Chapter Four

partial differential equation

(P.D.Es)

Chapter 4

Partial differential equations (P.D.Es):-

partial diff. eq., like ordinary diff. eq. are classified as either <u>Linear</u> or <u>non linear</u>.

A, B, C ثوابت   A, B, C..... متغيرات أو متغير

The <u>linear</u> second-order (P.D.E) is:-

$$\boxed{\underbrace{A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}}_{\text{second order}}+\underbrace{D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}}_{\text{First order}}+Fu=G(x,y)}$$

If $G(x,y)=0$, then this equation is homogenous otherwise, it is non homogenous.

This equation is called:- Elliptic if $B^2-4AC<0$,
parabolic if $B^2-4AC=0$ &
Hyperbolic if $B^2-4AC>0$

<u>Ex</u> classify the following equations:-

① $3\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial y} \Rightarrow 3\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial y}=0 \Rightarrow \begin{matrix}A=3\\B=C=0\end{matrix}$ $\therefore B^2-4AC=0$ $\therefore$ the eq. is parabolic

② $\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2} \Rightarrow \frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=0 \Rightarrow \begin{matrix}A=1\\B=0 \;\&\; C=-1\end{matrix} \Longrightarrow \begin{matrix}B^2-4AC\\0+4>0\end{matrix}$ $\therefore$ the eq. is hyperbolic

③ $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \Rightarrow \begin{matrix}A=C=1\\B=0\end{matrix} \Rightarrow \begin{matrix}B^2-4AC\\0-4<0\end{matrix}$ $\therefore$ the eq. is elliptic.

In parctical problems, the following types of equations are generally used:-

(i) One-dimensional **heat** flow :- $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ parabolic

(ii) One-dimensional **wave** equation:- $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ hyperbolic

(iii) Two-dimensional heat flow :- $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ elliptic
(**Laplace** equation).

(iv) Poisson's equation:- $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = h(x, y)$

(V) Radio equation :- $-\frac{\partial V}{\partial x} = L\frac{\partial I}{\partial t}, \quad \frac{\partial I}{\partial x} = C\frac{\partial V}{\partial t}$

Note

$\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$ 2-dim. heat flow

$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$ 2-dimensional wave equ.

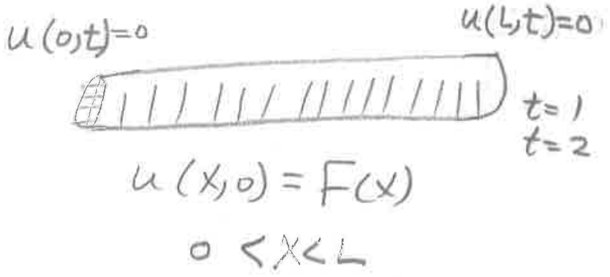
$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ — Laplace equ.

Definition:-

Boundary-value problems such as:-

solve $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ $\quad 0<x<L \quad , \boxed{t>0}$ شرط

subject to : B.c (Boundary condition)



$u(0,t)=0$

$u(L,t)=0 \;,\; \boxed{t>0}$ شرط في كل الحلول

* I.c (Initial condition)

$u(x,0)=f(x)$

$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) \quad , \quad 0<x<L$

---

and solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ $\quad 0<x<a \quad 0<y<b$

subject to B.C:

$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 \quad , \quad \left.\frac{\partial u}{\partial x}\right|_{x=a} = 0 \qquad 0<y<b$

$u(x,0)=0 \quad , \quad u(x,b)=f(x) \qquad 0<x<a$

are called boundary-value problems.

$u(0,t)=u(L,t)=0$

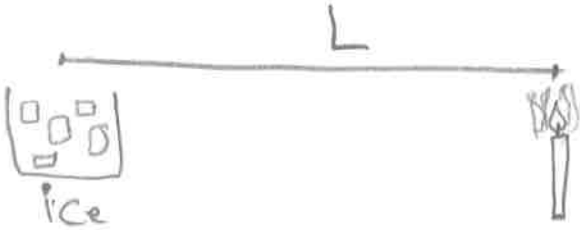
$u(0,t)=0$

$u(L,t)=3$

ملاحظة: اذا ال B.Cs = صفر ∴ homog. لل X

واذا أحد ال B.Cs ≠ صفر ∴ non-homog.

ولكن t و $0 \le t$ شرط



$u(0,t) = 10 \Rightarrow x = 0$

$t = t$

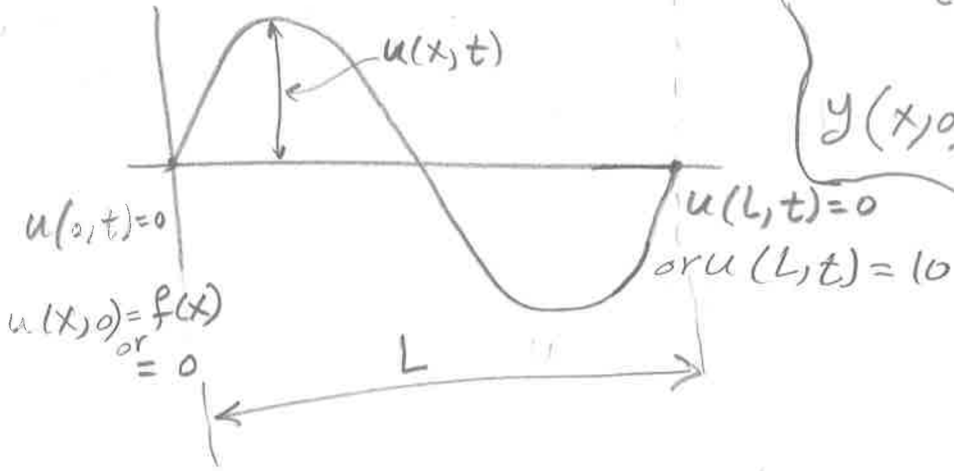
$u = 10$

$\left.\dfrac{\partial y}{\partial t}\right|_{(x,0)} = \sin 2x \Rightarrow t = 0$

$\dfrac{dy}{dt} = \sin 2x$

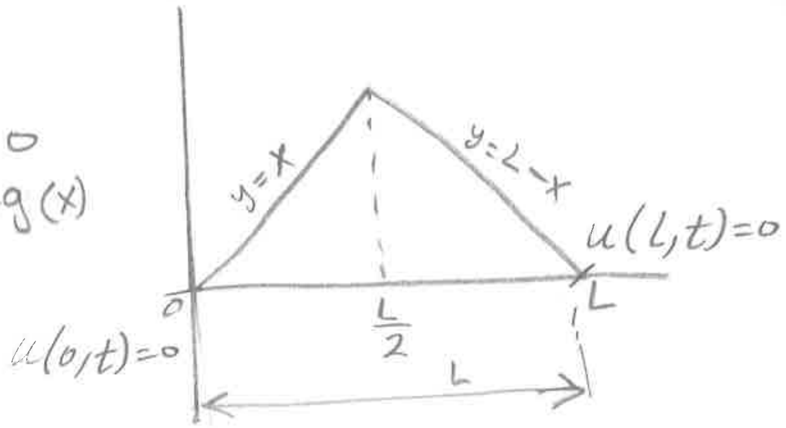
$y(x,0) = \sin 2x \Rightarrow t = 0$

$y = \sin 2x$

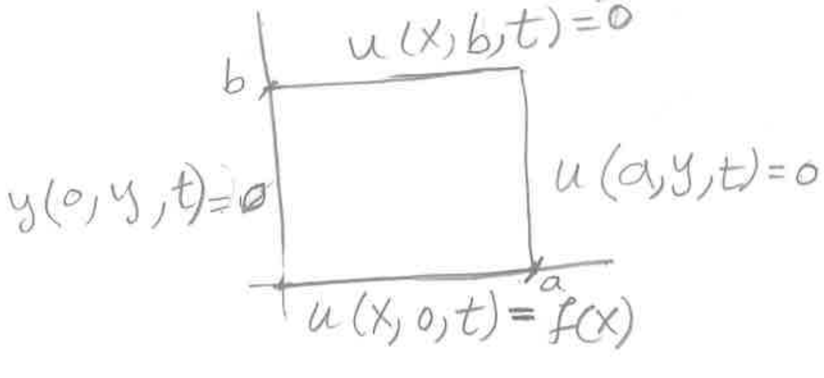


$u_t(x,0) = 0$

or $u_t(x,0) = g(x)$



$$u(x,0) = f(x) = \begin{cases} x & 0 < x < \frac{L}{2} \\ (L-x) & \frac{L}{2} < x < L \end{cases}$$



Solution of a P.D.E :

A solution of a linear partial differential equation

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+ \text{— — — —}$$

is a function $u(x,y)$ of two independent variables.

Method of separation Variables :-

In this method, we assume that the dependent variable is the product of two functions, each of which involves only one of the independent variables. So two ordinary differential equations are formed.

أي $u(x,y) = X_{(x)} \cdot Y_{(y)}$, to reduce a linear P.D.E in two variables to two O.D.Es (O.D.E ← P.D.E تحويل الى) (*)

$$\frac{\partial u}{\partial x} = u_x = X'Y \quad ; \quad \frac{\partial^2 u}{\partial x^2} = u_{xx} = X''Y$$

$$\frac{\partial u}{\partial y} = u_y = XY' \quad ; \quad \frac{\partial^2 u}{\partial y^2} = u_{yy} = XY''$$

Ex
$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \Rightarrow T''X = c^2 X'' T$$

$$\Rightarrow \frac{T''}{c^2 T} = \frac{X''}{X}$$

Ex $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0 \qquad u(x,0) = 4e^{-x}$

solution

let $u_{(x,y)} = X_{(x)} \cdot Y_{(y)}$

where X is a function of x only and Y is a function of y only.

