m1}\mathbf{H}_1 = (\mu_{r1} - 1)\,\mathbf{H}_1 = (5 - 1)(-2, 6, 4) \\ &= -8\mathbf{a}_x + 24\mathbf{a}_y + 16\mathbf{a}_z \text{ A/m}\end{aligned}$$

$$\begin{aligned}\mathbf{B}_1 &= \mu_1\mathbf{H}_1 = \mu_o\mu_{r1}\mathbf{H}_1 = 4\pi \times 10^{-7}(5)(-2, 6, 4) \\ &= -12.57\mathbf{a}_x + 37.7\mathbf{a}_y + 25.13\mathbf{a}_z\ \mu\text{Wb/m}^2\end{aligned}$$

(b) $$\begin{aligned}\mathbf{H}_{1n} &= (\mathbf{H}_1 \cdot \mathbf{a}_n)\mathbf{a}_n = \left[(-2, 6, 4) \cdot \frac{(-1, 1, 0)}{\sqrt{2}}\right]\frac{(-1, 1, 0)}{\sqrt{2}} \\ &= -4\mathbf{a}_x + 4\mathbf{a}_y\end{aligned}$$

But

$$\mathbf{H}_1 = \mathbf{H}_{1n} + \mathbf{H}_{1t}$$

Hence,

$$\begin{aligned}\mathbf{H}_{1t} &= \mathbf{H}_1 - \mathbf{H}_{1n} = (-2, 6, 4) - (-4, 4, 0) \\ &= 2\mathbf{a}_x + 2\mathbf{a}_y + 4\mathbf{a}_z\end{aligned}$$

Using the boundary conditions, we have

$$\mathbf{H}_{2t} = \mathbf{H}_{1t} = 2\mathbf{a}_x + 2\mathbf{a}_y + 4\mathbf{a}_z$$

$$\mathbf{B}_{2n} = \mathbf{B}_{1n} \rightarrow \mu_2\mathbf{H}_{2n} = \mu_1\mathbf{H}_{1n}$$

or

$$\mathbf{H}_{2n} = \frac{\mu_1}{\mu_2}\mathbf{H}_{1n} = \frac{5}{2}(-4\mathbf{a}_x + 4\mathbf{a}_y) = -10\mathbf{a}_x + 10\mathbf{a}_y$$

Thus

$$\mathbf{H}_2 = \mathbf{H}_{2n} + \mathbf{H}_{2t} = -8\mathbf{a}_x + 12\mathbf{a}_y + 4\mathbf{a}_z \text{ A/m}$$

and

$$\mathbf{B}_2 = \mu_2\mathbf{H}_2 = \mu_o\mu_{r2}\mathbf{H}_2 = (4\pi \times 10^{-7})(2)(-8, 12, 4)$$
$$= -20.11\mathbf{a}_x + 30.16\mathbf{a}_y + 10.05\mathbf{a}_z \ \mu\text{Wb/m}^2$$

PRACTICE EXERCISE 8.8

Region 1, described by $3x + 4y \geq 10$, is free space whereas region 2, described by $3x + 4y \leq 10$, is a magnetic material for which $\mu = 10\mu_o$. Assuming that the boundary between the material and free space is current free, find $\mathbf{B}_2$ if $\mathbf{B}_1 = 0.1\mathbf{a}_x + 0.4\mathbf{a}_y + 0.2\mathbf{a}_z$ Wb/m²

Answer: $-1.052\mathbf{a}_x + 1.264\mathbf{a}_y + 2\mathbf{a}_z$ Wb/m²

EXAMPLE 8.9

The xy-plane serves as the interface between two different media. Medium 1 ($z < 0$) is filled with a material whose $\mu_r = 6$, and medium 2 ($z > 0$) is filled with a material whose $\mu_r = 4$. If the interface carries current $(1/\mu_o)\,\mathbf{a}_y$ mA/m, and $\mathbf{B}_2 = 5\mathbf{a}_x + 8\mathbf{a}_z$ mWb/m², find $\mathbf{H}_1$ and $\mathbf{B}_1$.

Solution:

In the previous example $\mathbf{K} = 0$, so eq. (8.46) was appropriate. In this example, however, $\mathbf{K} \neq 0$, and we must resort to eq. (8.45) in addition to eq. (8.41). Consider the problem as illustrated in Figure 8.18. Let $\mathbf{B}_1 = (B_x, B_y, B_z)$ in mWb/m².

$$\mathbf{B}_{1n} = \mathbf{B}_{2n} = 8\mathbf{a}_z \rightarrow B_z = 8 \tag{8.8.1}$$

But

$$\mathbf{H}_2 = \frac{\mathbf{B}_2}{\mu_2} = \frac{1}{4\mu_o}(5\mathbf{a}_x + 8\mathbf{a}_z) \text{ mA/m} \tag{8.8.2}$$

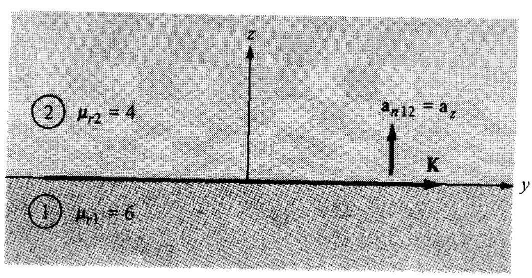


Figure 8.18 For Example 8.9.

and

$$\mathbf{H}_1 = \frac{\mathbf{B}_1}{\mu_1} = \frac{1}{6\mu_o}(B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z) \text{ mA/m} \tag{8.8.3}$$

Having found the normal components, we can find the tangential components using

$$(\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_{n12} = \mathbf{K}$$

or

$$\mathbf{H}_1 \times \mathbf{a}_{n12} = \mathbf{H}_2 \times \mathbf{a}_{n12} + \mathbf{K} \tag{8.8.4}$$

Substituting eqs. (8.8.2) and (8.8.3) into eq. (8.8.4) gives

$$\frac{1}{6\mu_o}(B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z) \times \mathbf{a}_z = \frac{1}{4\mu_o}(5\mathbf{a}_x + 8\mathbf{a}_z) \times \mathbf{a}_z + \frac{1}{\mu_o}\mathbf{a}_y$$

Equating components yields

$$B_y = 0, \qquad \frac{-B_x}{6} = \frac{-5}{4} + 1 \qquad \text{or} \qquad B_x = \frac{6}{4} = 1.5 \tag{8.8.5}$$

From eqs. (8.8.1) and (8.8.5),

$$\mathbf{B}_1 = 1.5\mathbf{a}_x + 8\mathbf{a}_z \text{ mWb/m}^2$$

$$\mathbf{H}_1 = \frac{\mathbf{B}_1}{\mu_1} = \frac{1}{\mu_o}(0.25\mathbf{a}_x + 1.33\mathbf{a}_z) \text{ mA/m}$$

and

$$\mathbf{H}_2 = \frac{1}{\mu_o}(1.25\mathbf{a}_x + 2\mathbf{a}_z) \text{ mA/m}$$

Note that H_{1x} is $(1/\mu_o)$ mA/m less than H_{2x} due to the current sheet and also that $B_{1n} = B_{2n}$.

PRACTICE EXERCISE 8.9

A unit normal vector from region 2 ($\mu = 2\mu_o$) to region 1 ($\mu = \mu_o$) is $\mathbf{a}_{n21} = (6\mathbf{a}_x + 2\mathbf{a}_y - 3\mathbf{a}_z)/7$. If $\mathbf{H}_1 = 10\mathbf{a}_x + \mathbf{a}_y + 12\mathbf{a}_z$ A/m and $\mathbf{H}_2 = H_{2x}\mathbf{a}_x - 5\mathbf{a}_y + 4\mathbf{a}_z$ A/m, determine

(a) $\mathbf{H}_{2x}$

(b) The surface current density $\mathbf{K}$ on the interface

(c) The angles $\mathbf{B}_1$ and $\mathbf{B}_2$ make with the normal to the interface.

Answer: (a) 5.833, (b) $4.86\mathbf{a}_x - 8.64\mathbf{a}_y + 3.95\mathbf{a}_z$ A/m, (c)76.27°, 77.62°.

8.8 INDUCTORS AND INDUCTANCES

A circuit (or closed conducting path) carrying current I produces a magnetic field $\mathbf{B}$ which causes a flux $\Psi = \int \mathbf{B} \cdot d\mathbf{S}$ to pass through each turn of the circuit as shown in Figure 8.19. If the circuit has N identical turns, we define the *flux linkage* λ as

$$\lambda = N\Psi \tag{8.50}$$

Also, if the medium surrounding the circuit is linear, the flux linkage λ is proportional to the current I producing it; that is,

$$\lambda \propto I$$

$$\text{or} \qquad \lambda = LI \tag{8.51}$$

where L is a constant of proportionality called the *inductance* of the circuit. The inductance L is a property of the physical arrangement of the circuit. A circuit or part of a circuit that has inductance is called an *inductor*. From eqs. (8.50) and (8.51), we may define inductance L of an inductor as the ratio of the magnetic flux linkage λ to the current I through the inductor; that is,

$$\boxed{L = \frac{\lambda}{I} = \frac{N\Psi}{I}} \tag{8.52}$$

The unit of inductance is the henry (H) which is the same as webers/ampere. Since the henry is a fairly large unit, inductances are usually expressed in millihenrys (mH).

