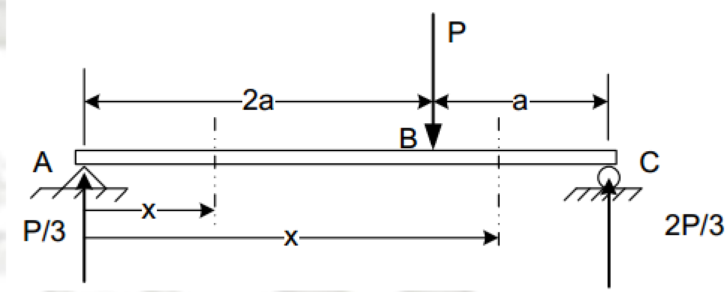
EXAMPLE 4-5

Determine the general formula for estimating the rotation and deflection at any section of the beams shown below. Also, find the maximum rotation and maximum deflection. Assume EI= constant.

Solution:



Part AB: $0 \le x \le 2a$;

$E\,I\,\frac{d^2y}{dx^2} = M(x);\; E\,I\,\frac{d^2y}{dx^2} = \frac{Px}{3}$

Integrate both sides of the above equation:

$$E\,I\,\frac{dy}{dx} = \frac{Px^2}{6} + C_1 \cdots\cdots (1)$$

Integrate both sides of the rotation equation:

$$E\,I\,y = \frac{Px^3}{18} + C_1x + C_2 \cdots\cdots (2)$$

Part BC: $2a \le x \le 3a$; $\quad E\,I\,\frac{d^2y}{dx^2} = \frac{Px}{3} - P(x-2a) = \frac{2P}{3}(3a-x)$

$$E\,I\,\frac{dy}{dx} = \frac{2P}{3}\left(3ax - \frac{x^2}{2}\right) + C_3 \cdots\cdots (3)$$

$$E\,I\,y = \frac{2P}{3}\left(\frac{3ax^2}{2} - \frac{x^3}{6}\right) + C_3x + C_4 \cdots\cdots (4)$$

The four constants are found by using two boundary conditions known as:

$@x = 0;\; y = 0 \;and\; @x = L;\; y = 0$

Also, two continuity conditions must be applied at point B:

$@\,point\,B\,(x = 2a);\; \frac{dy}{dx}\Big|_{AB} = \frac{dy}{dx}\Big|_{BC}$

$@\,point\,B\,(x = 2a);\; y|_{AB} = y|_{BC}$

From Equation 2:

$$0 = \frac{P(0)^3}{18} + C_1(0) + C_2 \Rightarrow C_2 = 0 \cdots (5)$$

From Equation 4:

$$0 = \frac{2P}{3}\left(\frac{3a(L)^2}{2} - \frac{(L)^3}{6}\right) + C_3(L) + C_4 \cdots (6)$$

From Equation 1 and Equation 3:

$$\frac{P(2a)^2}{6} + C_1 = \frac{2P}{3}\left(3a(2a) - \frac{(2a)^2}{2}\right) + C_3 \cdots (7)$$

From Equation 2 and Equation 4:

$$\frac{P(2a)^3}{18} + C_1(2a) + C_2 = \frac{2P}{3}\left(\frac{3a(2a)^2}{2} - \frac{(2a)^3}{6}\right) + C_3(2a) + C_4 \cdots (8)$$

Solving Equations (5-8) simultaneously, we get:

$$C_1 = -\frac{4Pa^2}{9} \qquad C_2 = 0 \qquad C_3 = -\frac{22Pa^2}{9} \qquad C_4 = \frac{4Pa^3}{3}$$

Equations (1-4) become:

Part AB: $0 \le x \le 2a$

$$E\,I\,\frac{dy}{dx} = \frac{Px^2}{6} - \frac{4Pa^2}{9} \cdots (9)$$

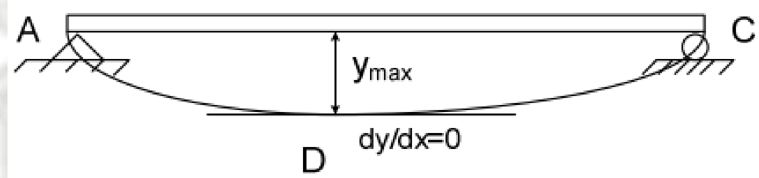
$$E\,I\,y = \frac{Px^3}{18} - \frac{4Pxa^2}{9} \cdots (10)$$

Part BC: $2a \le x \le 3a$

$$E\,I\,\frac{dy}{dx} = \frac{2P}{3}\left(3ax - \frac{x^2}{2}\right) - \frac{22Pa^2}{9} \cdots (11)$$

$$E\,I\,y = \frac{2P}{3}\left(\frac{3ax^2}{2} - \frac{x^3}{6}\right) - \frac{22Pxa^2}{9} + \frac{4Pa^3}{3} \cdots (12)$$

By inspecting the elastic curve, the maximum deflection occurs at point D, somewhere in the region AB. At point D the slope must be zero.



Therefore, from Equation 9:

$$0 = \frac{Px^2}{6} - \frac{4Pa^2}{9}$$

x= 1.633a

Substituting in Equation 10:

$$E\,I\,y_{max} = \frac{P(1.633a)^3}{18} - \frac{4P(1.633a)a^2}{9}$$

$$y_{max} = -0.484\frac{P(a)^3}{E\,I}$$

2- Macaulay’s Method (Discontinuity Functions or Singularity Functions):

The method of integration, used to find the equation of the elastic curve for a beam, is convenient if the load or internal moment can be expressed as a continuous function throughout the beam’s entire length. If several different loadings act on the beam, however, this method can become tedious to apply, because separate loading or moment functions must be written for each region of the beam.

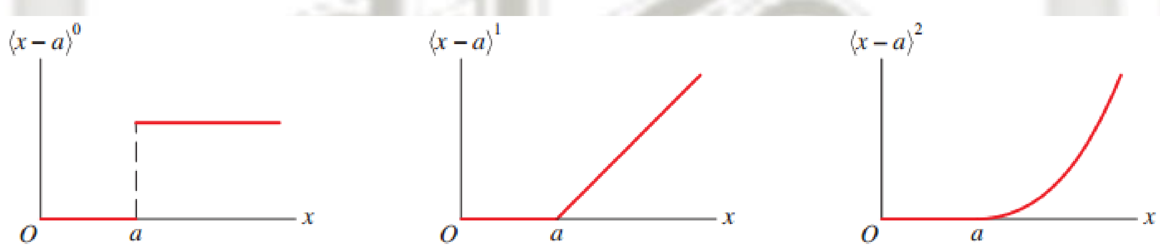
In this method, we will find the equation of the elastic curve using a single expression, either formulated directly from the loading on the beam, w = w(x), or from the beam’s internal moment, M = M(x). Then when this expression for w is substituted into EI d4y/dx4 = w(x) and integrated four times, or if the expression for M is substituted into EI d2y/dx2 =M(x) and integrated twice, the constants of integration will only have to be determined from the boundary conditions. Macaulay functions, **can be written in general form as:**

$$\langle x-a\rangle^n = \begin{cases} 0 & \text{when } x < a \\ (x-a)^n & \text{when } x \geq a \end{cases} \quad \text{for } n \geq 0 \ (n = 0,\ 1,\ 2, \ldots)$$

Here x represents the location of a point on the beam, and a is the location where the distributed loading begins. As stated by the equation, only when x ≥ a is $\langle x-a\rangle^n=(x-a)^n$; otherwise it is zero. Integration of the Macaulay function follows the same rules as for ordinary functions, i.e.

