Lecture#7: Stability

*Stability is the most important system specification. If a system is unstable, transient response and steady-state error are moot points.

*A system is stable if every bounded input yield a bounded output.
We call this statement the bounded-input, bounded-output (BIBO) definition of stability.

*the stability of a linear closed loop sys. can be determined from the locations of the closed loop poles in the s-plane, if any of these poles lie in the right half s-plane, then the sys. is said to be unstable.

* for any small sys., i.e 1st and 2nd order system poles of the sys. Can be determined easily, but when a large system i.e 4th order or higher, it is difficult to get its poles.

*unstable system: at least one or roots of char./ Eq. lie on right half s-plane(RH s-plane).

ROUTH'S STABILITY CRITERION

*the char-Eq. of a sys. Can be used to find whether the system is stable or not, howmany poles on the right or left s-plane (char-Eq.=1+open loop T.F=0$\Rightarrow 1+GH=0$)

$$a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

s^n	a_0	a_2	a_4	a_6	$\cdots$
s^{n-1}	a_1	a_3	a_5	a_7	$\cdots$
s^{n-2}	b_1	b_2	b_3	b_4	$\cdots$
s^{n-3}	c_1	c_2	c_3	c_4	$\cdots$
s^{n-4}	d_1	d_2	d_3	d_4	$\cdots$
.	.	.			
.	.	.			
.	.	.			
s^2	e_1	e_2			
s^1	f_1				
s^0	g_1				

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

.
.
.

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

$$c_3 = \frac{b_1 a_7 - a_1 b_4}{b_1}$$

.
.
.

$$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1}$$

$$d_2 = \frac{c_1 b_3 - b_1 c_3}{c_1}$$

.
.

*Routh criteria state that the number of roots with the positive real part is equal to the number of change in sign of coefficient of the first column of the array.

* if all elements sign (+ve) or (–ve) the sys is ***stable*** if there is a change in elements sign that means the sys. is ***unstable***.

Ex.(1): consider the char/Eq.= $s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$

s^4	1	3	5
s^3	2	4	0
s^2	1	5	
s^1	-6		
s^0	5		

*عدد ال root التي تقع بالجانب الأيمن بعدد التغير بالاشارة

* the number of changes in sign of the coefficients in the first column is 2. This means that there are two roots with positive real parts.

Routh-Hurwitz Criterion : Special Cases

Two special cases can occur : special case (1): The routh table sometimes will have a zero *only in the first column of a row*, or special case (2): the routh table sometimes will have an *entire row* that consists of zeros. Let us examine the first case.

Ex(2): Determine the stability of the closed loop transfer function $T(s)=\frac{10}{s^5+2s^4+3s^3+6s^2+5s+3}$

Solution : The Solution is shown in the table. Begin by assembling the Routh table down

s^5	1	3	5
s^4	2	6	3
s^3	~~0~~ ϵ	$\frac{7}{2}$	0
s^2	$\frac{6\epsilon-7}{\epsilon}$	3	0
s^1	$\frac{42\epsilon-49-6\epsilon^2}{12\epsilon-14}$	0	0
s^0	3	0	0

Label	First Column	$\epsilon = +$	$\epsilon = -$
s^5	1	+	+
s^4	2	+	+
s^3	~~0~~ ϵ	+	−
s^2	$\frac{6\epsilon-7}{\epsilon}$	−	+
s^1	$\frac{42\epsilon-49-6\epsilon^2}{12\epsilon-14}$	+	+
s^0	3	+	+

*the system is unstable, with two poles in the right half plane.

Ex(3): Determine the stability of the closed loop transfer function $T(s)=\frac{10}{s^5+2s^4+3s^3+6s^2+5s+3}$

s^5	1	6	8
s^4	~~7~~ 1	~~42~~ 6	~~56~~ 8
s^3	~~0~~ ~~4~~ 1	~~0~~ ~~12~~ 3	~~0~~ ~~0~~ 0
s^2	3	8	0
s^1	$\frac{1}{3}$	0	0
s^0	8	0	0

$$A(s) = s^4 + 6s^2 + 8$$

$$\frac{dA(s)}{dt} = 4s^3 + 12s$$

*special case(2): if all the coefficient, in any derived row are zero, this indicate that there are roots of equal magnitude lying opposite in the s-plane.