The inductance defined by eq. (8.52) is commonly referred to as *self-inductance* since the linkages are produced by the inductor itself. Like capacitances, we may regard inductance as a measure of how much magnetic energy is stored in an inductor. The magnetic energy (in joules) stored in an inductor is expressed in circuit theory as:

$$W_m = \frac{1}{2}LI^2 \tag{8.53}$$

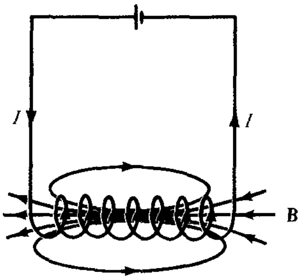


Figure 8.19 Magnetic field **B** produced by a circuit.

or

$$L = \frac{2W_m}{I^2} \tag{8.54}$$

Thus the self-inductance of a circuit may be defined or calculated from energy considerations.

If instead of having a single circuit we have two circuits carrying current I_1 and I_2 as shown in Figure 8.20, a magnetic interaction exists between the circuits. Four component fluxes Ψ_{11}, Ψ_{12}, Ψ_{21}, and Ψ_{22} are produced. The flux Ψ_{12}, for example, is the flux passing through circuit 1 due to current I_2 in circuit 2. If $\mathbf{B}_2$ in the field due to I_2 and S_1 is the area of circuit 1, then

$$\Psi_{12} = \int_{S_1} \mathbf{B}_2 \cdot d\mathbf{S} \tag{8.55}$$

We define the *mutual inductance* M_{12} as the ratio of the flux linkage $\lambda_{12} = N_1\Psi_{12}$ on circuit 1 to current I_2, that is,

$$M_{12} = \frac{\lambda_{12}}{I_2} = \frac{N_1\Psi_{12}}{I_2} \tag{8.56}$$

Similarly, the mutual inductance M_{21} is defined as the flux linkages of circuit 2 per unit current I_1; that is,

$$M_{21} = \frac{\lambda_{21}}{I_1} = \frac{N_2\Psi_{21}}{I_1} \tag{8.57a}$$

It can be shown by using energy concepts that if the medium surrounding the circuits is linear (i.e., in the absence of ferromagnetic material),

$$M_{12} = M_{21} \tag{8.57b}$$

The mutual inductance M_{12} or M_{21} is expressed in henrys and should not be confused with the magnetization vector $\mathbf{M}$ expressed in amperes/meter.

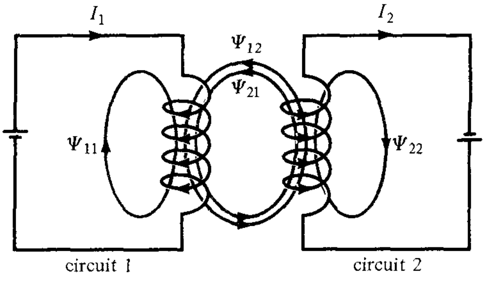


Figure 8.20 Magnetic interaction between two circuits.

We define the self-inductance of circuits 1 and 2, respectively, as

$$L_1 = \frac{\lambda_{11}}{I_1} = \frac{N_1\Psi_1}{I_1} \tag{8.58}$$

and

$$L_2 = \frac{\lambda_{22}}{I_2} = \frac{N_2\Psi_2}{I_2} \tag{8.59}$$

where $\Psi_1 = \Psi_{11} + \Psi_{12}$ and $\Psi_2 = \Psi_{21} + \Psi_{22}$. The total energy in the magnetic field is the sum of the energies due to L_1, L_2, and M_{12} (or M_{21}); that is,

$$\begin{aligned} W_m &= W_1 + W_2 + W_{12} \\ &= \frac{1}{2}L_1I_1^2 + \frac{1}{2}L_2I_2^2 \pm M_{12}I_1I_2 \end{aligned} \tag{8.60}$$

The positive sign is taken if currents I_1 and I_2 flow such that the magnetic fields of the two circuits strengthen each other. If the currents flow such that their magnetic fields oppose each other, the negative sign is taken.

As mentioned earlier, an inductor is a conductor arranged in a shape appropriate to store magnetic energy. Typical examples of inductors are toroids, solenoids, coaxial transmission lines, and parallel-wire transmission lines. The inductance of each of these inductors can be determined by following a procedure similar to that taken in determining the capacitance of a capacitor. For a given inductor, we find the self-inductance L by taking these steps:

1. Choose a suitable coordinate system.
2. Let the inductor carry current I.
3. Determine $\mathbf{B}$ from Biot–Savart's law (or from Ampere's law if symmetry exists) and calculate Ψ from $\Psi = \int \mathbf{B} \cdot d\mathbf{S}$.
4. Finally find L from $L = \dfrac{\lambda}{I} = \dfrac{N\Psi}{I}$.

The mutual inductance between two circuits may be calculated by taking a similar procedure.

In an inductor such as a coaxial or a parallel-wire transmission line, the inductance produced by the flux internal to the conductor is called the *internal inductance* L_{in} while that produced by the flux external to it is called *external inductance* L_{ext}. The total inductance L is

$$L = L_{\text{in}} + L_{\text{ext}} \tag{8.61}$$

Just as it was shown that for capacitors

$$RC = \frac{\varepsilon}{\sigma} \tag{6.35}$$

it can be shown that

$$\boxed{L_{\text{ext}}C = \mu\varepsilon} \tag{8.62}$$

Thus L_{ext} may be calculated using eq. (8.62) if C is known.

A collection of formulas for some fundamental circuit elements is presented in Table 8.3. All formulas can be derived by taking the steps outlined above.[3]

8.9 MAGNETIC ENERGY

Just as the potential energy in an electrostatic field was derived as

$$W_E = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\,dv = \frac{1}{2}\int \varepsilon E^2\,dv \tag{4.96}$$

we would like to derive a similar expression for the energy in a magnetostatic field. A simple approach is using the magnetic energy in the field of an inductor. From eq. (8.53),

$$W_m = \frac{1}{2}LI^2 \tag{8.53}$$

The energy is stored in the magnetic field **B** of the inductor. We would like to express eq. (8.53) in terms of **B** or **H**.

Consider a differential volume in a magnetic field as shown in Figure 8.21. Let the volume be covered with conducting sheets at the top and bottom surfaces with current ΔI.

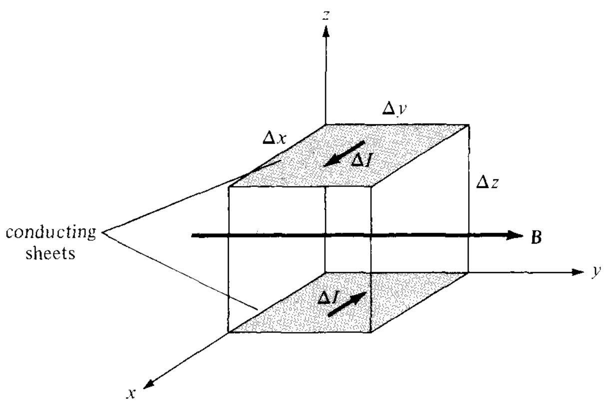


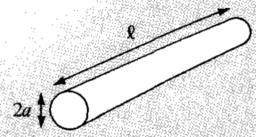
Figure 8.21 A differential volume in a magnetic field.

[3]Additional formulas can be found in standard electrical handbooks or in H. Knoepfel, *Pulsed High Magnetic Fields*. Amsterdam: North-Holland, 1970, pp. 312–324.

TABLE 8.3 A Collection of Formulas for Inductance of Common Elements

1. Wire

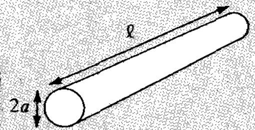
$$L = \frac{\mu_o \ell}{8\pi}$$



2. Hollow cylinder

$$L = \frac{\mu_o \ell}{2\pi}\left(\ln\frac{2\ell}{\mathbf{a}} - 1\right)$$

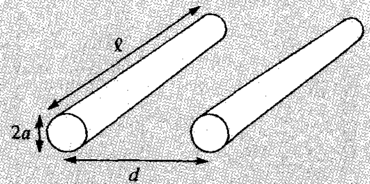
$$\ell \gg \mathbf{a}$$



3. Parallel wires

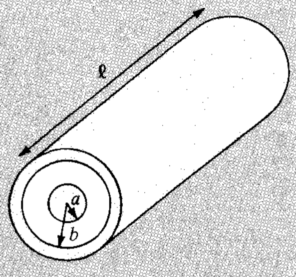
$$L = \frac{\mu_o \ell}{\pi}\ln\frac{d}{a}$$

$$\ell \gg d,\ d \gg a$$



4. Coaxial conductor

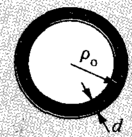
$$L = \frac{\mu_o \ell}{\pi}\ln\frac{b}{a}$$



5. Circular loop

$$L = \frac{\mu_o \ell}{2\pi}\left(\ln\frac{4\ell}{d} - 2.45\right)$$

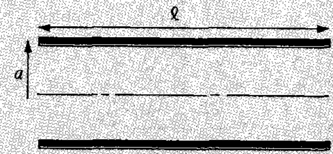
$$\ell = 2\pi\rho_o,\ \rho_o \gg d$$



6. Solenoid

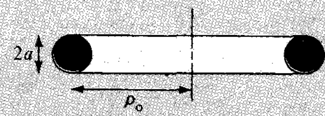
$$L = \frac{\mu_o N^2 S}{\ell}$$

$$\ell \gg a$$



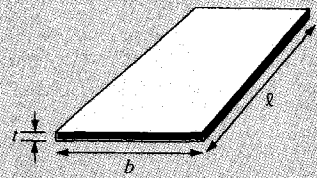
7. Torus (of circular cross section)

$$L = \mu_o N^2\left[\rho_o - \sqrt{\rho_o^2 - a^2}\right]$$



8. Sheet

$$L = \mu_o\, 2\ell\left(ln\frac{2\ell}{b + t} + 0.5\right)$$



We assume that the whole region is filled with such differential volumes. From eq. (8.52), each volume has an inductance

$$\Delta L = \frac{\Delta \Psi}{\Delta I} = \frac{\mu H \, \Delta x \, \Delta z}{\Delta I} \tag{8.63}$$

where $\Delta I = H \, \Delta y$. Substituting eq. (8.63) into eq. (8.53), we have

$$\Delta W_m = \frac{1}{2} \Delta L \, \Delta I^2 = \frac{1}{2} \mu H^2 \, \Delta x \, \Delta y \, \Delta z \tag{8.64}$$

or

$$\Delta W_m = \frac{1}{2} \mu H^2 \, \Delta v$$

The magnetostatic energy density w_m (in J/m^3) is defined as

$$w_m = \lim_{\Delta v \to 0} \frac{\Delta W_m}{\Delta v} = \frac{1}{2} \mu H^2$$

Hence,

$$w_m = \frac{1}{2} \mu H^2 = \frac{1}{2} B \cdot H = \frac{B^2}{2\mu} \tag{8.65}$$

Thus the energy in a magnetostatic field in a linear medium is

$$W_m = \int w_m \, dv$$

or

$$\boxed{W_m = \frac{1}{2} \int \mathbf{B} \cdot \mathbf{H} \, dv = \frac{1}{2} \int \mu H^2 \, dv} \tag{8.66}$$

which is similar to eq. (4.96) for an electrostatic field.

