16

Difference Equations and *z*-Transforms

Part A : Difference Equations

16.A.1 INTRODUCTION

Difference equations arise in the situations in which the discrete values of the independent variable involve. Many practical phenomena are modelled with the help of difference equations. In engineering, difference equations arise in control engineering, digital signal processing, electrical networks, etc. In social sciences, difference equations arise to study the national income of a country and then its variation with time, Cobweb phenomenon in economics, etc. Analogue to differential equation, difference equation is the most powerful instrument for the treatment of discrete processes.

16.A.2 DEFINITIONS

A **difference equation** is an equation which expresses a relation between an independent variable and the successive values of the dependent variable or the successive differences of the dependent variable.

For example, $$y_{x+3} + 2y_{x+2} - 3y_{x+1} + 5y_x = x^2 \quad ...(i)$$

$$\Delta^4 y_x - 3\Delta^3 y_x + 2\Delta^2 y_x + 5\Delta y_x + y_x = 3^x \quad ...(ii)$$

are two difference equations.

Since the differences are dsicrete values, eqn. (*i*) can be written in the following form :

$$y(x + 3) + 2y(x + 2) - 3y(x + 1) + 5y(x) = x^2 \quad ...(iii)$$

Without loss of generality, the presentation as given in eqn. (*iii*) will be considered in this chapter.

Order of a Difference Equation :

The difference between the largest and smallest arguments appearing in the difference equation is called its order.

e.g. The order of eqn. (*i*) (or, *iii*) is $(x + 3) - x = 3$. Whereas the order of eqn. (*ii*) can be determined only after operating the Δ operators on the functions.

Solution of a Difference Equation :

A solution of a difference equation is a relation between the independent variable and the dependent variable satisfying the equation.

e.g., The relation $y(x) = ca^x$ is a solution of the difference equation $y(x + 1) - ay(x) = 0$, $a \neq 1$ where c is an arbitrary constant.

The solution of a difference equation of order n shall generally contain n arbitrary constants.

A solution involving as many arbitrary constants as is the order of the equation, is called the **general solution**.

Any solution obtained from the general solution by assigning particular values to the arbitrary constants is called a **particular solution.**

In the above example, $y(x) = ca^x$ is the general solution and $y(x) = 3a^x$ is a particular solution.

16.A.3 FORMATION OF DIFFERENCE EQUATIONS

A difference equation is formed by eliminating the arbitrary constants from a relation giving the order of the equation is equal to the number of arbitrary constants. The following examples illustrate the formation of difference equations :

Example 1. Form the difference equation from

$$y(n) = A.3^n + B5^n$$

by eliminating the constants A and B.

We have,
$$y(n) = A.3^n + B.5^n$$
$$y(n + 1) = 3A.3^n + 5B.5^n$$
$$y(n + 2) = 27A.3^n + 125B.5^n$$

Eliminating A and B we obtain

$$\begin{vmatrix} y(n) & 1 & 1 \\ y(n+1) & 3 & 5 \\ y(n+2) & 27 & 125 \end{vmatrix} = 0$$

$$\Rightarrow \quad y(n + 2)\,\{5 - 3\} - y(n + 1)\,\{125 - 27\} + y(n)\,\{375 - 135\} = 0$$

$$\Rightarrow \quad 2y(n + 2) - 98y(n + 1) + 240y(n) = 0$$

which is the desired difference equation.

Example 2. Form the difference equation corresponding to the relation.

$$y = bx^2 - ax$$

We have,
$$y = bx^2 - ax \qquad ...(i)$$
$$\Delta y = b\Delta(x^2) - a.\Delta(x) = b\,.\,\{(x + 1)^2 - x^2\} - a\{x + 1 - x\}$$
$$\Rightarrow \quad \Delta y = b(2x - 1) - a \qquad ...(ii)$$
and
$$\Delta^2 y = 2b\{(x + 1) - x\} = 2b$$
$$\Rightarrow \quad b = \frac{1}{2}\,\Delta^2 y \qquad ...(iii)$$

From (ii) $$a = b(2x - 1) - \Delta y = \frac{1}{2} \,.\, \Delta^2 y \,.\, (2x - 1) - \Delta y \qquad ...(iv)$$

Substituting these values of a and b in (i) we obtain

$$y = \frac{x^2}{2} \,.\, \Delta^2 y - \frac{1}{2} \,.\, (2x - 1)\ x\Delta^2 y + x\Delta y$$

$$\Rightarrow \qquad (x^2 - x)\Delta^2 y - 2x\Delta y + 2y = 0$$

$$\Rightarrow \qquad (x^2 - x)\ y(x + 2) - 2x^2 y(x + 1) + (x^2 + x + 2)\ y(x) = 0$$

which is the desired difference equation.

PROBLEMS

1. Write the difference equations $\Delta^3 y_k + \Delta^2 y_k + \Delta y_k + y_k = 0$ in the subscript notation.
2. Find the difference equation for the equations

   (i) $y = A3^x + B(-2)^x$ (ii) $y = A2^n + n3^{n-1}$ (iii) $y = (A + Bn)\ 3^n$
3. Show that $y_k = \frac{1}{2}\ k(k - 1)$ is a solution of the difference equation $y_{k+1} - y_k = k$.

