2.3. HEAT TRANSFER FROM FINNED SURFACES

The rate of heat transfer from a surface at a temperature T_s to the surround in medium at T_∞ is given by Newton's law of cooling as

$$Q_{conv} = hA_s(T_s - T_\infty)$$

where $\boldsymbol{A_s}$ is the heat transfer surface area and $\boldsymbol{h}$ is the convection heat transfer coefficient.

When the temperatures $\boldsymbol{T_s}$ and $\boldsymbol{T_\infty}$ are fixed by design considerations, as is often the case, there are two ways to increase the rate of heat transfer: to increase the convection heat transfer coefficient $\boldsymbol{h}$ or to increase the surface area $\boldsymbol{A_s}$. Increasing $\boldsymbol{h}$ may require the installation of a pump or fan, or replacing the existing one with a larger one, but this approach may or may not be practical. Besides, it may not be adequate. The alternative is to increase the surface area by attaching to the surface ***extended surfaces called fins*** made of highly conductive materials such as aluminum.

Finned surfaces are commonly used in practice to enhance heat transfer, and they often increase the rate of heat transfer from a surface several fold. The car radiator shown in Fig. 2–19 is an example of a finned surface.

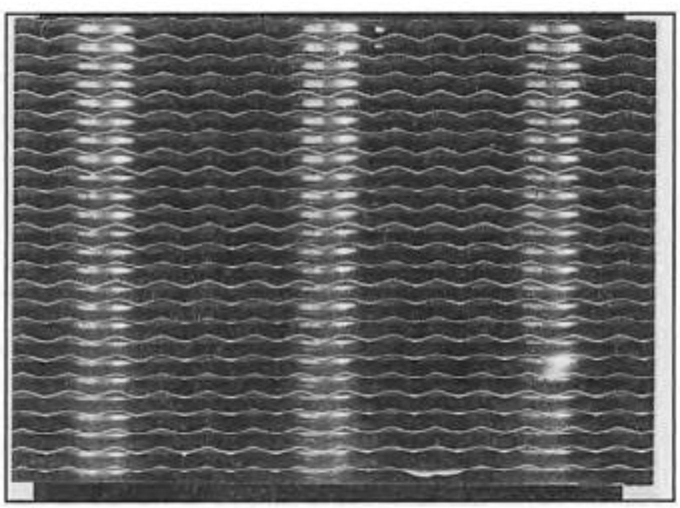

Figure 2-19 The thin plate fins of a car radiator greatly increase the rate of heat transfer to the air

• Fin Equation

In the analysis of fins, we consider

1. ***Steady*** operation with *no heat generation* in the fin
2. The thermal conductivity ***k*** of the material to remain constant.
3. The convection heat transfer coefficient ***h*** to be ***constant*** and ***uniform*** over the entire surface of the fin for convenience in the analysis.

Question: - ***Adding too many fins on a surface may actually decrease the overall heat transfer when the decrease in h offsets any gain resulting from the increase in the surface area.***

Consider a volume element of a fin at location ***x*** having a length of Δ***x*****,** cross sectional area of ***Ac***, and a perimeter of ***p***, as shown in figure 2–20.

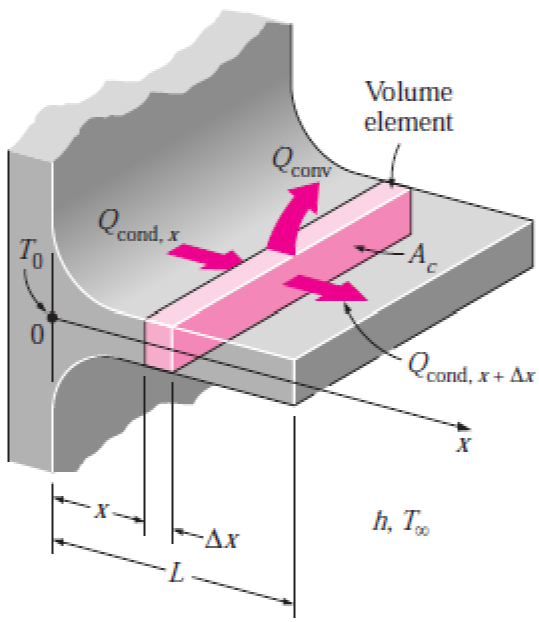

Figure 2-20 Volume element of a fin at location ***x*** having a length of Δx, cross-sectional area of *Ac*, and perimeter of *p*.