EXAMPLE 8.10

Calculate the self-inductance per unit length of an infinitely long solenoid.

Solution:

We recall from Example 7.4 that for an infinitely long solenoid, the magnetic flux inside the solenoid per unit length is

$$B = \mu H = \mu I n$$

where $n = N/\ell$ = number of turns per unit length. If S is the cross-sectional area of the solenoid, the total flux through the cross section is

$$\Psi = BS = \mu InS$$

Since this flux is only for a unit length of the solenoid, the linkage per unit length is

$$\lambda' = \frac{\lambda}{\ell} = n\Psi = \mu n^2 IS$$

and thus the inductance per unit length is

$$L' = \frac{L}{\ell} = \frac{\lambda'}{I} = \mu n^2 S$$

$$\boxed{L' = \mu n^2 S} \qquad \text{H/m}$$

PRACTICE EXERCISE 8.10

A very long solenoid with 2×2 cm cross section has an iron core ($\mu_r = 1000$) and 4000 turns/meter. If it carries a current of 500 mA, find

(a) Its self-inductance per meter
(b) The energy per meter stored in its field

Answer: (a) 8.042 H/m, (b) 1.005 J/m.

EXAMPLE 8.11

Determine the self-inductance of a coaxial cable of inner radius a and outer radius b.

Solution:

The self-inductance of the inductor can be found in two different ways: by taking the four steps given in Section 8.8 or by using eqs. (8.54) and (8.66).

Method 1: Consider the cross section of the cable as shown in Figure 8.22. We recall from eq. (7.29) that by applying Ampere's circuit law, we obtained for region 1 ($0 \leq \rho \leq a$),

$$\mathbf{B}_1 = \frac{\mu I \rho}{2\pi a^2} \mathbf{a}_\phi$$

and for region 2 ($a \leq \rho \leq b$),

$$\mathbf{B}_2 = \frac{\mu I}{2\pi\rho} \mathbf{a}_\phi$$

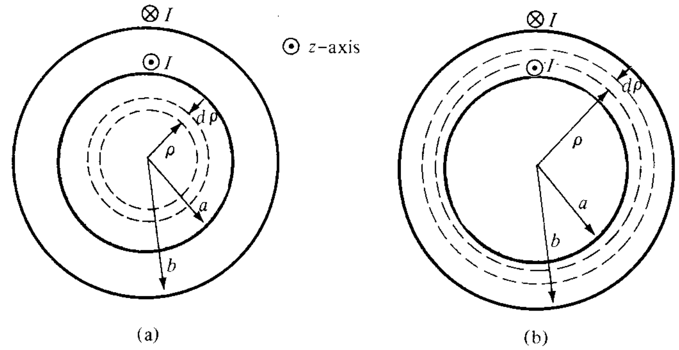


Figure 8.22 Cross section of the coaxial cable: **(a)** for region 1, $0 < \rho < a$, **(b)** for region 2, $a < \rho < b$; for Example 8.11.

We first find the internal inductance L_{in} by considering the flux linkages due to the inner conductor. From Figure 8.22(a), the flux leaving a differential shell of thickness $d\rho$ is

$$d\Psi_1 = B_1\, d\rho\, dz = \frac{\mu I \rho}{2\pi a^2}\, d\rho\, dz$$

The flux linkage is $d\Psi_1$ multiplied by the ratio of the area within the path enclosing the flux to the total area, that is,

$$d\lambda_1 = d\Psi_1 \cdot \frac{I_{\text{enc}}}{I} = d\Psi_1 \cdot \frac{\pi \rho^2}{\pi a^2}$$

because I is uniformly distributed over the cross section for d.c. excitation. Thus, the total flux linkages within the differential flux element are

$$d\lambda_1 = \frac{\mu I \rho\, d\rho\, dz}{2\pi a^2} \cdot \frac{\rho^2}{a^2}$$

For length ℓ of the cable,

$$\lambda_1 = \int_{\rho=0}^{a} \int_{z=0}^{\ell} \frac{\mu I \rho^3\, d\rho\, dz}{2\pi a^4} = \frac{\mu I \ell}{8\pi}$$

$$L_{\text{in}} = \frac{\lambda_1}{I} = \frac{\mu \ell}{8\pi} \tag{8.11.1}$$

The internal inductance per unit length, given by

$$\boxed{L'_{\text{in}} = \frac{L_{\text{in}}}{\ell} = \frac{\mu}{8\pi}} \quad \text{H/m} \tag{8.11.2}$$

is independent of the radius of the conductor or wire. Thus eqs. (8.11.1) and (8.11.2) are also applicable to finding the inductance of any infinitely long straight conductor of finite radius.

We now determine the external inductance L_{ext} by considering the flux linkages between the inner and the outer conductor as in Figure 8.22(b). For a differential shell of thickness $d\rho$,

$$d\Psi_2 = B_2\, d\rho\, dz = \frac{\mu I}{2\pi\rho}\, d\rho\, dz$$

In this case, the total current I is enclosed within the path enclosing the flux. Hence,

$$\lambda_2 = \Psi_2 = \int_{\rho=a}^{b} \int_{z=0}^{\ell} \frac{\mu I\, d\rho\, dz}{2\pi\rho} = \frac{\mu I \ell}{2\pi} \ln \frac{b}{a}$$

$$L_{ext} = \frac{\lambda_2}{I} = \frac{\mu\ell}{2\pi} \ln \frac{b}{a}$$

Thus

$$L = L_{in} + L_{ext} = \frac{\mu\ell}{2\pi}\left[\frac{1}{4} + \ln\frac{b}{a}\right]$$

or the inductance per length is

$$\boxed{L' = \frac{L}{\ell} = \frac{\mu}{2\pi}\left[\frac{1}{4} + \ln\frac{b}{a}\right]} \qquad \text{H/m}$$

Method 2: It is easier to use eqs. (8.54) and (8.66) to determine L, that is,

$$W_m = \frac{1}{2} L I^2 \qquad \text{or} \qquad L = \frac{2W_m}{I^2}$$

where

$$W_m = \frac{1}{2}\int \mathbf{B}\cdot\mathbf{H}\, dv = \int \frac{B^2}{2\mu}\, dv$$

Hence

$$\begin{aligned} L_{in} &= \frac{2}{I^2}\int \frac{B_1^2}{2\mu}\, dv = \frac{1}{I^2\mu}\iiint \frac{\mu^2 I^2 \rho^2}{4\pi^2 a^4}\, \rho\, d\rho\, d\phi\, dz \\ &= \frac{\mu}{4\pi^2 a^4}\int_0^{\ell} dz \int_0^{2\pi} d\phi \int_0^a \rho^3\, d\rho = \frac{\mu\ell}{8\pi} \end{aligned}$$

$$\begin{aligned} L_{ext} &= \frac{2}{I^2}\int \frac{B_2^2}{2\mu}\, dv = \frac{1}{I^2\mu}\iiint \frac{\mu^2 I^2}{4\pi^2 \rho^2}\, \rho\, d\rho\, d\phi\, dz \\ &= \frac{\mu}{4\pi^2}\int_0^{\ell} dz \int_0^{2\pi} d\phi \int_a^b \frac{d\rho}{\rho} = \frac{\mu\ell}{2\pi}\ln\frac{b}{a} \end{aligned}$$

and

$$L = L_{in} + L_{ext} = \frac{\mu \ell}{2\pi}\left[\frac{1}{4} + \ln\frac{b}{a}\right]$$

as obtained previously.

PRACTICE EXERCISE 8.11

Calculate the self-inductance of the coaxial cable of Example 8.11 if the inner conductor is made of an inhomogeneous material having $\mu = 2\mu_o/(1 + \rho)$.

Answer: $\dfrac{\mu_o \ell}{8\pi} + \dfrac{\mu_o \ell}{\pi}\left[\ln\dfrac{b}{a} - \ln\dfrac{(1+b)}{(1+a)}\right]$

EXAMPLE 8.12

Determine the inductance per unit length of a two-wire transmission line with separation distance d. Each wire has radius a as shown in Figure 6.37.

Solution:

We use the two methods of the last example.

