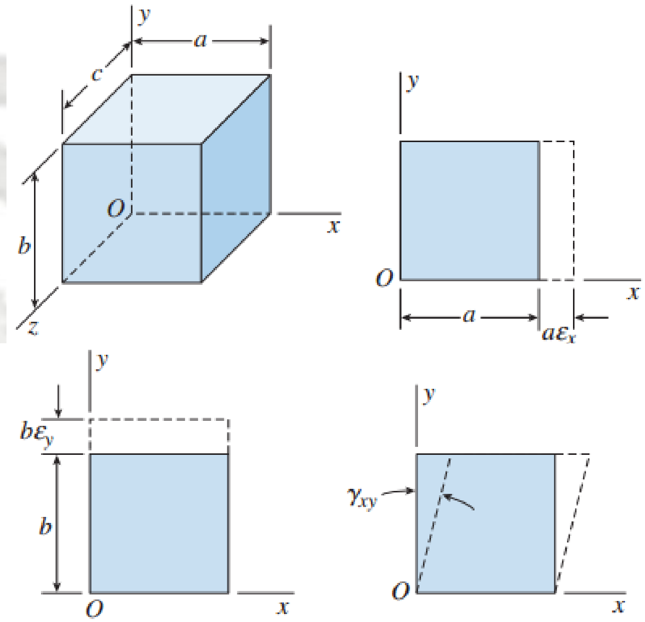
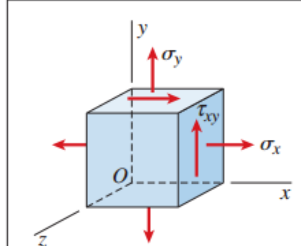
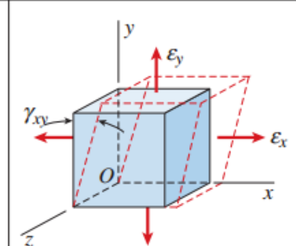
Plane Strain:

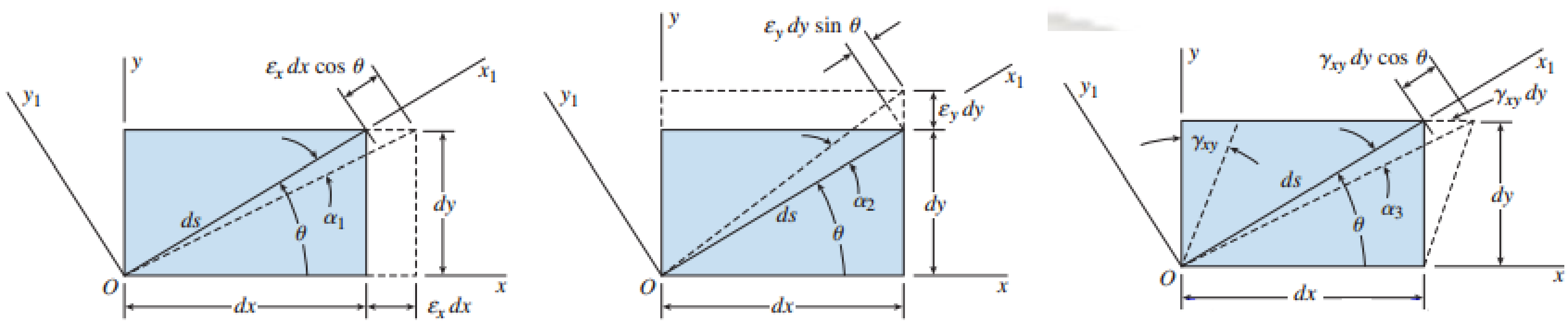
The strains at a point in a loaded structure vary according to the orientation of the axes, in a manner similar to that for stresses. Strains are customarily measured by strain gages; for example, gages are placed in aircraft to measure structural behavior during flight, and gages are placed in buildings to measure the effects of earthquakes. Since each gage measures the strain in one particular direction, it is usually necessary to calculate the strains in other directions by means of the transformation equations. Consider a small element of material having sides of lengths a, b, and c in the x, y, and z directions, respectively. If the only deformations are those in the xy plane, then three strain components may exist—the normal strain εx in the x direction, the normal strain εy in the y direction, and the shear strain γxy. An element of material subjected to these strains is said to be in a state of **plane strain.** The element in plane strain has no normal strain εz in the z direction and no shear strains γxz and γyz in the xz and yz planes, respectively.