$\therefore 3X'Y + 2XY' = 0 \Rightarrow 3X'Y = -2Y'X$

$\therefore \boxed{\frac{3X'}{X}} = \boxed{\frac{-2Y'}{Y}} = \text{Constant} = K$

(*) since the left-hand side is independent of y and is equal to right-hand side which is independent of x

$\therefore \frac{3X'}{X} = K, \quad \because X' = \frac{dX}{dx}$

$\therefore 3\frac{dX}{X\,dx} = K \Rightarrow 3\frac{dX}{X} = K\,dx \Rightarrow 3\int\frac{dX}{X} = \int K\,dx$

$\therefore 3\ln|X| = Kx + C_1 \Rightarrow \ln X = \frac{Kx + C_1}{3}$

$\Rightarrow X = e^{\frac{Kx}{3} + \frac{C_1}{3}} \Rightarrow X = e^{\frac{K}{3}x} \cdot e^{\frac{C_1}{3}} \Rightarrow \boxed{X = A e^{\frac{K}{3}x}}$

$\therefore \frac{-2Y'}{Y} = K, \quad \because Y' = \frac{dY}{dy}$

$\therefore -2\frac{dY}{Y\,dy} = K \Rightarrow \int\frac{dY}{Y} = \int\frac{-K}{2}\,dy$

$$\therefore \ln y = \frac{-k}{2} y + c_2 \Rightarrow y = e^{\frac{-k}{2} y + c_2} = e^{\frac{-k}{2} y} \cdot \underbrace{e^{c_2}}_{\to B}$$

$$\Rightarrow \boxed{y = B e^{\frac{-k}{2} y}}$$

$$\because u_{(x,y)} = X_{(x)} \cdot Y_{(y)} \Rightarrow \boxed{u_{(x,y)} = A e^{\frac{k}{3} x} \cdot B e^{-\frac{k}{2} y}}$$

let $C = A * B$

$$\therefore u_{(x,y)} = C e^{\frac{k}{3} x - \frac{k}{2} y}$$

نبحث عن قيمة الثابت C

B.C $\because u(x,0) = 4e^{-x} \Rightarrow \underline{x = x}$ & $\underline{y = 0}$

$$u(x,0) = 4e^{-x} = C e^{\frac{k}{3} x - 0} = C e^{\frac{k}{3} x}$$

$$\therefore \underline{4e^{-x}} = \underline{C e^{\frac{k}{3} x}}$$

من التشابه

$$\Rightarrow \boxed{C = 4} \text{ \& } e^{-x} = e^{\frac{k}{3} x} \Rightarrow \therefore -1 = \frac{k}{3} \Rightarrow \boxed{k = -3}$$

$$\therefore u_{(x,y)} = 4 e^{-x + \frac{3}{2} y}$$

هذا هو حل
المعادلة
وهو المطلوب

Ex Solve the p.d.e $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y} - 3u$

with the condition $u(0,y) = 2e^{3y}$

Solution

let $u(x,y) = X_{(x)} \cdot Y_{(y)}$ ------ ⊛

$\therefore \frac{\partial u}{\partial x} = X' \cdot Y \quad ; \quad \frac{\partial u}{\partial y} = X \cdot Y'$

$\because \frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y} - 3u \implies \boxed{X'Y = 4XY' - 3XY}\ (\div XY)$

$\therefore \frac{X'}{X} = 4\frac{Y'}{Y} - 3 = K$

$\therefore \int \frac{X'}{X} = \int K \Rightarrow \ln|X| = Kx + c_1 \Rightarrow X = e^{Kx+c_1} = e^{Kx} \cdot e^{c_1}$ A

$\therefore \boxed{X_{(x)} = Ae^{Kx}}$ --- sub. in ⊛

$\therefore 4\frac{Y'}{Y} - 3 = K \Rightarrow \frac{4Y'}{Y} = 3 + K \quad \therefore \int \frac{Y'}{Y} = \int \frac{3+K}{4}$

$\therefore \ln Y = \left(\frac{3+K}{4}\right)y + c_2 \Rightarrow Y_{(y)} = e^{\left(\frac{3+K}{4}\right)y} \cdot e^{c_2}$ B

$\therefore \boxed{Y_{(y)} = Be^{\left(\frac{3+K}{4}\right)y}}$ sub. in ⊛ $\quad \therefore u(x,y) = AB\, e^{Kx} \cdot e^{\left(\frac{3+K}{4}\right)y}$

نبحث عن قيمة AB و K

B.C's $\therefore u(0,y) = 2e^{3y} = AB\, e^{0}\, e^{\left(\frac{3+K}{4}\right)y} \xrightarrow[\text{التشابه}]{\text{من}} \begin{cases} \therefore AB = 2 \\ \frac{3+K}{4} = 3 \Rightarrow 12 = 3 + K \\ \therefore K = 9 \end{cases}$

$\Rightarrow u(x,y) = 2e^{9x} \cdot e^{3y}$ وهو المطلوب

Note

The second order linear homog. in which p and q are constant is take the form :-

$y'' + p y' + q y = 0$, to solve this eq, let's

$m^2 + pm + q = 0$

if

①	$m_1 \neq m_2$	$\therefore y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$
②	$m_1 = m_2 = m$	$\therefore y(x) = c_1 e^{mx} + c_2 x e^{mx}$ لمنع التكرار
③	$m_1 = a + bi$ $m_2 = a - bi$	$\therefore y(x) = e^{ax}[c_1 \cos bx + c_2 \sin bx]$

Ex $y'' + 4y' + 4y = 0$

$\therefore m^2 + 4m + 4 = 0 \Rightarrow (m+2)^2 = 0$

$\therefore m_1 = m_2 = -2 \Longrightarrow \boxed{y(x) = c_1 e^{-2x} + c_2 x e^{-2x}}$

ثم نجد من قيم الثوابت c_1 , c_2 من ال B.Cs .

Ex $\dfrac{d^2y}{dt^2} + 5y = 0$

$\therefore m^2 + 5 = 0 \Longrightarrow m_{1,2} = \mp\sqrt{5}\, i$

$\therefore a = 0$, $b = \mp\sqrt{5} \Rightarrow y(t) = e^{0x}[c_1 \cos\sqrt{5}t + c_2 \sin\sqrt{5}t]$

$\boxed{\therefore y(t) = c_1 \cos\sqrt{5}t + c_2 \sin\sqrt{5}t.}$

Super Postion theorem :-

If u_1 --- u_n are solution for D.E. and c_1 --- c_n are constant, then :-

$$c_1 u_1 + c_2 u_2 - - - c_n u_n = \sum_{n=0}^{\infty} c_n u_n$$ is also solution of D.E.

$$\therefore u(x,t) = \sum_{n=0}^{\infty} c_n u_n \rightarrow (\text{S.P.th.})$$

Note

$$\cos(n\pi) = (-1)^n$$

$$\sin(n\pi) = 0 \qquad n = 0, \mp 1, \mp 2, \mp 3 - - -$$

$$\cos\left(\frac{2n-1}{2}\right)\pi = 0$$

مهمة جداً
حفظ

Note

$$\grave{X} = \frac{dX}{dx}, \quad \grave{y} = \frac{dy}{dy}$$

$$\grave{T} = \frac{dT}{dt}$$

① one-dimensional heat flow equation:-

solution of a boundary-value problem by separation of variables. Consider a thin rod length L, with initial temp. $f(x)$ through out and whose ends are held at temp. zero for all time $t>0$ (homog.), then the temp. $u(x,t)$ in the rod is determined from the boundary-value problem.

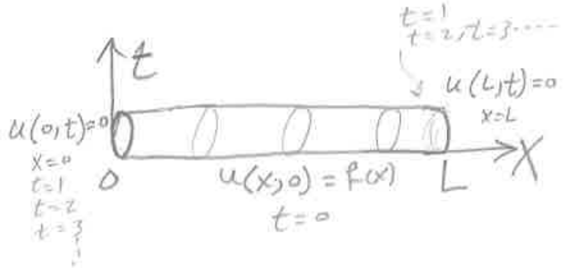
∴ الدالة متجانسة ويحدد الـ B.Cs في الـ heat eq. = ٣

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(0,t)=0 \quad , \quad u(L,t)=0$$
$$u(x,0)=f(x)$$

Solution ∴ الدالة متجانسة

let $u(x,t) = X_{(x)} \cdot T_{(t)}$



$$\therefore \frac{\partial u}{\partial t} = X \cdot T' \quad , \quad \frac{\partial^2 u}{\partial x^2} = X'' \cdot T$$

طريقة sep. Var. فصل المتغيرات

$$\Rightarrow X T' = c^2 X'' T \Longrightarrow \frac{T'}{T} = c^2 \frac{X''}{X}$$

⊗ X, T: are two independent variables.

أي X و T متغيرين منفصلين حيث اي تغير بـ T لا يؤثر على X والعكس صحيح.

$$\therefore \frac{T'}{c^2 T} = \frac{X''}{X} = K$$

(K: constant).

اما قيمة موجبة λ^2 او صفر او قيمة سالبة $-\lambda^2$

$$\Rightarrow K = \begin{cases} \lambda^2 \\ 0 \\ -\lambda^2 \end{cases}$$

⊗ حالة واحدة فقط هي الصحيحة لذا يجب ان نناقش كل حالة بعد تطبيق المعادلة و B.Cs.

λ^2: real separation constant

Case I $\boxed{\text{If } k=0}$

$$\because \frac{T'}{c^2T} = \frac{X''}{X} = 0 \Rightarrow \frac{T'}{c^2T} = 0 \Rightarrow T' = 0 \quad \therefore \int T'\,dt = 0 \quad \boxed{\therefore T = C}$$

$$\& \ \frac{X''}{X} = 0 \Rightarrow X'' = 0 \Rightarrow \therefore \int X''\,dx = 0 \quad \therefore X' = C_1 \quad \therefore \int X'\,dx = \int C_1$$

$$\boxed{\therefore X = C_1 X + C_2}$$

$$\because u(x,t) = X_{(x)} \cdot T_{(t)}$$

$$= (C_1 X + C_2)C \Rightarrow \boxed{u(x,t) = C_3 X + C_4}$$

نبحث عن قيم C_4 و C_3 من B.C's

B.C's

$$\because u(0,t) = 0 \Rightarrow 0 = 0 + C_4 \Rightarrow \boxed{C_4 = 0} \Rightarrow \boxed{u(x,t) = C_3 X}$$

$$\& \ \because u(L,t) = 0 \Rightarrow 0 = C_3 L + 0 \Rightarrow \boxed{C_3 = 0} \Rightarrow \boxed{u(x,t) = 0} \Rightarrow T.S$$

