

ROOT LOCUS TECHNIQUES

In this lecture you will learn the following :

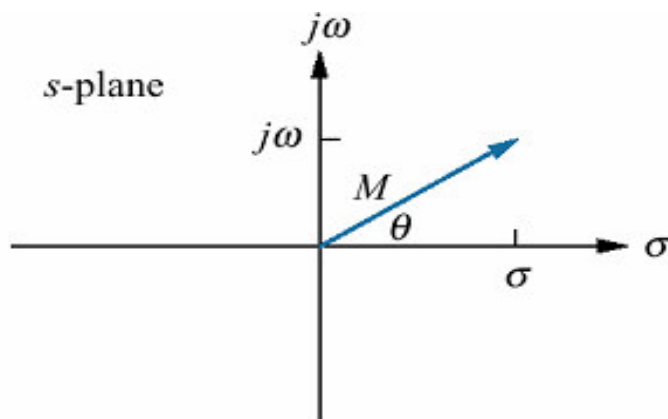
- ❖ The definition of a root locus
- ❖ How to sketch root locus
- ❖ How to use the root locus to find the poles of a closed loop system
- ❖ How to use root locus to describe the changes in transient response
- ❖ How to use root locus to design a parameter value to meet a desired transient response

Root locus is a graphical presentation of the closed loop poles as a system parameter is varied. The root locus also gives a graphic presentation of a system's stability. Before presenting root locus, let us review two concepts that we need for the ensuing discussion : (1) The Control System Problem, and (2) Complex Numbers and Their Representation as Vectors.

The Control System Problem : Consider a system transfer function:

$T(s)=KG(s)/[1+KG(s)H(s)]$. The poles of $T(s)$ are not immediately known without factoring the denominator, and they are a function of K . Since the system transient response and stability are dependent upon the poles of $T(s)$, we have no knowledge of the system performance unless we factor the denominator for specific values of K . The root locus will be used to give us a vivid picture of the poles of $G(s)$ as K varies.

Complex Numbers and Their Representation as Vectors : Any complex number, $\sigma+j\omega$, described in Cartesian coordinates can be graphically represented by a vector, as shown in the following figure. The complex numbers also can be described in the polar form with magnitude M , and angle θ , as $M\angle\theta$. If the complex number is substituting into



a complex function, $F(s)$, another complex number will result. For example, if $F(s) = (s+a)$, then substituting the complex number $s = \sigma+j\omega$ yields $F(s)=(\sigma+a)+j\omega$ which is another complex number. $F(s)$ has a zero at $-a$.

Defining the Root Locus

The root locus technique can be used to analyze and design the effect of a loop gain upon the system's transient response and stability. The closed loop transfer function for a system with a gain, K , is

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

From this equation, a pole, s , exists when the characteristic polynomial in the denominator becomes zero, or $KG(s)H(s) = -1 = 1\angle(2k+1)180^\circ \quad k=0, \pm 1, \pm 2, \pm 3, \dots$

Where -1 represented in polar form as $1\angle(2k+1)180^\circ$. Alternatively, a value of s is a closed loop pole if $|KG(s)H(s)|=1$ and $\angle KG(s)H(s) = (2k+1)180^\circ$

Sketching the Root Locus

The following five rules allow us to sketch the root locus using minimal calculations.

1.) Number of the branches : Each closed loop pole moves as the gain is varied. If we define a branch as the path that one pole traverses, then there will be one branch for each closed loop pole. Our first rule defines the number of branches of the root locus:

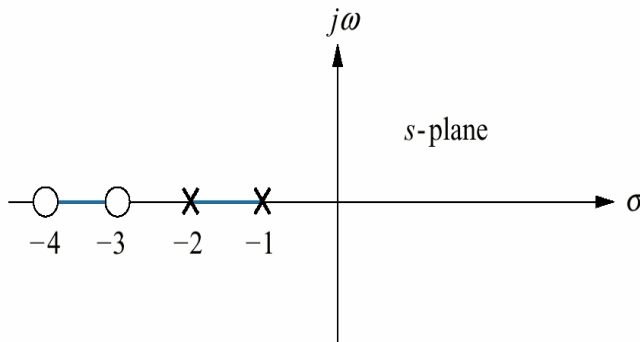
The number of the branches of the root locus are equal to the number of closed loop poles.

2.) Symmetry : Physically realizable system can not have the complex coefficients in their transfer functions. Thus, we conclude:

The root locus is symmetrical about the real axis.

3. Real-axis segment : Using some mathematical properties of root locus plot, we get

On the real axis, for $K > 0$ the root locus exist to the left of an odd number of real axis, finite open loop poles and/or finite open loop zeros.

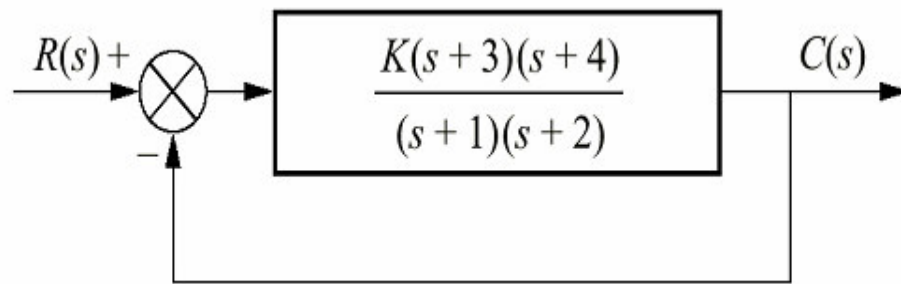


In this figure, the real axis segments of root locus are between -1 and -2 and between -3 and -4.

4. Starting and ending points : Consider the closed loop transfer function $T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$

As K approaches to zero, the closed loop poles approaches the poles of $G(s)H(s)$ which is the open loop transfer function. Similarly, as K approaches to infinity, the closed loop poles approaches the zeros of $G(s)H(s)$. We conclude that

The root locus begins at the finite and infinite poles of $G(s)H(s)$ and ends at the finite and infinite zeros of $G(s)H(s)$.



In this system, the root locus begins at the poles at -1 and -2 and ends at zeros at -3 and -4.

5. Behaviour at infinity : Every function of s has an equal number of poles and zeros. For example, consider the function

$$G(s) = \frac{s + 1}{(s + 2)(s + 3)}$$

This function has two poles at -2 and -3 and one zero at -1. Because the number of poles and zeros must be equal, this function has a zero at **infinity**.