NOTE: It should not be inferred from the similarities in the definitions of plane stress and plane strain that both occur simultaneously. In general, an element in plane stress will undergo a strain in the *z* direction, hence, it is **not** in plane strain. Also, an element in plane strain usually will have stresses σz acting on it because of the requirement that ; therefore, it is **not** in plane stress. Thus, under ordinary conditions plane stress and plane strain do not occur simultaneously

	Plane stress	Plane strain
		
Stresses	$\sigma_z = 0$ $\tau_{xz} = 0$ $\tau_{yz} = 0$ σ_x, σ_y, and τ_{xy} may have nonzero values	$\tau_{xz} = 0$ $\tau_{yz} = 0$ σ_x, σ_y, σ_z, and τ_{xy} may have nonzero values
Strains	$\gamma_{xz} = 0$ $\gamma_{yz} = 0$ ε_x, ε_y, ε_z, and γ_{xy} may have nonzero values	$\varepsilon_z = 0$ $\gamma_{xz} = 0$ $\gamma_{yz} = 0$ ε_x, ε_y, and γ_{xy} may have nonzero values

Transformation Equations for Plane Strain



The transformation equations for plane stress can also be used for the stresses in plane strain. Only, the symbol σ must be replaced by ε and Ƭ must be replaced by ɣ/2. Therefore, the transformation expressions of plane strain become:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\frac{\gamma_{x_1y_1}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

$$\gamma_{x_1y_1} = -(\varepsilon_x - \varepsilon_y)\sin 2\theta + \gamma_{xy}\cos 2\theta$$

$$\varepsilon_{x_1} + \varepsilon_{y_1} = \varepsilon_x + \varepsilon_y$$

Sign convention

$\boldsymbol{\varepsilon_x}$, $\boldsymbol{\varepsilon_y}$= +elongation

$\boldsymbol{\varepsilon_x}$, $\boldsymbol{\varepsilon_y}$= - contraction

γ_{xy} = + elongation for the diagonal of positive slope

θ = + counterclockwise

Corresponding Variables in the Transformation Equations for Plane Stress and Plane Strain

Stresses	Strains
σ_x	ε_x
σ_y	ε_y
τ_{xy}	$\gamma_{xy}/2$
σ_{x_1}	ε_{x_1}
$\tau_{x_1y_1}$	$\gamma_{x_1y_1}/2$

Principal Strains:

Principal strains exist on perpendicular planes with the principal angles θp calculated from the following equation:

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}$$

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \mp \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$\gamma_{xy} = 0$, No shear strain associated with the principal strains

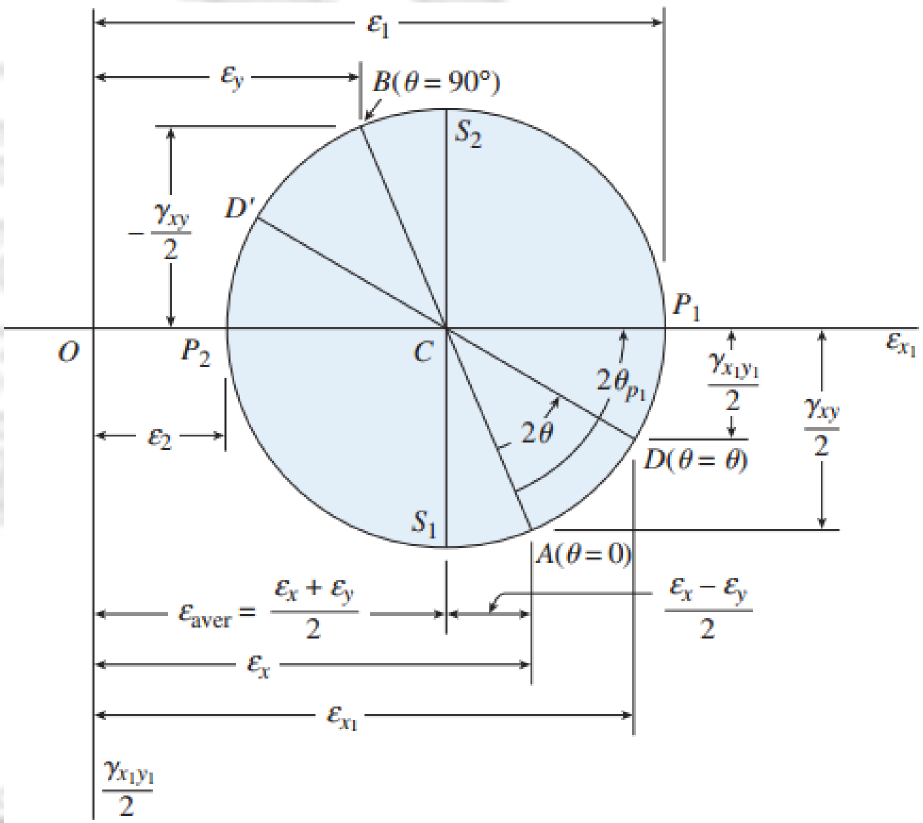
Maximum Shear Strains:

The maximum shear strains in the xy plane are associated with axes at 45° to the directions of the principal strains. The algebraically maximum shear strain (in the xy plane) is given by the following equation:

$$\frac{\gamma_{max}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\varepsilon_{aver} = \frac{\varepsilon_x + \varepsilon_y}{2}$$

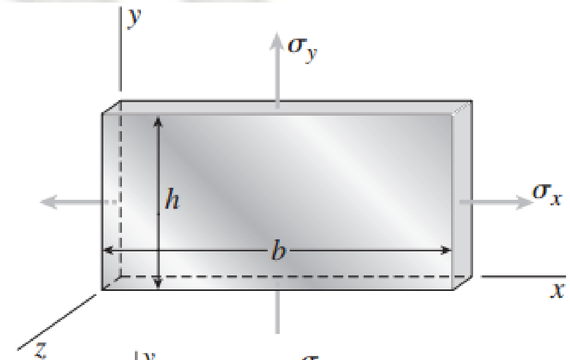
Mohr's Circle for Plane Strain:

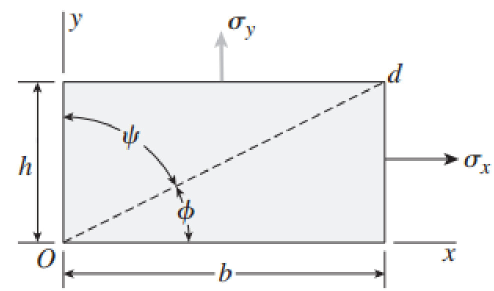


EXAMPLE 3-13

A thin rectangular plate in biaxial stress is subjected to stresses σx and σy as shown in the figure. Measurements show that the normal strains in the x and y directions are εx= 410x10E-6 and εy=-320x10E-6, respectively. If b=160 mm and h=60mm, determine the following quantities :

(a) the increase Δd in the length of diagonal Od.
(b) the change ΔΦ in the angle Φ between diagonal Od and the x axis.
(c) the change ΔΨ in the angle Ψ between diagonal Od and the y axis.





Solution:

$b = 160 \text{ mm} \quad h = 60 \text{ mm} \quad \varepsilon_x = 410 \times 10^{-6}$

$\varepsilon_y = -320 \times 10^{-6} \quad \gamma_{xy} = 0$

$\phi = \arctan \dfrac{h}{b} = 20.56°$

$L_d = \sqrt{b^2 + h^2} = 170.88 \text{ mm}$

(a) INCREASE IN LENGTH OF DIAGONAL

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

For $\theta = \phi = 20.56°$: $\varepsilon_{x_1} = 319.97 \times 10^{-6}$

$\Delta d = \varepsilon_{x_1} L_d = 0.0547 \text{ mm}$

(b) CHANGE IN ANGLE ϕ

$\alpha = -(\varepsilon_x - \varepsilon_y)\sin\theta\cos\theta - \gamma_{xy}\sin^2\theta$

For $\theta = \phi = 20.56°$: $\alpha = -240.0 \times 10^{-6}$ rad

Minus sign means line *Od* rotates clockwise (angle ϕ decreases).

$\Delta\phi = 240 \times 10^{-6}$ rad (decrease)

(c) CHANGE IN ANGLE ψ

Angle ψ increases the same amount that ϕ decreases.