(Trivial solution) حل تافه

Case II If $K = \lambda^2$ $(k>0)$

قيمة موجبة

$$\left(T' = \frac{dT}{dt}\right)$$

$$\because \frac{T'}{c^2T} = \frac{X''}{X} = \lambda^2 \Rightarrow \boxed{\frac{T'}{c^2T} = \lambda^2} \quad \therefore \int \frac{dT}{T} = \int c^2\lambda^2\,dt$$

$$\therefore \ln T = c^2\lambda^2 t + C_1 \Rightarrow T_{(t)} = e^{c^2\lambda^2 t + C_1} = e^{c^2\lambda^2 t} \cdot e^{C_1} \quad \left(\text{let } A_1 = e^{C_1}\right)$$

$$\boxed{\therefore T_{(t)} = A_1 e^{c^2\lambda^2 t}}$$

$$\& \ \boxed{\frac{X''}{X} = \lambda^2} \Rightarrow X'' - \lambda^2 X = 0 \Rightarrow \boxed{\therefore X_{(x)} = c_1 e^{\lambda x} + C_2 e^{-\lambda x}}$$

$$m^2 - \lambda^2 = 0 \Rightarrow m = \mp\lambda$$

$$\therefore u(x,t) = X_{(x)} \cdot T_{(t)}$$

$$= \left(c_1 e^{\lambda x} + c_2 e^{-\lambda x}\right) A_1 e^{c^2\lambda^2 t}$$

$$\therefore u(x,t) = e^{c^2\lambda^2 t}\left(A e^{\lambda x} + B e^{-\lambda x}\right)$$

⊛ نبحث عن قيم الثوابت A و B من الـ B.C's

⊛ ملاحظة :- عند التعويض بقيمة X بـ صفر إو L نعتبر حد الـ t ثابت $\neq$ صفر. والعكس صحيح بالنسبة للـ t

B.C's

عوض $x=0$

$$u(0,t) = 0 \Rightarrow e^{c^2\lambda^2 t}(A+B) = 0 \Rightarrow A+B=0 \quad \boxed{\therefore B = -A}$$

$$u(L,t) = 0 \Rightarrow e^{c^2\lambda^2 t}\left(A e^{\lambda L} - A e^{-\lambda L}\right) = 0$$

$$\Rightarrow A e^{c^2\lambda^2 t}\left(e^{\lambda L} - e^{-\lambda L}\right) = 0$$

either $e^{\lambda L} - e^{-\lambda L} = 0 \quad \therefore \lambda = 0 \Rightarrow u(x,t) = 0 \Rightarrow T.S$

or $A = 0,\ B = 0 \Rightarrow u(x,t) = 0 \Rightarrow T.S$

(Trivial solution).

case III If $K = -\lambda^2$ $(K<0)$

قيمة سالبة

$$\therefore \frac{T'}{c^2 T} = \frac{X''}{X} = -\lambda^2 \Rightarrow \frac{T'}{T} = -c^2\lambda^2 \therefore \int \frac{dT}{T} = -\int c^2\lambda^2\, dt$$

$$\Rightarrow \ln T = -c^2\lambda^2 t + c_1 \quad \boxed{\therefore T_{(t)} = A_1 e^{-c^2\lambda^2 t}} \quad \boxed{\text{where } A_1 = e^{c_1}}$$

$$\& \ \frac{X''}{X} = -\lambda^2 \Rightarrow X'' + \lambda^2 X = 0 \Rightarrow m^2 + \lambda^2 = 0 \quad \therefore m = \mp \lambda i$$

$$\boxed{\therefore X_{(x)} = c_1 \cos\lambda x + c_2 \sin\lambda x}$$

$$\therefore u(x,t) = X_{(x)} \cdot T_{(t)} \Rightarrow u(x,t) = \left(c_1 \cos\lambda x + c_2 \sin\lambda x\right) A_1 e^{-c^2\lambda^2 t}$$

$$\therefore u(x,t) = e^{-c^2\lambda^2 t}\left(A\cos\lambda x + B\sin\lambda x\right)$$

نبحث عن قيم الثوابت A و B من الـ B.C's

B.C's

① $u(0,t)=0 \Rightarrow e^{-c^2\lambda^2 t} * (A+0)=0 \Rightarrow Ae^{-c^2\lambda^2 t}=0 \Rightarrow A=0$

② $u(L,t)=0 \Rightarrow e^{-c^2\lambda^2 t} * (B\sin\lambda L)=0$

- $B=0$, $\sin\lambda L\neq 0$ $\Rightarrow u(x,t)=0 \Rightarrow$ T.S
- $B\neq 0$, $\sin\lambda L=0$

$\therefore \lambda L = n\pi$ $\quad n=0,\mp1,\mp2,\mp3,\ldots$

$$\Rightarrow \boxed{\lambda = \frac{n\pi}{L}}$$

$$\therefore u(x,t) = e^{-c^2\left(\frac{n^2\pi^2}{L^2}\right)t} * B\sin\left(\frac{n\pi}{L}\right)x$$

$$\therefore u(x,t) = Be^{-\frac{c^2n^2\pi^2 t}{L^2}} * \sin\frac{n\pi x}{L}$$

by s.p. theorem

$$\therefore u(x,t) = \sum_{n=0}^{\infty} B_n e^{-\frac{c^2n^2\pi^2 t}{L^2}} * \sin\frac{n\pi x}{L}$$

(*) نبحث عن قيمة الـ B_n عند $t=0$

B.C's

③ $$u(x,0)=f(x)=\sum_{n=0}^{\infty} B_n \sin\frac{n\pi x}{L}$$

$$\text{where } B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

حفظ

بالمقارنة مع Fourier series

Ex solve the following boundary value problem by using method of separation of variable :-

$u_t = 4 u_{xx}$ ... $u(0,t) = u(5,t) = 0$

$u(x,0) = f(x) = x$

عدد ال B.C's في ال heat eq = ٣

<u>Solution</u>

$\frac{\partial u}{\partial t} = 4 \frac{\partial^2 u}{\partial x^2}$

$\because u(0,t) = u(5,t) = 0$ الشروط متجانسة

let $\boxed{u(x,t) = X_{(x)} \cdot T_{(t)}}$

$\therefore u_x = X' T$, $u_{xx} = X'' \cdot T$

$u_t = X T'$

$\because u_t = 4 u_{xx} \Rightarrow X T' = 4 X'' T$ (by separation variables.)

$\therefore \frac{T'}{T} = 4 \frac{X''}{X} = k = \begin{cases} \lambda^2 \\ 0 \\ -\lambda^2 \end{cases}$

① If $k = \lambda^2 \Rightarrow \int \frac{T'}{T} = \int \lambda^2 \Rightarrow \ln T = \lambda^2 t + c$

$T = e^{\lambda^2 t + c} = e^{\lambda^2 t} \cdot C$ ← e^c ثابت

$\boxed{\therefore T = C e^{\lambda^2 t}}$

& $4 \frac{X''}{X} = \lambda^2 \Rightarrow 4X'' - \lambda^2 X = 0 \Rightarrow 4m^2 - \lambda^2 = 0$ $\boxed{\therefore m = \mp \frac{\lambda}{2}}$

$$\therefore \boxed{X = C_1 e^{\frac{\lambda}{2}x} + C_2 e^{-\frac{\lambda}{2}x}}$$

$$\because u(x,t) = X_{(x)}\,T_{(t)} \Rightarrow u(x,t) = C e^{\lambda^2 t}\left(C_1 e^{\frac{\lambda}{2}x} + C_2 e^{-\frac{\lambda}{2}x}\right)$$

$$\boxed{u(x,t) = e^{\lambda^2 t}\left(A e^{\frac{\lambda}{2}x} + B e^{-\frac{\lambda}{2}x}\right)}$$

نبحث عن قيم الثوابت A, B. من الـ B.Cs

B.Cs:

$\because u(0,t) = 0$ $\quad \therefore e^{\lambda^2 t}(A+B) = 0$ $\quad \therefore B = -A$

$u(5,t) = 0$ $\quad \therefore e^{\lambda^2 t}\left(A e^{\frac{5}{2}\lambda} - A e^{-\frac{5}{2}\lambda}\right) = 0 \Rightarrow \underbrace{A e^{\lambda^2 t}}_{\text{اما } = 0}\underbrace{\left(e^{\frac{5}{2}\lambda} - e^{-\frac{5}{2}\lambda}\right)}_{\text{or } = 0} = 0$

either $A e^{\lambda^2 t} = 0 \Rightarrow A = B = 0 \Rightarrow u(x,t) = 0 \Rightarrow$ T.S

or $\left(e^{\frac{5}{2}\lambda} - e^{-\frac{5}{2}\lambda}\right) = 0 \Rightarrow \lambda = 0 \Rightarrow u(x,t) = 0 \Rightarrow$ T.S

(2) If $k = 0$

$$\because \frac{T'}{T} = 0 \Rightarrow T' = 0 \Rightarrow \int T'\,dt = 0 \Rightarrow \boxed{T = C}$$

$$4\frac{X''}{X} = 0 \Rightarrow X'' = 0 \Rightarrow \int X'' = 0 \Rightarrow X' = C_1$$

$$\int X' = \int C_1 \Rightarrow \boxed{X = C_1 x + C_2}$$

$\because u(x,t) = X_x \cdot T_t$

$\therefore u(x,t) = C(C_1 x + C_2)$

$$\boxed{\therefore u(x,t) = C_3 x + C_4} \leftarrow$$

$C_3 = CC_1$, $C_4 = CC_2$

* نبحث عن قيم الثوابت C_3 , C_4 من الـ B.Cs

B.C's

$\because u(0,t)=0 \Rightarrow 0=0+C_4 \Rightarrow C_4=0$ $\boxed{\therefore u(x,t)=C_3 X}$

& $\because u(5,t)=0 \Rightarrow 0=5C_3+0 \Rightarrow C_3=0$ $\boxed{\therefore u(5,t)=0} \Rightarrow$ T.S

---

③ If $k=-\lambda^2$

$\therefore \frac{T'}{T}=4\frac{X''}{X}=-\lambda^2 \quad \Rightarrow \int \frac{T'}{T}=\int -\lambda^2$

$\therefore \ln T=-\lambda^2 t+c \Rightarrow T=e^{-\lambda^2 t+c} \Rightarrow T=e^{-\lambda^2 t}\cdot A_1$ $\quad e^c$

$\boxed{\therefore T=A_1 e^{-\lambda^2 t}}$

& $4\frac{X''}{X}=-\lambda^2 \Rightarrow 4X''+\lambda^2 X=0 \Rightarrow 4m^2+\lambda^2=0$

$\Rightarrow m_{1,2}=\mp\frac{\lambda}{2}i \quad \therefore \boxed{X=C_1\cos\frac{\lambda}{2}x+C_2\sin\frac{\lambda}{2}x}$

$\therefore u(x,t)=X_x\cdot T_t$

$\therefore u(x,t)=\left(C_1\cos\frac{\lambda}{2}x+C_2\sin\frac{\lambda}{2}x\right)\left(A_1 e^{-\lambda^2 t}\right)$

$u(x,t)=e^{-\lambda^2 t}\left(A\cos\frac{\lambda}{2}x+B\sin\frac{\lambda}{2}x\right)$ عوضنا عن ضرب الثوابت
A و B .

