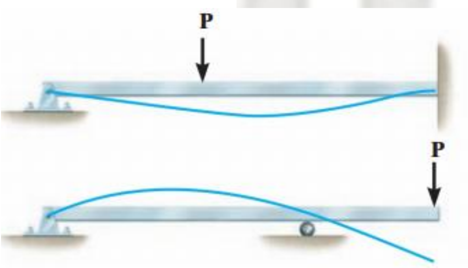
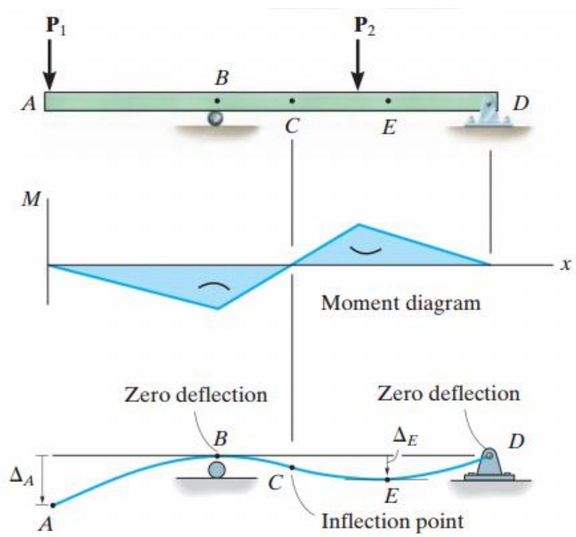
CHAPER FOUR – DEFLECTION OF BEAMS

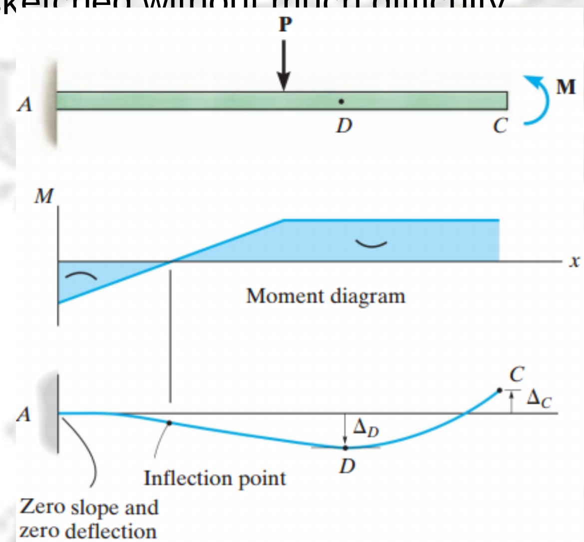
When a beam with a straight longitudinal axis is loaded by lateral forces, the axis is deformed into a curve, called the **deflection curve** of the beam. The calculation of deflections is an important part of structural analysis and design. Deflections are sometimes calculated in order to verify that they are within tolerable limits. Large deflections in buildings are unsightly and can cause cracks in ceilings and walls. In the design of machines and aircraft, specifications may limit deflections in order to prevent undesirable vibrations.

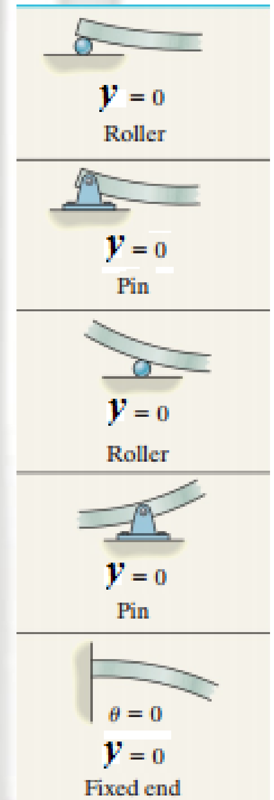
Elastic Deflection Curve

Before finding the slope or displacement, it is often helpful to sketch the deflected shape of the beam, which is represented by its **elastic curve.** This curve passes through the centroid of each cross section of the beam, and for most cases it can be sketched without much difficulty.







