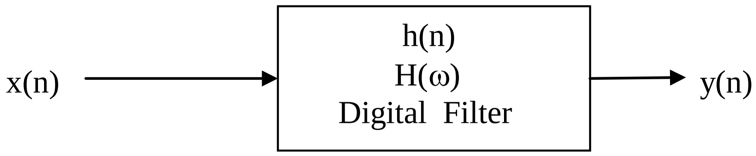
DIGITAL FILTER DESIGN

Introduction:



A digital filter is in fact a linear causal DSP system with impulse response h(n) and frequency response $H(\omega)= |H(\omega)| \angle \phi(\omega)$. Where the shape of $|H(\omega)|$ gives the type of the filter (lowpass, highpass,…..)(amplitude characteristics), and $\phi(\omega)$ gives its phase characteristics . Depending on the type of h(n), these filters are also classified into FIR and IIR. The properties and design procedures for these two classes are different. In this course, only FIR filter design is given for its simplicity.

FIR Digital Filter Design:

FIR filters are always stable having no feedback and with linear phase characteristics, i.e. $\phi(\omega) = -k\,\omega$ (k is a constant and the –sign indicates the phase lagging of the filter). Here ω is called a digital frequency (in rad) since it is normalized to the sampling frequency f_s of the A/D converter, i.e.:

$\omega = 2\pi f/f_s$ where f is the actual analogue frequency(in Hz) of the continuous signal x(t). This linear phase of the FIR filter is always preferred to avoid phase distortion at the output signal y(n).

Linear phase condition:

The FIR filter with impulse response h(n)={h(0),h(1),h(2),….h(N-1)} with N elements has a linear phase if these elements are symmetric about a midpoint α=(N-1)/2, i.e. h(n)=h(N-1-n), where n=0,1,2,…N-1. This midpoint may be an integer if N odd or a fraction if N is even:

For N even, then: if $H(\omega) = \sum_{n=0}^{N-1} h(n)e^{-j\omega n}$ splitting the summation then:

$$H(\omega) = \sum_{n=0}^{\frac{N}{2}-1} h(n)e^{-j\omega n} + \sum_{n=\frac{N}{2}}^{N-1} h(n)e^{-j\omega n}$$

Changing the variable of the 2nd sum and if m=N-1-n, then n=N-1-m:

$$H(\omega)=\sum_{n=0}^{\frac{N}{2}-1} h(n)e^{-j\omega n} + \sum_{m=\frac{N}{2}-1}^{0} h(N-1-m)e^{-j\omega(N-1-m)}$$

And since h(n)=h(N-1-n), then:

$$H(\omega)=\sum_{n=0}^{\frac{N}{2}-1} h(n)e^{-j\omega n} + \sum_{n=0}^{\frac{N}{2}-1} h(n)e^{-j\omega(N-1-n)} = \sum_{n=0}^{\frac{N}{2}-1} h(n)[e^{-j\omega n} + e^{-j\omega(N-1-n)}]$$

$$H(\omega)=e^{-j\omega(\frac{N-1}{2})}\sum_{n=0}^{\frac{N}{2}-1} h(n)[e^{-j\omega(n-\frac{N-1}{2})} + e^{j\omega(n-\frac{N-1}{2})}] = e^{-j\omega\alpha}\sum_{n=0}^{\frac{N}{2}-1} 2h(n)\cos(\omega(n-\alpha))$$

where α=(N-1)/2 (the midpoint), hence:

ϕ(ω) = -α ω (linear phase) and:

$$|H(\omega)| = \sum_{n=0}^{\frac{N}{2}-1} 2h(n)\cos(\omega(n-\alpha))$$ (sum is real)

<u>For N odd</u> and since α is an integer, then the center term for n=α is summed separately. With the same sub. m=N-1-n, then:

$$H(\omega)=h(\alpha)e^{-j\omega\alpha} + \sum_{n=0}^{\alpha-1} h(n)e^{-j\omega n} + h(n)e^{-j\omega(N-1-n)}$$

$$H(\omega)=h(\alpha)e^{-j\omega\alpha} + e^{-j\omega\alpha}\sum_{n=0}^{\alpha-1} h(n)[e^{-j\omega(n-\alpha)} + e^{j\omega(n-\alpha)}]$$

$$H(\omega)=e^{-j\omega\alpha}\{\quad h(\alpha)+\sum_{n=0}^{\alpha-1} 2h(n)\cos(\omega(n-\alpha))\quad\}$$

Hence: ϕ(ω) = -α ω (linear phase) and:

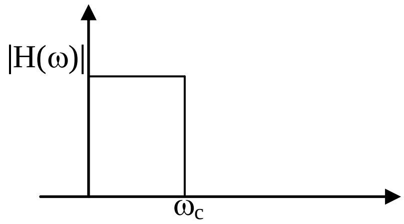
$$|H(\omega)| = h(\alpha)+\sum_{n=0}^{\alpha-1} 2h(n)\cos(\omega(n-\alpha))$$ (sum is real)

It should be noted, however, that N odd is mostly used since the symmetry will be around a real midpoint since α is integer.