This rule tells us the following conclusion :

The root locus approaches straight lines as the locus approaches the infinity. Further, the equation of the asymptotes is given by the real axis intercept, σ_a , and angle, θ_a , as follows :

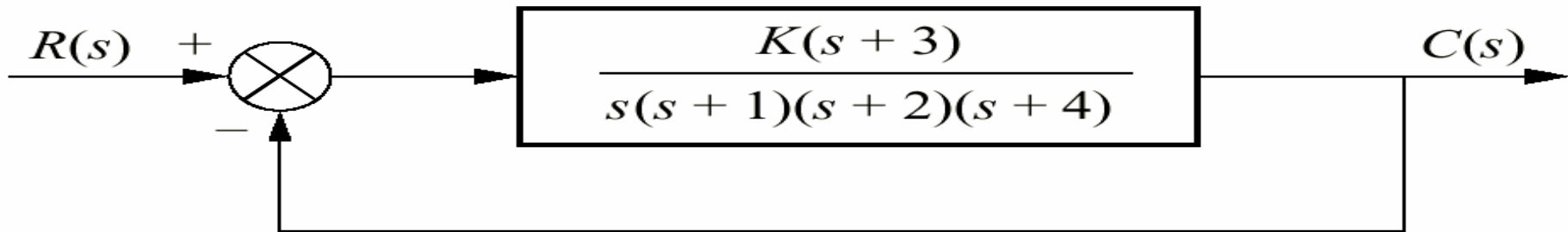
$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{(\text{number of finite poles}) - (\text{number of finite zeros})}$$

$$\theta_a = \frac{(2k + 1)\pi}{(\text{number of finite poles}) - (\text{number of finite zeros})}$$

where $k = 0, \pm 1, \pm 2, \pm 3, \dots$, and the angle is given in radians with respect to the positive extension of the real axis.

Let us demonstrate the concept with an example.

Example : Sketch the root locus for the system shown in figure.

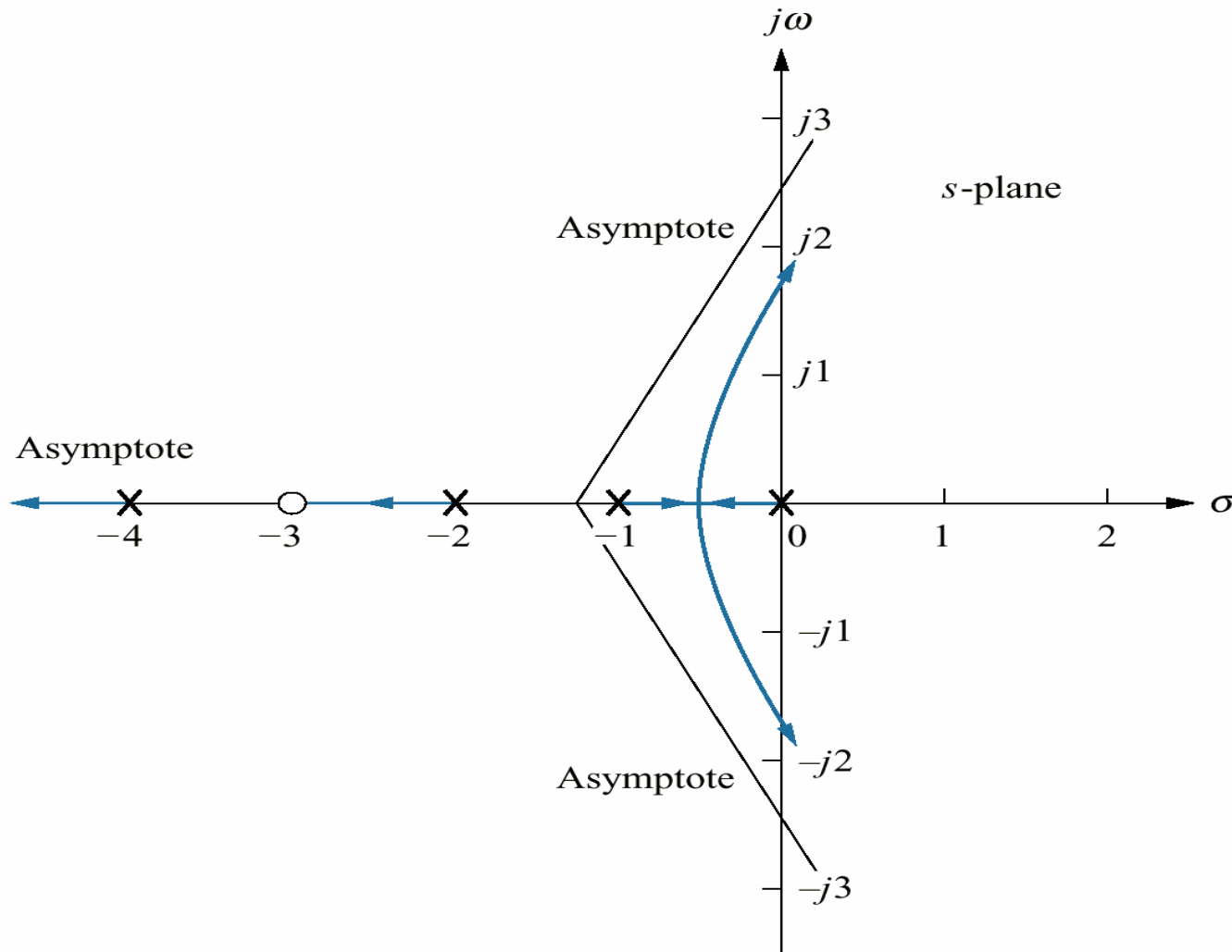


Solution : Begin by calculating the asymptotes :

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{(\text{number of finite poles}) - (\text{number of finite zeros})} = \frac{(-1 - 2 - 4) - (-3)}{4 - 1} = -\frac{4}{3}$$

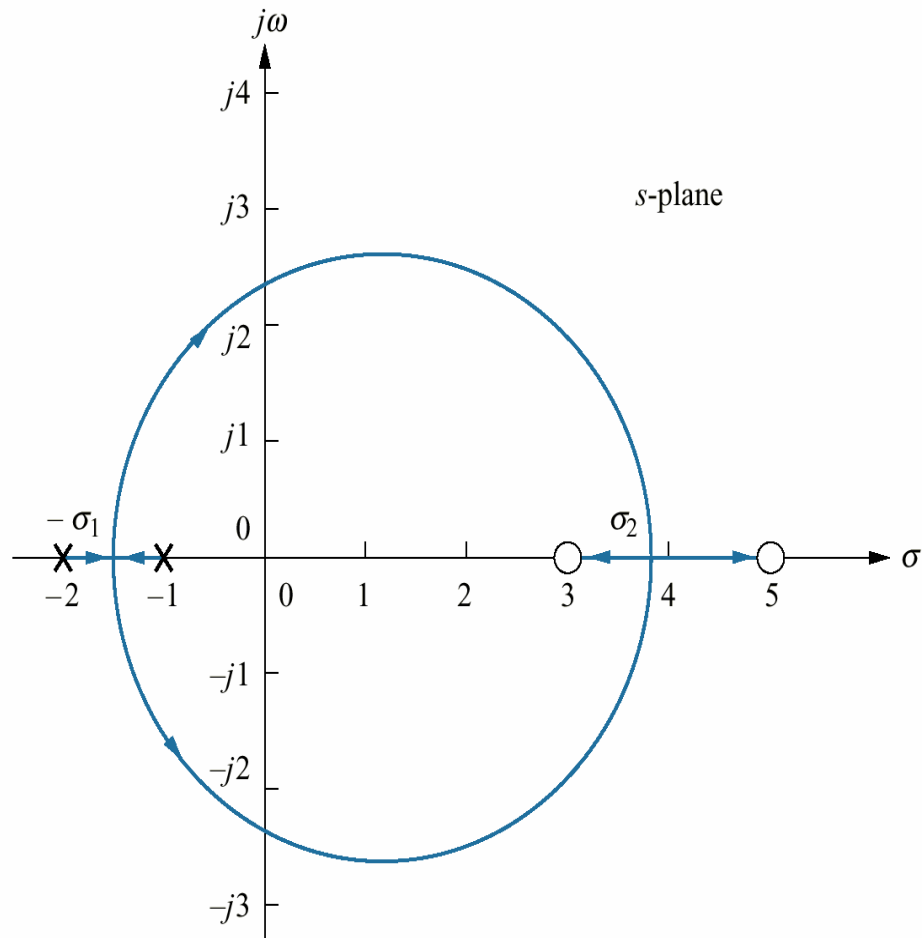
$$\theta_a = \frac{(2k + 1)\pi}{(\text{number of finite poles}) - (\text{number of finite zeros})} = \begin{aligned} &= \pi/3 \quad \text{for } k = 0 \\ &= \pi \quad \text{for } k = 1 \\ &= 5\pi/3 \quad \text{for } k = 2 \end{aligned}$$