ANSWERS

1. $y_{k+3} - 2y_{k+2} + 2y_{k+1} = 0$
2. (i) $y(x + 2) - y(x + 1) - 6y(x) = 0$ (ii) $y(n + 1) - 2y(n) = (n + 3)\ .\ 3^{n-1}$

   (iii) $y(n + 2) - 6y(n + 1) - 9y(n) = 0$

16.A.4 LINEAR DIFFERENCE EQUATIONS

A difference equation of the form

$$y(n + r) + k_1\, y(n + r - 1) + \ldots\ldots + k_r\, y(n) = g(n) \qquad ...(i)$$

where $k_1, k_2, \ldots\ldots, k_r$ are constants and $g(n)$ is functions of n or constant, is called linear difference equation with constant coefficients. If $g(n) = 0$, the equation is said to be homogeneous, otherwise, it is called non-homogeneous. Clearly equation (i) in homogeneous form can be rewritten as

$$(E^r + k_1\, E^{r-1} + \ldots\ldots + k_r)\ y(n) = 0 \qquad ...(ii)$$

where E is the shift operator such that $E^r.y(n) = y(n + r)$.

If $f(E) = (E^r + k_1\, E^{r-1} + \ldots\ldots + k_r$, then $f(E) = 0$ is called the auxiliary equation and $f(E)$ is called the characteristic function of (ii).

Here the results are similar to differential equation. However, the following results can easily be established.

(a) If the auxiliary equation has n distinct roots $\alpha_1, \alpha_2, \ldots\ldots, \alpha_r$, then the general solution of (ii) is

$$y(n) = c_1\, \alpha_1^{\,n} + c_2\, \alpha_2^{\,n} + \ldots\ldots + c_r\, .\, \alpha_r^{\,n}$$

where $c_1, c_2, \ldots\ldots, c_r$ are arbitrary constants.

(b) If the auxiliary equation has real repeated roots, say α_1 repeated p times, α_2 repeated q times, then the general solution of (ii) is

$$y(n) = (c_1 + c_2 n + \ldots\ldots + c_p n^{p-1})\ \alpha_1^{\,p} + (b_1 + b_2\, n + \ldots\ldots + b_q\, n^{q-1})\ \alpha_2^{\,q}\ .$$

(*c*) If the auxiliary equation has non-repeated complex roots, say two of them be $\alpha_1 = \alpha + i\beta$, $\alpha_2 = \alpha - i\beta$, then the general solution of (*ii*) is

$y(n) = r^n (c_1 \cos n\theta + c_2 \sin n\theta)$ where $r = \sqrt{\alpha^2 + \beta^2}$, $\theta = \tan^{-1}(\beta/\alpha)$ and c_1, c_2 are arbitrary constants.

(*d*) If the auxiliary equation has repeated complex roots, say $\alpha + i\beta$ and $\alpha - i\beta$ both repeated twice then the corresponding two terms of the general solution shall be

$$r^n [(c_1 + c_2 n) \cos n\theta + (c_3 + c_4 n) \sin n\theta]$$

Example 1. Solve the difference equation

$$16y(n + 2) - 8y(n + 1) + y(n) = 0.$$

The given equation can be written as

$$(16E^2 - 8E + 1)\, y(n) = 0$$

The auxiliary equation is $16E^2 - 8E + 1 = 0$

which has two equal roots 1/4, 1/4.

Thus the general solution is given by

$$y(n) = (c_1 + c_2 n)\, (1/4)^n, \quad \text{where } c_1, c_2 \text{ are arbitrary constants.}$$

Example 2. Solve the difference equation

$$y_{n+2} - 4y_{n+1} + 13y_n = 0.$$

The given equation can be written as

$$(E^2 - 4E + 13)y_n = 0$$

The auxiliary equation is $E^2 - 4E + 13 = 0$

which has complex roots $2 + 3i$, $2 - 3i$.

Thus the general solution is given by

$$y_n = r^n (c_1 \cos n\theta + c_2 \sin n\theta)$$

where $r = \sqrt{4 + 9} = \sqrt{13}$ and $\theta = \tan^{-1}(3/2)$.

Example 3. Solve $9y(n + 2) + 9y(n + 1) + 2y(n) = 0$ with $y(0) = 1$ and $y(1) = 1$.

The given equation can be written as

$$(9E^2 + 9E + 2)\, y(n) = 0$$

The auxiliary equation is $9E^2 + 9E + 2 = 0$

whose roots are $-1/3$, $-2/3$.

The general solution is

$$y(n) = c_1 (-1/3)^n + c_2(-2/3)^n$$

Now $y(0) = 1$ gives $c_1 + c_2 = 1$

$y(1) = 1$ gives $-c_1 - 2c_2 = 3$

Solving we obtain, $c_1 = 5$ and $c_2 = -4$.