Method 1: We determine L_{in} just as we did in the last example. Thus for region $0 \leq \rho \leq a$, we obtain

$$\lambda_1 = \frac{\mu I \ell}{8\pi}$$

as in the last example. For region $a \leq \rho \leq d - a$, the flux linkages between the wires are

$$\lambda_2 = \Psi_2 = \int_{\rho=a}^{d-a}\int_{z=0}^{\ell} \frac{\mu I}{2\pi\rho}\, d\rho\, dz = \frac{\mu I \ell}{2\pi}\ln\frac{d-a}{a}$$

The flux linkages produced by wire 1 are

$$\lambda_1 + \lambda_2 = \frac{\mu I \ell}{8\pi} + \frac{\mu I \ell}{2\pi}\ln\frac{d-a}{a}$$

By symmetry, the same amount of flux produced by current $-I$ in wire 2. Hence the total linkages are

$$\lambda = 2(\lambda_1 + \lambda_2) = \frac{\mu I \ell}{\pi}\left[\frac{1}{4} + \ln\frac{d-a}{a}\right] = LI$$

If $d \gg a$, the self-inductance per unit length is

$$\boxed{L' = \frac{L}{\ell} = \frac{\mu}{\pi}\left[\frac{1}{4} + \ln\frac{d}{a}\right]} \qquad \text{H/m}$$

Method 2: From the last example,

$$L_{in} = \frac{\mu \ell}{8\pi}$$

Now

$$\begin{aligned} L_{ext} &= \frac{2}{I^2} \int \frac{B^2\, dv}{2\mu} = \frac{1}{I^2 \mu} \iiint \frac{\mu^2 I^2}{4\pi^2 \rho^2} \rho\, d\rho\, d\phi\, dz \\ &= \frac{\mu}{4\pi^2} \int_0^{\ell} dz \int_0^{2\pi} d\phi \int_a^{d-a} \frac{d\rho}{\rho} \\ &= \frac{\mu \ell}{2\pi} \ln \frac{d-a}{a} \end{aligned}$$

Since the two wires are symmetrical,

$$\begin{aligned} L &= 2\,(L_{in} + L_{ext}) \\ &= \frac{\mu \ell}{\pi} \left[\frac{1}{4} + \ln \frac{d-a}{a} \right] \text{H} \end{aligned}$$

as obtained previously.

PRACTICE EXERCISE 8.12

Two #10 copper wires (2.588 mm in diameter) are placed parallel in air with a separation distance d between them. If the inductance of each wire is 1.2 μH/m, calculate

(a) L_{in} and L_{ext} per meter for each wire

(b) The separation distance d

Answer: (a) 0.05, 1.15 μH/m, (b) 40.79 cm.

EXAMPLE 8.13

Two coaxial circular wires of radii a and b ($b > a$) are separated by distance h ($h \gg a, b$) as shown in Figure 8.23. Find the mutual inductance between the wires.

Solution:

Let current I_1 flow in wire 1. At an arbitrary point P on wire 2, the magnetic vector potential due to wire 1 is given by eq. (8.21a), namely

$$\mathbf{A}_1 = \frac{\mu I_1 a^2 \sin\theta}{4r^2} \mathbf{a}_\phi = \frac{\mu I_1 a^2 b \mathbf{a}_\phi}{4[h^2 + b^2]^{3/2}}$$

If $h \gg b$

$$\mathbf{A}_1 \simeq \frac{\mu I_1 a^2 b}{4h^3} \mathbf{a}_\phi$$

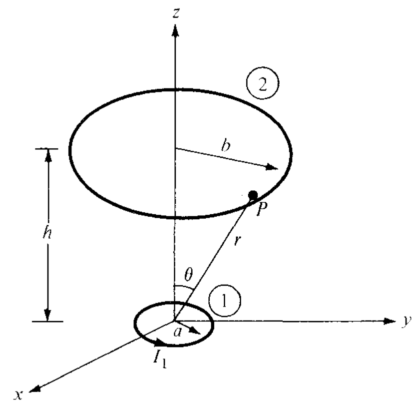


Figure 8.23 Two coaxial circular wires; for Example 8.13.

Hence,

$$\Psi_{12} = \oint \mathbf{A}_1 \cdot d\mathbf{l}_2 = \frac{\mu I_1 a^2 b}{4h^3} 2\pi b = \frac{\mu \pi I_1 a^2 b^2}{2h^3}$$

and

$$M_{12} = \frac{\Psi_{12}}{I_1} = \frac{\mu \pi a^2 b^2}{2h^3}$$

PRACTICE EXERCISE 8.13

Find the mutual inductance of two coplanar concentric circular loops of radii 2 m and 3 m.

Answer: 2.632 μH.

8.10 MAGNETIC CIRCUITS

The concept of magnetic circuits is based on solving some magnetic field problems using circuit approach. Magnetic devices such as toroids, transformers, motors, generators, and relays may be considered as magnetic circuits. The analysis of such circuits is made simple if an analogy between magnetic circuits and electric circuits is exploited. Once this is done, we can directly apply concepts in electric circuits to solve their analogous magnetic circuits.

The analogy between magnetic and electric circuits is summarized in Table 8.4 and portrayed in Figure 8.24. The reader is advised to pause and study Table 8.4 and Figure 8.24. First, we notice from the table that two terms are new. We define the *magnetomotive force* (mmf) $\mathcal{F}$ (in ampere-turns) as

$$\boxed{\mathcal{F} = NI = \oint \mathbf{H} \cdot d\mathbf{l}} \tag{8.67}$$

TABLE 8.4 Analogy between Electric and Magnetic Circuits

Electric	Magnetic
Conductivity σ	Permeability μ
Field intensity E	Field intensity H
Current $I = \int \mathbf{J} \cdot d\mathbf{S}$	Magnetic flux $\Psi = \int \mathbf{B} \cdot d\mathbf{S}$
Current density $J = \dfrac{I}{S} = \sigma E$	Flux density $B = \dfrac{\Psi}{S} = \mu H$
Electromotive force (emf) V	Magnetomotive force (mmf) $\mathcal{F}$
Resistance R	Reluctance $\mathcal{R}$
Conductance $G = \dfrac{1}{R}$	Permeance $\mathcal{P} = \dfrac{1}{\mathcal{R}}$
Ohm's law $R = \dfrac{V}{I} = \dfrac{\ell}{\sigma S}$	Ohm's law $\mathcal{R} = \dfrac{\mathcal{F}}{\Psi} = \dfrac{\ell}{\mu S}$
or $V = E\ell = IR$	or $\mathcal{F} = H\ell = \Psi\mathcal{R} = NI$
Kirchoff's laws:	Kirchhoff's laws:
$\Sigma I = 0$	$\Sigma \Psi = 0$
$\Sigma V - \Sigma RI = 0$	$\Sigma \mathcal{F} - \Sigma \mathcal{R} \Psi = 0$

The source of mmf in magnetic circuits is usually a coil carrying current as in Figure 8.24. We also define *reluctance* $\mathcal{R}$ (in ampere-turns/weber) as

$$\boxed{\mathcal{R} = \frac{\ell}{\mu S}} \tag{8.68}$$

where ℓ and S are, respectively, the mean length and the cross-sectional area of the magnetic core. The reciprocal of reluctance is *permeance* $\mathcal{P}$. The basic relationship for circuit elements is Ohm's law ($V = IR$):

$$\boxed{\mathcal{F} = \Psi\mathcal{R}} \tag{8.69}$$

Based on this, Kirchhoff's current and voltage laws can be applied to nodes and loops of a given magnetic circuit just as in an electric circuit. The rules of adding voltages and for

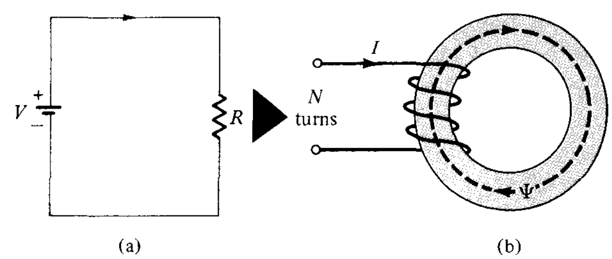


Figure 8.24 Analogy between **(a)** an electric circuit, and **(b)** a magnetic circuit.

combining series and parallel resistances also hold for mmfs and reluctances. Thus for n magnetic circuit elements in series

$$\Psi_1 = \Psi_2 = \Psi_3 = \cdots = \Psi_n \tag{8.70}$$

and

$$\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2 + \cdots + \mathcal{F}_n \tag{8.71}$$

For n magnetic circuit elements in parallel,

$$\Psi = \Psi_1 + \Psi_2 + \Psi_3 + \cdots + \Psi_n \tag{8.72}$$

and

$$\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = \cdots = \mathcal{F}_n \tag{8.73}$$

Some differences between electric and magnetic circuits should be pointed out. Unlike an electric circuit where current I flows, magnetic flux does not flow. Also, conductivity σ is independent of current density J in an electric circuit whereas permeability μ varies with flux density B in a magnetic circuit. This is because ferromagnetic (nonlinear) materials are normally used in most practical magnetic devices. These differences notwithstanding, the magnetic circuit concept serves as an approximate analysis of practical magnetic devices.