If the values for k continued to increase, the angles would begin repeat. The number of lines obtained equals the difference between the number of finite poles and the number of finite zeros. Rule 4 states that the root locus begins at the open loop poles and ends at the open loop zeros. For the example, there are more open loop poles than open loop zeros. Thus, there must be zeros at infinity. The asymptotes tell us how we get to these zeros at infinity.



This figure shows the complete root locus as well as the asymptotes that were just calculated. Notice that we have made use of all the rules learned so far. The real axis segments lie to the left of an odd number of poles and/or zeros. The locus starts at the open-loop poles and ends at the open-loop zeros. For this example, there is only one open loop finite zero and three infinite zeros. Rule 5, then, tells us that the three zeros at infinity are at the ends of the asymptotes.

6. Real Axis Breakaway and Break-in Points : Numerous root loci appear to break away from the real axis as the system poles move from the real axis to the complex plane. At other times the loci appear to return to the real axis as a pair of complex poles becomes real. This is illustrated in the figure.



Figure

Root locus example showing real-axis breakaway ($-\sigma_1$) and break-in points (σ_2)

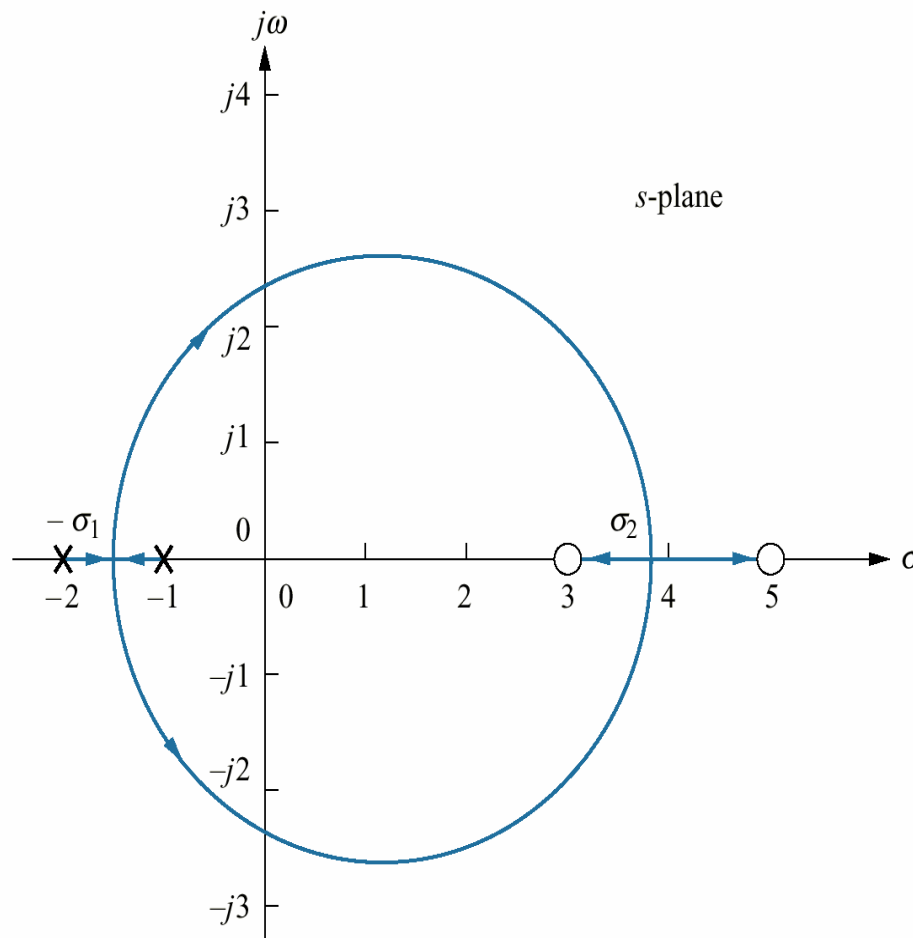
The figure shows a root locus leaving the real axis between -1 and -2 and returning to the real axis between +3 and +5. The point where the locus leaves the real axis, $-\sigma_1$, is called the **breakaway point**, and the point where the locus returns to the real axis, σ_2 , is called **break-in point**. At the breakaway or break-in point, the branches of the root locus form an angle of $180/n$ with the real axis, where n is the number of closed loop poles arriving or departing from the single breakaway or break-in point on the real axis. Thus for the poles shown in the figure the branches at the breakaway point form 90° angles with the real axis.

We now show how to find the breakaway and the break-in points. There is a simple method for finding the points at which the root locus breaks away from and breaks into real axis. The first method is to maximize and minimize the gain, K , using differential calculus. For all points on the root locus, because of the $KG(s)H(s) = -1$ as we pointed out before, $K = -1 / G(s)H(s)$. For points along the real axis segment of the root locus where breakaway and break-in points could exist, $s = \sigma$. Hence along the real axis ;