Hence the particular solution is

$$y(n) = 5\left(-\frac{1}{3}\right)^n - 4\left(-\frac{2}{3}\right)^n .$$

PROBLEMS

Solve the following difference equations :

1. $y(n + 2) + 6y(n + 1) + 25y(n) = 0$.

2. $y_{n+2} - 4y_{n+1} + 4y_n = 0$ with $y_0 = 1$, $y_1 = 3$.

3. $y(n + 4) + y_n = 0$.

4. $y(n + 2) - 2y(n + 1) - 3y(n) = 0$.

5. $y(n + 4) + 12y(n + 2) - 64y(n) = 0$.

6. $9y(n + 2) - 6y(n + 1) + y(n) = 0$ with $y(0) = 1$ and $y(1) = 1$.

7. $y_{n+3} + 5y_{n+2} + 8y_{n+1} + 4y_n = 0$ with $y(0) = 0$, $y(1) = -1$ and $y(2) = 2$.

8. $y_n = y_{n-1} + y_{n-2}$ with $y(1) = 0$, $y(2) = 1$ and $n > 2$.
(Fibonacci difference equation)

ANSWERS

1. $y(n) = (5)^n (c_1 \cos n\theta + c_2 \sin n\theta)$ where $\theta = \tan^{-1}(-4/3)$

2. $y_n = (n + 2)2^{n-1}$

3. $y(n) = c_1 \cos \dfrac{n\pi}{4} + c_2 \sin \dfrac{n\pi}{4} + c_3 \cos \dfrac{3n\pi}{4} + c_4 \sin \dfrac{3n\pi}{4}$.

4. $y(n) = c_1\, 3^n + c_2(-1)^n$

5. $y(n) = c_1\, 2^n + c_2(-2)^n + 4^n \left(c_3 \cos \dfrac{n\pi}{2} + c_4 \sin \dfrac{n\pi}{2}\right)$

6. $y(n) = 3^n \,.\, (1/3)^n$

7. $y(n) = -\dfrac{6}{5} \cdot (-1)^n + \left(\dfrac{6}{5} - \dfrac{1}{5} \cdot n\right) (-2)^n$

8. $y_n = \dfrac{5-\sqrt{5}}{10} \cdot \left(\dfrac{1+\sqrt{5}}{2}\right)^n + \dfrac{5+\sqrt{5}}{10} \left(\dfrac{1-\sqrt{5}}{2}\right)^n$

16.A.5 NON-HOMOGENEOUS LINEAR DIFFERENCE EQUATIONS

Similar to the method of ordinary differential equations, the general solution of a non-homogeneous linear difference equation is found by adding a particular solution called 'Particular Integral' (P.I.) of the non-homogeneous equation to the general solution called 'Complementary Function' (C.F.) of the corresponding homogeneous equation. Thus,

$$\text{general solution} = \text{C.F.} + \text{P.I.}$$

For causal system, the C.F. is referred as natural response and P.I. as forced response.

Consider equation (*i*) of the previous section (16.A.4) as

$$f(E) \,.\, y(n) = g(n) \qquad ...(i)$$

where
$$f(E) = E^r + k_1 E^{r-1} + \ldots\ldots + k_r$$

Then the particular integral is given by

$$\text{P.I.} = \frac{1}{f(E)} \,.\, g(n) \qquad ...(ii)$$

It can be evaluated by the method of operators. Various cases are given below :

Case I. When $g(n) = a^n$, a is a constant

$$\text{P.I.} = \frac{1}{f(E)} \,.\, a^n = \frac{1}{f(a)} \,.\, a^n \text{, provided } f(a) \neq 0$$

If $f(a) = 0$ then for the equation

(a) $(E - a)\, y(n) = a^n$, P.I. $= \dfrac{1}{(E-a)} \cdot a^n = n \cdot a^{n-1}$

(b) $(E - a)^2\, y(n) = a^n$, P.I. $= \dfrac{n(n-1)}{2!} \cdot a^{n-2}$

(c) $(E - a)^3\, y(n) = a^n$, P.I. $= \dfrac{n(n-1)(n-2)}{3!} \cdot a^{n-3}$

and so on.

Example 1. Solve $(E^2 - 5E + 6)\, y(n) = 4^n$.

Auxiliary equation, $E^2 - 5E + 6 = 0$

Its roots are 2 and 3.

$$\therefore \qquad \text{C.F.} = c_1\, 2^n + c_2\, 3^n$$

$$\text{P.I.} = \frac{1}{E^2 - 5E + 6} \cdot 4^n \qquad \text{(put } E = 4\text{)}$$

$$= \frac{1}{16 - 20 + 6} \cdot 4^n = \frac{1}{2} \cdot 4^n$$

Hence the general solution is given by

$$y(n) = c_1 \cdot 2^n + c_2 \cdot 3^n + \frac{1}{2} \cdot 4^n.$$

Example 2. Solve $(E^2 - 4E + 4)\, y(n) = 2^n$.