B.C's

$u(0,t)=0 \Rightarrow 0=e^{-\lambda^2 t}(A+0)=0 \Rightarrow Ae^{-\lambda^2 t}=0 \Rightarrow A=0$

$$u(5,t)=0 \Rightarrow e^{-\lambda^2 t}\left(0+B\sin\frac{\lambda}{2}(5)\right)=0$$

$$B\sin\frac{5\lambda}{2}=0$$

- $B=0$, $\sin\frac{5\lambda}{2}\neq 0$ $\Rightarrow u(x,t)=0 \Rightarrow T.S$
- $B\neq 0$, $\sin\frac{5\lambda}{2}=0$

$$\boxed{\therefore \frac{5\lambda}{2}=n\pi} \qquad n=0,\mp1,\mp2 ---$$

$$\boxed{\therefore \lambda=\frac{2n\pi}{5}}$$

$$\therefore u(x,t)=e^{-\frac{4n^2\pi^2}{25}t}\left(B\sin\frac{n\pi}{5}x\right)$$

by s.p. theorem

* يجب التحويل الى s.p.th قبل $t=0$ (B.CS)

$$u(x,t)=\sum_{n=0}^{\infty} B_n\, e^{-\frac{4n^2\pi^2 t}{25}} * \sin\frac{n\pi x}{5}$$

* الاوضعيت من حيث ان Bn نش $t=0$.

B.C's

$$\therefore u(x,0)=f(x)=x=\sum_{n=0}^{\infty} B_n * e^{0} * \sin\frac{n\pi x}{5}$$

$$\therefore x=\sum_{n=0}^{\infty} B_n\sin\frac{n\pi x}{5} \; ; \text{ From Fourier Series}$$

$$\therefore B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \Rightarrow B_n=\frac{2}{5}\int_0^5 x\sin\frac{n\pi x}{5}\,dx$$

u	dv
x (+)	$\sin\frac{n\pi x}{5}$
1 (−)	$-\left(\cos\frac{n\pi x}{5}\right)\frac{5}{n\pi}$
0	$-\left(\sin\frac{n\pi x}{5}\right)\frac{25}{n^2\pi^2}$

$$\therefore B_n=\frac{2}{5}\left[-\frac{5x}{n\pi}\cos\frac{n\pi x}{5}+\cancel{\frac{25}{n^2\pi^2}\sin\frac{n\pi x}{5}}^{\,0}\right]_0^5$$

$$\therefore B_n=\frac{-2}{\cancel{5}}\left[\frac{\cancel{25}^{\,5}}{n\pi}\cos n\pi\right]=\frac{-10}{n\pi}(-1)^n$$

The solution is :-

$$\therefore u(x,t) = \sum_{n=1}^{\infty} \frac{10}{n\pi}(-1)^{n+1} e^{\frac{-4n^2\pi^2 t}{25}} \cdot \sin\frac{n\pi x}{5}$$

وهو المطلوب

ولنفس السؤال السابق ولكن

$$u(x,0) = f(x) = x(\pi - x)$$

$$\therefore u(x,0) = f(x) = x(\pi - x) = \sum_{n=0}^{\infty} B_n * e^{0} * \sin\frac{n\pi x}{5}$$

$$\therefore x(\pi - x) = \sum_{n=0}^{\infty} B_n \sin\frac{n\pi x}{5} \quad \text{from F.S.}$$

$$\boxed{B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx} \Rightarrow B_n = \frac{2}{5}\int_0^5 x(\pi - x)\sin\frac{n\pi x}{5}\,dx$$

* تكامل عادي وكما في Fourier series لإيجاد قيمة B_n ثم نعوض في $u(x,t)$

ولنفس السؤال السابق ولكن

$$u(x,0) = f(x) = 3\sin 5\pi x - 8\sin 20\pi x$$

$$\therefore u(x,0) = 3\sin 5\pi x - 8\sin 20\pi x = \sum_{n=0}^{\infty} B_n * e^{0} * \sin\frac{n\pi x}{5}$$

$$\therefore 3\sin 5\pi x - 8\sin 20\pi x = \sum_{n=0}^{\infty} B_n \sin\frac{n\pi x}{5}$$

<u>طريقة الحل</u> اما Fourier series كما في الحالات السابقة أو استخدام طريقة أبسط وهي المقارنة وكما يلي :-

$$3\sin 5\pi x - 8\sin 20\pi x = B_1\sin\frac{\pi x}{5} + B_2\sin\frac{2\pi x}{5} + B_3\sin\frac{3\pi x}{5} + B_4\sin\frac{4\pi x}{5} + \cdots$$

$$B_1 = B_2 = B_3 = B_4 - - - - - - = 0$$

لس

$$\cancel{5\pi x} = \frac{n\cancel{\pi x}}{5} \Rightarrow n = 25 \ \& \ B_{25} = 3$$

$$20\cancel{\pi x} = \frac{n\cancel{\pi x}}{5} \Rightarrow n = 100 \ \& \ B_{100} = -8$$

$$u(x,t) = 3e^{\frac{-4(25)^2\pi^2 t}{25}} * \sin 5\pi x - 8\, e^{\frac{-4(100)^2\pi^2 t}{25}} * \sin 20\pi x$$

∴ The solution is:-

$$\therefore u(x,t) = 3e^{-100\pi^2 t} * \sin 5\pi x - 8e^{-1600\pi^2 t} * \sin 20\pi x$$

H.W

Find the temperature $u(x,t)$ in a bar of length π which is perfectly insulated everywhere including the ends $x=0$ and $x=\pi$. This leads to the conditions $u_x(0,t) = u_x(\pi,t) = 0$. Further the initial temperature is $f(x) = 3\cos x + 7\cos 2x$.

H.W

solve $u_t = 0.16\, u_{xx}$

$u(0,t) = u(100,t) = 0$, $u(x,0) = \begin{cases} 60 & 0 < x < 50 \\ 40 & 50 < x < 100 \end{cases}$

ans:- $u(x,t) = \sum_{n=0}^{\infty} A_n e^{-0.16 \frac{n^2 \pi^2 t}{100^2}} * \sin \frac{n\pi}{100} x$

$A_n = \frac{-120}{n\pi}\left[\cos \frac{n\pi}{2} - 1\right] - \frac{80}{n\pi}\left[(-1)^n - \cos \frac{n\pi}{2}\right]$

solution

② one-dimensional wave equation :-

Consider an elastic string tightly (بإحكام) stretched (مربوط) between two points O and A.
Let O be the origin and OA as x-axis. On giving a small displacement to the string, perpendicular to its length L (parallel to the y-axis). The vertical displacement is $u(x,t)$ at any time. The wave eq. :

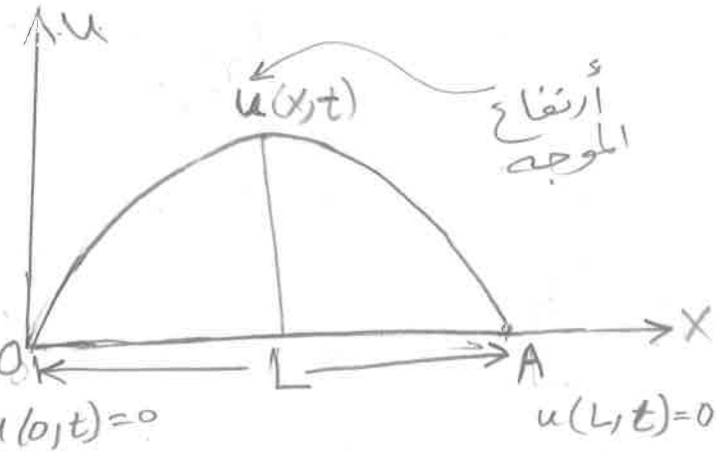
$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad \boxed{\begin{matrix} 0<x<L \\ t>0 \end{matrix}}$$

شرط

$u(0,t) = u(L,t) = 0$

$u(x,0) = f(x)$

$u_t(x,0) = g(x)$



$u(x,0) = f(x)$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = u_t(x,0) = g(x)$$

Σ = Wave eq. + B.C's صدار

$u(x,t)$: ارتفاع الموجه

$u_t(x,t)$: السرعة

Ex Solve the following :-

$$\frac{\partial^2 u}{\partial t^2} = 4\,\frac{\partial^2 u}{\partial x^2}$$

$u(0,t) = u(10,t) = 0$ [illegible]

$u(x,0) = x+2$

$u_t(x,0) = 0$

Solution

let $u(x,t) = X_{(x)} \cdot T_{(t)}$

$\because \dfrac{\partial^2 u}{\partial t^2} = X \cdot T''$ & $\dfrac{\partial^2 u}{\partial x^2} = X'' \cdot T$

$\because \dfrac{\partial^2 u}{\partial t^2} = 4\dfrac{\partial^2 u}{\partial x^2} \Rightarrow \left[X \cdot T'' = 4\, X'' \cdot T\right] \div 4XT$

$$\therefore \frac{T''}{4T} = \frac{X''}{X} = K = \begin{cases} \lambda^2 \rightarrow \text{T.S} \\ 0 \\ -\lambda^2 \end{cases}$$

---

① If $K = \lambda^2 \Rightarrow \dfrac{T''}{4T} = \lambda^2 \Rightarrow T'' - 4\lambda^2 T = 0$