Design procedure:

After studying linear phase condition and deriving the frequency response $H(\omega)$ of the filter, we discuss how to find $h(n)$ as symmetric around α taking N as an odd number.

Theoretically, to have ideal lowpass filter as shown: then, we must use infinite number of elements (N is very large), but due to the finite length of N, there will be a roll of and sidelobes at stopband region.



ω

To reduce these sidelobes in stopband, we use what is called **windows** .

Several types of windows are used, but the most commonly used are listed below:

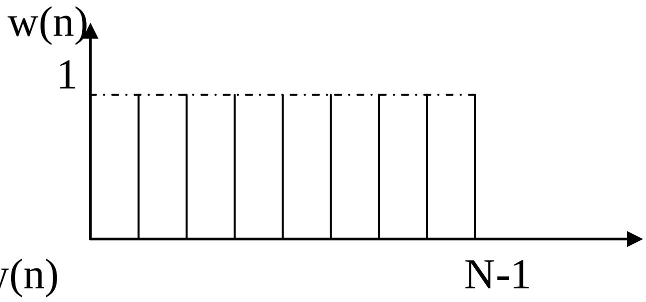
Window type	K-factor	Stopband ripple(Sidelobe Level SLL)
rectangular	2	-21dB
Bartlet	4	-25dB
Hanning	4	-44dB
Hamming	4	-53dB
Blackman	6	-74dB

Where the K-factor is a parameter used in finding N.

The definition of each window is given below:

1- Rectangular:

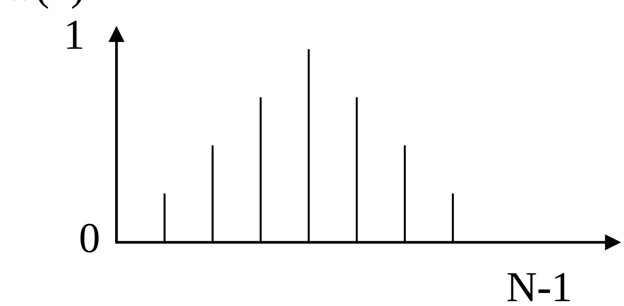
$$w(n)=\begin{cases} 1 & N-1 \geq n \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$



$\alpha=(N-1)/2$

2-Bartlet(triangular):

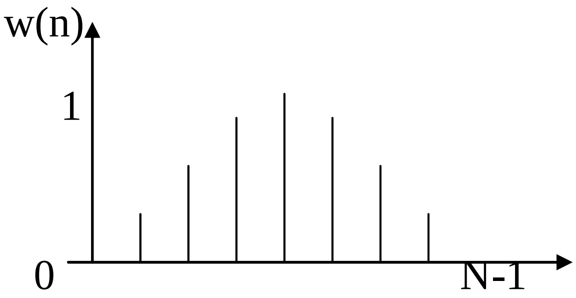
$$w(n)=\begin{cases} n/\alpha & \alpha \geq n \geq 0 \\ 2-(n/\alpha) & N-1 \geq n > \alpha \\ 0 & \text{elsewhere} \end{cases}$$



3-<u>Hanning</u>:

$$w(n) = 0.5 - 0.5\cos\left(\frac{2\pi n}{N-1}\right) \quad N-1 \geq n \geq 0$$

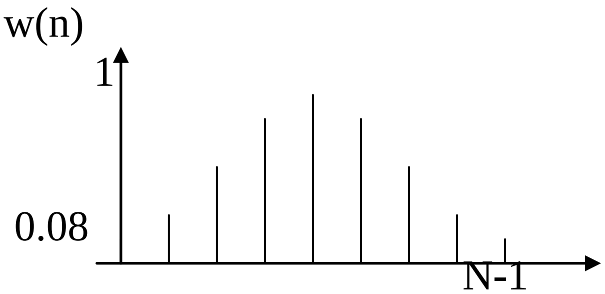
0 elsewhere



4-<u>Hamming</u>:

$$w(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right) \quad N-1 \geq n \geq 0$$