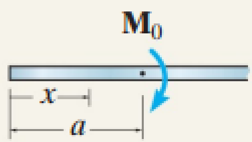
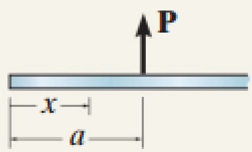
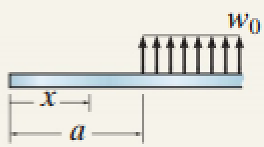
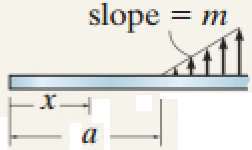
$$\int \langle x-a\rangle^n dx = = \frac{\langle x-a\rangle^{n+1}}{n+1} + C$$

Three Macaulay functions corresponding, respectively, to n = 0, n = 1, and n = 2 are plotted in the figure shown.



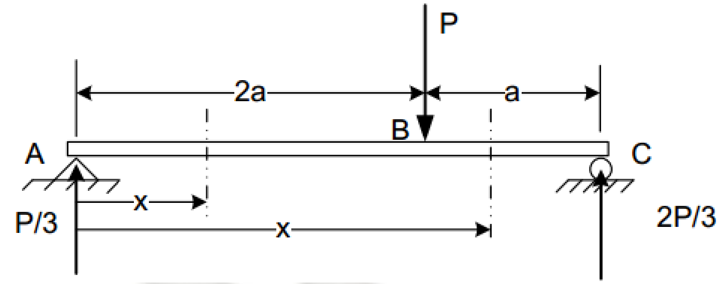
(a) $n = 0$ (b) $n = 1$ (c) $n = 2$

Macaulay functions for a uniform and triangular load are shown in table below. Using integration, the Macaulay functions for shear, V = ∫w(x) dx, and moment, M = ∫V dx, are also shown in the table.

Loading	Loading Function $w = w(x)$	Shear $V = \int w(x)dx$	Moment $M = \int V dx$
	$w = M_0\langle x-a\rangle^{-2}$	$V = M_0\langle x-a\rangle^{-1}$	$M = M_0\langle x-a\rangle^{0}$
	$w = P\langle x-a\rangle^{-1}$	$V = P\langle x-a\rangle^{0}$	$M = P\langle x-a\rangle^{1}$
	$w = w_0\langle x-a\rangle^{0}$	$V = w_0\langle x-a\rangle^{1}$	$M = \frac{w_0}{2}\langle x-a\rangle^{2}$
slope = m 	$w = m\langle x-a\rangle^{1}$	$V = \frac{m}{2}\langle x-a\rangle^{2}$	$M = \frac{m}{6}\langle x-a\rangle^{3}$

EXAMPLE 4-6

Resolve the **EXAMPLE 4-5**, by using discontinuity functions method



Solution:

$$E\,I\,\frac{d^2y}{dx^2} = M(x)$$

$$M(x) = \frac{P<x>^1}{3} - P<x-2a>^1$$

$$EI\frac{d^2y}{dx^2} = \frac{P<x>^1}{3} - P<x-2a>^1$$ Integrate both sides of the above equation:

$$EI\frac{dy}{dx} = \frac{P<x>^2}{6} - \frac{P<x-2a>^2}{2} + C_1 \cdots\cdots (13)$$ Integrate both sides of the rotation equation:

$$EIy = \frac{P<x>^3}{18} - \frac{P<x-2a>^3}{6} + C_1x + C_2$$ Apply the two boundary conditions known as:

$$@x=0;\ y=0\ and\ @x=3a;\ y=0 \qquad C_2 = 0$$

$$0 = \frac{P(3a)^3}{18} - \frac{P(3a-2a)^3}{6} + C_1(3a) \qquad C_1 = -\frac{4Pa^2}{9}$$

Therefore, the deflection equation of the whole beam can be written as:

$$EIy = \frac{P<x>^3}{18} - \frac{P<x-2a>^3}{6} - \frac{4Pa^2x}{9} \cdots\cdots (14)$$

Once again, by inspecting the elastic curve, the maximum deflection occurs at point D, somewhere in the region AB. At point D the slope must be zero.