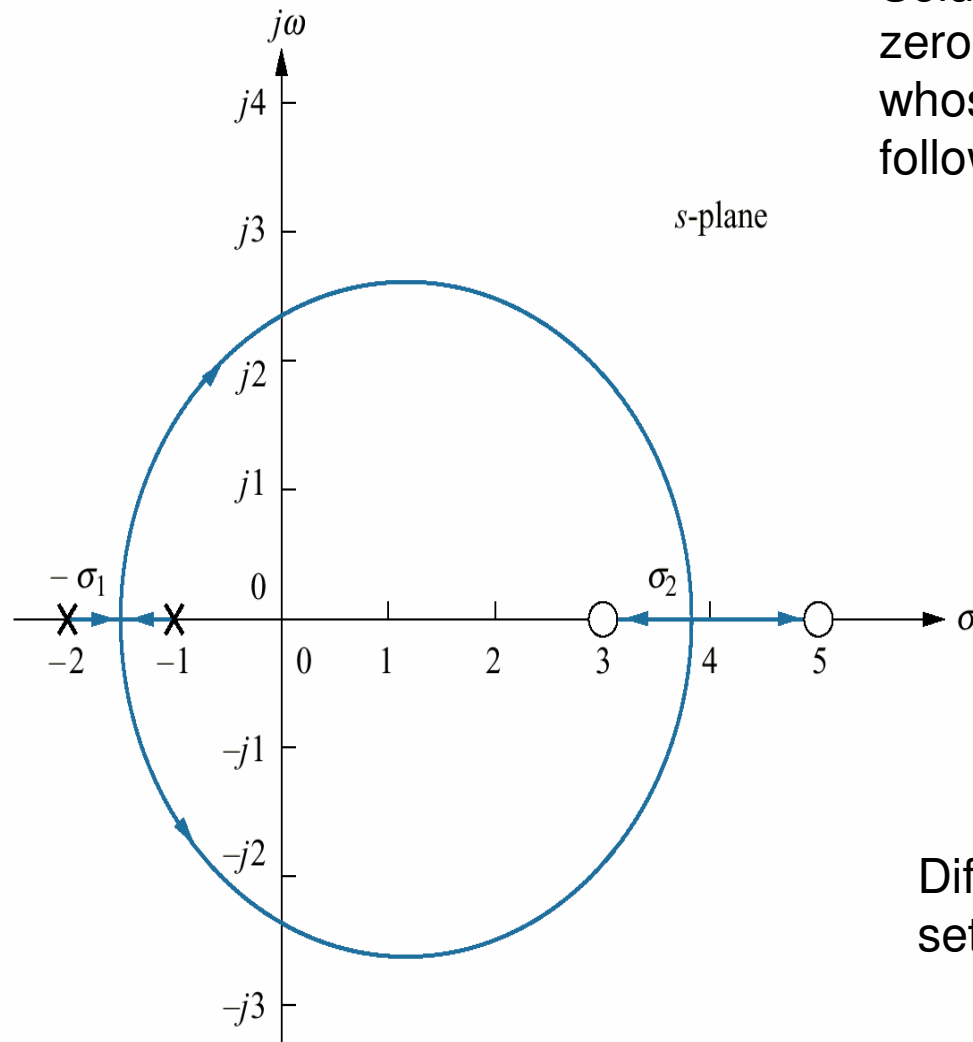
$$K = -\frac{1}{G(\sigma)H(\sigma)}$$

Hence if we differentiate this equation with respect to σ and set the derivative equal to zero, we can find the points of maximum and minimum gain and hence breakaway and break-in points.

Let us demonstrate with an example.



Example : Find the breakaway and break-in points for the root locus of the figure using differential calculus.



Solution : Using the open loop poles and zeros, we present the open loop system whose root locus is shown in figure as follows

$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)} = \frac{G(\sigma^2 - 8\sigma + 15)}{(\sigma^2 + 3\sigma + 2)} = -1$$

But for all points along the root locus, $KG(s)H(s) = -1$, and along the real axis, $s = \sigma$. Hence

$$\frac{K(\sigma^2 - 8\sigma + 15)}{\sigma^2 + 3\sigma + 2} = -1$$

Solving for K, we find

$$K = \frac{-(\sigma^2 + 3\sigma + 2)}{(\sigma^2 - 8\sigma + 15)}$$

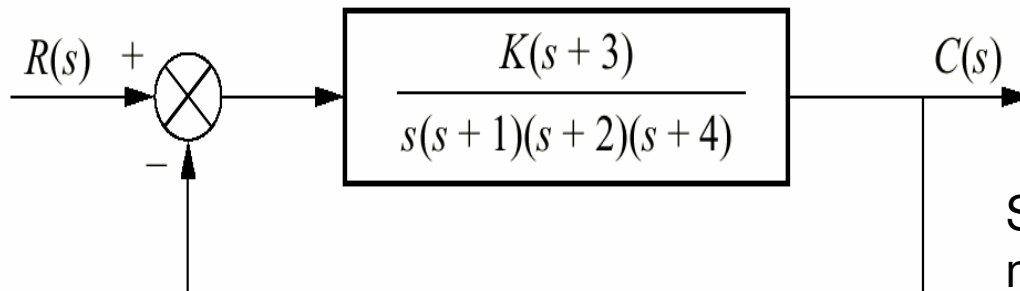
Differentiating K with respect to σ and setting the derivative equal to zero yields

$$\frac{dK}{d\sigma} = \frac{(11\sigma^2 - 26\sigma - 61)}{(\sigma^2 - 8\sigma + 15)^2}$$

Solving for σ , we find $\sigma = -1.45$ and 3.82 , which are the breakaway and break-in points.

7. $j\omega$ -Axis Crossings : The $j\omega$ axis crossing is a point on the root locus that separates the stable operation of the system from the unstable operation. The value of ω at the axis crossing yields the frequency of oscillation. To find $j\omega$,axis crossing, we can use the Routh-Hurwitz criterion as follows : Forcing a row of zeros in the Routh table will yield the gain; going back one row to the even polynomial equation and solving for the roots yields the frequency at the imaginary axis crossing.

Example : For the system of the following figure, find the frequency and gain, K , for which the root locus crosses the imaginary axis. For what range of K is the system stable?



s^4	1	14	$3K$
s^3	7	$8 + K$	
s^2	$90 - K$	$21K$	
s^1	$\frac{-K^2 - 65K + 720}{90 - K}$		
s^0	$21K$		

Solution : The closed loop transfer function for the system is

$$T(s) = \frac{K(s+3)}{s^4 + 7s^3 + 14s^2 + (8+K)s + 3K}$$

Simplifying some of the entries by multiplying any row by a constant, we obtain the Routh array shown in table. A complete row of zeros yields the possibility for imaginary axis roots. For positive value of gain only the s^1 row can yield a row of zeros. Thus,

$$-K^2 - 65K + 720 = 0$$

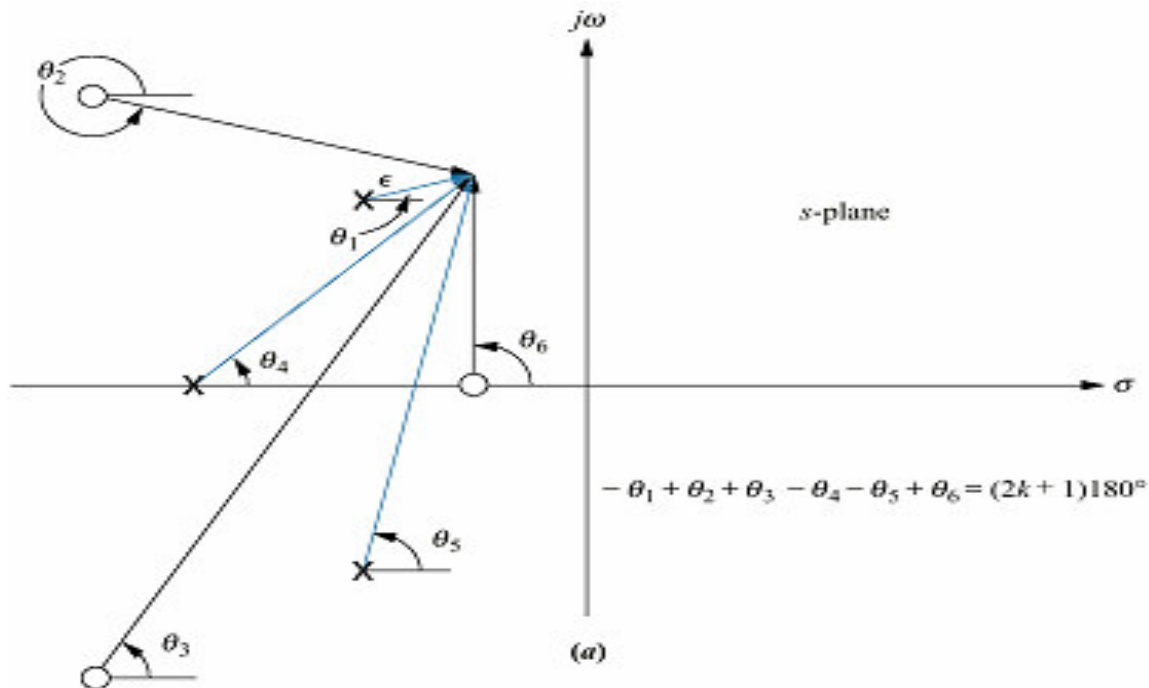
and $K = 9.65$. Forming the even polynomial by using the s^2 row with $K=9.65$