8.11 FORCE ON MAGNETIC MATERIALS

It is of practical interest to determine the force that a magnetic field exerts on a piece of magnetic material in the field. This is useful in electromechanical systems such as electromagnets, relays, rotating machines, and magnetic levitation. Consider, for example, an electromagnet made of iron of constant relative permeability as shown in Figure 8.25. The coil has N turns and carries a current I. If we ignore fringing, the magnetic field in the air gap is the same as that in iron ($B_{1n} = B_{2n}$). To find the force between the two pieces of iron, we calculate the change in the total energy that would result were the two pieces of the magnetic circuit separated by a differential displacement $d\mathbf{l}$. The work required to effect

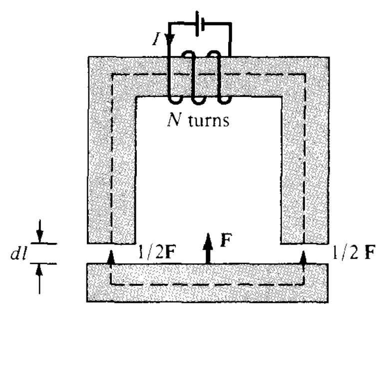


Figure 8.25 An electromagnet.

the displacement is equal to the change in stored energy in the air gap (assuming constant current), that is

$$-F\,dl = dW_m = 2\left[\frac{1}{2}\frac{B^2}{\mu_o}S\,dl\right] \quad (8.74)$$

where S is the cross-sectional area of the gap, the factor 2 accounts for the two air gaps, and the negative sign indicates that the force acts to reduce the air gap (or that the force is attractive). Thus

$$F = -2\left(\frac{B^2 S}{2\mu_o}\right) \quad (8.75)$$

Note that the force is exerted on the lower piece and not on the current-carrying upper piece giving rise to the field. The tractive force across a *single* gap can be obtained from eq. (8.75) as

$$\boxed{F = -\frac{B^2 S}{2\mu_o}} \quad (8.76)$$

Notice the similarity between eq. (8.76) and that derived in Example 5.8 for electrostatic case. Equation (8.76) can be used to calculate the forces in many types of devices including relays, rotating machines, and magnetic levitation. The tractive pressure (in N/m^2) in a magnetized surface is

$$p = \frac{F}{S} = \frac{B^2}{2\mu_o} = \frac{1}{2}BH \quad (8.77)$$

which is the same as the energy density w_m in the air gap.

EXAMPLE 8.14

The toroidal core of Figure 8.26(a) has $\rho_o = 10$ cm and a circular cross section with $a = 1$ cm. If the core is made of steel ($\mu = 1000\,\mu_o$) and has a coil with 200 turns, calculate the amount of current that will produce a flux of 0.5 mWb in the core.

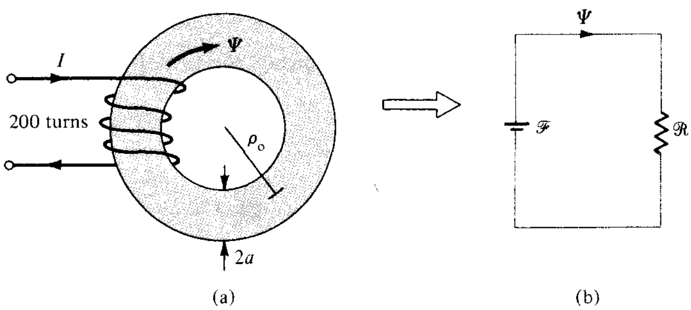


Figure 8.26 **(a)** Toroidal core of Example 8.14; **(b)** its equivalent electric circuit analog.

Solution:

This problem can be solved in two different ways: using the magnetic field approach (direct), or using the electric circuit analog (indirect).

Method 1: Since ρ_o is large compared with a, from Example 7.6,

$$B = \frac{\mu NI}{\ell} = \frac{\mu_o \mu_r NI}{2\pi \rho_o}$$

Hence,

$$\Psi = BS = \frac{\mu_o \mu_r NI\, \pi a^2}{2\pi \rho_o}$$

or

$$I = \frac{2\rho_o \Psi}{\mu_o \mu_r N a^2} = \frac{2(10 \times 10^{-2})(0.5 \times 10^{-3})}{4\pi \times 10^{-7}(1000)(200)(1 \times 10^{-4})}$$
$$= \frac{100}{8\pi} = 3.979 \text{ A}$$

Method 2: The toroidal core in Figure 8.26(a) is analogous to the electric circuit of Figure 8.26(b). From the circuit and Table 8.4.

$$\mathcal{F} = NI = \Psi \mathcal{R} = \Psi \frac{\ell}{\mu S} = \Psi \frac{2\pi \rho_o}{\mu_o \mu_r \pi a^2}$$

or

$$I = \frac{2\rho_o \Psi}{\mu_o \mu_r N a^2} = 3.979\ A$$

as obtained previously.

PRACTICE EXERCISE 8.14

A conductor of radius a is bent into a circular loop of mean radius ρ_o (see Figure 8.26a). If $\rho_o = 10$ cm and $2a = 1$ cm, calculate the internal inductance of the loop.

Answer: 31.42 nH.

EXAMPLE 8.15

In the magnetic circuit of Figure 8.27, calculate the current in the coil that will produce a magnetic flux density of 1.5 Wb/m^2 in the air gap assuming that $\mu = 50\mu_o$ and that all branches have the same cross-sectional area of 10 cm^2.

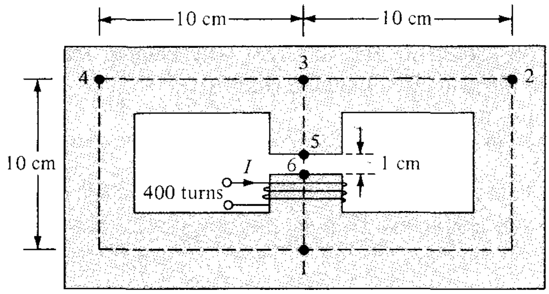


Figure 8.27 Magnetic circuit of Example 8.15.

Solution:

The magnetic circuit of Figure 8.27 is analogous to the electric circuit of Figure 8.28. In Figure 8.27, $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, and $\mathcal{R}_a$ are the reluctances in paths 143, 123, 35 and 16, and 56 (air gap), respectively. Thus

$$\mathcal{R}_1 = \mathcal{R}_2 = \frac{\ell}{\mu_o \mu_r S} = \frac{30 \times 10^{-2}}{(4\pi \times 10^{-7})(50)(10 \times 10^{-4})} = \frac{3 \times 10^8}{20\pi}$$

$$\mathcal{R}_3 = \frac{9 \times 10^{-2}}{(4\pi \times 10^{-7})(50)(10 \times 10^{-4})} = \frac{0.9 \times 10^8}{20\pi}$$

$$\mathcal{R}_a = \frac{1 \times 10^{-2}}{(4\pi \times 10^{-7})(1)(10 \times 10^{-4})} = \frac{5 \times 10^8}{20\pi}$$

We combine $\mathcal{R}_1$ and $\mathcal{R}_2$ as resistors in parallel. Hence,

$$\mathcal{R}_1 \| \mathcal{R}_2 = \frac{\mathcal{R}_1 \mathcal{R}_2}{\mathcal{R}_1 + \mathcal{R}_2} = \frac{\mathcal{R}_1}{2} = \frac{1.5 \times 10^8}{20\pi}$$

The total reluctance is

$$\mathcal{R}_T = \mathcal{R}_a + \mathcal{R}_3 + \mathcal{R}_1 \| \mathcal{R}_2 = \frac{7.4 \times 10^8}{20\pi}$$

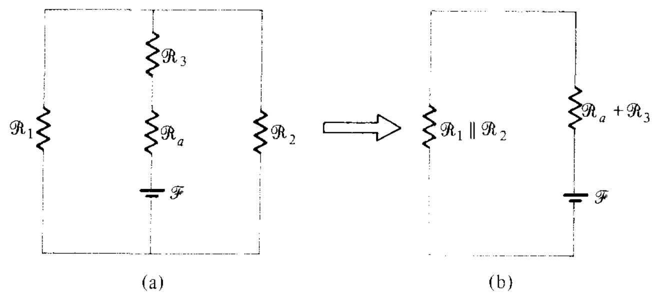


Figure 8.28 Electric circuit analog of the magnetic circuit in Figure 8.27.

The mmf is

$$\mathcal{F} = NI = \Psi_a \mathcal{R}_T$$

But $\Psi_a = \Psi = B_a S$. Hence

$$I = \frac{B_a S \mathcal{R}_T}{N} = \frac{1.5 \times 10 \times 10^{-4} \times 7.4 \times 10^8}{400 \times 20\pi}$$
$$= 44.16 \text{ A}$$

PRACTICE EXERCISE 8.15

The toroid of Figure 8.26(a) has a coil of 1000 turns wound on its core. If $\rho_o = 10$ cm and $a = 1$ cm, what current is required to establish a magnetic flux of 0.5 mWb

(a) If the core is nonmagnetic

(b) If the core has $\mu_r = 500$

Answer: (a) 795.8 A, (b) 1.592 A.

EXAMPLE 8.16

A U-shaped electromagnet shown in Figure 8.29 is designed to lift a 400-kg mass (which includes the mass of the keeper). The iron yoke ($\mu_r = 3000$) has a cross section of 40 cm^2 and mean length of 50 cm, and the air gaps are each 0.1 mm long. Neglecting the reluctance of the keeper, calculate the number of turns in the coil when the excitation current is 1 A.