Auxiliary equation $E^2 - 4E + 4 = 0$

Its roots are 2 and 2

$$\therefore \qquad \text{C.F.} = (c_1 + c_2 n) \cdot 2^n$$

$$\text{P.I.} = \frac{1}{E^2 - 4E + 4} \cdot 2^n = \frac{1}{(E-2)^2} \cdot 2^n \qquad \text{(case fails)}$$

$$= \frac{n(n-1)}{2!} \cdot 2^{n-2} \qquad \text{[using (b)]}$$

$$= \frac{n(n-1)}{8} \cdot 2^n .$$

Case II. 1. When g(n) = sin αn

$$\text{P.I.} = \frac{1}{f(E)} \cdot \sin \alpha n = \frac{1}{f(E)} \cdot \left[\frac{e^{i\alpha n} - e^{-i\alpha n}}{2i}\right]$$

$$= \frac{1}{2i}\left[\frac{1}{f(E)} \cdot e^{i\alpha n} - \frac{1}{f(E)} \cdot e^{-i\alpha n}\right]$$

$$= \frac{1}{2i}\left[\frac{1}{f(E)} \cdot a^n - \frac{1}{f(E)} \cdot b^n\right] \text{ where } a = e^{i\alpha} \text{ and } b = e^{-i\alpha}$$

Now it is similar to case I.

2. When g(n) = cos αn

$$\text{P.I.} = \frac{1}{f(E)} \cdot \cos \alpha n = \frac{1}{f(E)}\left[\frac{e^{i\alpha n} + e^{-i\alpha n}}{2}\right]$$

$$= \frac{1}{2} \cdot \left[\frac{1}{f(E)} \cdot e^{i\alpha n} + \frac{1}{f(E)} \cdot e^{-i\alpha n}\right]$$

$$= \frac{1}{2}\left[\frac{1}{f(E)} \cdot a^n + \frac{1}{f(E)} \cdot b^n\right] \text{, as before.}$$

Now it is similar to case I.

Example 3. Solve $2y(n + 2) + 3y(n + 1) + y(n) = \cos 2n$.

The given equation can be written as

$$(2E^2 + 3E + 1)\, y(n) = \cos 2n$$

∴ Auxiliary Equation is $2E^2 + 3E + 1 = 0$

whose roots are – 1 and – 1/2

∴ $$\text{C.F.} = c_1 (-1)^n + c_2 (-1/2)^n$$

$$\text{P.I.} = \frac{1}{2E^2 + 3E + 1} \cdot \cos 2n = \frac{1}{2E^2 + 3E + 1} \cdot \left(\frac{e^{i2n} + e^{-i2n}}{2}\right)$$

$$= \frac{1}{2} \cdot \left[\frac{1}{2E^2 + 3E + 1} \cdot e^{i2n} + \frac{1}{2E^2 + 3E + 1} \cdot e^{-i2n}\right]$$

$$= \frac{1}{2} \cdot \left[\frac{1}{2e^{i4} + 3e^{i2} + 1} \cdot e^{i2n} + \frac{1}{2e^{-i4} + 3e^{-i2} + 1} \cdot e^{-i2n}\right]$$

$$= \frac{1}{2} \cdot \frac{(2e^{-i4} + 3e^{-i2} + 1)\, e^{i2n} + (2e^{i4} + 3e^{i2} + 1) \cdot e^{-i2n}}{(2e^{i4} + 3e^{i2} + 1)(2e^{-i4} + 3e^{-i2} + 1)}$$

$$= \frac{1}{2} \cdot \frac{2\left(e^{-(4-2n)i} + e^{(4-2n)i}\right) + 3\left(e^{-(2-2n)i} + e^{(2-2n)i}\right) + \left(e^{2ni} + e^{-2ni}\right)}{2(e^{i4} + e^{-i4}) + 9(e^{i2} + e^{-i2}) + 12}$$

$$= \frac{1}{2} \cdot \frac{2\cos(4-2n) + 3\cos(2-2n) + \cos 2n}{2\cos 4 + 9\cos 2 + 12}$$

Case III. When g(n) = n^p

$$\text{P.I.} = \frac{1}{f(E)} \cdot n^p = \frac{1}{f(1+\Delta)} \cdot n^p$$

The above P.I. is evaluated in two steps.

1. Using Binomial theorem, expand $[f(1 + \Delta)]^{-1}$ upto the term Δ^p,

2. Express n^p in the factorial form and operate the expansion terms on it.

Example 4. Solve $y(n + 2) - y(n + 1) - 2y(n) = n^2$

The given equation can be written as

$$(E^2 - E - 2)\, y(n) = n^2.$$

Auxiliary equation, $E^2 - E - 2 = 0$

whose roots are – 1 and 2

$$\therefore \qquad \text{C.F.} = c_1 (-1)^n + c_2 . 2^n$$

$$\therefore \qquad \text{P.I.} = \frac{1}{E^2 - E - 2} . n^2 = \frac{1}{(1+\Delta)^2 - (1+\Delta) - 2} . n^2$$

$$= \frac{1}{\Delta^2 + \Delta - 2} . n^2 = -\frac{1}{2}\left[1 - \left(\frac{\Delta^2 + \Delta}{2}\right)\right]^{-1} . n^2$$