$\Delta\psi = 240 \times 10^{-6}$ rad (increase)

EXAMPLE 3-14

A thin square plate in biaxial stress is subjected to stresses σx and σy as shown in the figure. Measurements show that the normal strains in the x and y directions are εx= 845x10E-6 and εy=211x10E-6, respectively. If the width of the plate b=225 mm, determine the following quantities :

(a) the increase Δd in the length of diagonal Od.

(b) the change ΔΦ in the angle Φ between diagonal Od and the x axis.

(c) the shear strain ɣ associated with diagonals Od and cf (that is, find the decrease in angle ced).

Solution:

$b = 225$ mm $\quad \varepsilon_x = 845 \times 10^{-6}$

$\varepsilon_y = 211 \times 10^{-6} \quad \phi = 45° \quad \gamma_{xy} = 0$

$L_d = b\sqrt{2} = 318.2$ mm

(a) Increase in length of diagonal

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

For $\theta = \phi = 45°$: $\varepsilon_{x_1} = 528 \times 10^{-6}$

$\Delta d = \varepsilon_{x_1} L_d = 0.168$ mm

(b) Change in angle ϕ

$$\alpha = -(\varepsilon_x - \varepsilon_y)\sin\theta\cos\theta - \gamma_{xy}\sin^2\theta$$

For $\theta = \phi = 45°$: $\alpha = -317 \times 10^{-6}$ rad

Minus sign means line *Od* rotates clockwise (angle ϕ decreases).

$\Delta\phi = 317 \times 10^{-6}$ rad (decrease)

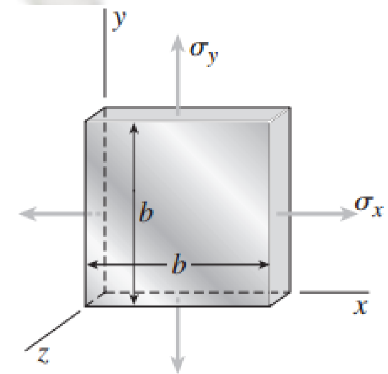
(c) Shear strain between diagonals

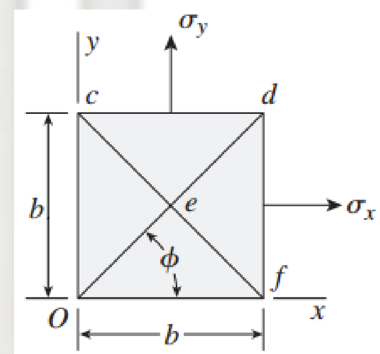
$$\frac{\gamma_{x_1y_1}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

For $\theta = \phi = 45°$: $\gamma_{x_1y_1} = -634 \times 10^{-6}$ rad

(Negative strain means angle *ced* increases)

$\gamma = -634 \times 10^{-6}$ rad





EXAMPLE 3-15

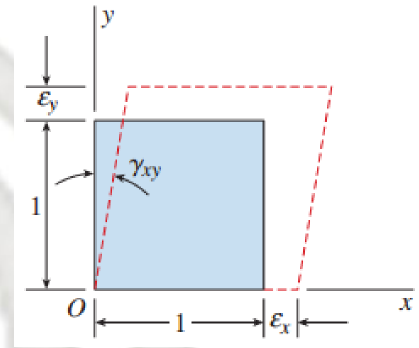
An element of material in plane strain is subjected to strains εx= 480x10E-6,εy=70x10E-6 and ɣxy=420x10E-6. Determine the following quantities:

(a) the strains for an element oriented at an angle θ=75°.

(b) the principal strains

(c) the maximum shear strains.

Show the results on sketches of properly oriented



Solution:

$$\varepsilon_x = 480 \times 10^{-6} \qquad \varepsilon_y = 70 \times 10^{-6}$$

$$\gamma_{xy} = 420 \times 10^{-6}$$

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\frac{\gamma_{x_1y_1}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

$$\varepsilon_{y_1} = \varepsilon_x + \varepsilon_y - \varepsilon_{x_1}$$

For $\theta = 75°$:

$$\varepsilon_{x_1} = 202 \times 10^{-6} \qquad \gamma_{x_1y_1} = -569 \times 10^{-6}$$

$$\varepsilon_{y_1} = 348 \times 10^{-6}$$

PRINCIPAL STRAINS

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= 275 \times 10^{-6} \pm 293 \times 10^{-6}$$