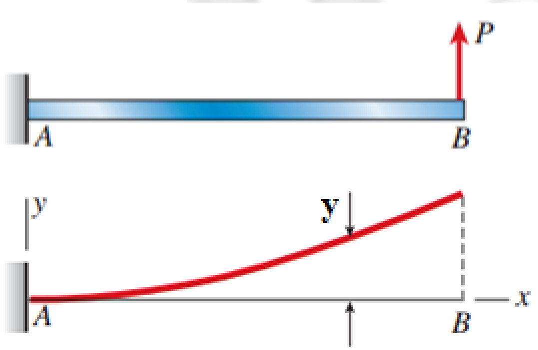


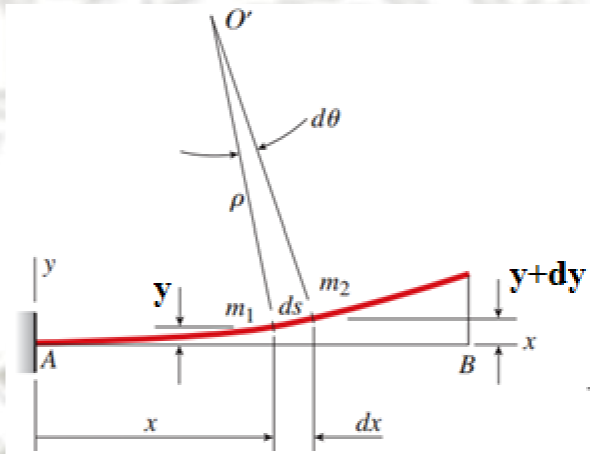
1- Double Integration Method

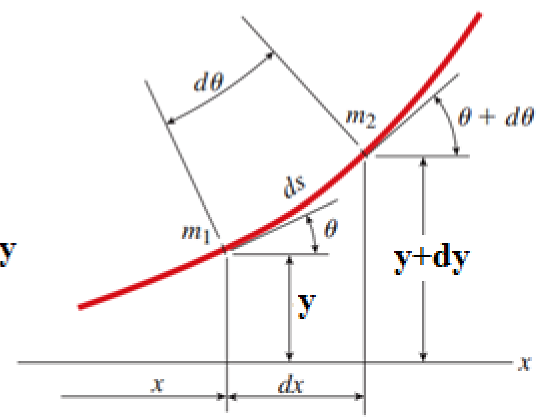
In this method, for each segment, we integrate the differential equation of the bending-moment twice to obtain a slope equation , a deflection equation , and two constants of integration.

Differential Equations of the Deflection Curve:

consider a cantilever beam with a concentrated load acting upward at the free end. The axis of the beam deforms into a curve, as shown in the figure. The reference axes have their origin at the fixed end of the beam, with the x axis directed to the right and the y axis directed upward.







$$\rho \, d\theta = ds \qquad \kappa = \frac{1}{\rho} = \frac{d\theta}{ds} \qquad \frac{d\mathbf{y}}{dx} = \tan\theta \qquad \cos\theta = \frac{dx}{ds} \qquad \sin\theta = \frac{d\mathbf{y}}{ds}$$

The changes in the structures are so small as to be unnoticed by a casual observer. Consequently, the deflection curves of most beams and columns have very small angles of rotation, very small deflections, and very small curvatures. Under these conditions we can make some mathematical approximations that greatly simplify beam analysis.

$$ds \approx dx \qquad \theta \approx \tan\theta = \frac{dy}{dx} \qquad \frac{d\theta}{dx} = \frac{d^2y}{dx^2} \qquad \kappa = \frac{1}{\rho} = \frac{d\theta}{dx}$$

If the material of a beam is **linearly elastic** and follows Hooke's law, the curvature is: $\kappa = \frac{1}{\rho} = \frac{M}{EI}$

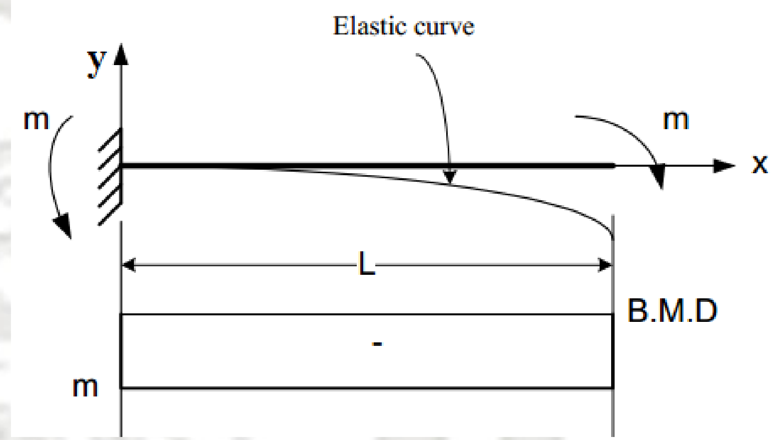
Thus, the basic **differential equation of the deflection curve** of a beam is:

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \qquad \boxed{EI\frac{d^2y}{dx^2} = M(x)}$$

EXAMPLE 4-1

Determine the general formula for estimating the rotation and deflection at any section of the beams shown below. Also, find the maximum rotation and maximum deflection. Assume EI= constant.