* there is no change in sign that's mean there is no pole lie in the right half plane.

note: when there is a zero row that mean either

i) one or more than one pair of roots lie on jw-axis.

ii) one or more than one pair of roots symmetrically on real axis.

To find root of auxiliary Eq. , then $A(s) = s^4 + 6s^2 + 8 = 0$

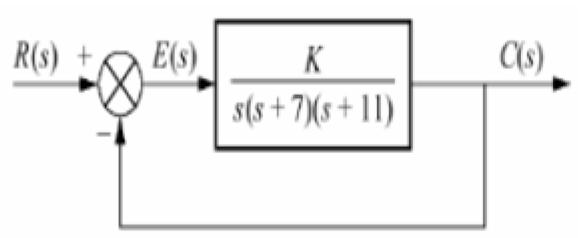
$$(s^2 + 4)(s^2 + 2) = 0 \Rightarrow (s^2 + 4) = 0 \,, (s^2 + 2) = 0$$

$$s_{1,2} = \pm 2j, \; s_{3,4} = \pm\sqrt{2}\,j$$

Ex(4): Find the range of the gain K, for the system shown below that will cause the system to be stable, unstable and marginally stable, find also the value of freq. for marginally stable. Assume K > 0 .

Solution : First find the closed-loop transfer function

$$T(s) = \frac{K}{s^3 + 18s^2 + 77s + K}$$



s^3	1	77
s^2	18	K
s^1	$\frac{1386 - K}{18}$	
s^0	K	

*Since K assumed positive, we see that all elements in the first column are always positive except the s^1 row. This entry can be positive, zero, or negative, depending upon the value of K.

i) If K<1386, all terms in the first column will be positive, and since there are no sign changes, the system will have three poles at the left half plane and be *stable*.

ii) If K>1386, the s^1 term in the first column is negative. There are two sign changes, indicating that the system has two right half plane poles and one left half plane pole, which makes the system *unstable*.

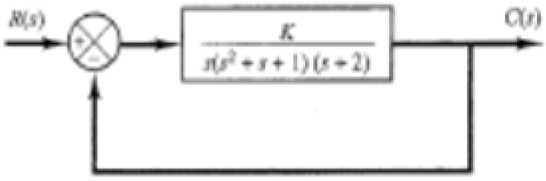
iii) If K=1386, the system is marginally stable (critical stable), we have an entire row of zeros, which could signify $j\omega$ poles. Returning the s^2 row and replacing K with 1386, we form the even polynomial A(s)=18s2+1386.

* in order to obtain the freq., we replace s by $j\omega$, and make A(s) =0 , then

$$-18\omega^2 + 1386 = 0 \Rightarrow \omega = \pm 8.779,$$

H.W: 1) Comment on stability for Char/Eq.= s^3+2s^2+s+2.

2)find the value of K such that the sys. is stable, and sys. Given sustained oscillations, also find the value of freq. of sustained oscillations.



STABILITY IN STATE SPACE

Ex.(5): For the system given below, find out how many poles are in the left half plane, in the right half plane, and on the $j\omega$-axis

$$\dot{x} = \begin{bmatrix} 0 & 3 & 1 \\ 2 & 8 & 1 \\ -10 & -5 & -2 \end{bmatrix} x + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

Solution : First form (sI-A) :

$$(sI - A) = \begin{bmatrix} s & -3 & -1 \\ -2 & s-8 & -1 \\ 10 & 5 & s+2 \end{bmatrix}$$

Now find the det(sI-A). det(sI-A) = $s^3-6s^2-7s-52$

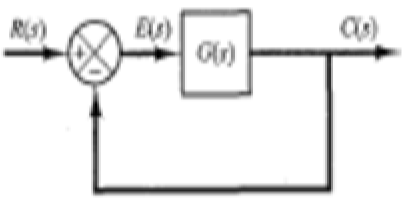
Using this polynomial, form the Routh table.

s^3	1	-7
s^2	~~-6~~ -3	~~-52~~ -26
s^1	~~$-\frac{47}{3}$~~ -1	~~8~~ 0
s^0	-26	

Since there is one sign change in the first column, the system has one right half plane pole and two left half plane poles. It is therefore unstable.

Steady state errors

*the basic control system described by the following block diagram.