$$= -\frac{1}{2}\left[1 + \left(\frac{\Delta^2 + \Delta}{2}\right) + \left(\frac{\Delta^2 + \Delta}{2}\right)^2 + \ldots\ldots\right] . n^2$$

$$= -\frac{1}{2}\left[1 + \frac{\Delta^2}{2} + \frac{\Delta}{2} + \frac{\Delta^2}{4} + \ldots\ldots\right] n^2 = -\frac{1}{2}\left[1 + \frac{\Delta}{2} + \frac{3}{4}\Delta^2 + \ldots\ldots\right] n^2$$

$$= -\frac{1}{2}\left[1 + \frac{\Delta}{2} + \frac{3}{4}\Delta^2 + \ldots\ldots\right]\left(n^{(2)} + n^{(1)}\right)$$

$$= -\frac{1}{2}\left[\left\{n^{(2)} + n^{(1)}\right\} + \frac{1}{2}\left\{2n^{(1)} + 1\right\} + \frac{3}{4}\left\{2.1.n^{(0)}\right\}\right]$$

$$= -\frac{1}{2}(n^2 + n + 2)$$

Hence the general solution is given by

$$y(n) = c_1(-1)^n + c_2 2^n - \frac{1}{2}(n^2 + n + 2)$$

Case IV. When $g(n) = a^n . G(n)$, $G(n)$ being a polynomial of degree n and a is a constant.

$$\text{P.I.} = \frac{1}{f(E)} . a^n G(n) = a^n . \frac{1}{f(aE)} . G(n)$$

which is evaluated using case III.

Example 5. Solve $y(n + 2) + y(n + 1) + y(n) = n . 2^n$.

The given equation can be written as

$$(E^2 + E + 1)\, y(n) = n . 2^n.$$

Auxiliary equation is $E^2 + E + 1 = 0$

whose roots are $\dfrac{-1 \pm \sqrt{3}i}{2}$.

$$\therefore \qquad \text{C.F.} = c_1 \cos n\theta + c_2 \sin n\theta \text{ where } \theta = \tan^{-1}\left(-\sqrt{3}\right)$$

$$\text{P.I.} = \frac{1}{E^2 + E + 1} . n2^n = 2^n . \frac{1}{(2E)^2 + 2E + 1} . n$$

$$= 2^n . \frac{1}{4E^2 + 2E + 1} . n = 2^n . \frac{1}{4(1+\Delta)^2 + 2(1+\Delta) + 1} . n$$

$$= 2^n \cdot \frac{1}{7+10\Delta+4\Delta^2} \cdot n = \frac{2^n}{7}\left[1+\frac{10\Delta+4\Delta^2}{7}\right]^{-1} \cdot n$$

$$= \frac{2^n}{7}\left[1-\frac{10\Delta+4\Delta^2}{7}\right]n = \frac{2^n}{7}\left[n-\frac{10}{7}\right]$$

Hence the general solution is given by

$$y(n) = (5/2)^{n/2}\,(c_1 \cos n\theta + c_2 \sin n\theta) + \frac{2^n}{7}\left(n-\frac{10}{7}\right) \text{ where } \theta = \tan^{-1}(-3).$$

PROBLEMS

Solve the following difference equations :

1. $(E^2 - 5E + 6)\, y(n) = n + 2^n$

2. $y(n+2) - 4y(n+1) + 3y(n) = 5^n$

3. $y_{n+2} - 5y_{n+1} + 6y_n = 36$

4. $y_{n+1} - 2y_n = n + 1$

5. $y(n+2) + 4y(n+1) + 4 = n$

6. $(E^2 - 2E + 5)\, y(n) = 3.2^n - 5.6^n$

7. $y(n+3) + y(n+2) - 8y(n+1) - 12y(n) = 2n^2 + 5$

8. $y(n+2) - y(n+1) - 2y(n) = n^2$

9. $y(n+2) - 4y(n+1) + 4y(n) = 3n + 2^n$

10. $y_{n+2} - 2 \cos a.y_{n+1} + y_n = \cos an$, a is a constant

11. $y(n+2) - 7y(n+1) + 12y(n) = \cos n$

12. $y(n+2) - 6y(n+1) + 8y(n) = 3n^2 + 2$

13. $y_{n+2} - 6y_{n+1} + 8y_n = 3n^2 + 2 - 5.3^n$

14. $y_{n+3} - 5y_{n+2} + 8y_{n+1} - 4y_n = n.2^n$

15. $(E^2 - 5E + 6)\, y(n) = 4^n(n^2 - n + 5)$

16. $y(n+2) - 4y(n+1) + 4y(n) = n^2 \cdot 2^n$

ANSWERS

1. $y(n) = c_1 2^n + c_2 3^n + \frac{1}{4}(2n+2) - n2^{n-1}$

2. $y(n) = c_1 + c_2.3^n + 5^n/8$

3. $y_n = c_1 \cdot 2^n + c_2 .3^n + 18$

4. $y(n) = c_1.2^n - (n+2)$

5. $y(n) = c_1(-4)^n + \frac{5n-6}{25}$

6. $y(n) = 5^{n/2} \cdot (c_1 \cos n\theta + c_2 \sin n\theta) + \frac{3}{5} \cdot 2^n - \frac{5}{29} \cdot 6^n$, where $\theta = \tan^{-1}(2)$

