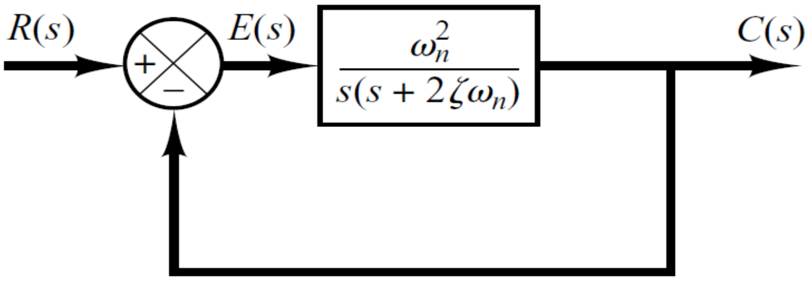
Lecture06

Response of second order systems

*A general second-order system is characterized by the following transfer function.



$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \Rightarrow \quad = \frac{b}{s^2 + as + b}$$

1+GH=0 is called the characteristic equation

* $\omega_n = \sqrt{b}$ is referred to as *the un-damped natural frequency* of the second order system, which is the frequency of oscillation of the system without damping.

* $\xi = \dfrac{a}{2\sqrt{b}}$ referred to as *the damping ratio* of the second order system, which is a measure of the degree of resistance to change in the system output.

<u>Example#1:</u> Determine the un-damped natural frequency and damping ratio of the following second order system. $\dfrac{C(s)}{R(s)} = \dfrac{4}{s^2 + 2s + 4}$

<u>Sol</u>: Compare the numerator and denominator of the given transfer function with the general 2^{nd} order transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{4}{s^2 + 2s + 4}$$

$$\omega_n = \sqrt{4} = 2,\ 2\zeta\omega_n = 2 \Rightarrow \zeta = 0.5$$

*** the second order system have two poles; these poles are** $p_{1,2} = -\omega_n\zeta \mp \omega_n\sqrt{\zeta^2 - 1}$

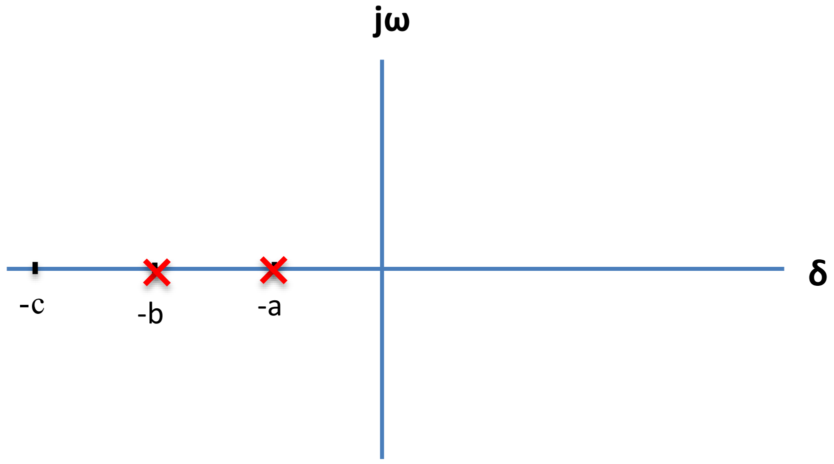
1-real &distinct 2-real & same 3-complex conjugate 4-imaginary pole

*note: 1) the transient &steady state time response to a 2^{nd} order system depend on the values of the 2 roots of the char-eq.