$$\varepsilon_1 = 568 \times 10^{-6} \qquad \varepsilon_2 = -18 \times 10^{-6}$$

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = 1.0244$$

$$2\theta_p = 45.69° \quad \text{and} \quad 225.69°$$

$$\theta_p = 22.85° \quad \text{and} \quad 112.85°$$

For $\theta_p = 22.85°$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$= 568 \times 10^{-6}$$

$$\therefore\ \theta_{p_1} = 22.8° \qquad \varepsilon_1 = 568 \times 10^{-6}$$

$$\theta_{p_2} = 112.8° \qquad \varepsilon_2 = -18 \times 10^{-6}$$

MAXIMUM SHEAR STRAINS

$$\frac{\gamma_{max}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} = 293 \times 10^{-6}$$

$$\gamma_{max} = 587 \times 10^{-6}$$

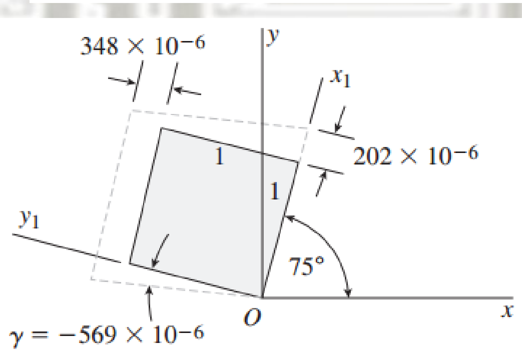
$$\theta_{s_1} = \theta_{p_1} - 45° = -22.2° \text{ or } 157.8°$$

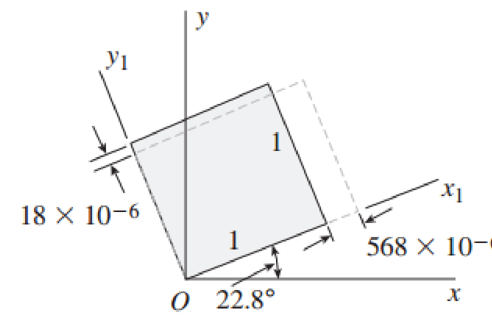
$$\gamma_{max} = 587 \times 10^{-6}$$

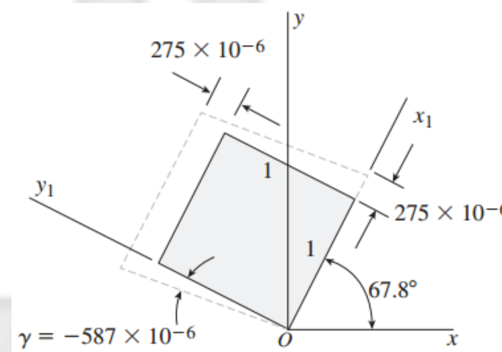
$$\theta_{s_2} = \theta_{s_1} + 90° = 67.8°$$

$$\gamma_{min} = -587 \times 10^{-6}$$

$$\varepsilon_{aver} = \frac{\varepsilon_x + \varepsilon_y}{2} = 275 \times 10^{-6}$$







EXAMPLE 3-16

Using Mohr's circle, solve the **Example 3-15**.

Solution:

$\varepsilon_x = 480 \times 10^{-6} \qquad \varepsilon_y = 70 \times 10^{-6}$

$\gamma_{xy} = 420 \times 10^{-6} \qquad \dfrac{\gamma_{xy}}{2} = 210 \times 10^{-6} \qquad \theta = 75^\circ$

$$R = \sqrt{(205 \times 10^{-6})^2 + (210 \times 10^{-6})^2}$$
$$= 293.47 \times 10^{-6}$$

$$\alpha = \arctan \frac{210}{205} = 45.69^\circ$$

$$\beta = \alpha + 180^\circ - 2\theta = 75.69^\circ$$

POINT C: $\varepsilon_{x_1} = 275 \times 10^{-6}$

POINT D $(\theta = 75^\circ)$:

$$\varepsilon_{x_1} = 275 \times 10^{-6} - R\cos\beta = 202 \times 10^{-6}$$
$$\frac{\gamma_{x_1y_1}}{2} = -R\sin\beta = -284.36 \times 10^{-6}$$
$$\gamma_{x_1y_1} = -569 \times 10^{-6}$$