We obtain $(90-K)s^2 + 21K = 80.35s^2 + 202.7 = 0$

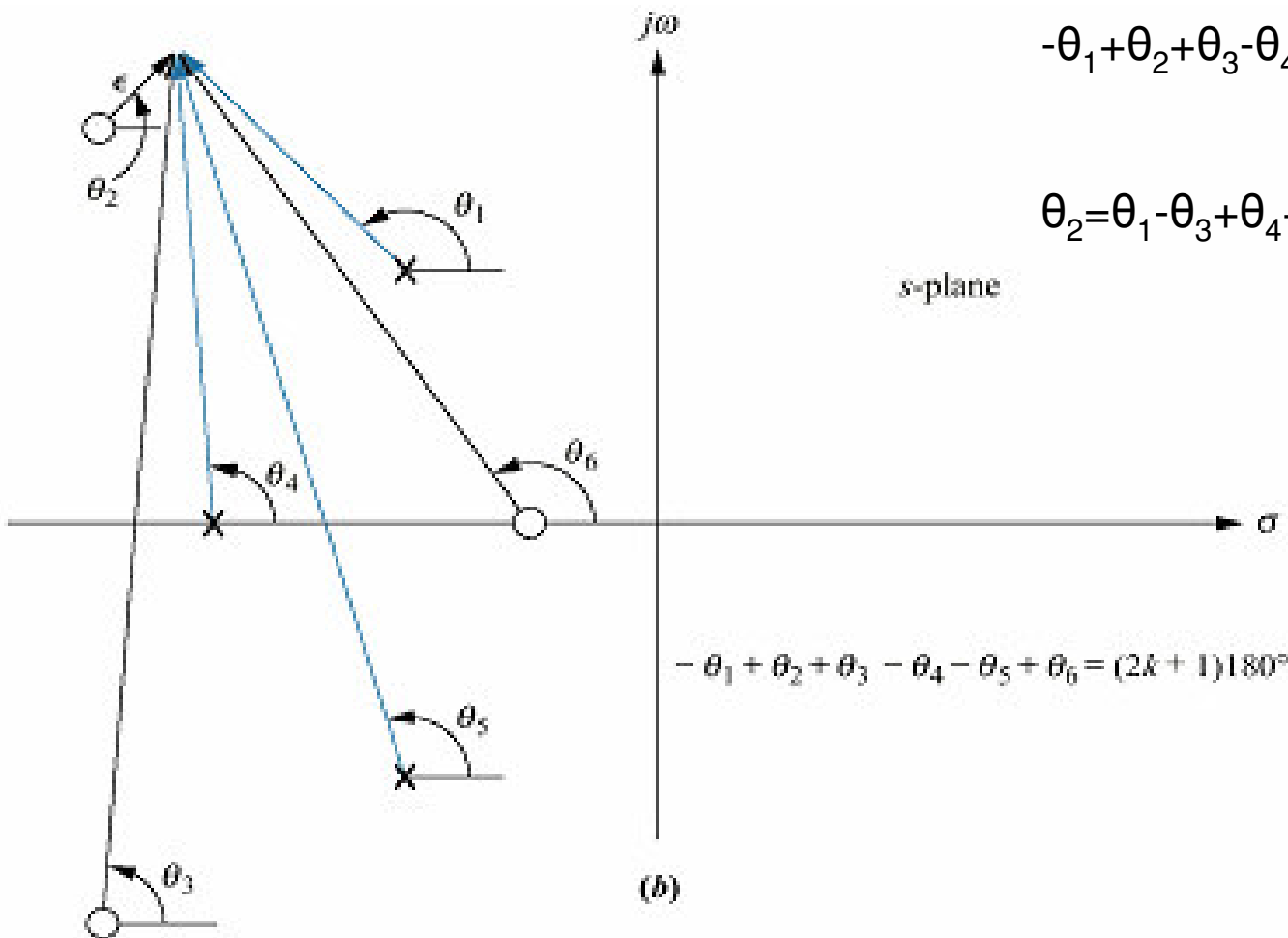
and s is found to be $\pm j1.59$. Thus the root locus crosses the $j\omega$ axis at $\pm j1.59$ at a gain of 9.65. We conclude that the system is stable for $0 \leq K \leq 9.65$.

8. Angles of Departure and Arrival : In order to sketch the root locus more accurately, we want to calculate the root locus departure angle from the complex poles and the arrival angle to the complex zeros. If we assume a point on the root locus ϵ close to the point, we assume all angles drawn from all other poles and zeros are drawn directly to the pole that is near the point. Thus, the only unknown angle in the sum is the angle drawn from the pole that is ϵ close. We can solve for this unknown angle, which is also the angle of departure from this complex pole. Hence from the figure(a)

$$-\theta_1 + \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 = (2k+1)180^\circ \text{ or } \theta_1 = \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 - (2k+1)180^\circ$$



If we assume a point on the root locus ϵ close to a complex zero, the sum of angles drawn from all finite poles and zeros to this point is an odd multiple of 180° . Except for the zero that is ϵ close to the point, we can assume all angles drawn from all other poles and zeros are drawn directly to the zero that is near the point. Thus, the only unknown angle in the sum is the angle drawn from the zeros that is ϵ close. We can solve for this unknown angle, which is also the angle of arrival to this complex zero. Hence from the figure(b) or