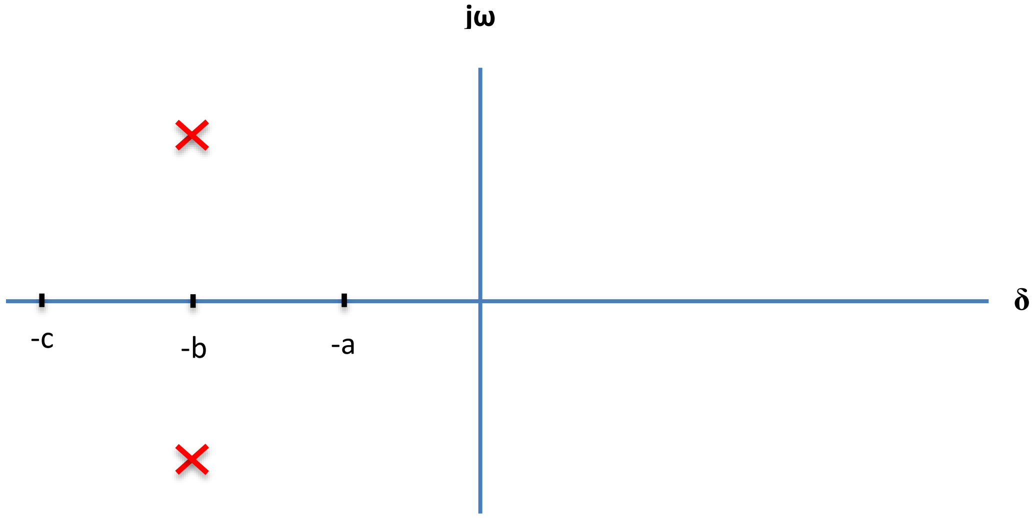
2) the roots of the char-eq called (poles) 3) the factor of s^2 should be (1).

* the dynamic behavior of second order system can be described by ζ and ω_n. According the value of ζ, a second-order system can be set into one of the four categories:

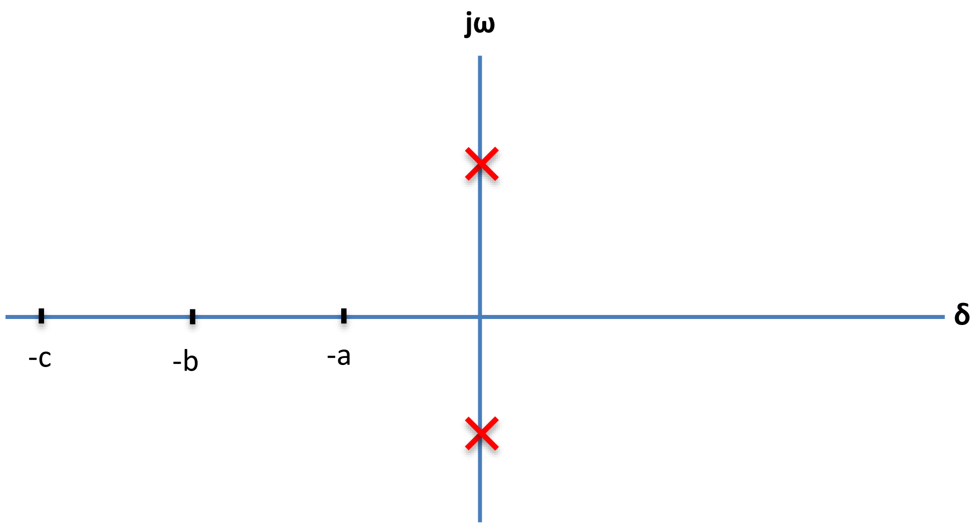
1-*Overdamped* - when the system has two real distinct poles (ζ >1).