Under steady conditions, the energy balance on this volume element can be expressed as

$$\begin{pmatrix}\text{Rate of } \textit{heat}\\ \textit{conduction} \text{ into}\\ \text{the element at } x\end{pmatrix} = \begin{pmatrix}\text{Rate of } \textit{heat}\\ \textit{conduction} \text{ from the}\\ \text{element at } x+\Delta x\end{pmatrix} + \begin{pmatrix}\text{Rate of } \textit{heat}\\ \textit{convection} \text{ from}\\ \text{the element}\end{pmatrix}$$

Or

$$Q_{cond,x} = Q_{cond,x+\Delta x} + Q_{conv}$$

Where

$$Q_{conv} = h(p\Delta x)(T - T_\infty)$$

Substituting and dividing by Δx, we obtain

$$\frac{Q_{cond,x+\Delta x} - Q_{cond,x}}{\Delta x} + hp(T - T_\infty) = 0 \qquad 2\text{-}39$$

Taking the limit as $\Delta x \longrightarrow 0$ gives

$$\frac{dQ_{cond}}{dx} + hp(T - T_\infty) = 0 \qquad 2\text{-}40$$

From Fourier's law of heat conduction, we have

$$Q_{cond} = -kA_c \frac{dT}{dx} \qquad 2\text{-}41$$

where $\boldsymbol{A_c}$ is the cross-sectional area of the fin at location $\boldsymbol{x}$. Substitution of this relation into Eq. 2–40 gives the differential equation governing heat transfer in fins,

$$\frac{d}{dx}\left(kA_c \frac{dT}{dx}\right) - hp(T - T_\infty) = 0 \qquad 2\text{-}42$$

In general, the cross-sectional area $\boldsymbol{A_c}$ and the perimeter $\boldsymbol{p}$ of a fin vary with $\boldsymbol{x}$, which makes this differential equation difficult to solve. ***In the special case of <u>constant cross section and constant thermal conductivity</u>, the differential equation 2–42 reduces to***

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \qquad 2\text{-}43$$

Where

$$m^2 = \frac{hp}{kA_c} \qquad 2\text{-}44$$

and $\theta = T - T_\infty$ is the ***temperature excess***. At the fin base we have $\theta_b = T_b - T_\infty$.

Equation 2-43 is a linear, homogeneous, second-order differential equation with constant coefficients. the solution functions of the differential equation above are the exponential functions e^{ax} or e^{-ax} or constant multiples of them. Therefore, the general solution of the differential equation Eq. 2-43 is

$$\boldsymbol{\theta(x) = C_1 e^{mx} + C_2 e^{-mx}} \qquad 2\text{-}45$$

where C_1 and C_2 are arbitrary constants whose values are to be determined from the boundary conditions at the base and at the tip of the fin.

Note: we need only two conditions to determine C_1 and C_2 uniquely.

At the fin base we have a ***specified temperature*** boundary condition, expressed as

$\theta(0) = \theta_b = T_b - T_\infty$ ⟶ ***boundary condition at fin base***

At the fin tip we have several possibilities, including specified temperature, negligible heat loss (idealized as an insulated tip), convection, and combined convection and radiation as shown in figure 2–21.

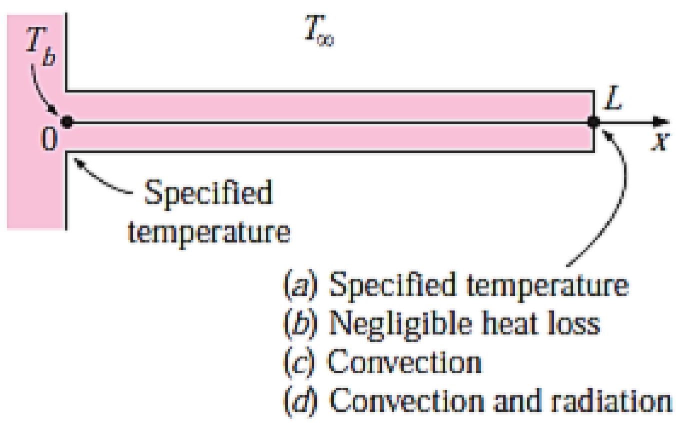

Figure 2-21 Boundary conditions at the fin base and the fin tip

Now, we consider each case separately

1- Infinitely Long Fin ($T_{fin\ tip} = T_\infty$)