0 elsewhere

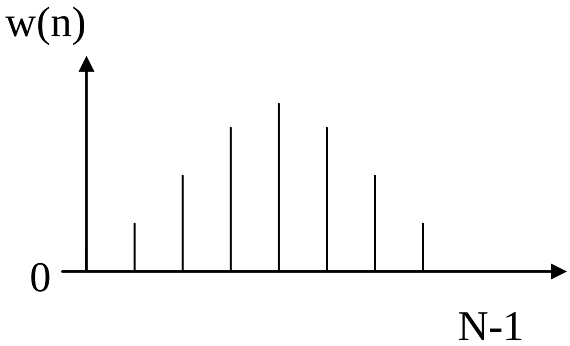


5-<u>Blackman</u>:

$$w(n) = 0.42 - 0.5\cos\left(\frac{2\pi n}{N-1}\right) + 0.08\cos\left(\frac{4\pi n}{N-1}\right)$$

For $N-1 \geq n \geq 0$

0 elsewhere



Now to find the overall impulse response h(n) with windowing, then:

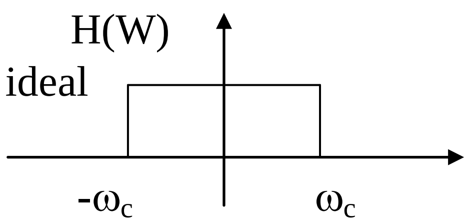
$h(n) = h_d(n)\, w(n)$

where $h_d(n)$ depends on filter type (LFP,HPF,BPF,BSF..) and w(n) is usually chosen according to the required sidelobe level (SLL)at stopband .

To find $h_d(n)$, then:

1- <u>For LPF with cutoff at $\omega = \omega_c$</u>:

Since even symmetry in ω, then:

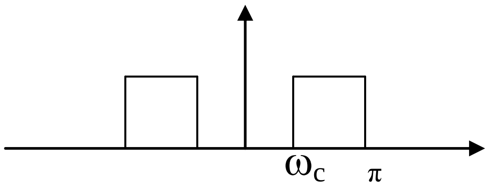


$$h_d(n) = \frac{1}{\pi}\int_0^{w_c} \cos(\omega(n-\alpha))\, d\omega$$

$$h_d(n) = \frac{\sin\{\omega_c(n-\alpha)\}}{\pi(n-\alpha)} \quad \text{For } N-1 \geq n \geq 0,\ n \neq \alpha$$

And $h_d(\alpha) = \omega_c/\pi$

2- <u>For HPF with cutoff at ω =ω_c</u>:



$$h_d(n) = \frac{1}{\pi}\int_{\omega_c}^{\pi} \cos(\omega(n-\alpha))d\omega$$

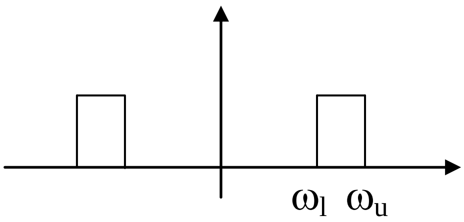
$$h_d(n) = -\frac{\sin\{\omega_c(n-\alpha)\}}{\pi(n-\alpha)}$$ For $N-1 \geq n \geq 0,\ n \neq \alpha$

And $h_d(\alpha)=1-(\omega_c/\pi)$

(note: the maximum used digital frequency is π
Since $f_s(min)=2\,f$ according to Nyquist rate)

3- <u>For HPF with lower and upper cutoff frequencies ω_ℓ and ω_u</u>:

$$h_d(n) = \frac{1}{\pi}\int_{\omega_\ell}^{\omega_u} \cos(\omega(n-\alpha))d\omega$$



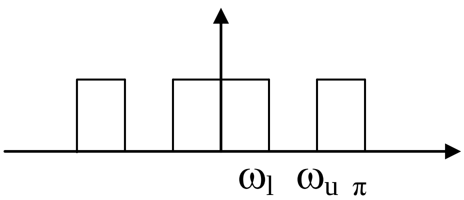
$$h_d(n) = \frac{\sin\{\omega_u(n-\alpha)\} - \sin\{\omega_\ell(n-\alpha)\}}{\pi(n-\alpha)}$$

for $N-1 \geq n \geq 0,\ n \neq \alpha$

And $h_d(\alpha) = (\omega_u - \omega_\ell)/\pi$

4- <u>For BSF with lower and upper cutoff frequencies ω_ℓ and ω_u</u>:

$$h_d(n) = \frac{1}{\pi}\int_{\omega_\ell}^{\omega_u} \cos(\omega(n-\alpha))d\omega$$



$$h_d(n) = \frac{\sin\{\omega_\ell(n-\alpha)\} - \sin\{\omega_u(n-\alpha)\}}{\pi(n-\alpha)}$$

for $N-1 \geq n \geq 0,\ n \neq \alpha$

And $h_d(\alpha) = (\pi - \omega_u + \omega_\ell)/\pi$

In general, for all above types of filters with N odd, then:
$\phi(\omega) = -\alpha\,\omega$ (linear phase) and:

$$|H(\omega)| = h(\alpha) + \sum_{n=0}^{\alpha-1} 2h(n)\cos(\omega(n-\alpha))$$

<u>Next, we discuss how to find N</u>: usually, N gives filter roll of (how fast is the transition from passband to stopband or how is the obtained response is closed to ideal response).

In general: greatest odd integer

$$N = \frac{2\pi\ (K \quad factor)}{(\omega_2 - \omega_1)}$$

where k_1 and k_2 are gain values at ω_1 and ω_2 and k_1 is the -3dB level and k_2 is usually taken as the maximum SLL.

<u>Ex</u>: Design and realize a linear phase digital lowpass filter (LPF) having 3dB cutoff frequency of 7.5KHz. and stopband attenuation of at least 40dB at 35KHz. Find the difference equation and the frequency response of this filter. Use f_s = 100KHz.

<u>Solution</u>:

This is FIR filter since linear phase is required.

Since k_2= -40dB, this gives maxi SLL from which type of window is chosen according to to previous window-type table where Hanning window is chosen since its max. SLL is -44dB,

Next, we find N:

If f_1=7.5KHz, then $\omega_1=2\pi(7.5)/100=0.15\pi$ rad (recall that $\omega=2\pi f/f_s$)

Similarly if f_2=35KHz, then $\omega_2=2\pi(35)/100=0.7\pi$ rad.

Then: $$N = \frac{2\pi\ (K \quad factor)}{(\omega_2 - \omega_1)} = \frac{2\pi\ (4)}{0.7\pi - 0.15\pi} = 14.5 \quad = 15$$

(K factor=4 for Hanning window)

Next we find the midpoint α=(N-1)/2=(15-1)/2=7

Here $\omega_c = \omega_1 = 0.15\pi$ rad (-3dB point).

For Hanning window $w(n) = 0.5 - 0.5\cos(\frac{2\pi n}{N-1}) = 0.5 - 0.5\cos(\frac{n\pi}{7})$

For LPF, then $h_d(n) = \frac{\sin\{\omega_c(n-\alpha)\}}{\pi(n-\alpha)} = \frac{\sin(0.15\pi(n-7))}{\pi(n-7)}$, $14 \geq n \geq 0$, $n \neq 7$,

and $h_d(7)=\omega_c/\pi=0.15$.

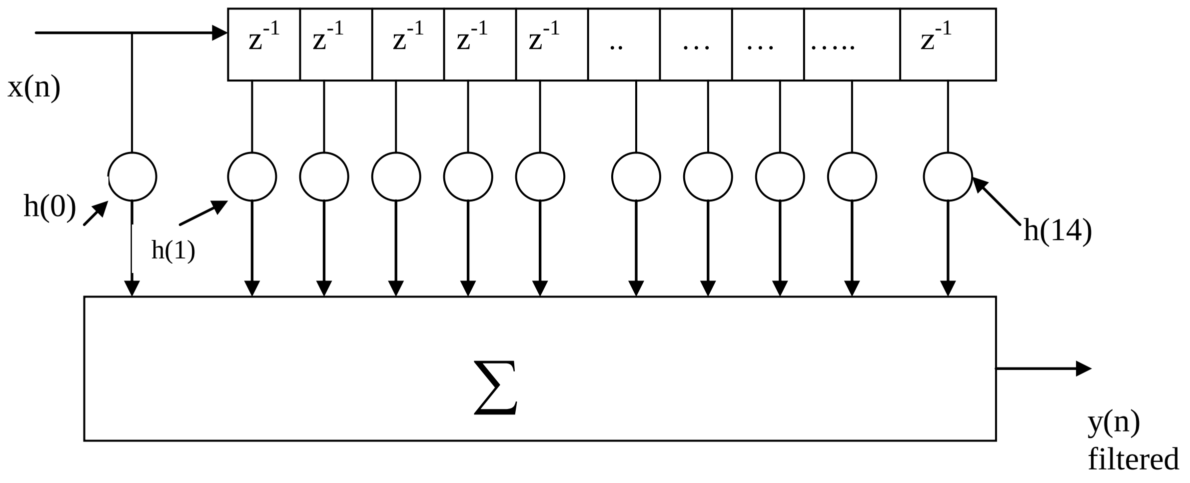
And the overall impulse response $h(n)=h_d(n)\,w(n)$