Solution:

The tractive force across the two air gaps must balance the weight. Hence

$$F = 2\frac{(B_a^2 S)}{2\mu_o} = mg$$

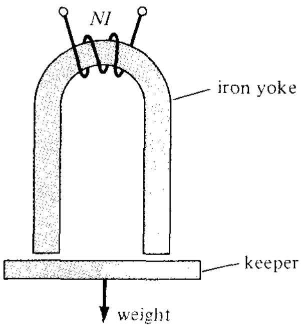


Figure 8.29 U-shaped electromagnet; for Example 8.16.

or

$$B_a^2 = \frac{mg\mu_o}{S} = \frac{400 \times 9.8 \times 4\pi \times 10^{-7}}{40 \times 10^{-4}}$$

$$B_a = 1.11 \text{ Wb/m}^2$$

But

$$\mathcal{F} = NI = \Psi(\mathcal{R}_a + \mathcal{R}_i)$$

$$\mathcal{R}_a = \frac{\ell_a}{\mu S} = \frac{2 \times 0.1 \times 10^{-3}}{4\pi \times 10^{-7} \times 40 \times 10^{-4}} = \frac{6 \times 10^6}{48\pi}$$

$$\mathcal{R}_i = \frac{\ell_i}{\mu_o \mu_r S} = \frac{50 \times 10^{-2}}{4\pi \times 10^{-7} \times 3000 \times 40 \times 10^{-4}} = \frac{5 \times 10^6}{48\pi}$$

$$\mathcal{F}_a = \frac{\mathcal{R}_a}{\mathcal{R}_a + \mathcal{R}_i}\mathcal{F} = \frac{6}{6+5}NI = \frac{6}{11}NI$$

Since

$$\mathcal{F}_a = H_a\ell_a = \frac{B_a\ell_a}{\mu_o}$$

$$N = \frac{11}{6}\frac{B_a\ell_a}{\mu_o I} = \frac{11 \times 1.11 \times 0.1 \times 10^{-3}}{6 \times 4\pi \times 10^{-7} \times 1}$$

$$N = 162$$

PRACTICE EXERCISE 8.16

Find the force across the air gap of the magnetic circuit of Example 8.15.

Answer: 895.2 N.

SUMMARY

1. The Lorentz force equation

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) = m\frac{d\mathbf{u}}{dt}$$

relates the force acting on a particle with charge Q in the presence of EM fields. It expresses the fundamental law relating EM to mechanics.

2. Based on the Lorentz force law, the force experienced by a current element $I d\mathbf{l}$ in a magnetic field $\mathbf{B}$ is

$$d\mathbf{F} = I\, d\mathbf{l} \times \mathbf{B}$$

From this, the magnetic field $\mathbf{B}$ is defined as the force per unit current element.

3. The torque on a current loop with magnetic moment **m** in a uniform magnetic field **B** is

$$\mathbf{T} = \mathbf{m} \times \mathbf{B} = IS\mathbf{a}_n \times \mathbf{B}$$

4. A magnetic dipole is a bar magnet or a small filamental current loop; it is so called due to the fact that its **B** field lines are similar to the **E** field lines of an electric dipole.
5. When a material is subjected to a magnetic field, it becomes magnetized. The magnetization **M** is the magnetic dipole moment per unit volume of the material. For linear material,

$$\mathbf{M} = \chi_m \mathbf{H}$$

where χ_m is the magnetic susceptibility of the material.
6. In terms of their magnetic properties, materials are either linear (diamagnetic or paramagnetic) or nonlinear (ferromagnetic). For linear materials,

$$\mathbf{B} = \mu\mathbf{H} = \mu_o\mu_r\mathbf{H} = \mu_o(1 + \chi_m)\mathbf{H} = \mu_o(\mathbf{H} + \mathbf{M})$$

where μ = permeability and $\mu_r = \mu/\mu_o$ = relative permeability of the material. For nonlinear material, $B = \mu(H)\,H$, that is, μ does not have a fixed value; the relationship between B and H is usually represented by a magnetization curve.
7. The boundary conditions that **H** or **B** must satisfy at the interface between two different media are

$$\mathbf{B}_{1n} = \mathbf{B}_{2n}$$

$$(\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_{n12} = \mathbf{K} \quad \text{or} \quad \mathbf{H}_{1t} = \mathbf{H}_{2t} \quad \text{if } \mathbf{K} = 0$$

where $\mathbf{a}_{n12}$ is a unit vector directed from medium 1 to medium 2.
8. Energy in a magnetostatic field is given by

$$W_m = \frac{1}{2}\int \mathbf{B} \cdot \mathbf{H}\, dv$$

For an inductor carrying current I

$$W_m = \frac{1}{2}LI^2$$

Thus the inductance L can be found using

$$L = \frac{\int \mathbf{B} \cdot \mathbf{H}\, dv}{I^2}$$

9. The inductance L of an inductor can also be determined from its basic definition: the ratio of the magnetic flux linkage to the current through the inductor, that is,

$$L = \frac{\lambda}{I} = \frac{N\Psi}{I}$$

Thus by assuming current I, we determine $\mathbf{B}$ and $\Psi = \int \mathbf{B} \cdot d\mathbf{S}$, and finally find $L = N\Psi/I$.

10. A magnetic circuit can be analyzed in the same way as an electric circuit. We simply keep in mind the similarity between

$$\mathcal{F} = NI = \oint \mathbf{H} \cdot d\mathbf{l} = \Psi\mathcal{R} \qquad \text{and} \qquad V = IR$$

that is,

$$\mathcal{F} \leftrightarrow V, \Psi \leftrightarrow I, \mathcal{R} \leftrightarrow R$$

Thus we can apply Ohms and Kirchhoff's laws to magnetic circuits just as we apply them to electric circuits.

11. The magnetic pressure (or force per unit surface area) on a piece of magnetic material is

$$P = \frac{F}{S} = \frac{1}{2}BH = \frac{B^2}{2\mu_o}$$

where B is the magnetic field at the surface of the material.

REVIEW QUESTIONS

8.1 Which of the following statements are not true about electric force $\mathbf{F}_e$ and magnetic force $\mathbf{F}_m$ on a charged particle?

(a) $\mathbf{E}$ and $\mathbf{F}_e$ are parallel to each other whereas $\mathbf{B}$ and $\mathbf{F}_m$ are perpendicular to each other.
(b) Both $\mathbf{F}_e$ and $\mathbf{F}_m$ depend on the velocity of the charged particle.
(c) Both $\mathbf{F}_e$ and $\mathbf{F}_m$ can perform work.
(d) Both $\mathbf{F}_e$ and $\mathbf{F}_m$ are produced when a charged particle moves at a constant velocity.
(e) $\mathbf{F}_m$ is generally small in magnitude compared to $\mathbf{F}_e$.
(f) $\mathbf{F}_e$ is an accelerating force whereas $\mathbf{F}_m$ is a purely deflecting force.

8.2 Two thin parallel wires carry currents along the same direction. The force experienced by one due to the other is

(a) Parallel to the lines
(b) Perpendicular to the lines and attractive
(c) Perpendicular to the lines and repulsive
(d) Zero

8.3 The force on differential length $d\mathbf{l}$ at point P in the conducting circular loop in Figure 8.30 is

(a) Outward along OP
(b) Inward along OP

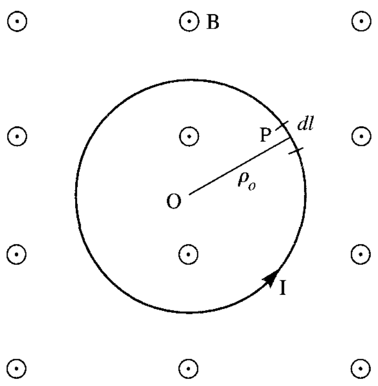


Figure 8.30 For Review Questions 8.3 and 8.4.

(c) In the direction of the magnetic field
(d) Tangential to the loop at P

8.4 The resultant force on the circular loop in Figure 8.30 has the magnitude of

(a) $2\pi\rho_o IB$
(b) $\pi\rho_o^2 IB$
(c) $2\rho_o IB$
(d) Zero

8.5 What is the unit of magnetic charge?

(a) Ampere-meter square
(b) Coulomb
(c) Ampere
(d) Ampere-meter

8.6 Which of these materials requires the least value of magnetic field strength to magnetize it?

(a) Nickel
(b) Silver
(c) Tungsten
(d) Sodium chloride

8.7 Identify the statement that is not true of ferromagnetic materials.

(a) They have a large χ_m.
(b) They have a fixed value of μ_r.
(c) Energy loss is proportional to the area of the hysteresis loop.
(d) They lose their nonlinearity property above the curie temperature.

8.8 Which of these formulas is wrong?

(a) $B_{1n} = B_{2n}$
(b) $B_2 = \sqrt{B_{2n}^2 + B_{2t}^2}$
(c) $H_1 = H_{1n} + H_{1t}$
(d) $\mathbf{a}_{n21} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{K}$, where $\mathbf{a}_{n21}$ is a unit vector normal to the interface and directed from region 2 to region 1.

8.9 Each of the following pairs consists of an electric circuit term and the corresponding magnetic circuit term. Which pairs are not corresponding?

(a) V and $\mathcal{F}$
(b) G and $\mathcal{P}$
(c) ε and μ
(d) IR and $H\mathcal{R}$
(e) $\Sigma I = 0$ and $\Sigma \Psi = 0$

8.10 A multilayer coil of 2000 turns of fine wire is 20 mm long and has a thickness 5 mm of winding. If the coil carries a current of 5 mA, the mmf generated is

(a) 10 A-t
(b) 500 A-t
(c) 2000 A-t
(d) None of the above

Answers: 8.1 b,c, 8.2b, 8.3a, 8.4d, 8.5d, 8.6a, 8.7b, 8.8c, 8.9c,d, 8.10a.

PROBLEMS

8.1 An electron with velocity $\mathbf{u} = (3\mathbf{a}_x + 12\mathbf{a}_y - 4\mathbf{a}_z) \times 10^5$ m/s experiences no net force at a point in a magnetic field $\mathbf{B} = 10\mathbf{a}_x + 20\mathbf{a}_y + 30\mathbf{a}_z$ mWb/m^2. Find **E** at that point.

8.2 A charged particle of mass 1 kg and charge 2 C starts at the origin with velocity $10\mathbf{a}_z$ m/s in a magnetic field $\mathbf{B} = 1\mathbf{a}_x$ Wb/m^2. Find the location and the kinetic energy of the particle at $t = 2$ s.