Therefore, from Equation 13: $$0 = \frac{Px^2}{6} - \frac{4Pa^2}{9}$$

x= 1.633a

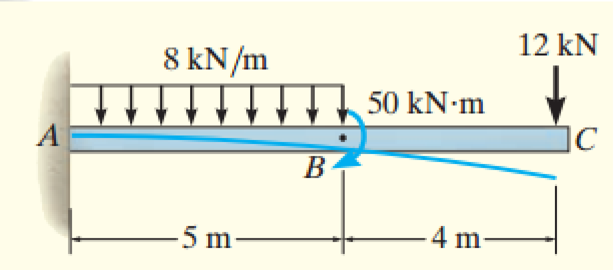
Substituting in Equation 14: $$E\,I\,y_{max} = \frac{P(1.633a)^3}{18} - \frac{4P(1.633a)a^2}{9}$$

$$y_{max} = -0.484\frac{P(a)^3}{E\,I}$$

EXAMPLE 4-7

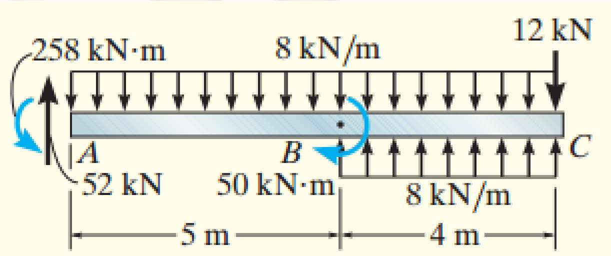
Determine the equation of the elastic curve for the cantilevered beam shown in the figure by using discontinuity functions method. EI is constant.

Solution:



$$M(x) = -258\langle x-0\rangle^0 + 52\langle x-0\rangle^1 - \frac{1}{2}(8)\langle x-0\rangle^2 + 50\langle x-5\rangle^0 + \frac{1}{2}(8)\langle x-5\rangle^2$$

$$= (-258 + 52x - 4x^2 + 50\langle x-5\rangle^0 + 4\langle x-5\rangle^2)\ \mathrm{kN\cdot m}$$



$$EI\frac{d^2v}{dx^2} = -258 + 52x - 4x^2 + 50\langle x-5\rangle^0 + 4\langle x-5\rangle^2$$

$$EI\frac{dv}{dx} = -258x + 26x^2 - \frac{4}{3}x^3 + 50\langle x-5\rangle^1 + \frac{4}{3}\langle x-5\rangle^3 + C_1$$

$$EI\,v = -129x^2 + \frac{26}{3}x^3 - \frac{1}{3}x^4 + 25\langle x-5\rangle^2 + \frac{1}{3}\langle x-5\rangle^4 + C_1x + C_2$$

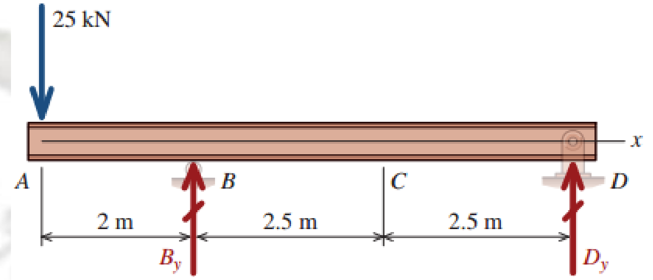
Since $dv/dx = 0$ at $x = 0$, $C_1 = 0$; and $v = 0$ at $x = 0$, so $C_2 = 0$. Thus,

$$v = \frac{1}{EI}\left(-129x^2 + \frac{26}{3}x^3 - \frac{1}{3}x^4 + 25\langle x-5\rangle^2 + \frac{1}{3}\langle x-5\rangle^4\right)\ \mathrm{m}$$

EXAMPLE 4-8

Determine the equation of the elastic curve for the cantilevered beam shown in the figure by using discontinuity functions method. Assume a constant value of *EI* = 17 × 103 kN.m2.