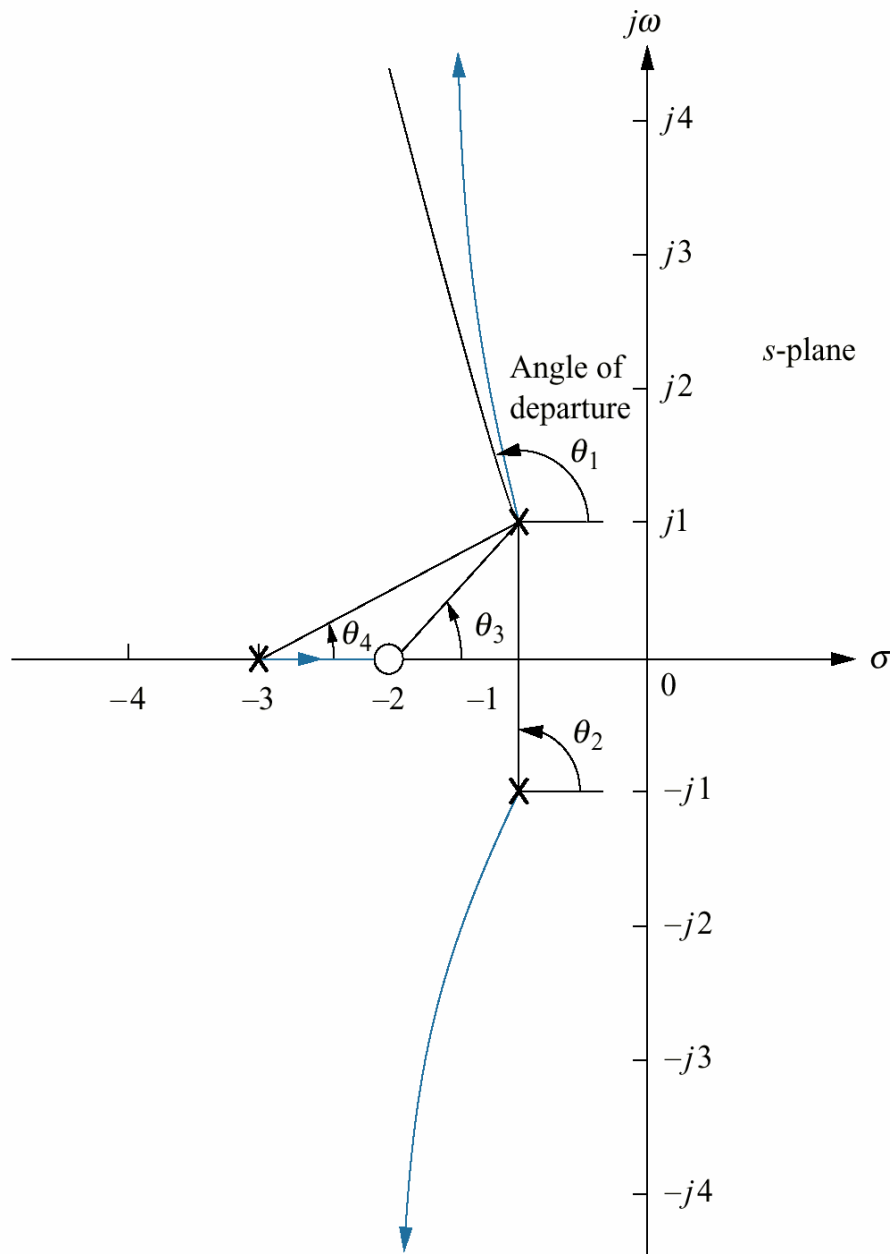
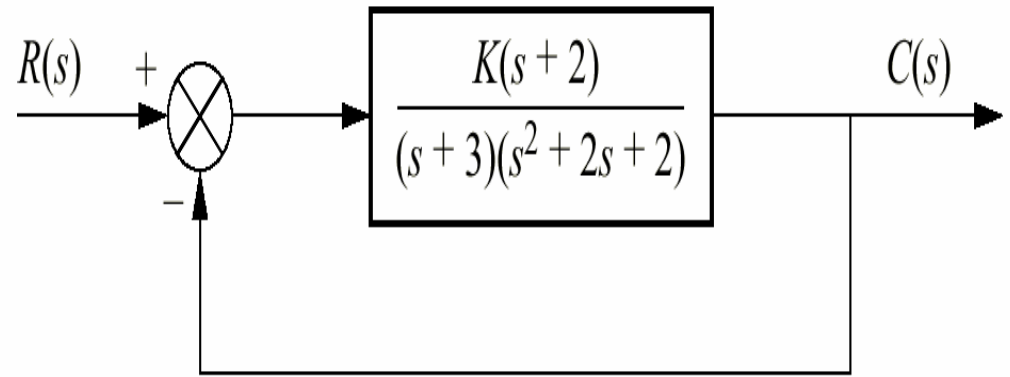
$$-\theta_1 + \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 = (2k+1)180^\circ$$

or

$$\theta_2 = \theta_1 - \theta_3 + \theta_4 + \theta_5 + \theta_6 + (2k+1)180^\circ$$

$$-\theta_1 + \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 = (2k+1)180^\circ$$

Example : Given the unity feedback system of following figure, find the angle of departure from the complex poles and sketch the root locus.



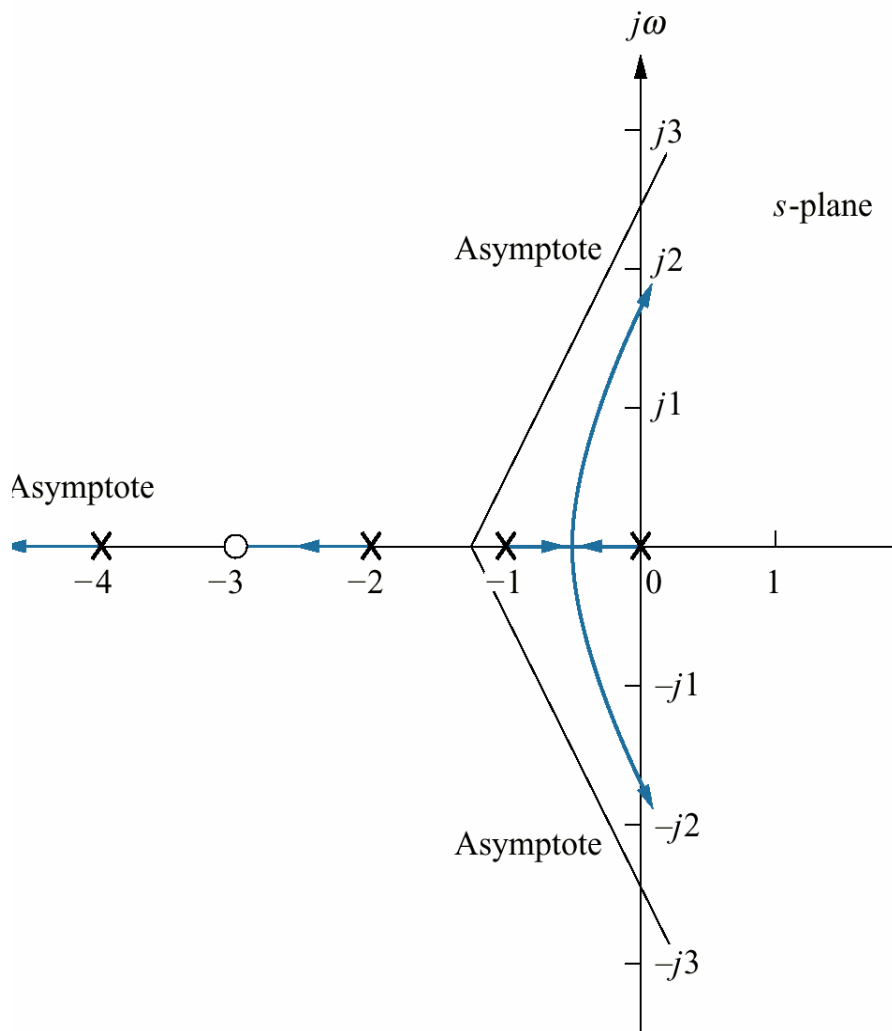
Solution : Using the poles and zeros of $G(s) = \frac{(s+2)}{(s+3)(s^2+2s+2)}$ as plotted in the figure, we calculate the sum of angles drawn to a point ϵ close to the complex pole, $-1+j1$, in the second quadrant. Thus,

