ROOT-LOCUS ANALYSIS

Lecture 12: Examples of Root Locus Plots

Example: Given that

$$KG(s) = \frac{K}{s(s+2)(s+4)}$$

Sketch the root locus of $1 + KG(s) = 0$ and compute the value of K that will yield a "dominant" second order behavior with a damping ratio, $\zeta = 0.7$.

We have $n = 3$ and $m = 0$.
Open loop zero: none
Open loop poles: $s = 0, -2, -4$

Rule 1: The loci start from $K = 0$ at the OL poles
Rule 2: The loci end at $K \to \infty$ at 3 infinite zeros at $|\infty|$
Rule 3: The number of loci is 3, as $n = 3$
Rule 4: Root loci are symmetrical with respect to real axis

Rule 5: There are (3 – 0) = 3 asymptotes. The angles of asymptotes are given by:

$$\theta_j = \frac{(2j+1)\pi}{3-0} \qquad ; j = 0,1,2$$

$$= \frac{\pi}{3}, \frac{3\pi}{3}, \frac{5\pi}{3}$$

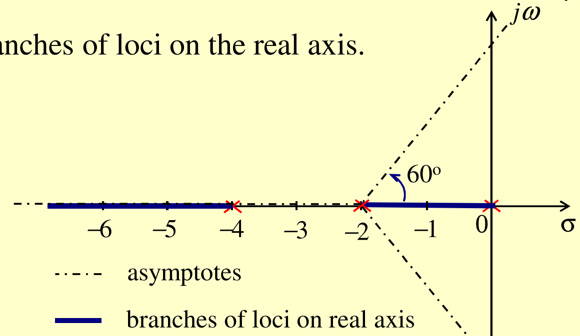
i.e. $\theta_j = 60°, 180°, 300° (-60°)$

Rule 6: The point of intersection of the asymptotes is

$$\sigma_c = \frac{\sum \text{OL\_poles} - \sum \text{OL\_zeros}}{n-m}$$

$$= \frac{(0-2-4)-(-0)}{3-0} = -2.0$$

Rule 7: Identify branches of loci on the real axis.



Rule 9: Crossing the imaginary axis.

The C.E. is $s^3 + 6s^2 + 8s + K = 0$ $\left(\frac{Y(s)}{R(s)} = \frac{K}{s^3 + 6s^2 + 8s + K}\right)$

s^3	1	8	0
s^2	6	K	
s^1	$\frac{48-K}{6}$		
s^0	K		

When $K = 48$ in the s^1 row we get an all zero row.
The Auxiliary Equation is

$$6s^2 + 48 = 0$$

$$s = \pm j2\sqrt{2}$$

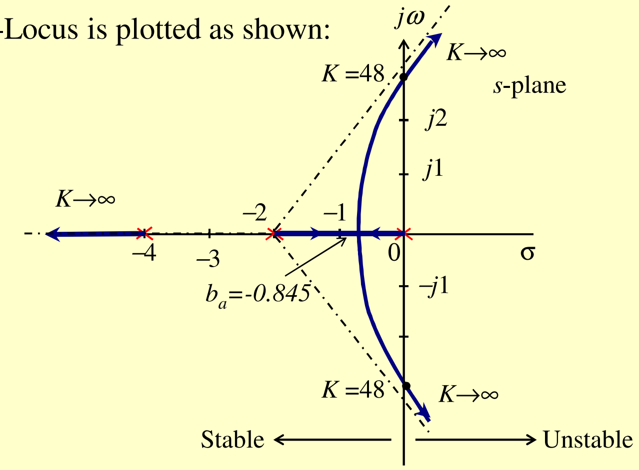
Rule 10: The breakaway point is the solution of $\frac{dK}{ds} = 0$

$$K = -(s^3 + 6s^2 + 8s)$$

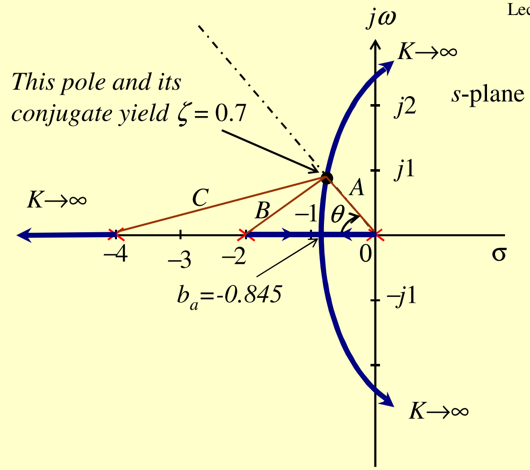
$$\frac{dK}{ds} = -(3s^2 + 12s + 8) = 0$$

Solving, $s = -0.845$ is the valid breakaway point for $K > 0$.

The Root-Locus is plotted as shown:



With $\zeta = 0.7$, we have $\theta = \cos^{-1}\zeta = 45.6^{\circ}$. One of the poles that will give the required ζ is given by the intersection of the root locus with the straight line that has angle θ w.r.t. the negative real axis.