For a sufficiently long fin of *uniform* cross section (***Ac* = constant**), the temperature of the fin at the fin tip will approach the environment temperature T_∞ and thus θ will approach zero. That is,

Boundary conditions are

at x= 0 → T = To → $C_1 = \theta_o$

at x = ∞ → T = T∞ → $C_2 = 0$

$$\frac{\theta}{\theta_o} = e^{-mx}$$

$$\frac{T_x - T_\infty}{T_o - T_\infty} = e^{-mx}$$

Note: the temperature along the fin in this case decreases *exponentially* from *Tb* to *T*∞, as shown in figure 2–22. The steady rate of *heat transfer* from the entire fin can be determined from Fourier's law of heat conduction.

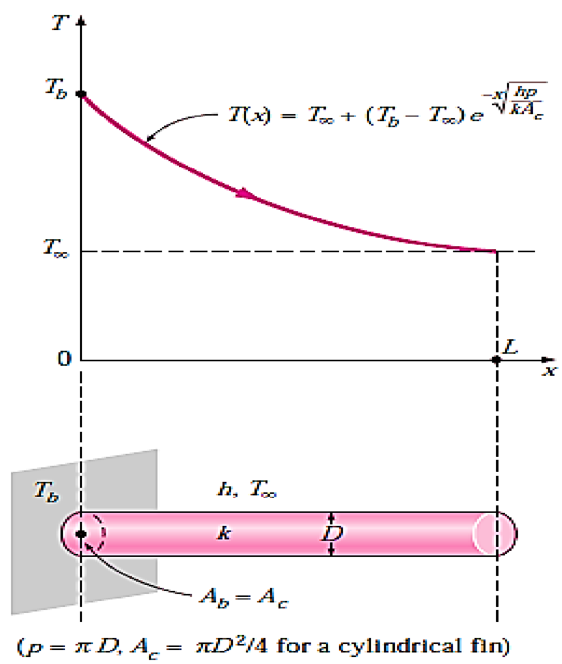

Figure 2-22 a long circular fin of uniform cross section and the variation of temperature along it

$$Q = -kA\left.\frac{dT}{dx}\right|_{x=0}$$

$$\theta = \theta_o e^{-mx}$$

$$\left.\frac{dT}{dx}\right|_{x=0} = -m\theta_O e^{-mx}$$

$$= -m\theta_O$$

$$Q = mkA\theta_o \qquad , \quad m^2 = \frac{hp}{kA}$$

$$Q = \sqrt{hpkA}\,\theta_o$$

2- Negligible Heat Loss from the Fin Tip (Insulated fin tip, $Q_{fin\ tip} = 0$)

Fins are not likely to be so long that their temperature approaches the surrounding temperature at the tip. A more realistic situation is for heat transfer from the fin tip to be negligible since the heat transfer from the fin is proportional to its surface area, and the surface area of the fin tip is usually a negligible fraction of the total fin area. Then the fin tip can be assumed to be insulated, and the condition at the fin tip can be expressed as

Boundary conditions are

at x = 0 ⟹ T = To

at x = L ⟹ $\frac{dT}{dx} = 0$

$$\frac{T_x - T_\infty}{T_O - T_\infty} = \frac{cosh[m(L-x)]}{cosh(mL)}$$

$$Q = \sqrt{hpkA}\,\theta_o \tanh(mL)$$

3- Convection (or Combined Convection and Radiation) from Fin Tip

The fin tips, in practice, are exposed to the surroundings, and thus the proper boundary condition for the fin tip is convection that also includes the effects of radiation. The fin equation can still be solved in this case using the convection at the fin tip as the second boundary condition, but the analysis becomes more involved, and it results in rather lengthy expressions for the temperature distribution and the heat transfer. Yet, in general, the fin tip area is a small fraction of the total fin surface area, and thus the complexities involved can hardly justify the improvement in accuracy.