7. $y(n) = c_1.3^n + (c_2 + c_3 n)(-2)^n - \frac{n^2}{9} + \frac{n}{27} - \frac{17}{54}$

8. $y(n) = c_1(-1)^n + c_2(2)^n - \frac{1}{2}(n^2 + n + 2)$

9. $y(n) = (c_1 + c_2 n)2^n + 6 + 3n + \frac{1}{8} n^2 \cdot 2n$

10. $y_{(n)} = c_1 \cos an + c_2 \sin an + \frac{n \sin (n-1)\, a}{2 \sin a}$

11. $y(n) = \frac{1}{170}[18 \cos n - 7 \sin n]$

12. $y(n) = c_1.2^n + c_2 4^n + n^2 + \frac{8}{3} n + \frac{44}{9}$

13. $y_n = c_1 2^n + c_2 4^n + n^2 + \frac{8}{3} n + \frac{44}{9} + 5.3^n$

14. $y_n = c_1 + (c_2 + c_3 n).2^n + \frac{n}{24}(n-1)(n-2) \cdot 2^n$

15. $y(n) = c_1.2^n + c_2.3^n + 4^n(n^2 - 13n + 61)/2$

16. $y(n) = (c_1 + c_2 n)2^n + \frac{2^n}{48}(n^4 - 4n^3 + 5n^2 - 2)$

16.A.6 SIMULTANEOUS DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

Analogue to the method for solving simultaneous ordinary differential equations with constant coefficients the simultaneous difference equations with constant coefficients are solved. The method is illustrated with the following example :

Example. Solve $y(n+1) - y(n) + 2x(n+1) = 0$ and

$$x(n+1) - x(n) - 2y(n) = 2^n.$$

Given equations in symbolic form are

$$(E-1)\, y(n) + 2E\, x(n) = 0 \qquad \text{...}(i)$$

$$-2y(n) + (E-1)\, x(n) = 2^n \qquad \text{...}(ii)$$

Multiply (i) by $(E-1)$, (ii) by $2E$ and subtracting we obtain

$$\Rightarrow \quad (E+1)^2\, y(n) = 2E(2^n) = -2^{n+2} \qquad \text{...}(iii)$$

Auxiliary equation is $(E+1)^2 = 0$. Its roots are $-1, -1$

$$\therefore \quad \text{C.F.} = (c_1 + c_2 n)(-1)^n.$$

$$\text{P.I.} = \frac{-1}{(E+1)^2} \cdot 2^{n+2} = \frac{-1}{9} \cdot 2^{n+2}$$

Therefore

$$y(n) = (c_1 + c_2 n)(-1)^n - \frac{1}{9} \cdot 2^{n+2} \qquad \text{...}(iv)$$

From (ii), $\quad -2(c_1 + c_2\, n)(-1)^n + \dfrac{2^{n+3}}{9} + (E-1)\, x(n) = 2^n$

$$\Rightarrow \quad (E-1)\, x(n) = 2^n + 2(c_1 + c_2\, n)(-1)^n - \frac{2^{n+3}}{9}.$$

$$\Rightarrow \quad x(n) = \left[c_1 + c_2\left(n - \frac{1}{2}\right)\right](-1)^n + \frac{2^n}{q} \qquad \text{...}(v)$$

$\therefore$ Eqn. (iv) and (v) give the general solution.

PROBLEMS

Solve :

1. $y(n+1) + x(n) - 3y(n) = n,$

$3y(n) + x(n+1) - 5x(n) = 4^n$. with $y(1) = 2$ and $x(1) = 0$.

2. $y(n+1) - 3y(n) - 2x(n) + n = 0$

$x(n+1) - 2x(n) - y(n) - n = 0$, with $y(0) = 0$, $x(0) = 3$.

ANSWERS

1. $y(n) = \dfrac{133}{100} \cdot 2^n - \dfrac{1}{61} \cdot 6^n + 4^{n-1} - \dfrac{4}{5} n - \dfrac{19}{25}$

$x(n) = \dfrac{133}{100} \cdot 2^n + \dfrac{1}{20} \cdot 6^n - 4^{n-1} - \dfrac{3}{5} n - \dfrac{35}{25}$

2. $y(n) = 2 \,.\, 4^n - 2 - \frac{1}{2}\, n(n-1),$

$x(n) = 4^n + 2 + \frac{1}{2}\, n \,.\, (n+1)$

PART B : Z-TRANSFORMS

16.B.1 Introduction, 16.B.2 Some Standard *z*-Transforms, 16.B.3 Properties of *z*-Transforms, 16.B.4 Initial Value and Final Value Theorems, 16.B.5 Inverse *z*-Transforms, 16.B.6 Inverse *z*-Transforms by Power Series Method, 16.B.7 Inverse z-Transform by Partial Fractions Method, 16.B.8 Inverse *z*-Transform by Integral Method, 16.B.9 Application to Difference Equations.