The overall transfer function for this system is given by

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

$$E(s) = R(s) - B(s)$$

$$= R(s) - C(s)H(s)$$

$$= R(s) - E(s)G(s)H(s)$$

$$E(s) = \frac{R(s)}{1 + G(s)H(s)}$$

$$e_{ss} = \lim_{s \to 0} sE(s)$$

$$= \lim_{s \to 0} \frac{sR(s)}{1 + G(s)H(s)}$$

<u>1) for step i/p (position i/p):</u> the steady state error of a unit step i/p is

$$R(s) = \frac{1}{s}$$

$$e_{ss} = \lim_{s \to 0} \left[\frac{s}{1 + G(s)H(s)} \cdot \frac{1}{s} \right]$$

$$= \lim_{s \to 0} \frac{1}{1 + G(s)H(s)}$$

$$= \frac{1}{1 + \lim_{s \to 0} G(s)H(s)}$$

$$= \frac{1}{1 + K_p}$$

$K_p = \lim_{s \to 0} G(s)H(s)$ is position error coefficient constant

*for step i/p with amplitude =A , $e_{s.s} = (A/1+k_p)$

<u>2) for ramp input (velocity i/p):</u>

$$R(s) = \frac{1}{s^2}$$

$$e_{ss} = \lim_{s \to 0} \left[\frac{s}{1 + G(s)H(s)} \cdot \frac{1}{s^2} \right]$$

$$= \lim_{s\to 0} \frac{1}{s[1 + G(s)H(s)]}$$

$$= \frac{1}{\lim_{s\to 0} sG(s)H(s)}$$

$$= \frac{1}{K_v}$$

$$K_v = \lim_{s\to 0} sG(s)H(s)$$

is called the *static velocity error constant.*

3) for acceleration (or parabolic i/p):

$$R(s) = \frac{1}{s^3}$$

$$e_{ss} = \lim_{s\to 0} \left[\frac{s}{1 + G(s)H(s)} \cdot \frac{1}{s^3} \right]$$

$$= \lim_{s\to 0} \frac{1}{s^2[1 + G(s)\, H(s)]}$$

$$= \frac{1}{\lim_{s\to 0} s^2 G(s)H(s)}$$

$$= \frac{1}{K_a}$$

$$K_a = \lim_{s\to 0} s^2 G(s)H(s)$$

is called the *static acceleration constant.*

The results, so far obtained, are summarised in Table 6.6. The following points are apparent from the table:

1. Ramp and parabolic inputs should not be applied to type 0 system.
2. Parabolic input should not be applied to type 1 system.
3. For type ≥ 3, $e_{ss} = 0$ for all inputs

Two more points which are not apparent from Table 6.6 are as follows:

1. e_{ss} for a linear combination of inputs can be superimposed.
2. The method fails for sinusoidal inputs, because the final value theorem cannot be applied in that case.

Table 6.6 Steady-state error for different types and different inputs

Type	Input		
	Step	*Ramp*	*Parabolic*
	$e_{ss} = \frac{1}{1+K_p}$	$e_{ss} = \frac{1}{K_v}$	$e_{ss} = \frac{1}{K_a}$
0	$\frac{1}{1+K}$	∞	∞
1	0	$\frac{1}{K}$	∞
2	0	0	$\frac{1}{K}$
3	0	0	0

EXAMPLE 6.18 The open-loop transfer function of a unity feedback system is given by

$$G(s) = \frac{100}{s^2(s+4)(s^2+5s+25)}$$

Find static error coefficients and the steady-state error of the system when it is subjected to an input of $r(t) = 2 + 4t + 2t^2$.

Solution: Static error coefficients are

$$K_p = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0} \frac{100}{s^2(s+4)(s^2+5s+25)} = \infty$$

$$K_v = \lim_{s\to 0} sG(s)H(s) = \lim_{s\to 0} \frac{100}{s(s+4)(s^2+5s+25)} = \infty$$

$$K_a = \lim_{s\to 0} s^2G(s)H(s) = \lim_{s\to 0} \frac{100}{(s+4)(s^2+5t+25)} = \frac{100}{(4)(25)} = 1$$

Taking the Laplace transform of the input, we get

$$R(s) = \frac{2}{s} + \frac{4}{s^2} + \frac{4}{s^3}$$

$$e_{ss} = \frac{2}{1+K_p} + \frac{4}{K_v} + \frac{4}{K_a}$$

$$e_{ss} = \frac{2}{1+\infty} + \frac{4}{\infty} + \frac{4}{1} = 4$$