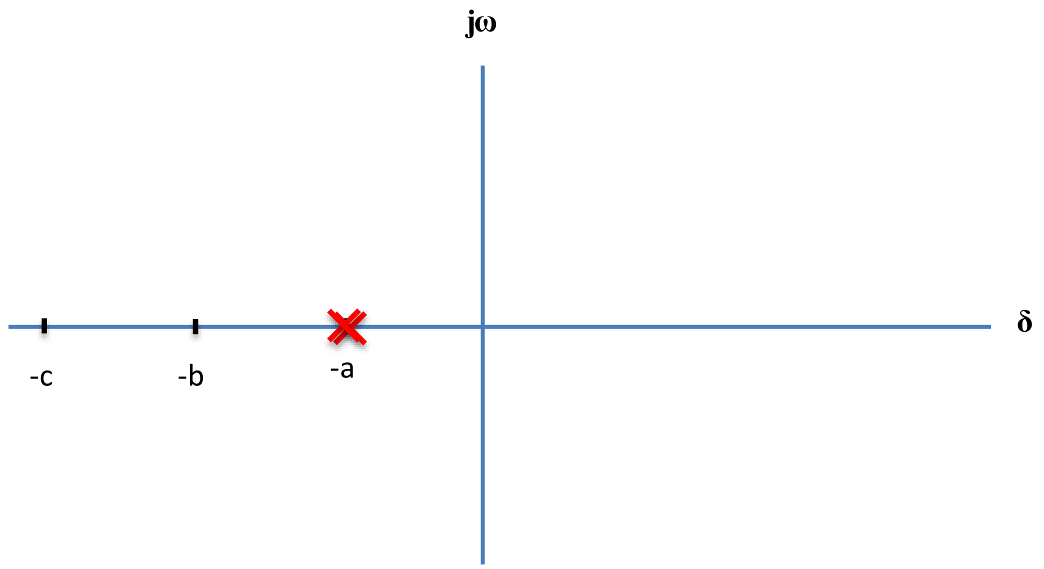
2- *Underdamped* - when the system has two complex conjugate poles (0< ζ <1).



3. *Undamped* - when the system has two imaginary poles (ζ = 0).



4. *Critically damped* - when the system has two real but equal poles ($\zeta = 1$).



* for any RLC circuit the system will be 2nd order system

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1) the time response for 2nd order system with ζ =1 (critical damped).

* in this case, the system response is called critically damped, the closed loop poles are real & equal, where $p_{1,2} = -\omega_n\zeta \mp \omega_n\sqrt{\zeta^2 - 1}$ with ζ =1 these poles becomes: $p_{1,2} = -\omega_n$

therefore $\dfrac{C(s)}{R(s)} = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \dfrac{\omega_n^2}{s^2 + 2s + \omega_n^2} = \dfrac{\omega_n^2}{(s+\omega_n)^2}$

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B_1}{(s+\omega_n)^2} + \frac{B_2}{(s+\omega_n)}, \quad A = \frac{\omega_n^2}{s(s+\omega_n)^2}.s \;=1$$

$$A = \frac{\omega_n^2}{s(s+\omega_n)^2}.(s+\omega_n)^2 \;= -\omega_n \quad , A = \frac{d}{ds}\left(\frac{\omega_n^2}{s(s+\omega_n)^2}.(s+\omega_n)^2\right) = -1$$

$$C(s) = \frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{(s+\omega_n)} \Rightarrow C(t) = 1 - \omega_n t e^{-\omega_n t} - e^{-\omega_n t} \quad \text{, at ant } t \ge 0$$

* the response doesn't oscillated

t	$C(t)$
0	
1	
2	
3	
4	

H.W: Find the response $c(t)$ for $C(s)=\dfrac{s+3}{s(s+2)^2}$, and find $c_{s.s}(t)$ and $e_{s.s}(t)$

Ans: $c(t)=0.75-(t+0.75)e^{-2t}$, $c_{s.s}(t)=\lim_{t\to\infty} c(t)=0.75$, $e_{s.s}(t)=\lim_{t\to\infty} e(t)=0.25$

2) the time response for 2nd order system with ζ >1 (over damped).