$\therefore m^2 - 4\lambda^2 = 0 \Rightarrow (m-2\lambda)(m+2\lambda) = 0$

فرق بين مربعين

$\Rightarrow m_{1,2} = \mp 2\lambda$

$$\therefore \boxed{T = C_1 e^{2\lambda t} + C_2 e^{-2\lambda t}}$$

$\dfrac{X''}{X} = \lambda^2 \Rightarrow X'' - \lambda^2 X = 0 \quad \therefore m^2 - \lambda^2 = 0$

$\therefore (m-\lambda)(m+\lambda) = 0$

$\therefore m_{1,2} = \mp\lambda$.

$$\therefore \boxed{X = C_3 e^{\lambda x} + C_4 e^{-\lambda x}}$$

$$\therefore u(x,t) = X_{(x)} T_{(t)} = \left(C_3 e^{\lambda x} + C_4 e^{-\lambda x}\right)\left(C_1 e^{2\lambda t} + C_2 e^{-2\lambda t}\right) \text{ --- } ①$$

* نبحث عن قيم الثوابت C_1, C_2, C_3, C_4

B.C's

$$u(0,t) = (C_3 + C_4)(C_1 e^{2\lambda t} + C_2 e^{-2\lambda t}) = 0$$

$$\Rightarrow C_3 + C_4 = 0 \Rightarrow \boxed{C_3 = -C_4} \quad \circledast$$

$$\& \; u(10,t) = (C_3 e^{10\lambda} - C_3 e^{-10\lambda})(C_1 e^{2\lambda t} + C_2 e^{-\lambda t}) = 0$$

$$\therefore u(10,t) = C_3 (e^{10\lambda} - e^{-10\lambda})(C_1 e^{2\lambda t} + C_2 e^{-\lambda t}) = 0$$

$$\Rightarrow C_3 (e^{10\lambda} - e^{-10\lambda}) = 0$$

either: $e^{10\lambda} - e^{-10\lambda} = 0 \quad \therefore \lambda = 0$
$\Rightarrow u(x,t) = 0 \Rightarrow$ T.S.

or: $C_3 = 0 \quad \therefore C_4 = 0$
$\Rightarrow u(x,t) = 0 \Rightarrow$ T.S.

---

② If $k = 0 \Rightarrow \ddot{T} = 0 \Rightarrow m^2 = 0$
$\therefore m_1 = m_2 = 0$

$$\boxed{\therefore T = C_1 + C_2 t}$$

$$\& \; \ddot{X} = 0 \Rightarrow m^2 = 0 \Rightarrow \boxed{X = C_3 + C_4 X}$$

$$\therefore u(x,t) = X_{(x)} \cdot T_{(t)} = (C_3 + C_4 x)(C_1 + C_2 t) \quad \circledast$$

$$\therefore u(x,t) = (C_3 + C_4 x)(C_1 + C_2 t)$$

نبحث عن قيم الثوابت C_1, C_2, C_3, C_4 ②

B.C's

$$\therefore u(0,t) = (C_3 + 0)(C_1 + C_2 t) = 0 \Rightarrow C_3 = 0$$

$$\& \; u(10,t) = (0 + 10 C_4)(C_1 + C_2 t) = 0 \Rightarrow C_4 = 0$$

$$\therefore u(x,t) = 0 \Rightarrow \text{T.S.}$$

③ If $k = -\lambda^2 \Rightarrow \frac{T''}{4T} = -\lambda^2 \Rightarrow T'' + 4\lambda^2 T = 0$

$m^2 + 4\lambda^2 = 0$

$\therefore m_{1,2} = \mp 2\lambda i$

$$\boxed{\therefore T = c_1 \cos 2\lambda t + c_2 \sin 2\lambda t}$$

& $\frac{X''}{X} = -\lambda^2 \Rightarrow X'' + \lambda^2 X = 0 \Rightarrow m^2 + \lambda^2 = 0$

$\therefore m_{1,2} = \mp \lambda i$

$$\therefore \boxed{X = c_3 \cos \lambda x + c_4 \sin \lambda x}$$

$\because u(x,t) = X_{(x)} \cdot T_{(t)}$ ----- ③

$\therefore u(x,t) = (c_3 \cos \lambda x + c_4 \sin \lambda x)(c_1 \cos 2\lambda t + c_2 \sin 2\lambda t)$

نبحث عن قيم الثوابت c_1 , c_2 , c_3 , c_4 .

B.C's الشرط الاول

$\because u(0,t) = 0 = (c_3 + 0)(c_1 \cos 2\lambda t + c_2 \sin 2\lambda t)$

$$\Rightarrow \boxed{c_3 = 0}$$

الشرط الثاني

$u(10,t) = 0 = (0 + c_4 \sin 10\lambda)(c_1 \cos 2\lambda t + c_2 \sin 2\lambda t)$

$\Rightarrow c_4 \sin 10\lambda = 0$ $\xrightarrow{\text{either}} c_4 = 0 \Rightarrow$ T.S.

$\xrightarrow{\text{or}} \sin 10\lambda = 0$

$\therefore 10\lambda = n\pi \qquad n = 0, \mp 1, \mp 2 ---$

$$\boxed{\therefore \lambda = \frac{n\pi}{10}}$$

$$\therefore u(x,t) = \left(C_4 \sin \frac{n\pi}{10}x\right)\left(C_1 \cos \frac{n\pi}{5}t + C_2 \sin \frac{n\pi}{5}t\right)$$

للمعادلة الجديدة بعد التعويض ب $C_3 = 0$, $\lambda = \frac{n\pi}{10}$ في المعادلة (3)

$$\therefore u(x,t) = C_1 C_4 \sin \frac{n\pi}{10}x \cos \frac{n\pi}{5}t + C_2 C_4 \sin \frac{n\pi}{10}x \cdot \sin \frac{n\pi}{5}t$$

let $B = C_1 C_4$ & $A = C_2 C_4$

$$\therefore u(x,t) = B \sin \frac{n\pi}{10}x \cos \frac{n\pi}{5}t + A \sin \frac{n\pi}{10}x \sin \frac{n\pi}{5}t$$

By S.P. theorem:-

$$u(x,t) = \left(\sum_{n=0}^{\infty} B_n \sin \frac{n\pi}{10}x \cos \frac{n\pi}{5}t\right) + \left(\sum_{n=0}^{\infty} A_n \sin \frac{n\pi}{10}x \sin \frac{n\pi}{5}t\right) \quad (4)$$

الآن تعتبر المعادلة رقم (4) هي المعادلة النهائية والتي بها يتم تحقيق الشرطين الثالث والرابع لإيجاد قيم A_n و B_n .

الشرط الثالث

$$\therefore u(x,0) = f(x) = x + 2$$

$$\therefore u(x,0) = x + 2 = \left(\sum_{n=0}^{\infty} B_n \sin \frac{n\pi}{10}x * 1\right) + (0)$$

$$\therefore x + 2 = \sum_{n=0}^{\infty} B_n \sin \frac{n\pi}{10}x$$

from Fourier series

$$\therefore B_n = \frac{2}{10}\int_0^{10} (x+2) \sin \frac{n\pi}{10}x \, dx$$

u	dv
$x+2$	$\sin \frac{n\pi x}{10}$
1	$-\left(\cos \frac{n\pi x}{10}\right)\left(\frac{10}{n\pi}\right)$
0	$-\left(\sin \frac{n\pi x}{10}\right)\left(\frac{10}{n\pi}\right)^2$

$$\therefore B_n = \frac{1}{5}\left[(x+2)\left(\frac{-10}{n\pi}\right)\cos \frac{n\pi x}{10} + \left(\frac{10}{n\pi}\right)^2 \sin \frac{n\pi x}{10}\right]_0^{10}$$

$$\therefore B_n = \frac{1}{5}\left[\frac{-120}{n\pi}\cdot(-1)^n + \frac{20}{n\pi}\right]$$

$$\boxed{\therefore B_n = \frac{4}{n\pi}\left[6(-1)^{n+1} + 1\right]}$$

الشرط الرابع

$$\therefore u_t(x,0) = 0 \Rightarrow u_t = \frac{\partial u}{\partial t} = \overbrace{\left(B_n \sin\frac{n\pi}{10}x\right)}^{\text{ثابت}}\left(-\sin\frac{n\pi}{5}t\right)\left(\frac{n\pi}{5}\right)$$

t متغير
x ثابت

$$+\underbrace{\left(A_n \sin\frac{n\pi}{10}x\right)}_{\text{ثابت}}\left(\cos\frac{n\pi}{5}t\right)\left(\frac{n\pi}{5}\right)$$

$$\therefore u_t(x,0) = 0 = 0 + \left(A_n \sin\frac{n\pi}{10}x\right) * 1 * \left(\frac{n\pi}{5}\right)$$

$$\therefore A_n\left(\frac{n\pi}{5}\right)\sin\frac{n\pi}{10}x = 0 \Longrightarrow A_n = 0$$

* عند التعويض بقيمة $t=0$ نعتبر حد ال x ثابت

$\therefore$ The solution is :- (from eq. 4)

$$\boxed{u(x,t) = \sum_{n=0}^{\infty}\frac{4}{n\pi}\left(6(-1)^{n+1}+1\right)\sin\frac{n\pi}{10}x\cos\frac{n\pi}{5}t}$$