***8.3** A particle with mass 1 kg and charge 2 C starts from rest at point (2, 3, −4) in a region where $\mathbf{E} = -4\mathbf{a}_y$ V/m and $\mathbf{B} = 5\mathbf{a}_x$ Wb/m^2. Calculate

(a) The location of the particle at $t = 1$ s
(b) Its velocity and K.E. at that location

8.4 A −2-mC charge starts at point (0, 1, 2) with a velocity of $5\mathbf{a}_x$ m/s in a magnetic field $\mathbf{B} = 6\mathbf{a}_y$ Wb/m^2. Determine the position and velocity of the particle after 10 s assuming that the mass of the charge is 1 gram. Describe the motion of the charge.

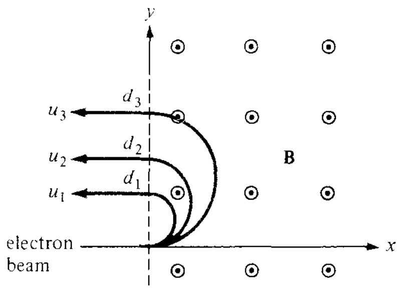


Figure 8.31 For Problem 8.5.

***8.5** By injecting an electron beam normally to the plane edge of a uniform field $B_o\mathbf{a}_z$, electrons can be dispersed according to their velocity as in Figure 8.31.

(a) Show that the electrons would be ejected out of the field in paths parallel to the input beam as shown.

(b) Derive an expression for the exit distance d above entry point.

8.6 Given that $\mathbf{B} = 6x\mathbf{a}_x - 9y\mathbf{a}_y + 3z\mathbf{a}_z$ Wb/m^2, find the total force experienced by the rectangular loop (on $z = 0$ plane) shown in Figure 8.32.

8.7 A current element of length 2 cm is located at the origin in free space and carries current 12 mA along $\mathbf{a}_x$. A filamentary current of $15\mathbf{a}_z$ A is located along $x = 3$, $y = 4$. Find the force on the current filament.

***8.8** Three infinite lines L_1, L_2, and L_3 defined by $x = 0, y = 0$; $x = 0, y = 4$; $x = 3, y = 4$, respectively, carry filamentary currents -100 A, 200 A, and 300 A along $\mathbf{a}_z$. Find the force per unit length on

(a) L_2 due to L_1

(b) L_1 due to L_2

(c) L_3 due to L_1

(d) L_3 due to L_1 and L_2. State whether each force is repulsive or attractive.

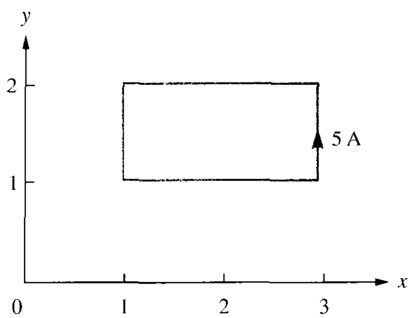


Figure 8.32 For Problem 8.6.

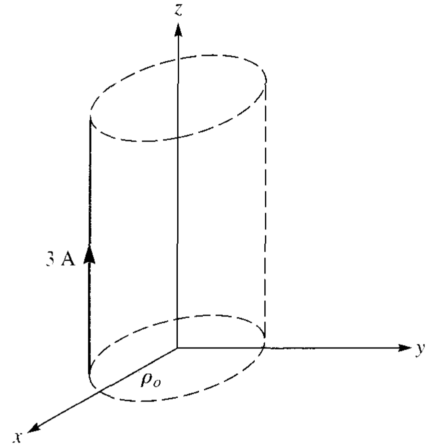


Figure 8.33 For Problem 8.9.

8.9 A conductor 2 m long carrying 3A is placed parallel to the z-axis at distance $\rho_o = 10$ cm as shown in Figure 8.33. If the field in the region is $\cos(\phi/3)\,\mathbf{a}_\rho$ Wb/m^2, how much work is required to rotate the conductor one revolution about the z-axis?

***8.10** A conducting triangular loop carrying a current of 2 A is located close to an infinitely long, straight conductor with a current of 5 A, as shown in Figure 8.34. Calculate (a) the force on side 1 of the triangular loop and (b) the total force on the loop.

***8.11** A three-phase transmission line consists of three conductors that are supported at points A, B, and C to form an equilateral triangle as shown in Figure 8.35. At one instant, conductors A and B both carry a current of 75 A while conductor C carries a return current of 150 A. Find the force per meter on conductor C at that instant.

***8.12** An infinitely long tube of inner radius a and outer radius b is made of a conducting magnetic material. The tube carries a total current I and is placed along the z-axis. If it is exposed to a constant magnetic field $B_o\mathbf{a}_\rho$, determine the force per unit length acting on the tube.

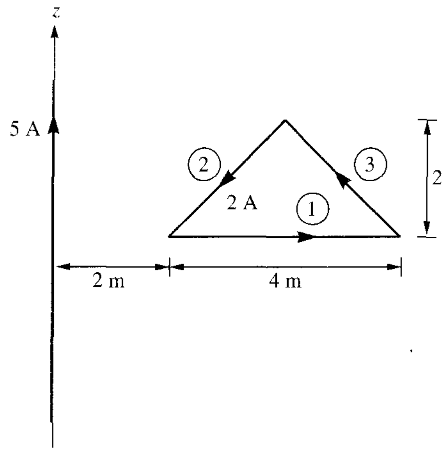


Figure 8.34 For Problem 8.10.

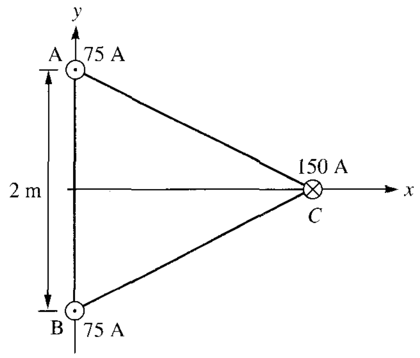


Figure 8.35 For Problem 8.11.

***8.13** An infinitely long conductor is buried but insulated from an iron mass ($\mu = 2000\mu_o$) as shown in Figure 8.36. Using image theory, estimate the magnetic flux density at point P.

8.14 A galvanometer has a rectangular coil of side 10 by 30 mm pivoted about the center of the shorter side. It is mounted in radial magnetic field so that a constant magnetic field of 0.4 Wb/m^2 always acts across the plane of the coil. If the coil has 1000 turns and carries current 2 mA, find the torque exerted on it.

8.15 A small magnet placed at the origin produces $\mathbf{B} = -0.5\mathbf{a}_z$ mWb/m^2 at (10, 0, 0). Find $\mathbf{B}$ at

(a) (0, 3, 0)
(b) (3, 4, 0)
(c) (1, 1, −1)

8.16 A block of iron ($\mu = 5000\mu_o$) is placed in a uniform magnetic field with 1.5 Wb/m^2. If iron consists of 8.5×10^{28} atoms/m^3, calculate: (a) the magnetization $\mathbf{M}$, (b) the average magnetic current.

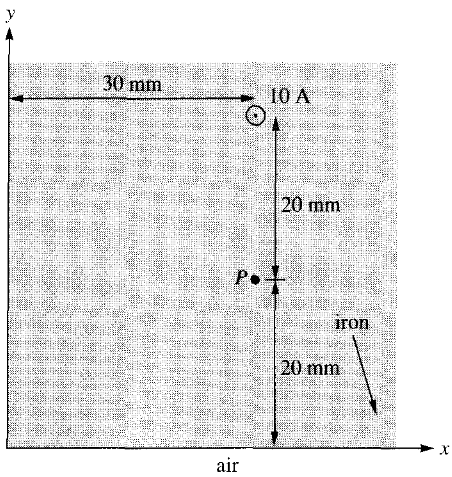


Figure 8.36 For Problem 8.13.

8.17 In a certain material for which $\mu = 6.5\mu_o$,

$$\mathbf{H} = 10\mathbf{a}_x + 25\mathbf{a}_y - 40\mathbf{a}_z \text{ A/m}$$

find

(a) The magnetic susceptibility χ_m of the material
(b) The magnetic flux density **B**
(c) The magnetization **M**,
(d) The magnetic energy density

8.18 In a ferromagnetic material ($\mu = 4.5\mu_o$),

$$\mathbf{B} = 4y\mathbf{a}_z \text{ mWb/m}^2$$

calculate: (a) χ_m, (b) **H**, (c) **M**, (d) $\mathbf{J}_b$.

8.19 The magnetic field intensity is $H = 1200$ A/m in a material when $B = 2$ Wb/m^2. When H is reduced to 400 A/m, $B = 1.4$ Wb/m^2. Calculate the change in the magnetization M.

8.20 An infinitely long cylindrical conductor of radius a and permeability $\mu_o\mu_r$ is placed along the z-axis. If the conductor carries a uniformly distributed current I along $\mathbf{a}_z$ find **M** and $\mathbf{J}_b$ for $0 < \rho < a$.

8.21 If $\mathbf{M} = \dfrac{k_o}{a}(-y\mathbf{a}_x + x\mathbf{a}_y)$ in a cube of size a, find $\mathbf{J}_b$. Assume k_o is a constant.

***8.22** (a) For the boundary between two magnetic media such as is shown in Figure 8.16, show that the boundary conditions on the magnetization vector are

$$\frac{M_{1t}}{\chi_{m1}} - \frac{M_{2t}}{\chi_{m2}} = K \qquad \text{and} \qquad \frac{\mu_1}{\chi_{m1}} m_{1n} = \frac{\mu_2}{\chi_{m2}} M_{2n}$$

(b) If the boundary is not current free, show that instead of eq. (8.49), we obtain

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2}\left[1 + \frac{K\mu_2}{B_2 \sin\theta_2}\right]$$

8.23 If $\mu_1 = 2\mu_o$ for region 1 ($0 < \phi < \pi$) and $\mu_2 = 5\mu_o$ for region 2 ($\pi < \phi < 2\pi$) and $\mathbf{B}_2 = 10\mathbf{a}_\rho + 15\mathbf{a}_\phi - 20\mathbf{a}_z$ mWb/m^2. Calculate: (a) $\mathbf{B}_1$, (b) the energy densities in the two media.