The overdamped system is corresponds to ζ >1 , the response doesn't oscillated , the closed loop poles are real put unequal($s_1 \neq s_2$), $s_{1,2}=-\omega_n\zeta \mp \omega_n\sqrt{\zeta^2-1}$

$$\frac{C(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{\omega_n^2}{(s+\omega_n\zeta+\omega_n\sqrt{\zeta^2-1})(s+\omega_n\zeta-\omega_n\sqrt{\zeta^2-1})}=\frac{\omega_n^2}{s_1.s_2}$$

$$C(s)=\frac{\omega_n^2}{s(s_1.s_2)}=\frac{A}{s}+\frac{B}{s_1}+\frac{C}{s_2},\quad C(t)=1+\frac{\omega_n}{2\sqrt{\zeta^2-1}}.\left[\frac{1}{s_1}e^{-s_1t}-\frac{1}{s_2}e^{-s_2t}\right]$$

$$C(t)=1+\frac{\omega_n}{2\sqrt{\zeta^2-1}}.\left[\frac{e^{-\omega_n\zeta+\omega_n\sqrt{\zeta^2-1}\,t}}{-\omega_n\zeta+\omega_n\sqrt{\zeta^2-1}}-\frac{e^{-\omega_n\zeta-\omega_n\sqrt{\zeta^2-1}\,t}}{-\omega_n\zeta-\omega_n\sqrt{\zeta^2-1}}\right],$$

3) the time response for 2nd order system with $0<\zeta<1$ (under damped).

* in this case, the system response is called under damped, the closed loop poles are complex conjucate and lie in the left half s-plane, the transient response is oscillatory.

$p_{1,2}=-\omega_n\zeta \mp j\omega_n\sqrt{1-\zeta^2}$ with unit step input R(s)=1/s

$$C(s)=\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}=\frac{A}{s}+\frac{Bs+C}{(s^2+2\zeta\omega_n s+\omega_n^2)},A=1,B=-1,C=-2\zeta\omega_n$$

$$C(s)==\frac{1}{s}+\frac{-s-2\zeta\omega_n}{(s^2+2\zeta\omega_n s+\omega_n^2)}=\frac{1}{s}-\frac{1}{(s^2+2\zeta\omega_n s+\omega_n^2)}-\frac{2\zeta\omega_n}{(s^2+2\zeta\omega_n s+\omega_n^2)}$$

$$C(s)=\frac{1}{s}-\frac{s}{s^2+2\zeta\omega_n s+\zeta^2\omega_n^2+\omega_n^2-\zeta^2\omega_n^2}-\frac{2\zeta\omega_n}{s^2+2\zeta\omega_n s+\zeta^2\omega_n^2+\omega_n^2-\zeta^2\omega_n^2}$$

$$C(s)=\frac{1}{s}-\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}-\frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}$$

$$C(s)=\frac{1}{s}-\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}-\frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}$$

*Where $\omega_d=\omega_n\sqrt{1-\zeta^2}$, is the frequency of transient oscillations and is called damped natural frequency.

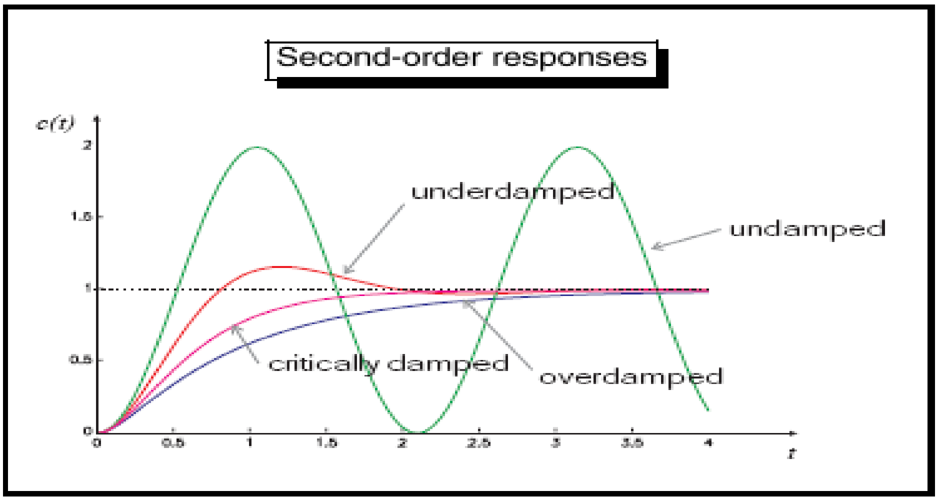
$$C(s)=\frac{1}{s}-\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}-\frac{\frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2+\omega_d^2}=\frac{1}{s}-\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}-\frac{\zeta}{\sqrt{1-\zeta^2}}\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}$$