16.B.1 INTRODUCTION

The *z*-transform is named as a letter of the alphabet rather than a famous mathematician. A method for solving linear constant coefficient difference equations by Laplace transforms was introduced to graduate engineering students by Gardner and Barnes in the early 1940s. They applied their procedure which was based on jump functions to transmission lines, and applications involving Bessel functions. This approach is quite complicated and in a separate attempt to simplify matters, a transform of a sampled signal or sequence was defined in 1947 by W. Hurewicz as which was later denoted in 1952 as a ''Z-transform'' by a sampled-data control group at Colombia University.

In any case, it is presumably not an accident that the *z*-transform was invented at about the same time as digital computers. However *z*-transform can be viewed as a mathematical operation that takes a set of points (which represents time sequence in discrete-time systems) and transforms them into a set of complex numbers.

Definitions : Given the sequence $x(n)$, the *z*-transform is defined as

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) \,.\, z^{-n} \qquad ...(i)$$

where z is taken to be a complex variable.

This expression is sometimes referred to as the two-sided *z*-transform since the summation is over all the integers.

If $x(n) = 0$ for $n < 0$, then the *z*-transform is

$$Z\{x(n)\} = X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n} \qquad ...(ii)$$

This expression is sometimes referred to as the one-sided *z*-transform. In this text the discussions will be confined to one-sided *z*-transform, so we call it simply as *z*-transform.

Clearly *z*-transform exists only for the values of *z* for which the series of eqn. (*i*) or (*ii*) converges. The series of Eqn. (*ii*) is said to converge absolutely when the series of real numbers

$$\sum_{n=0}^{\infty} |\, x(n)\, z^{-n} \,| \qquad ...(iii)$$

converges. It is also well known that a series converges absolutely also converges.

Let us apply ratio test to Eqn. (*ii*)

$$\operatorname*{Lt}_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = \operatorname*{Lt}_{n\to\infty}\left|\frac{x(n+1)z^{-n-1}}{x(n)z^{-n}}\right| = \operatorname*{Lt}_{n\to\infty}\left|\frac{x(n+1)}{x(n)}\right|.|z|^{-1}$$

The convergence criterion requires that

$$|z|^{-1}\operatorname*{Lt}_{n\to\infty}\left|\frac{x(n+1)}{x(n)}\right| < 1$$

and so we obtain the region of convergence as

$$|z| > \operatorname*{Lt}_{n\to\infty}\left|\frac{x(n+1)}{x(n)}\right| = R.\text{(say)}$$

Thus the series in Eqn. (*ii*) converges out side the circle with centre at origin and radius R.

16.B.2 SOME STANDARD Z-TRANSFORMS

(*a*) Discrete Unit impulse defined as

$$\delta(n) = x(n) = 1 \quad , \; n = 0$$
$$= 0 \quad , \; \text{otherwise}$$

Here $X(z) = 1$ since only the first term exists in Eqn. (*ii*)

(*b*) Discrete Unit step defined as

$$u(n) = x(n) = 1 \quad , \; n \ge 0$$
$$= 0 \quad , \; \text{otherwise}$$

Here $$X(z) = \sum_{n=0}^{\infty} . z^{-n} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

(*c*) If $x(n) = a^n$ then

$$X(z) = \sum_{n=0}^{\infty} a^n . z^{-n} = \sum_{n=0}^{\infty} (a/z)^n = \frac{1}{1-a/z} = \frac{z}{z-a} \;, \; |a/z| < 1$$

(*d*) If $x(n) = n^p$ ($n \ge 0$ and $p > 0$) then

$$X(z) = \sum_{n=0}^{\infty} n^p . z^{-n} = z\sum_{n=0}^{\infty} n^{p-1}(nz^{-n-1})$$

$$= -z . \frac{d}{dz}\left(\sum_{n=0}^{\infty} n^{p-1} z^{-n}\right) = -z . \frac{d}{dz}\{Z(n^{p-1})\}$$

If $p = 1$, then $Z(n) = \dfrac{z}{(z-1)^2}$ ($x(n) = n$ is called discrete unit ramp)

If $p = 2$, $\quad Z(n^2) = \dfrac{z^2+z}{(z-1)^3}$

If $p = 3$, $\quad Z(n^3) = \dfrac{z^3+4z^2+z}{(z-1)^4}$

16.B.3 PROPERTIES OF Z-TRANSFORMS

(*a*) **Linearity Property :**

If $x_1(n)$ and $x_2(n)$ be two discrete functions then

$$Z\{\alpha\, x_1(n) + \beta\, x_2(n)\} = \alpha\, Z\{x_1(n)\} + \beta\, Z\{x_2(n)\}$$

where α and β are scalars.

(*b*) **Shift Property :**

1. If $Z\{x(n)\} = X(z)$ and m is a positive integer then

$$Z\{x(n-m)\} = z^{-m} \,.\, X(z)$$

Here the discrete function $x(n)$ is shifted to the right. This property is useful to solve difference equations.