ولنفس السؤال السابق ولكن بتغيير الشرط الرابع حيث

$$u_t(x,0) = g(x) = \begin{cases} x & 0<x<5 \\ x-10 & 5<x<10 \end{cases}$$

$$\because U_t = \sum_{n=0}^{\infty} B_n \sin\frac{n\pi}{10}x\left(-\sin\frac{n\pi}{5}t\right)\left(\frac{n\pi}{5}\right) + \sum_{n=0}^{\infty} A_n \sin\frac{n\pi}{10}x\left(\cos\frac{n\pi}{5}t\right)\left(\frac{n\pi}{5}\right)$$

$$\because U_t(x,0) = g(x) = \begin{cases} x & 0<x<5 \\ x-10 & 5<x<10 \end{cases}$$

$$\therefore U_t(x,0) = g(x) = 0 + \sum_{n=0}^{\infty} A_n \sin\frac{n\pi}{10}x) * 1 * \left(\frac{n\pi}{5}\right)$$

$$\boxed{\therefore g(x) = \sum_{n=0}^{\infty} A_n\left(\frac{n\pi}{5}\right)\sin\frac{n\pi}{10}x}$$ from F.S

$$A_n\left(\frac{n\pi}{5}\right) = \frac{2}{10}\int_0^{10} g(x)\sin\frac{n\pi}{10}x$$

$$\therefore A_n\left(\frac{n\pi}{5}\right) = \frac{1}{5}\left[\int_0^5 x\sin\frac{n\pi}{10}x\,dx + \int_5^{10}(x-10)\sin\frac{n\pi}{10}x\,dx\right]$$

$$\Rightarrow A_n = \checkmark$$

والحل العام هو المعادلة رقم 4
بعد التعويض بـ Bn و An

H.W

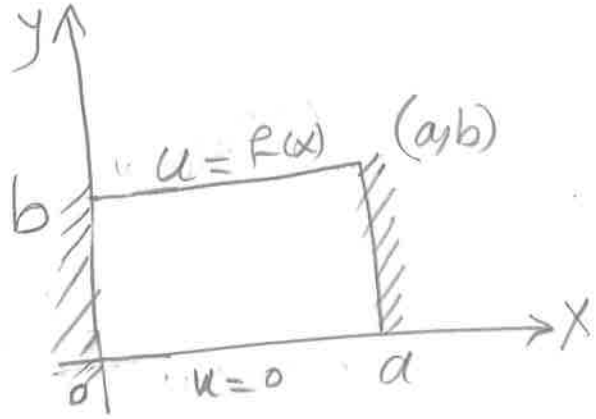
Vibration of an elastic is governed by the partial differential equation $U_{tt} = U_{xx}$; The Length of the string is π and the ends are fixed. The initial deflection is zero and the initial velocity is $U_t(x,0) = 6\sin 2x + 7\sin 5x - 4\sin 10x$. Find the deflection of the vibrating string for $t > 0$?

ans: $u(x,t) = 3\sin 2x \sin 2t + \frac{7}{5}\sin 5x \sin 5t - \frac{2}{5}\sin 10x * \sin 10t$

Solution

③ Two-dimensional heat flow (Laplace equation):-

Suppose we wish to find the steady-state temperature ($t=c$) $u(x,y)$ in a rectangular plate whose vertical edges are insulate, when no heat escape from the lateral face of the plate, we solve the boundary value problem.



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0<x<a, \quad 0<y<b$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=a} = 0 \qquad 0<y<b$$

$$u(x,0) = 0, \quad u(x,b) = f(x) \qquad 0<x<a$$

solution

let $u(x,y) = X_{(x)} \cdot Y_{(y)}$

$\therefore \left[X''Y + Y''X = 0 \right] \div XY \Rightarrow \frac{X''}{X} + \frac{Y''}{Y} = 0$

$\therefore \frac{X''}{X} = -\frac{Y''}{Y} = K$

If $K \geq 0$ then $u(x,y) = 0$
$\Rightarrow$ Trivial solution (T.S).
so, $K = -\lambda^2$.

$\therefore \frac{X''}{X} = -\lambda^2 \Rightarrow X'' + \lambda^2 X = 0$
$\therefore m^2 + \lambda^2 = 0$
$\therefore m^2 = -\lambda^2 \Rightarrow m_{1,2} = \mp \lambda i \Rightarrow$ Complex no.

$$\boxed{X_{(x)} = C_1 \cos\lambda x + C_2 \sin\lambda x}$$

$\therefore -\frac{Y''}{Y} = -\lambda^2 \Rightarrow Y'' - \lambda^2 Y = 0 \Rightarrow m^2 = \lambda^2 \quad \therefore \underbrace{m_{1,2} = \mp\lambda}_{\text{real no.}}$

$$\boxed{\therefore Y = C_3 e^{\lambda y} + C_4 e^{-\lambda y}}$$

$\therefore u(x,y) = X_{(x)} \cdot Y_{(y)}$

$$\boxed{\therefore u(x,y) = (C_1 \cos\lambda x + C_2 \sin\lambda x)(C_3 e^{\lambda y} + C_4 e^{-\lambda y})}$$

$$\boxed{\therefore u_x = (-C_1\lambda \sin\lambda x + C_2\lambda \cos\lambda x)(C_3 e^{\lambda y} + C_4 e^{-\lambda y})}$$

B.C 1

$\therefore u_x(0,y) = 0 = (0 + C_2\lambda * 1)\cancel{(C_3 e^{\lambda y} + C_4 e^{-\lambda y})}$

$\therefore C_2\lambda = 0 \Longrightarrow \boxed{C_2 = 0}$

B.C.2

& $u_x(a,y) = 0 = (-C_1\lambda \sin\lambda a + \overset{C_2=0}{0})\cancel{(C_3 e^{\lambda y} + C_4 e^{-\lambda y})}$

$\therefore C_2\lambda \sin\lambda a = 0 \Longrightarrow \underset{\Rightarrow \text{T.S}}{C_2\lambda \neq 0} \quad \therefore \sin\lambda a = 0$

$\therefore \sin n\pi = 0$

$\therefore n\pi = \lambda a \rightarrow \boxed{\lambda = \frac{n\pi}{a}}$

$\therefore u(x,y) = (C_1 \cos\frac{n\pi}{a}x)(C_3 e^{\frac{n\pi}{a}y} + C_4 e^{-\frac{n\pi}{a}y})$

$A = C_1C_3$

$B = C_1C_4$

$$\boxed{\therefore u(x,y) = A e^{\frac{n\pi}{a}y} \cos\frac{n\pi}{a}x + B e^{-\frac{n\pi}{a}y} \cos\frac{n\pi}{a}x}$$

By s.p. theorem :-

$$\boxed{u(x,y) = \sum_{n=0}^{\infty} A_n e^{\frac{n\pi}{a}y} \cos\frac{n\pi}{a}x + \sum_{n=0}^{\infty} B_n e^{-\frac{n\pi}{a}y} \cos\frac{n\pi}{a}x}$$

$\therefore$ B.C. 3

$$\therefore u(x,0) = 0 = \sum_{n=0}^{\infty} A_n * 1 * \cos\frac{n\pi x}{a} + \sum_{n=0}^{\infty} B_n * 1 * \cos\frac{n\pi x}{a}$$

$$\therefore \sum_{n=0}^{\infty} \cos\frac{n\pi x}{a}\left(A_n + B_n\right) = 0 \Rightarrow \boxed{A_n = -B_n}$$

$$\therefore u(x,y) = \sum_{n=0}^{\infty} \cos\frac{n\pi x}{a}\left(-B_n e^{\frac{n\pi}{a}y} + B_n e^{-\frac{n\pi}{a}y}\right)$$

$$\therefore u(x,y) = \sum_{n=0}^{\infty} B_n \cdot \cos\frac{n\pi x}{a}\left(-e^{\frac{n\pi}{a}y} + e^{-\frac{n\pi}{a}y}\right)$$

B.C. 4

$$\therefore u(x,b) = f(x) = \sum_{n=0}^{\infty} B_n \cos\frac{n\pi x}{a}\left(-e^{\frac{n\pi b}{a}} + e^{-\frac{n\pi b}{a}}\right)$$

$$\therefore f(x) = \sum_{n=0}^{\infty} -B_n\left(e^{\frac{n\pi b}{a}} - e^{-\frac{n\pi b}{a}}\right)\cos\frac{n\pi x}{a}$$

where:- $\sinh\frac{n\pi b}{a} = \frac{e^{\frac{n\pi b}{a}} - e^{-\frac{n\pi b}{a}}}{2}$

$$\therefore f(x) = \sum_{n=0}^{\infty}\left(-2B_n \sinh\frac{n\pi b}{a}\right)\cos\frac{n\pi x}{a}$$

From Fourier series :-

$$\therefore \left(-2B_n \sinh\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi}{a}x\,dx$$

$\therefore$ ثم نكمل حل السؤال حسب $f(x)$ الموجودة في السؤال ونجد B_n ثم A_n حيث $B_n = -A_n$

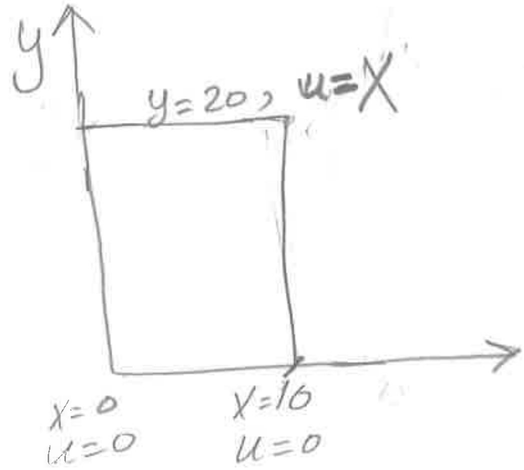
وهو المطلوب

Ex/ Solve the Laplace equation:-

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ where the boundary conditions

are $u(0,y) = u(10,y) = 0$ & initial conditions are

$u(x,0) = 0$, $u(x,20) = X$.



solution

let $u(x,y) = X_{(x)} \cdot Y_{(y)}$

$\therefore \left[\ddot{X} Y + \ddot{Y} X = 0 \right] \div XY$

$\therefore \frac{X''}{X} + \frac{Y''}{Y} = 0 \Rightarrow \frac{X''}{X} = \frac{-Y''}{Y} = K$

If $K \geq 0$ then $u(x,y) = 0 \Rightarrow$ Trivial solution.

so, $K < 0 \Rightarrow K = -\lambda^2$

$\therefore \frac{X''}{X} = -\lambda^2 \Rightarrow \boxed{X_{(x)} = C_1 \cos \lambda x + C_2 \sin \lambda x}$

$\frac{-Y''}{Y} = -\lambda^2 \Rightarrow \boxed{Y_{(y)} = C_3 e^{\lambda y} + C_4 e^{-\lambda y}}$

$\therefore u(x,y) = (C_1 \cos \lambda x + C_2 \sin \lambda x)(C_3 e^{\lambda y} + C_4 e^{-\lambda y})$

B.C.1

$$u(0,y) = (c_1 * 1 + 0)\cancel{(c_3 e^{\lambda y} + c_4 e^{-\lambda y})} = 0$$

$\therefore c_1 = 0$

B.C.2

$$u(10,y) = (c_2 \sin 10\lambda)\cancel{(c_3 e^{\lambda y} + c_4 e^{-\lambda y})} = 0$$

$\therefore c_2 \sin 10\lambda = 0 \to c_2 = 0 \Rightarrow$ T.S.