$$c(t)=1-e^{-\zeta\omega_n t}\cos\omega_d t-\frac{\zeta}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_d t \Rightarrow c(t)=1-e^{-\zeta\omega_n t}\left[\cos\omega_d t+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

$$(t)=1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}[\sin(\omega_d t+\beta)] \qquad \beta=\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

If $\zeta=0$=, the $\omega_d=\omega_n\sqrt{1-\zeta^2}=\omega_n$, $c(t)=1-\cos\omega_n t$

*note : the system response depending on the value of ζ as shown in Figure below.



*** Time-Domain Specification:** The transient response of a practical control system often exhibits damped oscillations before reaching steady state. In specifying the transient-response characteristics of a control system to a unit-step input, it is common to specify the following:

1. Delay time, t_d **2.** Rise time, $\mathbf{t_r}$ **3.** Peak time, $\mathbf{t_p}$
4. Maximum overshoot, $\mathbf{M_p}$ **5.** Settling time, $\mathbf{t_s}$

*These specifications are defined in what follows and are shown graphically in Figure below.

1. Delay time, t_d: The delay time is the time required for the response to reach half the final value the very first time.

2. Rise time, t_r : The rise time is the time required for the response to rise from 10% to 90%, 5% to 95%, or 0% to 100% of its final value. For underdamped secondorder systems, the 0% to 100% rise time is normally used. For overdamped systems, the 10% to 90% rise time is commonly used.

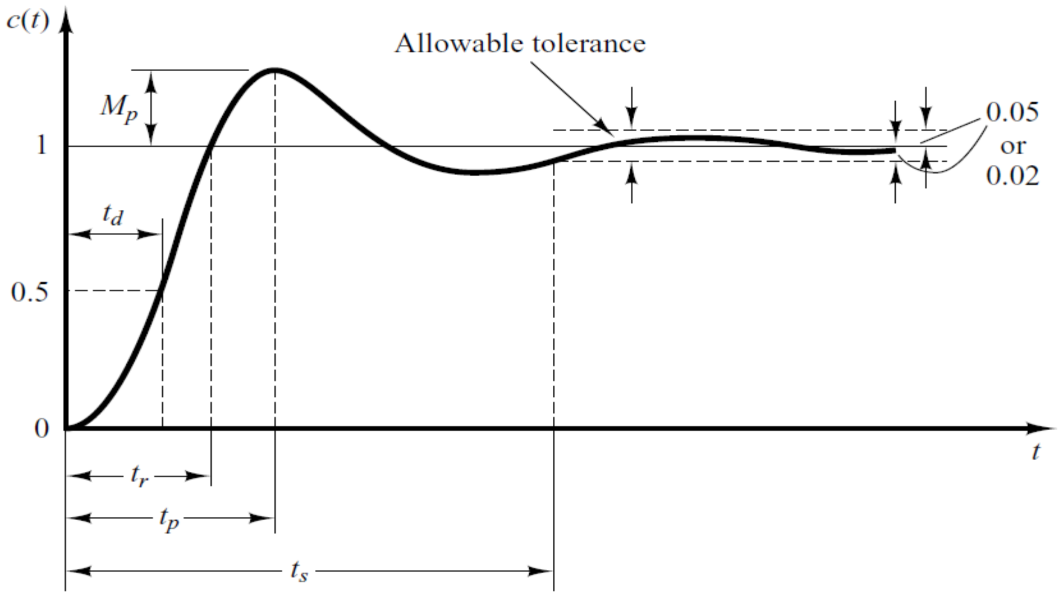
3. Peak time, t_p: The peak time is the time required for the response to reach the first peak of the overshoot.