If the root-locus is sketched to scale, the value of K that will yield the required ζ can be computed from the measured values of A, B and C, i.e.

$$K_1 = A \times B \times C$$

Example: Construct the Root-Locus for a system with open-loop transfer function

$$KG(s)H(s) = \frac{K(s+3)}{s(s+5)(s+6)(s^2+2s+2)}$$

That is, $n = 5$ and $m = 1$.

Open loop zero: $s = -3$
Open loop poles: $s = 0, -5, -6, \ -1 \pm j1$

Rule 1: The loci start from $K = 0$ at the OL poles

Rule 2: The loci end at $K \to \infty$ at the OL zero and 4 infinite zeros at $|\infty|$

Rule 3: The number of loci is 5, as $n = 5$

Rule 4: Root loci are symmetrical with respect to real axis

Rule 5: There are (5 – 1) = 4 asymptotes. The angles of asymptotes are given by:

$$\theta_j = \frac{(2j+1)\pi}{4} \qquad ; j = 0,1,2,3.$$

$$= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

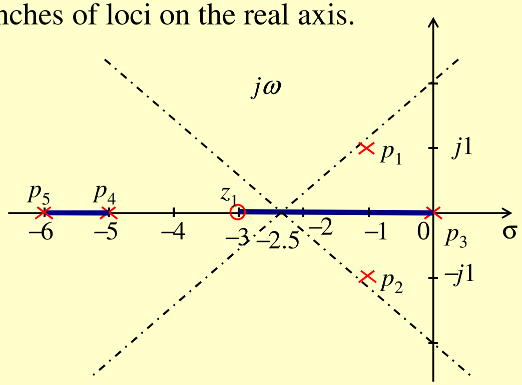
i.e. $\theta_j = 45°, 135°, 225° (-135°), 315° (-45°)$

Rule 6: The point of intersection of the asymptotes is

$$\sigma_c = \frac{\sum \text{OL\_poles} - \sum \text{OL\_zeros}}{n-m}$$

$$= \frac{(0-5-6-1+j1-1-j1)-(-3)}{5-1} = -2.5$$

Rule 7: Identify branches of loci on the real axis.



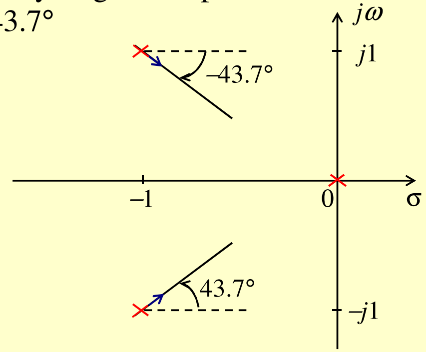
Rule 8: The angle of departure from the complex pole at $(-1 + j1)$ is given by:

$$\theta_d = 180° + \left\{ \tan^{-1}\left(\frac{1}{3-1}\right) - \left[\tan^{-1}(-1) + \tan^{-1}\left(\frac{1}{5-1}\right) + \tan^{-1}\left(\frac{1}{6-1}\right) + 90° \right] \right\}$$

$$= 180° + \{26.6° - [135° + 14° + 11.3° + 90°]\}$$

$$= -43.7°$$

By conjugate symmetry, angle of departure from the complex pole at $(-1 - j1)$ is $\theta_d = +43.7°$



Rule 9: The points of intersection of root locus with the imaginary axis using the Routh-Hurwitz Criterion.
The Characteristic Equation is

$$s(s+5)(s+6)(s^2+2s+2) + K(s+3) = 0$$

$$s^5 + 13s^4 + 54s^3 + 82s^2 + (60+K)s + 3K = 0$$

s^5	1	54	$(60 + K)$	0
s^4	13	82	$3K$	0
s^3	47.7	$(60 + 0.77K)$	0	
s^2	$(65.6 - 0.21K)$	$3K$		
s^1	$\frac{3940 - 105K - 0.16K^2}{65.6 - 0.21K}$			
s^0	$3K$			

For stability,

$65.6 - 0.21K > 0 \quad \Rightarrow \quad K < 309$
$3940 - 105K - 0.16K^2 > 0 \quad \Rightarrow \quad K < 35$
$3K > 0 \Rightarrow \quad K > 0$

i.e. $\quad 0 < K < 35$

Substitute $K = 35$ in the s^1 row gives an all zero row.