or $\to \sin 10\lambda = 0 \Rightarrow \because \sin n\pi = 0$

$$\Rightarrow n\pi = 10\lambda \Rightarrow \boxed{\lambda = \frac{n\pi}{10}}$$

$$\therefore u(x,y) = \left(c_2 \sin \frac{n\pi}{10}x\right)\left(c_3 e^{\frac{n\pi}{10}y} + c_4 e^{-\frac{n\pi}{10}y}\right)$$

$$\therefore u(x,y) = A e^{\frac{n\pi}{10}y} \sin\frac{n\pi}{10}x + B e^{-\frac{n\pi}{10}y} \sin\frac{n\pi}{10}x$$

By S.P. theorem.

$$\therefore u(x,y) = \sum_{n=0}^{\infty} A_n e^{\frac{n\pi}{10}y} \sin\frac{n\pi}{10}x + \sum_{n=0}^{\infty} B_n e^{-\frac{n\pi}{10}y} \sin\frac{n\pi}{10}x$$

B.C.3

$$\therefore u(x,0) = 0 = \sum_{n=0}^{\infty} A_n * 1 * \sin\frac{n\pi}{10}x + \sum_{n=0}^{\infty} B_n * 1 * \sin\frac{n\pi}{10}x$$

$$\Rightarrow \sum_{n=0}^{\infty} (A_n + B_n)\cancel{\sin\frac{n\pi}{10}x} = 0 \Rightarrow \boxed{A_n = -B_n}$$

$$\therefore u(x,y) = \sum_{n=0}^{\infty} \sin\frac{n\pi}{10}x \left(-B_n e^{\frac{n\pi y}{10}} + B_n e^{-\frac{n\pi y}{10}}\right)$$

$$\therefore u(x,y) = \sum_{n=0}^{\infty} -B_n \sin\frac{n\pi}{10}x \left(\underbrace{e^{\frac{n\pi y}{10}} - e^{-\frac{n\pi y}{10}}}_{=2\sinh\frac{n\pi y}{10}}\right)$$

$$\Rightarrow \boxed{u(x,y) = \sum_{n=0}^{\infty} -2B_n \sin\frac{n\pi}{10}x \cdot \sinh\frac{n\pi}{10}y} \text{ ------ } (*)$$

B.C.4 $\because u(x,20) = f(x) = x = \sum_{n=0}^{\infty} -2B_n \sin\frac{n\pi}{10}x \sinh 2n\pi$

$$\therefore f(x) = x = \sum_{n=0}^{\infty} \left(-2B_n \sinh 2n\pi\right) \sin\frac{n\pi}{10}x$$

From Fourier series:-

$$\therefore \left(-2B_n \sinh 2n\pi\right) = \frac{2}{10}\int_0^{10} x \sin\frac{n\pi}{10}x \, dx$$

u	dv
x	$\sin\frac{n\pi}{10}x$
1 (+)	$-\cos\frac{n\pi}{10}x\left(\frac{10}{n\pi}\right)$
0 (−)	$-\sin\frac{n\pi x}{10}\left(\frac{10}{n\pi}\right)^2$

$$\therefore -2B_n \sinh 2n\pi = \frac{1}{5}\left[\frac{-10x}{n\pi}\cos\frac{n\pi x}{10} + \left(\frac{10}{n\pi}\right)^2 \sin\frac{n\pi x}{10}\right]_0^{10}$$

$$\therefore B_n = \frac{-1}{10\sinh 2n\pi}\left[\frac{-100}{n\pi}(-1)^n\right] \Rightarrow \boxed{B_n = \frac{10(-1)^n}{n\pi \sinh 2n\pi}}$$

$$\because u(x,y) = \sum_{n=0}^{\infty} -B_n \sin\frac{n\pi}{10}x \sinh\frac{n\pi}{10}y$$

$$\boxed{\therefore u(x,y) = \sum_{n=1}^{\infty} \left(\frac{10(-1)^{n+1}}{n\pi \sinh 2n\pi} \sin\frac{n\pi}{10}x\right) \sinh\frac{n\pi}{10}y}$$

وهو المطلوب

ولنفس السؤال السابق ولكن

الشرط الرابع

$$u(x,20) = \sin \frac{n\pi}{10} x$$

$$\therefore f(x) = \sin \frac{n\pi}{10} x = \sum_{n=0}^{\infty} \left(-2B_n \sinh 2n\pi\right) * \sin \frac{n\pi}{10} x$$

نحل اما Fourier series أو الطريقة السريعة

$$\therefore \cancel{\sin \frac{n\pi}{10} x} = \sum_{n=0}^{\infty} \left(-2B_n \sinh 2n\pi\right) \cancel{\sin \frac{n\pi}{10} x}$$

$$\boxed{\therefore B_n = \frac{-1}{2\sinh 2n\pi}}$$

OR Fourier Series:-

$$\therefore -2B_n \sinh 2n\pi = \frac{2}{10} \int_0^{10} \overbrace{\sin \frac{n\pi}{10} x}^{f(x)} * \sin \frac{n\pi}{10} x$$

$$\therefore -2B_n \sinh 2n\pi = \frac{1}{5} \int_0^{10} \sin^2 \frac{n\pi}{10} x$$

$$\therefore -2B_n \sinh 2n\pi = \frac{1}{5} \int_0^{10} \frac{1 - \cos \frac{2n\pi}{10} x}{2}$$

$$\therefore -2B_n \sinh 2n\pi = \frac{1}{10}\left[x - \sin \frac{n\pi}{5} x\right]_0^{10} \Rightarrow B_n = \frac{-1}{2\sinh 2n\pi}$$

Non-homogenous :-

Ex

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$$

أو واحد منهم صفر

$u(0,t)=7$, $u(6,t)=10$

$u(x,0) = \sin \pi x$

Solution

$$\boxed{\begin{aligned} &\text{let } u(x,t) = V(x,t) + \overset{\text{non-homog. تابع لل}}{\underset{\text{x}}{G(x)}} \\ &u_t = V_t \\ &u_{xx} = V_{xx} + G''(x) \end{aligned}}$$

⊗ للتحويل من الغير متجانسة الى المتجانسة نفرض ⇐

$\because \frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$

$\therefore V_t = 4(V_{xx} + G''(x)) \Rightarrow \boxed{\boxed{V_t = 4V_{xx}} + 4\underbrace{G''(x)}_{=0}}$

To convertible into homogenous ($V_t = 4V_{xx}$) must be $G''(x) = 0$

⊛ للتحويل الى الـ homog. يجب ان تكون الحالة

$V_t = 4V_{xx}$

$V(0,t) = V(6,t) = 0$

$V(x,0)$ ونغض النظر عنه حيث يجب أيجادها حسب الحالة وكما سيوضح ....

$\Rightarrow G''(x) = 0 \Rightarrow m^2 = 0$

$\therefore m_{1,2} = 0$

$$\boxed{\therefore G(x) = C_1 x + C_2}$$

$\because u(x,t) = V(x,t) + G(x)$

$\therefore$ B.C.1

$$u(0,t) = V(0,t) + G(0) = 7$$

To homog. $\Rightarrow \underbrace{V(0,t) = 0}_{\text{لازم للتجانس}} \Rightarrow \boxed{G(0) = 7}$

& B.C.2

$$u(6,t) = V(6,t) + G(6) = 10$$

To homog. $\Rightarrow \underbrace{V(6,t) = 0}_{\text{لازم للتجانس}} \Rightarrow \boxed{G(6) = 10}$

$\therefore G(0) = 7$ & $G(6) = 10$; $G(x) = C_1 x + C_2$

$\Rightarrow G(0) = 7 = C_1 * 0 + C_2 \Rightarrow \boxed{C_2 = 7}$

&

$G(6) = 10 = 6C_1 + 7 \Rightarrow \boxed{C_1 = \frac{1}{2}}$

$\Rightarrow \boxed{G(x) = \frac{1}{2}x + 7}$

B.C.3

$\therefore u(x,0) = \boxed{V(x,0) + G(x) = \sin\pi x}$

$\therefore V(x,0) = \sin\pi x - G(x)$

$\boxed{\therefore V(x,0) = \sin\pi x - \frac{x}{2} - 7}$ $\Leftarrow$ المعادلة المراد الحصول عليها للتحويل من غير متجانسة الى متجانسة

$\therefore$ The homogenous equation is:-

$$\boxed{V_t = 4V_{xx} \qquad \begin{array}{l} V(0,t) = V(6,t) = 0 \\ V(x,0) = \sin\pi x - \frac{x}{2} - 7 \end{array}}$$

let $V(x,t) = X_{(x)} \cdot T_{(t)}$

$\therefore \left[\dot{T}X = 4\ddot{X}T\right] \div XT$

$\therefore \dfrac{\dot{T}}{T} = \dfrac{4\ddot{X}}{X} = K$

If $K \geq 0 \Rightarrow V(x,t)=0 \Rightarrow T.S$
$\therefore K < 0 \Rightarrow K = -\lambda^2$

$\therefore \dfrac{\dot{T}}{T} = -\lambda^2 \Rightarrow \ln T = -\lambda^2 t + C_1$

$\therefore T = Ce^{-\lambda^2 t}$ & $\dfrac{4\ddot{X}}{X} = -\lambda^2$

$\therefore m^2 + \dfrac{\lambda^2}{4} = 0$
$\therefore m_{1,2} = \mp \dfrac{\lambda}{2} i$

$\therefore$ $\boxed{X = C_1\cos\lambda x + C_2 \sin\lambda x}$

$$\boxed{\therefore V(x,t) = \left(A\cos\tfrac{\lambda}{2}x + B\sin\tfrac{\lambda}{2}x\right)e^{-\lambda^2 t}}$$

B.C's

$V(0,t) = 0 = Ae^{-\lambda^2 t} = 0 \Rightarrow \boxed{A=0}$

$V(6,t) = 0 = B\sin 3\lambda = 0$ $\boxed{\therefore \lambda = \dfrac{n\pi}{3}}$

$\therefore V(x,t) = \left(B\sin\dfrac{n\pi}{6}x\right)e^{-\frac{n^2\pi^2}{9}t}$

By S.P. theorem :-

$\therefore V(x,t) = \sum_{n=0}^{\infty}\left(B_n\sin\dfrac{n\pi x}{6}\right)e^{-\frac{n^2\pi^2}{9}t}$