Note: A practical way of accounting for the heat loss from the fin tip is to replace the *fin length L* in the relation for the *insulated tip* case by a **corrected length** defined as (Fig. 2–20)

Corrected fin length: $$L_C = L + \frac{A_C}{P}$$

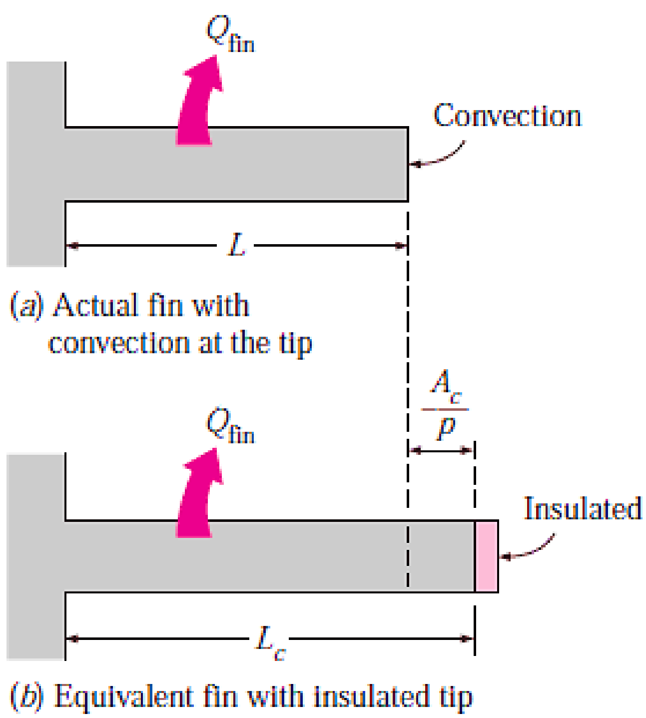

Figure 2-20 Corrected fin length *Lc* is defined such that heat transfer from a fin of length *Lc* with insulated tip is equal to heat transfer from the actual fin of length *L* with convection at the fin tip.

Note: Using the proper relations for *Ac* and *p*, the corrected lengths for rectangular and cylindrical fins are easily determined to be

$$L_{C,rectangular\ fin} = L + \frac{t}{2} \quad , L_{C,cylinderical\ fin} = L + \frac{D}{4}$$

where *t* is the thickness of the rectangular fins and *D* is the diameter of the cylindrical fins.

Boundary conditions are

at x = 0 ⇒ T = To

at x = L ⇒ $Q_{cond,x=L} = Q_{conv,end\ face}$

$$-kA\frac{dT}{dx} = h_{end}A(T - T_\infty)$$

$$\frac{T_x - T_\infty}{T_o - T_\infty} = \frac{cosh[m(L-x)] + \frac{h_{end}}{mk} sinh[m(L-x)]}{cosh(mL) + \frac{h_{end}}{mk} sinh[mL]}$$

$$q = \sqrt{hpKA}\,\theta_o \frac{\sinh[mL] + \frac{h_e}{mK}\cosh[mL]}{\cosh[mL] + \frac{h_e}{mK}\sinh[mL]}$$

• Fin Efficiency

In reality, the temperature of the fin will drop along the fin, and thus the heat transfer from the fin will be less because of the decreasing temperature difference $T_x - T_\infty$ toward the fin tip, as shown in Fig. 2–21.