POINT D' $(\theta = 165^\circ)$:

$$\varepsilon_{x_1} = 275 \times 10^{-6} + R\cos\beta = 348 \times 10^{-6}$$
$$\frac{\gamma_{x_1y_1}}{2} = R\sin\beta = 284.36 \times 10^{-6}$$
$$\gamma_{x_1y_1} = 569 \times 10^{-6}$$

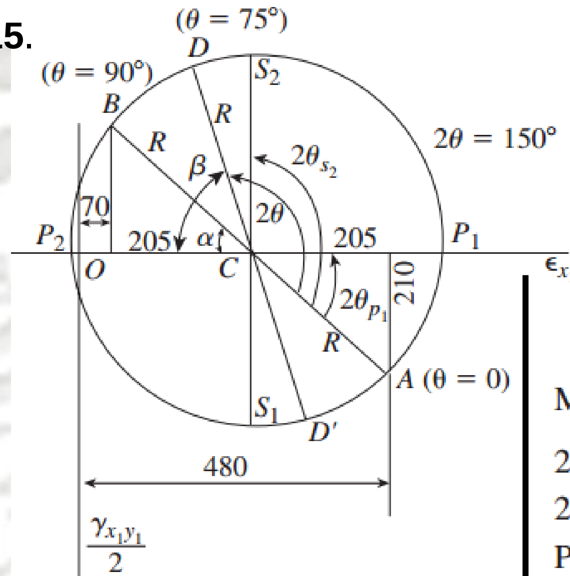
PRINCIPAL STRAINS

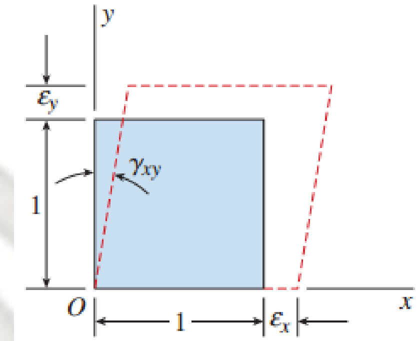
$2\theta_{p_1} = \alpha = 45.69^\circ \qquad \theta_{p_1} = 22.8^\circ$

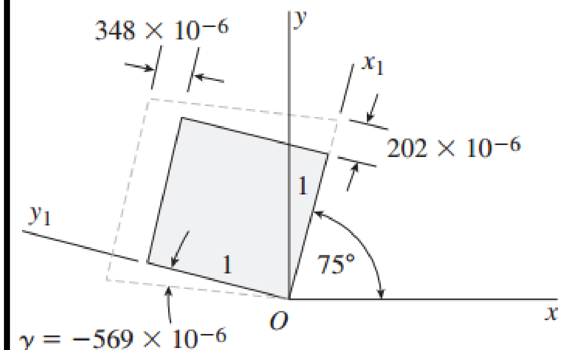
$2\theta_{p_2} = 2\theta_{p_1} + 180^\circ = 225.69^\circ \qquad \theta_{p_2} = 112.8^\circ$

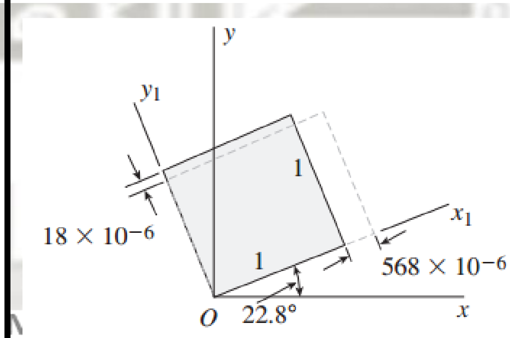
Point P_1: $\varepsilon_1 = 275 \times 10^{-6} + R = 568 \times 10^{-6}$

Point P_2: $\varepsilon_2 = 275 \times 10^{-6} - R = -18 \times 10^{-6}$









MAXIMUM SHEAR STRAINS

$2\theta_{s_2} = 90^\circ + \alpha = 135.69^\circ \qquad \theta_{s_2} = 67.8^\circ$

$2\theta_{s_1} = 2\theta_{s_2} + 180^\circ = 315.69^\circ \qquad \theta_{s_1} = 157.8^\circ$

Point S_1: $\varepsilon_{aver} = 275 \times 10^{-6}$

$\gamma_{max} = 2R = 587 \times 10^{-6}$

Point S_2: $\varepsilon_{aver} = 275 \times 10^{-6}$

$\gamma_{min} = -587 \times 10^{-6}$