B.C.3

$\therefore V(x,0) = \sin\pi x - \dfrac{x}{2} - 7 = \sum_{n=0}^{\infty} B_n \sin\dfrac{n\pi x}{6} * 1$

$\therefore$ From Fourier Series

$\therefore B_n = \dfrac{2}{6}\int_0^6 \left(\sin\pi x - \dfrac{x}{2} - 7\right)\sin\dfrac{n\pi}{6}x$

$$\therefore B_n = \frac{1}{3}\left[\int_0^6 \sin\pi x \sin\frac{n\pi x}{6}\,dx - \int_0^6 \frac{x}{2}\sin\frac{n\pi}{6}x\,dx - 7\int_0^6 \sin\frac{n\pi x}{6}\,dx\right]$$

$$\therefore B_n = \frac{1}{3}\left[\int_0^6 \frac{1}{2}\left[\cos\left(1-\frac{n}{6}\right)\pi x - \cos\left(1+\frac{n}{6}\right)\pi x\right] - \frac{1}{2}\left[\frac{-6x}{n\pi}\cos\frac{n\pi x}{6} + \cancel{\left(\frac{6}{n\pi}\right)^2\sin\frac{n\pi x}{6}}\right]_0^6 + 7\left[\frac{6}{n\pi}\cos\frac{n\pi x}{6}\right]_0^6\right.$$

u	dv
x ⊕	$\sin\frac{n\pi}{6}x$
1 ⊖	$-\cos\frac{n\pi}{6}x\left(\frac{6}{n\pi}\right)$
0	$-\sin\frac{n\pi}{6}x\left(\frac{6}{n\pi}\right)^2$

$$\therefore B_n = \frac{1}{3}\left[\frac{1}{2\pi}\left(\frac{6}{6-n}\right)\sin\left(\frac{6-n}{6}\right)\pi x - \frac{1}{2\pi}\left(\frac{6}{6+n}\right)\sin\left(\frac{6+n}{6}\right)\pi x\right]_{x=0}^{x=6} -$$

$$+\frac{1}{\cancel{6}}\left(\frac{\cancel{6}}{n\pi}\left[(-1)^n\right]\right) + \frac{\overset{14}{\cancel{42}}}{\cancel{3}n\pi}\left[(-1)^n - 1\right]$$

$$\therefore B_n = \frac{20}{n\pi}(-1)^n - \frac{14}{n\pi} \qquad n \neq 6$$

If $n = 6 \Longrightarrow B_6 = \frac{1}{3}\int_0^6\left(\sin^2\pi x - \frac{x}{2}\sin\pi x - 7\sin\pi x\right)dx$

$$\therefore B_6 = \frac{1}{3}\left[\int_0^6 \frac{1-\cos 2\pi x}{2} - \frac{1}{2}\left[\frac{-x}{\pi}\cos\pi x + \cancel{\frac{\sin\pi x}{\pi^2}}^{\,0}\right]_0^6 + 7\frac{\cos\pi x}{\pi}\right]_0^6$$

u	dv
x ⊕	$\sin\pi x$
1 ⊖	$-\frac{\cos\pi x}{\pi}$
0	$-\frac{\sin\pi x}{\pi^2}$

$$\therefore B_6 = \frac{1}{3}\left[\frac{x}{2} - \cancel{\frac{\sin 2\pi x}{4\pi}}^{\,0}\right]_0^6 - \frac{1}{6}\left[\frac{-6}{\pi}\right] + \cancel{\frac{7}{3\pi}(1-1)}^{\,0}$$

$$\therefore \boxed{B_6 = 1 + \frac{1}{\pi}}$$

$$\therefore V(x,t) = \sum_{n=0}^{5}\left[\frac{15}{n\pi}\left[(-1)^n - 1\right]\sin\frac{n\pi}{6}x\right] + \left(1+\frac{1}{\pi}\right)\sin\pi x$$

$$+ \sum_{n=7}^{\infty}\left(\frac{15}{n\pi}\right)\left[(-1)^n - 1\right]\sin\frac{n\pi}{6}x\Big]$$

$$\therefore u(x,t) = V(x,t) + G(x)$$

$$\therefore u(x,t) = V(x,t) + \overbrace{\tfrac{1}{2}x + 7}^{G(x)}$$

وهو المطلوب

H.W

Solve $u_t = 4u_{xx}$ $\quad u(0,t)=0$

$u(10,t)=50$

$u(x,0)=100$

ans: $\boxed{u(x,t)=\sum_{n=0}^{\infty} \frac{100}{n\pi}\left(2-(-1)^n\right) e^{\frac{-n^2\pi^2}{25}t} * \sin\frac{n\pi}{10}x+5x}$

Ex solve $\dfrac{\partial^2 u}{\partial t^2} = 9\dfrac{\partial^2 u}{\partial x^2} + x$

$$u(0,t) = u(8,t) = 3$$
$$u(x,0) = \sin \pi x + 3$$
$$u_t(x,0) = 0$$

solution $\because$ non-homogenous

$$\boxed{\begin{gathered}\therefore \text{let } u(x,t) = V(x,t) + G(x) \\ u_t = v_t \\ \therefore u_{tt} = V_{tt} \;\&\; u_{xx} = V_{xx} + G''(x)\end{gathered}}$$

$$\because \frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 u}{\partial x^2} + x$$

$$\therefore V_{tt} = 9\left(V_{xx} + G''(x)\right) + x$$

$$\therefore \boxed{V_{tt} = 9V_{xx}} + \underline{9G''(x) + x}$$

to homog $\therefore 9G''(x) + x = 0$

$$\therefore 9G''(x) + x = 0 \Rightarrow \int G''(x) = \int \frac{-x}{9}$$

$$\therefore \int G'(x) = \int \frac{-x^2}{18} + \int C_1 \qquad \boxed{\therefore G(x) = \frac{-x^3}{54} + C_1 x + C_2}$$

$$\because u(x,t) = V(x,t) + G(x)$$

to homog. $V_{tt} = 9V_{xx}$

$V(0,t) = V(8,t) = 0$

$\therefore$ **B.C. 1**

لازم للتجانس $=0$

$$u(0,t) = \boxed{V(0,t)} + G(0) = 3 \Rightarrow G(0) = 3$$

$$u(8,t) = \boxed{V(8,t)} + G(8) = 3 \Rightarrow G(8) = 3$$

$\therefore\ G(x) = \frac{-x^3}{54} + C_1 x + C_2$

$\therefore\ G(0) = 3 = 0 + 0 + C_2 \Rightarrow \boxed{C_2 = 3}$

$G(8) = \cancel{3} = \frac{-512}{54} + 8C_1 + \cancel{3} \Rightarrow C_1 = \frac{512}{54 * 8}$

$\therefore\ \boxed{C_1 = \frac{32}{27}}$

$\boxed{\therefore\ G(x) = \frac{-x^3}{54} + \frac{32}{27}x + 3}$

B.C3

$u(x,0) = \boxed{\sin\pi x + 3 = V(x,0) + G(x)}$

$\therefore\ V(x,0) = \sin\pi x + 3 - G(x)$

$V(x,0) = \sin\pi x + \cancel{3} + \frac{x^3}{54} - \frac{32}{27}x - \cancel{3}$

$\boxed{\therefore\ V(x,0) = \sin\pi x + \frac{x^3}{54} - \frac{32x}{27}}$

& $u_t = V_t(x,0) = 0$

$\therefore$ The homog. eq. is :-

$V_{tt} = 9V_{xx}$ $\quad$ $V(0,t) = V(8,t) = 0$

$V(x,0) = \sin\pi x + \frac{x^3}{54} - \frac{32x}{27}$

$V_t(x,0) = 0$

let $V(x,t) = X_{(x)} \cdot T_{(t)}$

$\therefore \left[T'' X = 9 X'' T \right] \div XT$

$\therefore \frac{T''}{T} = \frac{9X''}{X} = K$ If $K \geq 0 \Rightarrow V(x,t) = 0$ $\Rightarrow T.S$

$\therefore K < 0 = -\lambda^2$

$\therefore \frac{T''}{9T} = \frac{X''}{X} = -\lambda^2$

أكمل الحل كواجب

$\therefore \frac{T''}{9T} = -\lambda^2 \Rightarrow m^2 = -9\lambda^2 \Rightarrow m_{1,2} = \mp 3\lambda i \Rightarrow T = C_1 \cos 3\lambda t + C_2 \sin 3\lambda t$

$\frac{X''}{X} = -\lambda^2 \Rightarrow m^2 = -\lambda^2 \Rightarrow m_{1,2} = \mp \lambda i \Rightarrow X = C_3 \cos \lambda x + C_4 \sin \lambda x$

$V(x,t) = (C_1 \cos 3\lambda t + C_2 \sin 3\lambda t)(C_3 \cos \lambda x + \underline{C_4} \sin \lambda x)$

B.C1 $V(0,t) = 0 = C_3$

B.C2 $V(8,t) = 0 \Rightarrow C_4 \sin 8\lambda = 0 \Rightarrow 8\lambda = n\pi \Rightarrow \boxed{\lambda = \frac{n\pi}{8}}$

$\therefore V(x,t) = \sum_{n=0}^{\infty} A_n \sin \frac{n\pi}{8} x \cos \frac{3n\pi}{8} t + \sum_{n=0}^{\infty} B_n \sin \frac{n\pi}{8} x \sin \frac{3n\pi}{8} t$

B.C3 $V_t(x,t) = 0 = \sum_{n=0} A_n \sin \frac{n\pi}{8} x \left(\sin \frac{3n\pi}{8} t\right)\left(\frac{3n\pi}{8}\right) + \sum_{n=0}^{\infty} B_n \sin \frac{n\pi}{8} x \cos \frac{3n\pi}{8} t \left(\frac{3n\pi}{8}\right)$

$\therefore V_t(x,0) = 0 = \sum_{n=0} B_n \sin \frac{n\pi}{8} x \left(\frac{3n\pi}{8}\right) \Rightarrow \boxed{B_n = 0}$

$\therefore V(x,t) = \sum_{n=0}^{\infty} A_n \sin \frac{n\pi}{8} x \cos \frac{3n\pi}{8} t$

B.C4 $V(x,0) = \sin \pi x + \frac{x^3}{54} - \frac{32x}{27} = \sum_{n=0}^{\infty} A_n \sin \frac{n\pi}{8} x$