$$-\theta_1 - \theta_2 + \theta_3 - \theta_4 =$$

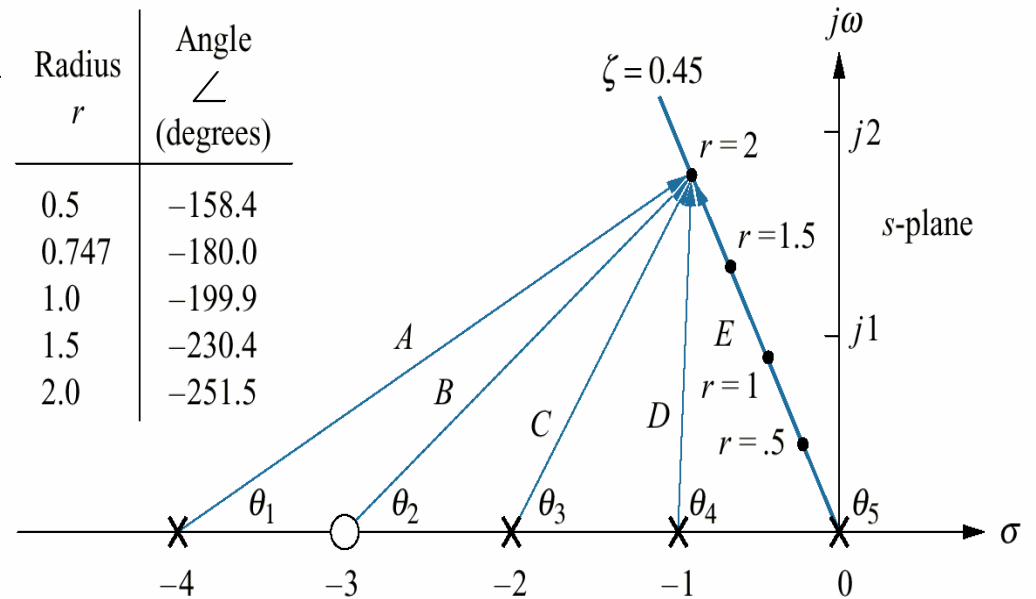
$$-\theta_1 - 90^\circ + \tan^{-1}(1/1) - \tan^{-1}(1/2) = 180^\circ$$

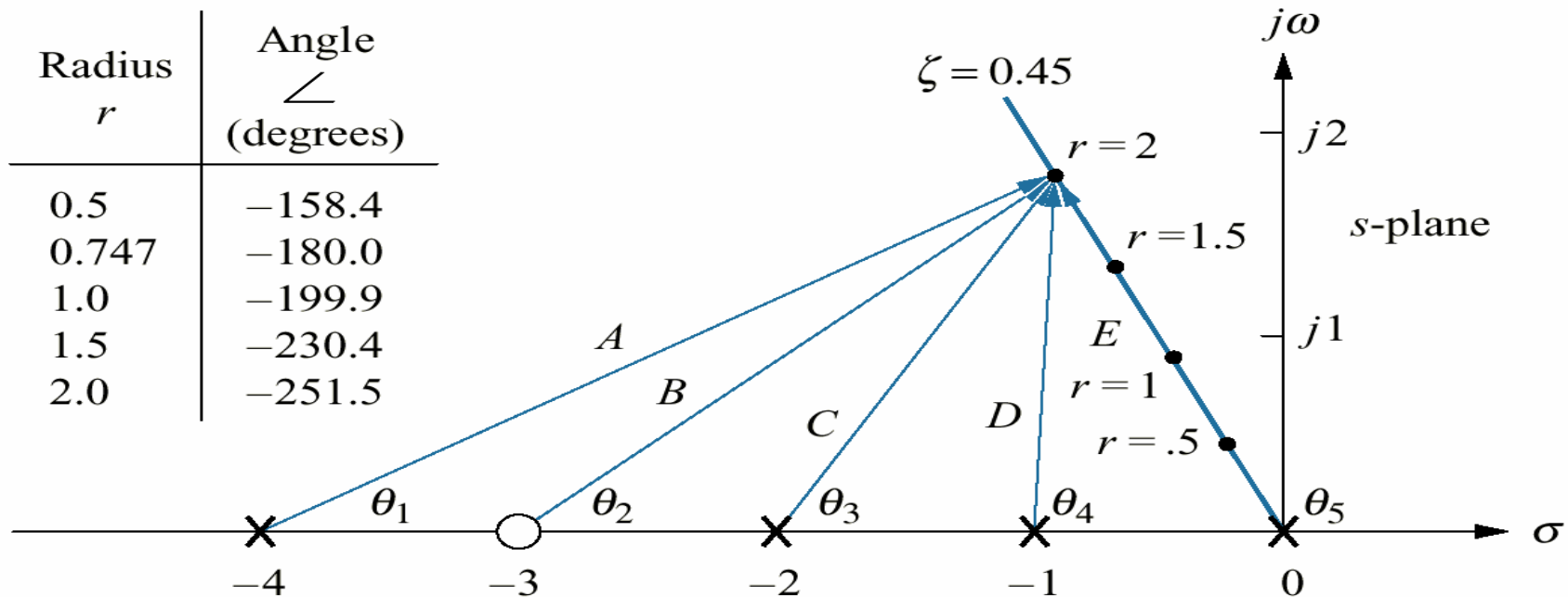
From which $\theta_1 = -251.6^\circ = 108.4^\circ$.

9. Plotting and Calibrating the Root Locus : Once we sketch the root locus, we may want to accurately locate points on the root locus as well as find their associated gain. For example, we might want to know the exact coordinates of the root locus as it crosses the radial line representing 20% overshoot. Further, we may also want the values of gain at this point. Consider the root locus shown in following figure.



Let us assume we want to find the exact point at which the root locus crosses the 0.45 damping ratio line and gain at that point. Other figure shows the system's open loop poles and zeros along with $\zeta=0.45$.





If a few test points along the $\zeta=0.45$ line are selected, we can evaluate their angular sum and locate that point where the angles add up to an odd multiple of 180° . It is at this point that the root locus exists. We can find the value of the gain, K , at this point. Selecting the point at radius 2 ($r=2$) on the $\zeta=0.45$ line, we add the angles of zeros and subtract the angles of poles, obtaining $\theta_2 - \theta_1 - \theta_3 - \theta_4 - \theta_5 = -215.5^\circ$. Since the sum is not equal to an odd multiple of 180° , the point at radius 2 is not on the root locus. Proceeding similarly for this point at radius=1.5, 1, 0.747 and 0.5, we obtain the table shown in figure. From the table we see that the point at radius 0.747 is on the root locus, since the angles add up to -180° . The gain, K , at this point is found with

$$K = \frac{|A||B||C||D|}{|B|} = 1.71$$

In summary, we search a given line for the point yielding a summation of angles (zero angles- pole angles) equal to an odd multiple of 180° . We conclude that the point is on the root locus. The gain at this point is then found by multiplying the pole lengths drawn to that point and dividing by the product of the zero lengths drawn to that point.