Solution:



$$E\,I\,\frac{d^2y}{dx^2} = M(x)$$

$$E\,I\,\frac{d^2y}{dx^2} = -m$$

Integrate both sides of the above equation:

$$E\,I\,\frac{dy}{dx} = -mx + C_1$$

Apply the Boundary Conditions (B.C):

$$@x = 0;\ \frac{dy}{dx} = 0 => C_1 = 0$$

$$E\,I\,\frac{dy}{dx} = -mx \quad \cdots Rotation\ equation$$

$$@x = L;\ \left(\frac{dy}{dx}\right)_{max} = \theta_{max} = -\frac{m\,L}{E\,I}$$

Integrate both sides of the rotation equation:

$$E\,I\,y = -\frac{mx^2}{2} + C_2$$

Apply the Boundary Conditions (B.C):

$$@x = 0;\ y = 0 => C_2 = 0$$

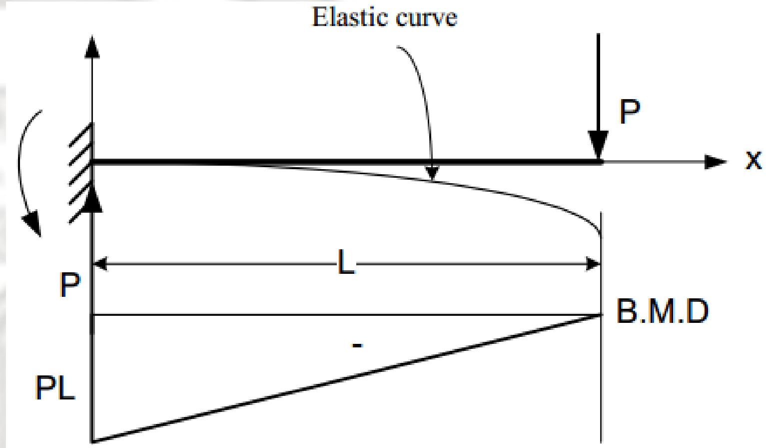
$$E\,I\,y = -\frac{mx^2}{2} \cdots Deflection\ equation$$

$$@x = L;\ y_{max} = \Delta_{max} = -\frac{m\,L^2}{2E\,I}$$

EXAMPLE 4-2

Determine the general formula for estimating the rotation and deflection at any section of the beams shown below. Also, find the maximum rotation and maximum deflection. Assume EI= constant.

Solution:



$$E\,I\,\frac{d^2y}{dx^2} = M(x)$$

$$E\,I\,\frac{d^2y}{dx^2} = -PL + Px$$

Integrate both sides of the above equation:

$$E\,I\,\frac{dy}{dx} = -PLx + \frac{Px^2}{2} + C_1$$

Apply the Boundary Conditions (B.C):

$$@x = 0;\ \frac{dy}{dx} = 0 => C_1 = 0$$

$$E\,I\,\frac{dy}{dx} = -PLx + \frac{Px^2}{2} \quad \cdots Rotation\ equation$$

$$@x = L;\ \left(\frac{dy}{dx}\right)_{max} = \theta_{max} = -\frac{PL^2}{2E\,I}$$

Integrate both sides of the rotation equation:

$$E\,I\,y = -\frac{PLx^2}{2} + \frac{Px^3}{6} + C_2$$

Apply the Boundary Conditions (B.C):

$$@x = 0;\ y = 0 => C_2 = 0$$

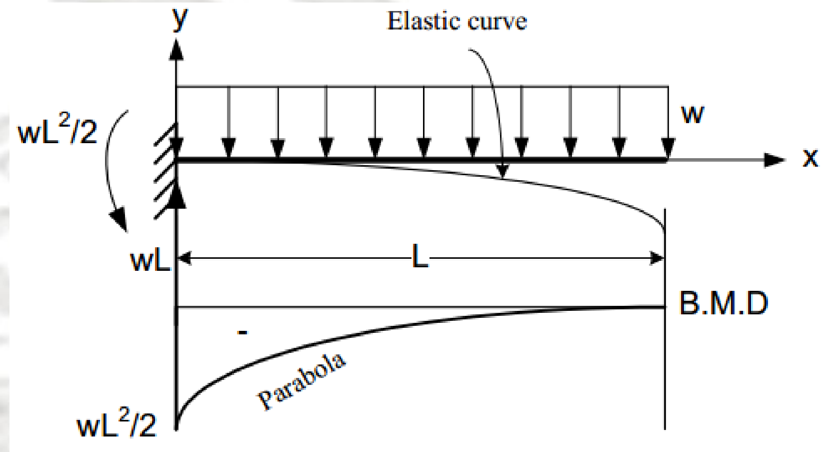
$$E\,I\,y = -\frac{PLx^2}{2} + \frac{Px^3}{6} \cdots Deflection\ equation$$