The Auxiliary Equation is

$$[65.6 - 0.21(35)]s^2 + 3(35) = 0$$

$58.2s^2 + 105 = 0 \quad \Rightarrow \quad s = \pm j1.343$

The root loci intersect the imaginary axis at $s = \pm j1.343$

Rule 10: The breakaway point is the solution of $\frac{dK}{ds} = 0$

$$K = \frac{-\left(s^5 + 13s^4 + 54s^3 + 82s^2 + 60s\right)}{(s+3)}$$

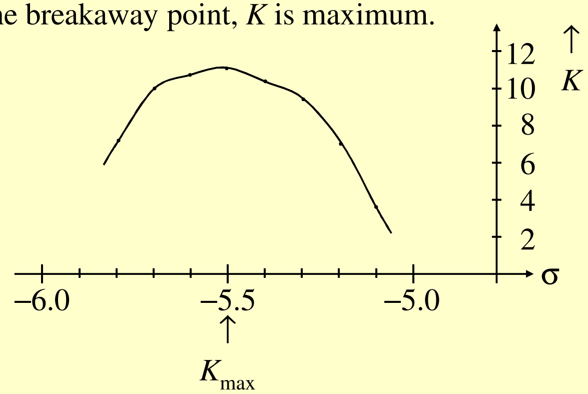
$$\frac{dK}{ds} = \frac{-\left(5s^4 + 52s^3 + 162s^2 + 164s + 60\right)}{(s+3)} + \frac{\left(s^5 + 13s^4 + \cdots + 60s\right)}{(s+3)^2} = 0$$

Solving, s = -5.52 is the breakaway point.

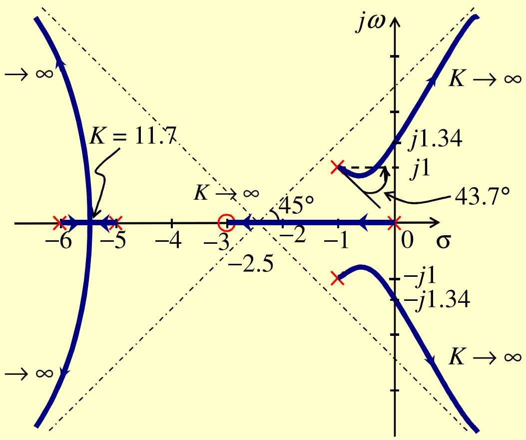
Alternatively, plot K vs s

s	−5.1	−5.2	−5.3	−5.4	−5.5	−5.6	−5.7	−5.8
K	3.89	7.05	9.43	10.9	11.6	11.4	10.2	7.9

At $s = -5.5$, the breakaway point, K is maximum.



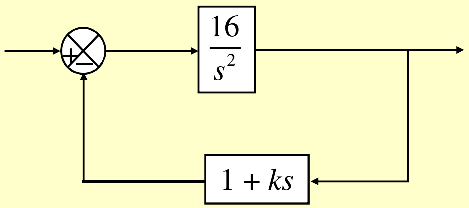
From the information obtained, the root-locus is sketched as shown:



K not a Multiplying Factor

If K is not a multiplying factor, some modification of the Characteristic Equation is required for constructing the root loci.

Example Consider the system



The Characteristic Equation for the system is

$$1+\frac{16(1+ks)}{s^2}=0$$

Sketch the root locus w.r.t. k. From the root locus, compute the value of k that will give a damping ratio $\zeta = 0.707$.

Since k is not a multiplying factor, we modify the C.E. such that k appears as a multiplying factor.

The C.E. is $s^2 + 16 + 16ks = 0$

Rewrite it as $1+\frac{16ks}{s^2+16}=0$

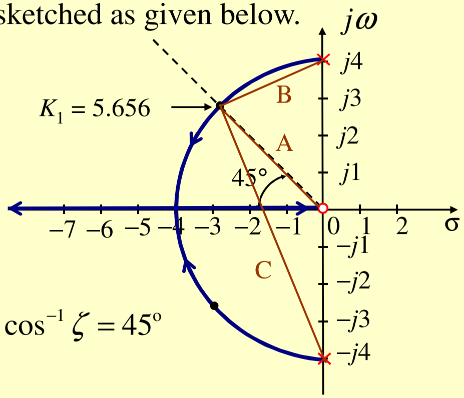
Define $K = 16k$, then the above equation becomes

$$1+\frac{Ks}{s^2+16}=0$$

Open loop poles: $p_{1,2} = \pm j4.0$

Open loop zero: $z_1 = 0$

The root locus can be easily sketched as given below.



With $\zeta = 0.707$, we have $\theta = \cos^{-1}\zeta = 45^{o}$

From the root-locus, the value of K that will yield the required ζ can be computed from the measured values of A, B and C, i.e.

$$K_1 = \frac{B \times C}{A} = \frac{3.061 \times 7.391}{4} = 5.656$$

But $K = 16k$, hence $k = 0.3535$.

Information Obtainable from the Root-Locus

(1) The stability condition for any value of K, or other system parameter against which the root-locus is plotted.

(2) The limits of K for which the system is stable.

(3) The effect of variation of K on the performance potential of the system.

(4) The actual characteristic equation for any value of K.

(5) Evaluation of performance criteria such as undamped and damped natural frequencies, damping ratio and the transient response exponential terms.