8.24 The interface $2x + y = 8$ between two media carries no current. If medium 1 ($2x + y \geq 8$) is nonmagnetic with $\mathbf{H}_1 = -4\mathbf{a}_x + 3\mathbf{a}_y - \mathbf{a}_z$ A/m. Find: (a) the magnetic energy density in medium 1, (b) $\mathbf{M}_2$ and $\mathbf{B}_2$ in medium 2 ($2x + y \leq 8$) with $\mu = 10\mu_o$, (c) the angles $\mathbf{H}_1$ and $\mathbf{H}_2$ make with the normal to the interface.

8.25 The interface $4x - 5z = 0$ between two magnetic media carries current $35\mathbf{a}_y$ A/m. If $\mathbf{H}_1 = 25\mathbf{a}_x - 30\mathbf{a}_y + 45\mathbf{a}_z$ A/m in region $4x - 5z \leq 0$ where $\mu_{r1} = 5$, calculate $\mathbf{H}_2$ in region $4x - 5z \geq 0$ where $\mu_{r2} = 10$.

8.26 The plane $z = 0$ separates air ($z \geq 0$, $\mu = \mu_o$) from iron ($z \leq 0$, $\mu = 200\mu_o$). Given that

$$\mathbf{H} = 10\mathbf{a}_x + 15\mathbf{a}_y - 3\mathbf{a}_z \text{ A/m}$$

in air, find **B** in iron and the angle it makes with the interface.

8.27 Region $0 \leq z \leq 2$ m is filled with an infinite slab of magnetic material ($\mu = 2.5\mu_o$). If the surfaces of the slab at $z = 0$ and $z = 2$, respectively, carry surface currents $30\mathbf{a}_x$ A/m and $-40\mathbf{a}_x$ A/m as in Figure 8.37, calculate **H** and **B** for

(a) $z < 0$
(b) $0 < z < 2$
(c) $z > 2$

8.28 In a certain region for which $\chi_m = 19$,

$$\mathbf{H} = 5x^2yz\mathbf{a}_x + 10xy^2z\mathbf{a}_y - 15xyz^2\mathbf{a}_z \text{ A/m}$$

How much energy is stored in $0 < x < 1, 0 < y < 2, -1 < z < 2$?

8.29 The magnetization curve for an iron alloy is approximately given by $B = \frac{1}{3}H + H^2\mu$ Wb/m^2. Find: (a) μ_r when $H = 210$ A/m, (b) the energy stored per unit volume in the alloy as H increases from 0 to 210 A/m.

***8.30** (a) If the cross section of the toroid of Figure 7.15 is a square of side a, show that the self-inductance of the toroid is

$$L = \frac{\mu_o N^2 a}{2\pi} \ln\left[\frac{2\rho_o + a}{2\rho_o - a}\right]$$

(b) If the toroid has a circular cross section as in Figure 7.15, show that

$$L = \frac{\mu_o N^2 a^2}{2\rho_o}$$

where $\rho_o \gg a$.

8.31 When two parallel identical wires are separated by 3 m, the inductance per unit length is 2.5 μH/m. Calculate the diameter of each wire.

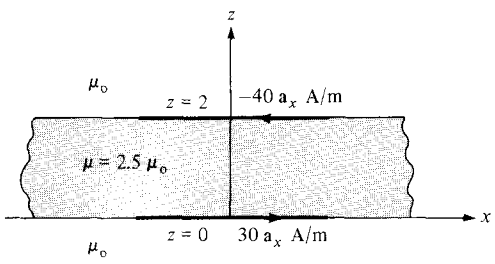


Figure 8.37 For Problem 8.27.

8.32 A solenoid with length 10 cm and radius 1 cm has 450 turns. Calculate its inductance.

8.33 The core of a toroid is 12 cm^2 and is made of material with $\mu_r = 200$. If the mean radius of the toroid is 50 cm, calculate the number of turns needed to obtain an inductance of 2.5 H.

8.34 Show that the mutual inductance between the rectangular loop and the infinite line current of Figure 8.4 is

$$M_{12} = \frac{\mu b}{2\pi} \ln \left[\frac{a + \rho_o}{\rho_o} \right]$$

Calculate M_{12} when $a = b = \rho_o = 1$ m.

***8.35** Prove that the mutual inductance between the closed wound coaxial solenoids of length ℓ_1 and ℓ_2 ($\ell_1 \gg \ell_2$), turns N_1 and N_2, and radii r_1 and r_2 with $r_1 \simeq r_2$ is

$$M_{12} = \frac{\mu N_1 N_2}{\ell_1} \pi r_1^2$$

8.36 A cobalt ring ($\mu_r = 600$) has a mean radius of 30 cm. If a coil wound on the ring carries 12 A, calculate the number of turns required to establish an average magnetic flux density of 1.5 Wb/m in the ring.

8.37 Refer to Figure 8.27. If the current in the coil is 0.5 A, find the mmf and the magnetic field intensity in the air gap. Assume that $\mu = 500\mu_o$ and that all branches have the same cross-sectional area of 10 cm^2.

8.38 The magnetic circuit of Figure 8.38 has current 10 A in the coil of 2000 turns. Assume that all branches have the same cross section of 2 cm^2 and that the material of the core is iron with $\mu_r = 1500$. Calculate R, $\mathcal{F}$, and Ψ for

(a) The core
(b) The air gap

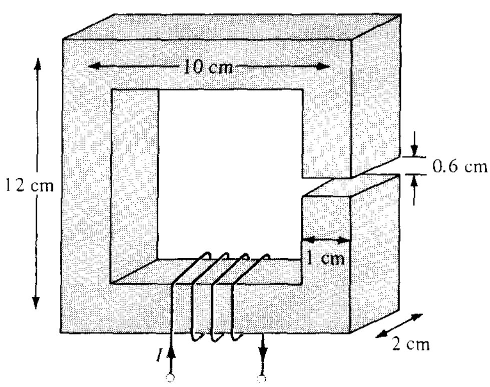


Figure 8.38 For Problem 8.38.

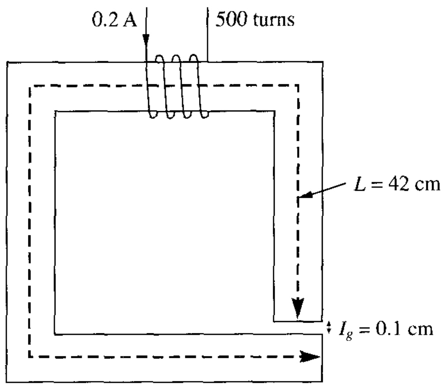


Figure 8.39 For Problem 8.39.

8.39 Consider the magnetic circuit in Figure 8.39. Assuming that the core ($\mu = 1000\mu_o$) has a uniform cross section of 4 cm^2, determine the flux density in the air gap.

8.40 An electromagnetic relay is modeled as shown in Figure 8.40. What force is on the armature (moving part) of the relay if the flux in the air gap is 2 mWb? The area of the gap is 0.3 cm^2, and its length 1.5 mm.

8.41 A toroid with air gap, shown in Figure 8.41, has a square cross section. A long conductor carrying current I_2 is inserted in the air gap. If $I_1 = 200$ mA, $N = 750$, $\rho_o = 10$ cm, $a = 5$ mm, and $\ell_a = 1$ mm, calculate

(a) The force across the gap when $I_2 = 0$ and the relative permeability of the toroid is 300

(b) The force on the conductor when $I_2 = 2$ mA and the permeability of the toroid is infinite. Neglect fringing in the gap in both cases.

8.42 A section of an electromagnet with a plate below it carrying a load is shown in Figure 8.42. The electromagnet has a contact area of 200 cm^2 per pole with the middle pole having a winding of 1000 turns with $I = 3$ A. Calculate the maximum mass that can be lifted. Assume that the reluctance of the electromagnet and the plate is negligible.

8.43 Figure 8.43 shows the cross section of an electromechanical system in which the plunger moves freely between two nonmagnetic sleeves. Assuming that all legs have the same cross-sectional area S, show that

$$\mathbf{F} = -\frac{2N^2I^2\mu_o S}{(a + 2x)^2}\mathbf{a}_x$$

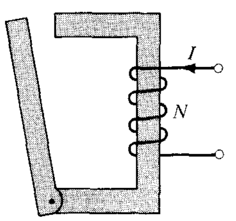


Figure 8.40 For Problem 8.40.

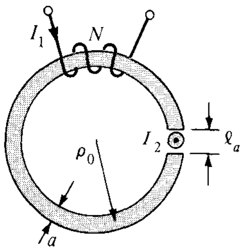


Figure 8.41 For Problem 8.41.

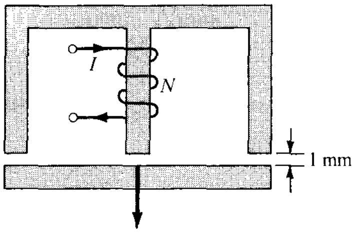


Figure 8.42 For Problem 8.42.

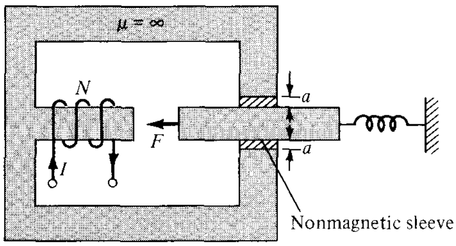


Figure 8.43 For Problem 8.43.