Proof. By definition

$$Z\{x(n-m)\} = \sum_{n=0}^{\infty} x(n-m) \,.\, z^{-n}$$

$$= x(0)\, z^{-m} + x(1)\, z^{-(m+1)} + \ldots\ldots \infty$$

(Since $x(n-m) = 0$ for $n = 0, 1, \ldots\ldots (m-1)$).

$$= z^{-m} \sum_{n=0}^{\infty} x(n)\, z^{-n} = z^{-m} \,.\, X(z)$$

2. If $Z\{x(n)\} = X(z)$ and m is a positive integer then

$$Z\{x(n+m)\} = z^{m}[Z\,\{x(n)\} - x(0) - x(1)\, z^{-1} \ldots\ldots - x(m-1)\, z^{-(m-1)}]$$

Here the discrete function $x(n)$ is shifted to the left.

Proof. By definition

$$z\{x(n+m)\} = \sum_{n=0}^{\infty} x(n+m)\, z^{-n} = z^{m} \,.\, \sum_{n=0}^{\infty} x(n+m)\, z^{-(n+m)}$$

$$= z^{m} \,.\left[\sum_{n=0}^{\infty} x(n) \,.\, z^{-m} - \sum_{n=0}^{m-1} x(n) \,.\, z^{-n}\right]$$

$$= z^{m}\, [Z\,\{x(n)\} - x(0) - x(1)\, z^{-1} \ldots\ldots - x(m-1)\, z^{-(m-1)}]$$

(*c*) **Damping Rule :**

If $Z\,\{x(n)\} = X(z)$ then $Z\{a^{-n} \,.\, x(n)\} = X(az)$

Proof. By definition

$$Z\,\{a^{-n}\, x(n)\} = \sum_{n=0}^{\infty} a^{-n}\, x(n) \,.\, z^{-n} = \sum_{n=0}^{\infty} x(n) \,.\, (az)^{-n}$$

$$= X(az)$$

Similarly, $Z\{a^{n}\, x(n)\} = X(z/a)$

(*d*) **Convolution :**

If $Z\{x(n)\} = X(z)$ and $Z\{y(n)\} = Y(z)$ then

$$Z\{x(n) * y(n)\} = X(z) \,.\, Y(z)$$

Proof. By definition

$$Z\{x(n) * y(n)\} = \sum_{n=0}^{\infty} \left[\sum_{m=0}^{n} x(n-m)\, y(m) \right] z^{-n}$$

$$= \left[\sum_{n=0}^{\infty} x(n)\, z^{-n} \right] . \left[\sum_{n=0}^{\infty} y(n)\, z^{-n} \right] = X(z) \,.\, Y(z)$$

(Using Cauchy product from Calculus)

We see that the z-transform of the convolution of two discrete functions is the product of the z-transforms of the discrete functions.

(*e*) **Differentiation**

It is known that a convergent power series can be differentiated termwise within its region of convergence. Since $X(z)$ is a function of z^{-1}, it is generally easier to differentiate with respect to z^{-1}.

Since $$X(z) = \sum_{n=0}^{\infty} x(n) \,.\, z^{-n}$$

$\therefore$ $$\frac{dX(z)}{dz^{-1}} = \sum_{n=0}^{\infty} x(n) \,.\, n \,.\, (z^{-1})^{n-1}$$

$\Rightarrow$ $$z^{-1} \,.\, \frac{dX(z)}{dz^{-1}} = \sum_{n=0}^{\infty} nx(n) \,.\, z^{-n} \quad \Rightarrow \quad Z\,\{n\, x(n)\} = z^{-1} \,.\, \frac{dX(z)}{dz^{-1}}$$

Similarly, $Z\{n(n-1)x\,(n)\} = z^2 \dfrac{d^2 X(z)}{d(z^{-1})^2}$.

16.B.4 INITIAL VALUE AND FINAL VALUE THEOREMS

1. Initial Value Theorem :

If $$Z\{x(n)\} = X(z), \text{ then } x(0) = \operatorname*{Lt}_{z\to\infty} X(z),$$

$$x(1) = \operatorname*{Lt}_{z\to\infty} \{z\,[X(z) - x(0)]\},$$

$$x(2) = \operatorname*{Lt}_{z\to\infty} [z^2\,[X(z) - x(0) - x(1)\, z^{-1}]\}$$

and so on.

Proof is obvious.

2. Final Value Theorem :

If $Z\{x(n)\} = X(z)$, then

$$\operatorname*{Lt}_{n\to\infty} x(n) = \operatorname*{Lt}_{z\to 1} (z-1) \,.\, X(z)$$

Proof. Here $Z\{x(n+1) - x(n)\} = \displaystyle\sum_{n=0}^{\infty} \{x(n+1) - x(n)\}\, z^{-n}$

$\Rightarrow$ $$Z\{x(n+1)\} - Z\{x(n)\} = \sum_{n=0}^{\infty} \{x(n+1) - x(n)\}\, z^{-n}$$