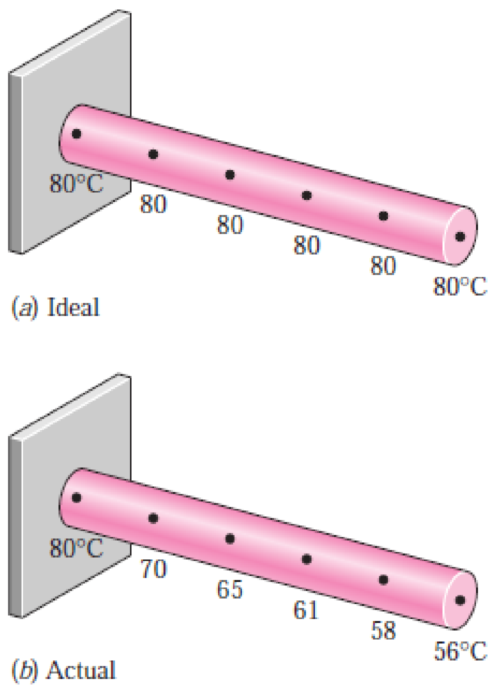

Figure 2-21 Ideal and actual temperature distribution in a fin

To account for the effect of this decrease in temperature on heat transfer, we define a **fin efficiency** as

$$fin\ efficiency = \frac{\text{Actual heat transfer rate from the fin}}{\text{Ideal heat transfer rate from the fin if the entire fin were at base temperature}}$$

$$\eta_{fin} = \frac{Q_{fin}}{Q_{fin,max}}$$

Case I: **Infinitely Long Fin**

$$\eta_{fin} = \frac{\sqrt{hpkA}\,(T_o - T_\infty)}{h(PL)\,(T_o - T_\infty)}$$

$$= \sqrt{\frac{hPkA}{h^2P^2}}\,\frac{1}{L}$$

$$\eta_{fin} = \frac{1}{mL}$$

Case2: **Negligible Heat Loss from the Fin Tip (Insulated fin tip)**

$$\eta_{fin} = \frac{\sqrt{hpkA}\,(T_o - T_\infty)\tanh(mL)}{h(PL)\,(T_o - T_\infty)}$$

$$\eta_{fin} = \frac{\tanh(mL)}{mL}$$

Case3: **Convection (or Combined Convection and Radiation) from Fin Tip**

$$\eta_{fin} = \frac{\tanh(mL_c)}{mL_c}$$

- **Fins Efficiency from Chart**

Fin efficiency relations are developed for fins of various profiles and are plotted in figure 2–22 for fins on a ***plain surface*** and in figure 2–23 for ***circular fins*** of constant thickness. The fin surface area associated with each profile is also given on each figure. For most fins of constant thickness encountered in practice, the fin thickness $\boldsymbol{t}$ is too small relative to the fin length $\boldsymbol{L}$, and thus the fin tip area is negligible.

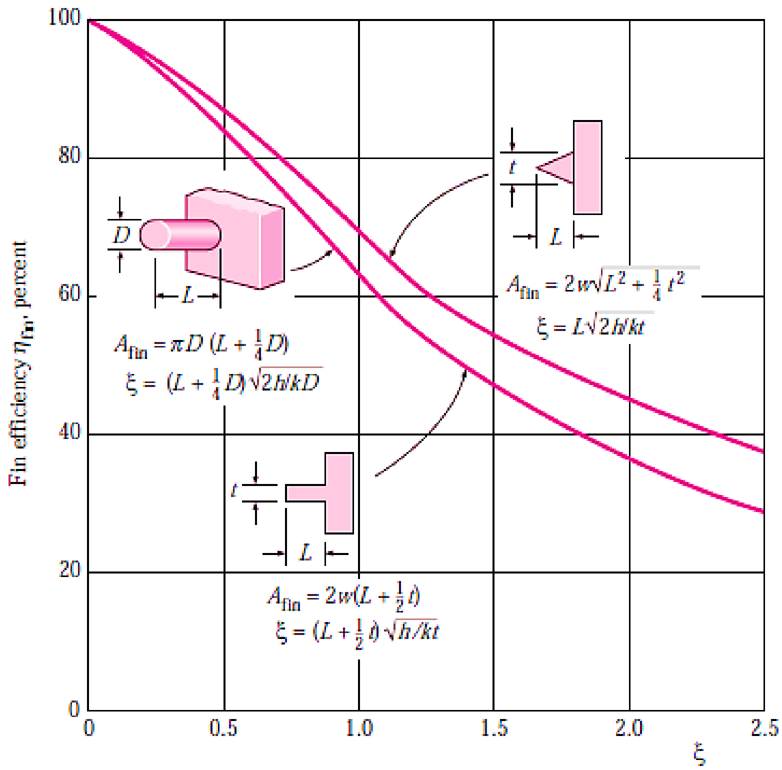

Figure 2-22 Efficiency of circular, rectangular, and triangular fins on a plain surface of width *w* (from Gardner).