$$@x = L;\ y_{max} = \Delta_{max} = -\frac{P\,L^3}{3E\,I}$$

EXAMPLE 4-3

Determine the general formula for estimating the rotation and deflection at any section of the beams shown below. Also, find the maximum rotation and maximum deflection. Assume EI= constant.

Solution:



$$E\,I\,\frac{d^2y}{dx^2} = M(x)$$

$$E\,I\,\frac{d^2y}{dx^2} = -\frac{wL^2}{2} - \frac{wx^2}{2} + wLx$$

Integrate both sides of the above equation:

$$E\,I\,\frac{dy}{dx} = -\frac{wL^2x}{2} - \frac{wx^3}{6} + \frac{wLx^2}{2} + C_1$$

Apply the Boundary Conditions (B.C):

$$@x = 0;\ \frac{dy}{dx} = 0 => C_1 = 0$$

$$E\,I\,\frac{dy}{dx} = -\frac{wL^2x}{2} - \frac{wx^3}{6} + \frac{wLx^2}{2} \cdots Rotation\ equation$$

$$@x = L;\ \left(\frac{dy}{dx}\right)_{max} = \theta_{max} = -\frac{wL^3}{6E\,I}$$

Integrate both sides of the rotation equation:

$$E\,I\,y = -\frac{wL^2x^2}{4} - \frac{wx^4}{24} + \frac{wLx^3}{6} + C_2$$

Apply the Boundary Conditions (B.C):

$$@x = 0;\ y = 0 => C_2 = 0$$

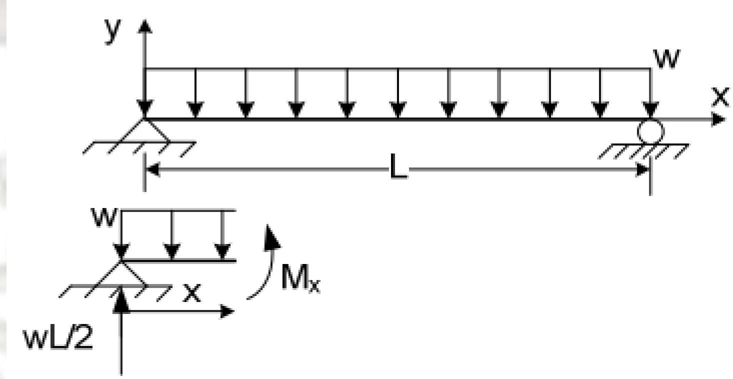
$$E\,I\,y = -\frac{wL^2x^2}{4} - \frac{wx^4}{24} + \frac{wLx^3}{6} \cdots Deflection\ equation$$

$$@x = L;\ y_{max} = \Delta_{max} = -\frac{w\,L^4}{8E\,I}$$

EXAMPLE 4-4

Determine the general formula for estimating the rotation and deflection at any section of the beams shown below. Also, find the maximum rotation and maximum deflection. Assume EI= constant.

Solution:



$$E\,I\,\frac{d^2y}{dx^2} = M(x)$$

$$E\,I\,\frac{d^2y}{dx^2} = \frac{wLx}{2} - \frac{wx^2}{2}$$

Integrate both sides of the above equation:

$$E\,I\,\frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} + C_1$$

Apply the Boundary Conditions (B.C):

$$@x = \frac{L}{2};\ \frac{dy}{dx} = 0 => C_1 = -\frac{wL^3}{24}$$

$$E\,I\,\frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} - \frac{wL^3}{24} \cdots Rotation\ equation$$

$$@x = 0\ or\ x = L;\ \left(\frac{dy}{dx}\right)_{max} = \theta_{max} = -\frac{wL^3}{24E\,I}$$

Integrate both sides of the rotation equation:

$$E\,I\,y = \frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24} + C_2$$

Apply the Boundary Conditions (B.C):

$$@x = 0;\ y = 0 => C_2 = 0$$

$$E\,I\,y = \frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24} \cdots Deflection\ equation$$

$$@x = \frac{L}{2};\ y_{max} = \Delta_{max} = -\frac{5w\,L^4}{384E\,I}$$

Very popular value of maximum deflection in a simply supported beam