4. Maximum (percent) overshoot, M_p: The maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by

Maximum percent overshoot = $\%MP \equiv \dfrac{C(t_p) - C(\infty)}{C(\infty)} * 100\%$

The amount of the maximum (percent) overshoot directly indicates the relative stability of the system.

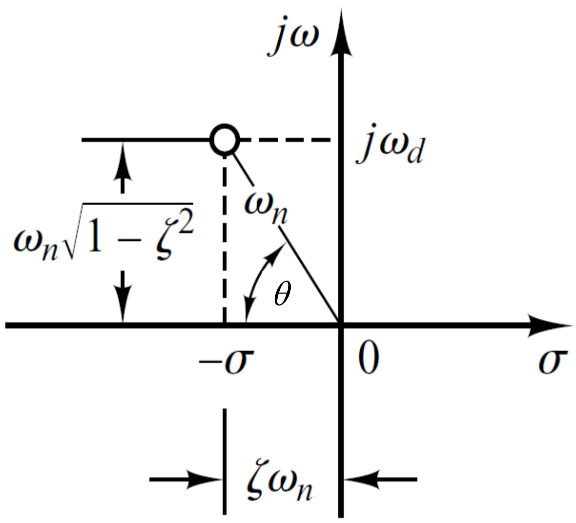
5. Settling time, t_s: The settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%). The settling time is related to the largest time constant of the control system.



***Settling time (2%) criterion: Time consumed in exponential decay up to 98% of the input.** $t_s = 4T = \dfrac{4}{\zeta\omega_n}$.

*** Settling time (5%) criterion: Time consumed in exponential decay up to 95% of the input.**

$$t_s = 3T = \frac{3}{\zeta\omega_n}$$



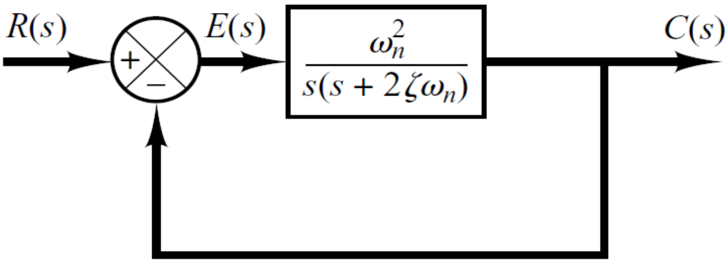
$$t_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - \theta}{\omega_n \sqrt{1 - \zeta^2}}$$

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}$$

$$\theta = \cos^{-1}(\xi)$$

Example: Consider the system shown in following figure, where damping ratio is 0.6 and natural undamped frequency is 5 rad/sec. Obtain the rise time t_r, peak time t_p, maximum overshoot M_p, and settling time 2% and 5% criterion t_s when the system is subjected to a unit-step input.



Sol: $t_r = \dfrac{3.141-\theta}{\omega_n\sqrt{1-\zeta^2}}$, $\theta = \tan^{-1}(\dfrac{\omega_n\sqrt{1-\zeta^2}}{\zeta\omega_n}) = 0.93\,\text{rad}$, $t_r = \dfrac{3.141-0.93}{5\sqrt{1-0.6^2}} = 0.55s$

$$t_p = \frac{\pi}{\omega_d} = \frac{3.141}{4} = 0.785s \ , \ t_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.6\times5} = 1.33s \text{ based on } 2\%$$

$$t_s = \frac{3}{\zeta\omega_n} = \frac{3}{0.6\times5} = 1s \text{ based on } 5\% \ , \ M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}\times100 = e^{-\frac{3.141\times0.6}{\sqrt{1-0.6^2}}}\times100 = 9.5\%$$