SUMMARY

Basic rules for the sketching the root locus :

Number of branches : The number of branches of the root locus equals the number of closed loop poles.

Symmetry : The root locus is symmetrical about the real axis.

Real axis segment : On the real axis, for $K > 0$ the root locus exists to the left of an odd number of real-axis, finite open loop poles and/or finite open loop zeros.

Starting and ending points : The root locus begins at the finite and the infinite poles of $G(S)H(S)$ and ends at the finite and infinite zeros of $G(S)H(s)$.

Behaviour at infinity : The root locus approaches straight lines as asymptotes as the root locus approaches infinity. Further, the equations of the asymptotes are given by the real axis intercept and angle in radians as follows :

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{(\text{number of finite poles}) - (\text{number of finite zeros})}$$

$$\theta_a = \frac{(2k + 1)\pi}{(\text{number of finite poles}) - (\text{number of finite zeros})}$$

where $k = 0, \pm 1, \pm 2, \pm 3, \pm 4, \dots$

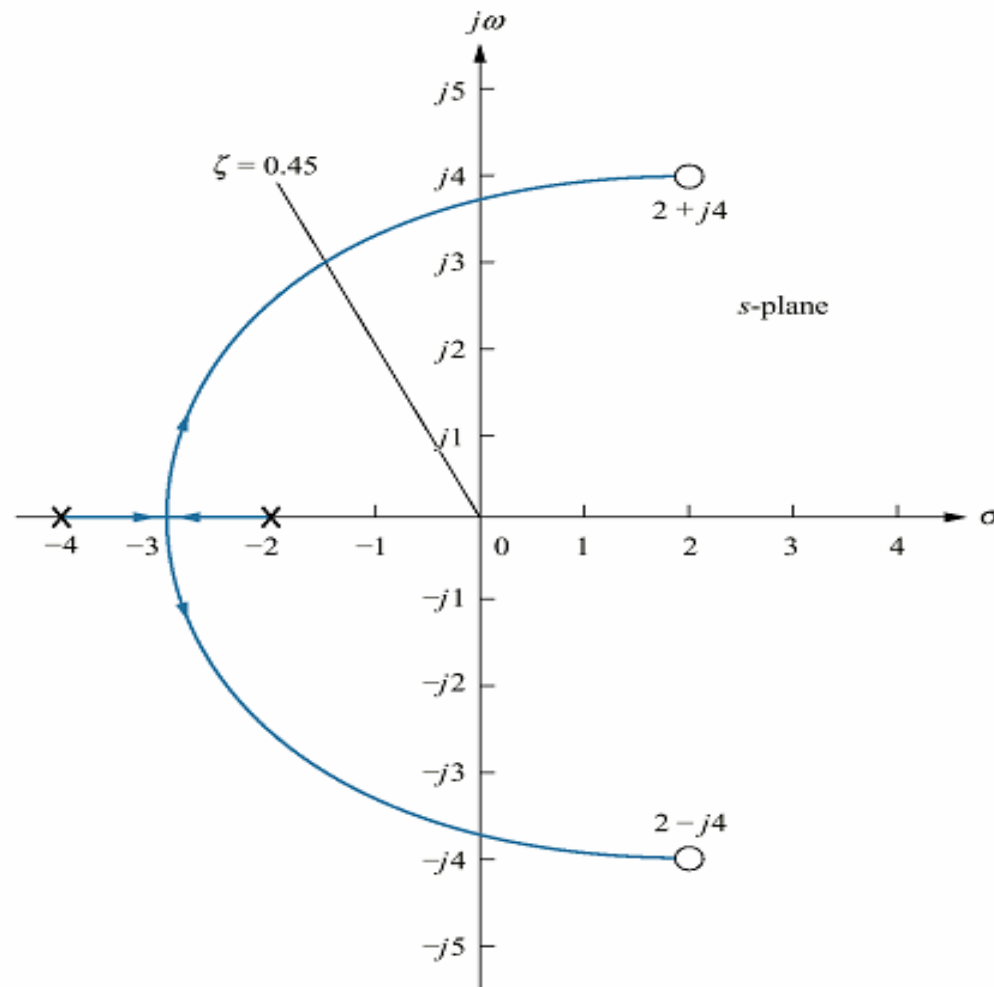
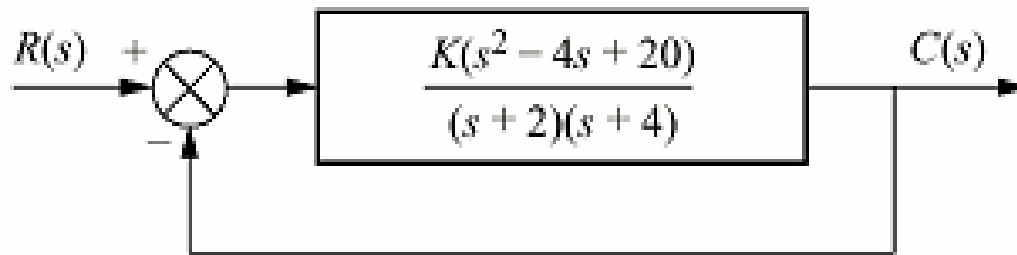
Additional Rules for Refining the Sketch

Real axis breakaway and break-in points : The root locus breaks away from the real axis at a point where the gain is maximum and breaks into the real axis at a point where the gain is minimum.

Calculation of $j\omega$ axis crossings : The root locus crosses the $j\omega$ -axis at the point where $\angle G(s)H(s) = (2k + 1)180^\circ$. Routh-Hurwitz or a search of the $j\omega$ -axis for $(2k+1)180^\circ$ can be used to find the $j\omega$ axis crossing.

Angles of departure and arrival : The root locus departs from complex open loop poles and arrives at complex open loop zeros at angles that can be calculated as follows. Assume a point ϵ close to the complex pole or zero. Add all angles drawn from all open loop poles and zeros to this point. The sum equals $(2k+1)180^\circ$. The only unknown angle is that drawn from the ϵ close pole or zero, since the vector drawn from all other poles and zeros can be considered drawn to the complex pole or zero that is ϵ close to the point. Solving for the unknown angle yields the angle of departure or arrival.

Example : Sketch the root locus for the system shown in figure and find the breakaway point on the real axis and the range of K within which the system is stable.



Solution : The root locus of the system is shown in following figure. The real axis segment is found to be between -2 and -4. The root locus starts at the open loop poles and ends at the open loop zeros. The root locus crosses the $j\omega$ -axis at $\pm j3.9$. To find breakaway point use the root locus algorithm to search the real axis between -2 and -4 for the point that yields maximum gain. Naturally all points will have the sum of their angles equal to an odd multiple of 180° . A maximum gain of 0.0248 is found to at the point -2.88. Therefore the breakaway point is between the open loop poles on the real axis at -2.88. Therefore the system is stable for K between 0 and 15.