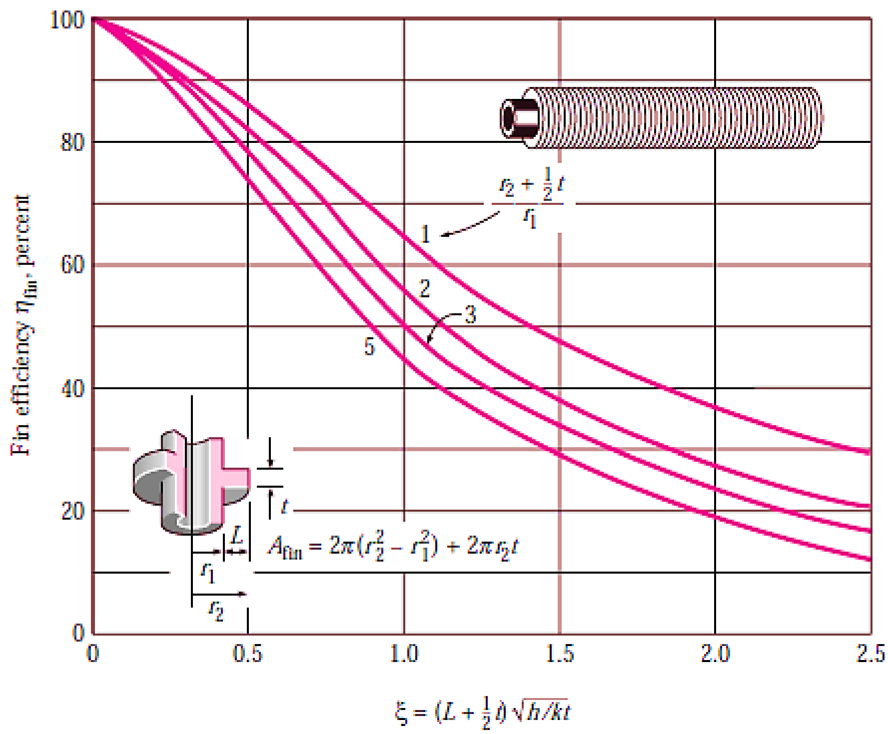

Figure 2-23 Efficiency of circular fins of length L and constant thickness t (from Gardner).

• Fin Effectiveness

Fins are used to *enhance* heat transfer, and the use of fins on a surface cannot be recommended unless the enhancement in heat transfer justifies the added cost and complexity associated with the fins. In fact, there is no assurance that adding fins on a surface will ***enhance*** heat transfer. The performance of the fins is judged on the basis of the enhancement in heat transfer relative to the no-fin case see figure 2-24.

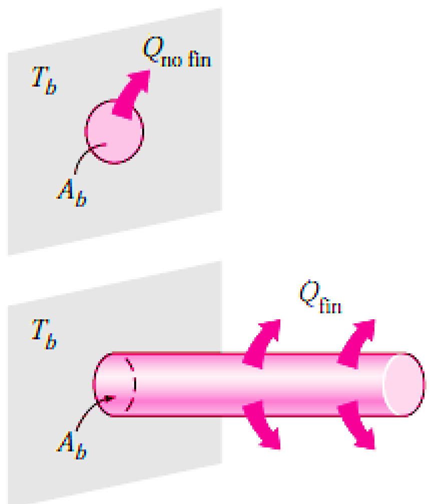

Figure 2-24 The effectiveness of a fin

The performance of fins expressed in terms of the ***fin effectiveness*** (ε_{fin}) is defined as

$$\varepsilon_{fin} = \frac{Q_{with\ fin}}{Q_{without\ fin}} = \frac{Q_{with\ fin}}{hA_b(T_b - T_\infty)}$$

Here, A_b is the cross-sectional area of the fin at the base and $Q_{without\ fin}$ represents the rate of heat transfer from this area if no fins are attached to the surface.