Solution:



$$EI\frac{d^2y}{dx^2} = M(x) = -25\text{ kN}\langle x-0\text{ m}\rangle^1 + 35\text{ kN}\langle x-2\text{ m}\rangle^1$$

$$EI\frac{dy}{dx} = -\frac{25\text{ kN}}{2}\langle x-0\text{ m}\rangle^2 + \frac{35\text{ kN}}{2}\langle x-2\text{ m}\rangle^2 + C_1$$

$$EI\,y = -\frac{25\text{ kN}}{6}\langle x-0\text{ m}\rangle^3 + \frac{35\text{ kN}}{6}\langle x-2\text{ m}\rangle^3 + C_1x + C_2$$

at $x = 2$ m , $y = 0$

$$-\frac{25\text{ kN}}{6}(2\text{ m})^3 + \frac{35\text{ kN}}{6}(0\text{ m})^3 + C_1(2\text{ m}) + C_2 = 0$$

at $x = 7$, $y = 0$

$$-\frac{25\text{ kN}}{6}(7\text{ m})^3 + \frac{35\text{ kN}}{6}(5\text{ m})^3 + C_1(7\text{ m}) + C_2 = 0$$

$C_1 = 133.3333\text{ kN}\cdot\text{m}^2$ and $C_2 = -233.3333\text{ kN}\cdot\text{m}^3$

$$EI\frac{dy}{dx} = -\frac{25\text{ kN}}{2}\langle x-0\text{ m}\rangle^2 + \frac{35\text{ kN}}{2}\langle x-2\text{ m}\rangle^2 + 133.3333\text{ kN}\cdot\text{m}^2$$

$$EI\,y = -\frac{25\text{ kN}}{6}\langle x-0\text{ m}\rangle^3 + \frac{35\text{ kN}}{6}\langle x-2\text{ m}\rangle^3 + (133.3333\text{ kN}\cdot\text{m}^2)\,x - 233.3333\text{ kN}\cdot\text{m}^3$$

(a) Beam Deflection at *A*

At the tip of the overhang, where $x = 0$ m, the beam deflection is

$$EIv_A = -\frac{25\text{ kN}}{6}\langle x - 0\text{ m}\rangle^3 + \frac{35\text{ kN}}{2}\langle x - 2\text{ m}\rangle^3 + (133.3333\text{ kN}\cdot\text{m}^2)x - 233.3333\text{ kN}\cdot\text{m}^3$$

$$= -233.3333\text{ kN}\cdot\text{m}^3$$

$$\therefore v_A = -\frac{233.3333\text{ kN}\cdot\text{m}^3}{17\times10^3\text{ kN}\cdot\text{m}^2} = -0.013725\text{ m} = 13.73\text{ mm}\downarrow$$

(b) Beam Deflection at *C*

At C, where $x = 4.5$ m, the beam deflection is

$$EIv_C = -\frac{25\text{ kN}}{6}(4.5\text{ m})^3 + \frac{35\text{ kN}}{6}(2.5\text{ m})^3 + (133.3333\text{ kN}\cdot\text{m}^2)(4.5\text{ m}) - 233.3333\text{ kN}\cdot\text{m}^3$$