$$h\,(n)=\frac{\sin(0.15\pi(n-7))}{\pi(n-7)}[0.5-0.5\cos(\frac{n\pi}{7})] \qquad \text{for } 14\geq n\geq 0,\ n\neq 7$$

And $h(7)=h_d(7)= 0.15$ since $w(7)=1$.
Due to symmetry around n=7, only h(n) values at n=0,1,2,3,4,5,6 need to be calculated.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
h(n)	0	0.0008	0.0083	0.029	0.064	0.1	0.13	0.15	0.13	0.1	0.064	0.029	0.0083	0.0008	0

The realization of this filter is done using a tapped delay line:



We can also find the difference equation as:
$y(n)=x(n)\,h(0)+x(n-1)\,h(1)+x(n-2)\,h(n-2)+\ldots\ldots\ldots\ldots+x(n-14)\,h(14)$
and the frequency response: $\phi(\omega) = -7\,\omega$ rad

$$|\,H(\omega)\,|= 0.15+\sum_{n=0}^{6}2h(n)\cos(\omega(n-7))$$

<u>H.W</u>: Repeat previous example for a HPF with 3dB cutoff at 0.7π rad and stopband attenuation of at least 50dB at 0.2π rad.

<u>Example</u>: Find the impulse response of a digital linear phase bandstop filter (BSF) having: 1- 3dB cutoff at 20Hz 1400Hz. 2- stopband attenuation of at least 40dB at 660Hz and 750Hz. Use f_s=3KHz.

<u>Solution</u>: since linear phase, then this is FIR filter.
For 40dB SLL, we use Hanning window,

[K factor=4 and $w(n) = 0.5 - 0.5\cos(\frac{2\pi n}{N-1})$]

For lowpass section:

$$N_1 = \frac{2\pi\ (K \quad factor)}{(\omega_2 - \omega_1)} = \frac{2\pi\ (4)}{\frac{2\pi(660)}{3000} - \frac{2\pi(20)}{3000}} = 18.75 \quad = 19$$

For highpass section:

$$N_2 = \frac{2\pi\ (K \quad factor)}{(\omega_2 - \omega_1)} = \frac{2\pi\ (4)}{\frac{2\pi(1400)}{3000} - \frac{2\pi(750)}{3000}} = 18.46 \quad = 19$$

When we design a filter with two section we usually use:

$N=\max(N_1,N_2)=\max(19,19)=19$. This gives $\alpha=9$.

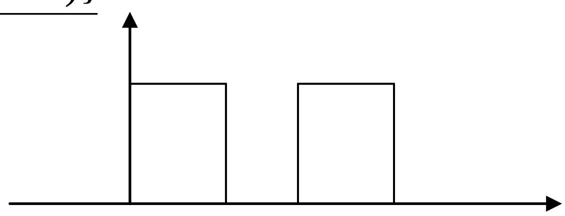
The ω_u and ω_ℓ are the 3dB cutoff of the highpass and lowpass sections respectively, $\omega_u=2\pi(1400)/3000=(14/15)\pi$ rad

And $\omega_\ell= 2\pi(20)/3000=\pi/75$ rad

And $h_d(n) = \dfrac{\sin\{(\pi/75)(n-9)\} - \sin\{(14/15)\pi(n-9)\}}{\pi(n-9)}$

for $18 \geq n \geq 0,\ n \neq 9$

And $h_d(9)= [\pi-(14/15)\pi+(\pi/75)]/\pi=0.08$



and

$$h(n) = [0.5 - 0.5\cos(n\pi/9)][\frac{\sin\{(\pi/75)(n-9)\} - \sin\{(14/15)\pi(n-9)\}}{\pi(n-9)}]$$

$h(9)=0.08$ since $w(9)=1$.

n	0	1	2	3	4	5	6	7	8	9	10
h(n)	0	0.0015	-0.003	0.0159	-0.0173	0.0424	-0.0368	0.0689	-0.0512	0.08	-0.0512

11	12	13	14	15	16	17	18
0.0689	-0.0368	0.0424	-0.0173	0.0159	-0.003	0.0015	0