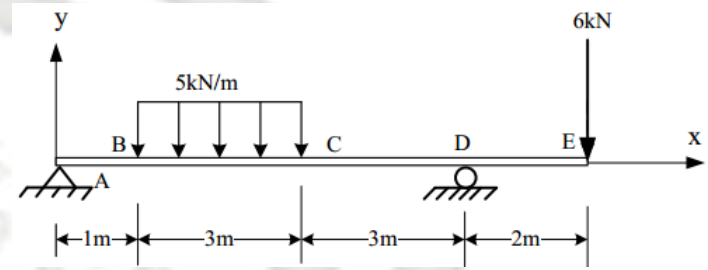
$$= 78.1249\text{ kN}\cdot\text{m}^3$$

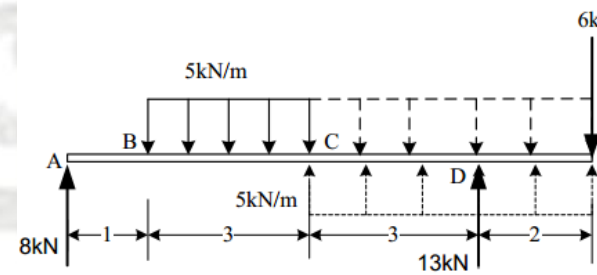
$$\therefore v_C = \frac{78.1249\text{ kN}\cdot\text{m}^3}{17\times10^3\text{ kN}\cdot\text{m}^2} = 0.004596\text{ m} = 4.60\text{ mm}\uparrow$$

EXAMPLE 4-9

Using the singularity functions, determine the deflection equation of the overhanging beam loaded as shown below. Also, find the value of EIy at the mid-span between the supports.

Solution:





$$M(x) = 8x - \frac{5<x-1>^2}{2} + \frac{5<x-4>^2}{2} + 13<x-7>$$

Therefore, the differential curvature equation for the elastic curve becomes:

$$EI\frac{d^2y}{dx^2} = 8x - \frac{5<x-1>^2}{2} + \frac{5<x-4>^2}{2} + 13<x-7>$$

Integrate both sides of the above equation twice gives:

$$EI\frac{dy}{dx} = 4x^2 - \frac{5<x-1>^3}{6} + \frac{5<x-4>^3}{6} + \frac{13<x-7>^2}{2} + C_1$$

$$EIy = \frac{4x^3}{3} - \frac{5<x-1>^4}{24} + \frac{5<x-4>^4}{24} + \frac{13<x-7>^3}{6} + C_1x + C_2$$

To find the value of C_2, we apply the B.C. @x=0; y=o. That gives C_2=0.

Also, to obtain C_1, we note that @x=7; y=0. This gives:

$$0 = \frac{4(7)^3}{3} - \frac{5(6)^4}{24} + \frac{5(3)^4}{24} + 0 + C_1(7) + 0 \qquad C_1 = -29.17$$

$$EIy = \frac{4x^3}{3} - \frac{5<x-1>^4}{24} + \frac{5<x-4>^4}{24} + \frac{13<x-7>^3}{6} - 29.17x$$

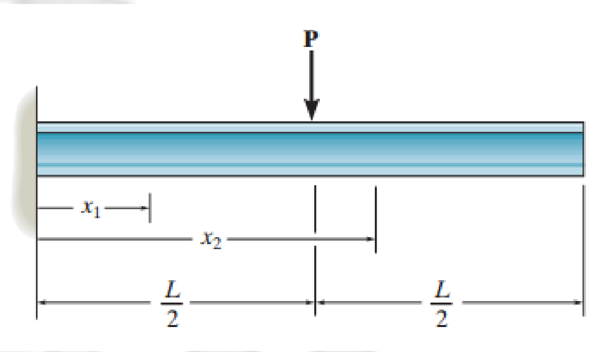
In order to obtain the midspan deflection, the value x=3.5m is substituted in the deflection equation and ignoring the negative values of the pointed brackets $<x-4>^4$ and $<x-7>^3$. We find:

$$EIy = \frac{4(3.5)^3}{3} - \frac{5<3.5-1>^4}{24} - 29.17(3.5)$$

$$y = -\frac{53}{EI}kN - m^3$$

H.W 1

Determine the equations of the elastic curve for the beam using the and coordinates.x1 x2. EI is constant.



H.W 2

Determine the equations of the elastic curve for the beam using the x coordinate. Specify the slope at A and maximum deflection. EI is constant.