3.7 VARIGNON'S THEOREM

The distributive property of vector products can be used to determine the moment of the resultant of several *concurrent forces*. If several forces $\mathbf{F}_1$, $\mathbf{F}_2$, . . . are applied at the same point A (Fig. 3.14), and if we denote by $\mathbf{r}$ the position vector of A, it follows immediately from Eq. (3.5) of Sec. 3.4 that

$$\mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2 + \cdots) = \mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2 + \cdots \tag{3.14}$$

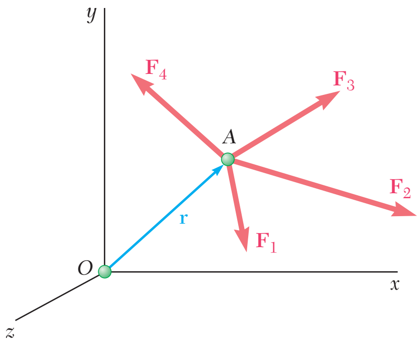


Fig. 3.14

In words, *the moment about a given point O of the resultant of several concurrent forces is equal to the sum of the moments of the various forces about the same point O*. This property, which was originally established by the French mathematician Varignon (1654–1722) long before the introduction of vector algebra, is known as *Varignon's theorem.*

The relation (3.14) makes it possible to replace the direct determination of the moment of a force $\mathbf{F}$ by the determination of the moments of two or more component forces. As you will see in the next section, $\mathbf{F}$ will generally be resolved into components parallel to the coordinate axes. However, it may be more expeditious in some instances to resolve $\mathbf{F}$ into components which are not parallel to the coordinate axes (see Sample Prob. 3.3).

3.8 RECTANGULAR COMPONENTS OF THE MOMENT OF A FORCE

In general, the determination of the moment of a force in space will be considerably simplified if the force and the position vector of its point of application are resolved into rectangular x, y, and z components. Consider, for example, the moment $\mathbf{M}_O$ about O of a force $\mathbf{F}$ whose components are F_x, F_y, and F_z and which is applied at a point A of coordinates x, y, and z (Fig. 3.15). Observing that the components of the position vector $\mathbf{r}$ are respectively equal to the coordinates x, y, and z of the point A, we write

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{3.15}$$
$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} \tag{3.16}$$

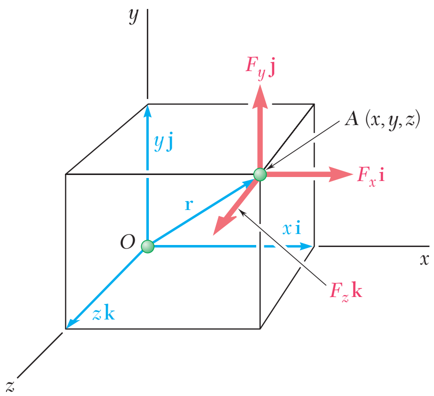


Fig. 3.15

Substituting for $\mathbf{r}$ and $\mathbf{F}$ from (3.15) and (3.16) into

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} \tag{3.11}$$

and recalling the results obtained in Sec. 3.5, we write the moment $\mathbf{M}_O$ of $\mathbf{F}$ about O in the form

$$\mathbf{M}_O = M_x\mathbf{i} + M_y\mathbf{j} + M_z\mathbf{k} \tag{3.17}$$

where the components M_x, M_y, and M_z are defined by the relations

$$\begin{aligned} M_x &= yF_z - zF_y \\ M_y &= zF_x - xF_z \\ M_z &= xF_y - yF_x \end{aligned} \tag{3.18}$$

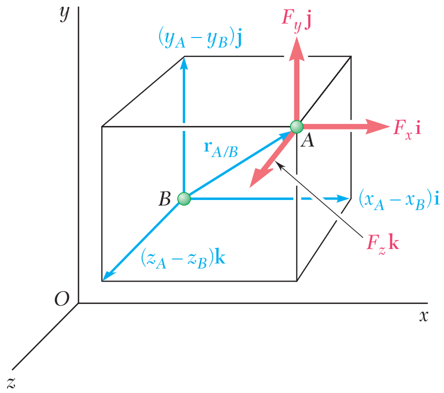


Fig. 3.16

As you will see in Sec. 3.11, the scalar components M_x, M_y, and M_z of the moment $\mathbf{M}_O$ measure the tendency of the force $\mathbf{F}$ to impart to a rigid body a motion of rotation about the x, y, and z axes, respectively. Substituting from (3.18) into (3.17), we can also write $\mathbf{M}_O$ in the form of the determinant

$$\mathbf{M}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} \tag{3.19}$$

To compute the moment $\mathbf{M}_B$ about an arbitrary point B of a force $\mathbf{F}$ applied at A (Fig. 3.16), we must replace the position vector $\mathbf{r}$ in Eq. (3.11) by a vector drawn from B to A. This vector is the *position vector of A relative to B* and will be denoted by $\mathbf{r}_{A/B}$. Observing that $\mathbf{r}_{A/B}$ can be obtained by subtracting $\mathbf{r}_B$ from $\mathbf{r}_A$, we write

$$\mathbf{M}_B = \mathbf{r}_{A/B} \times \mathbf{F} = (\mathbf{r}_A - \mathbf{r}_B) \times \mathbf{F} \tag{3.20}$$

or, using the determinant form,

$$\mathbf{M}_B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_{A/B} & y_{A/B} & z_{A/B} \\ F_x & F_y & F_z \end{vmatrix} \tag{3.21}$$

where $x_{A/B}$, $y_{A/B}$, and $z_{A/B}$ denote the components of the vector $\mathbf{r}_{A/B}$:

$$x_{A/B} = x_A - x_B \qquad y_{A/B} = y_A - y_B \qquad z_{A/B} = z_A - z_B$$

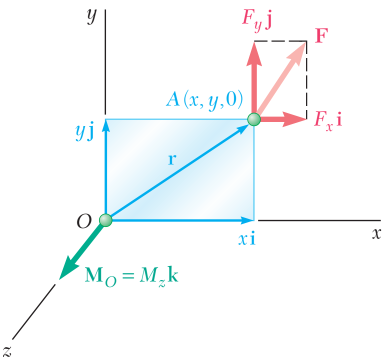


Fig. 3.17

In the case of *problems involving only two dimensions,* the force $\mathbf{F}$ can be assumed to lie in the xy plane (Fig. 3.17). Setting $z = 0$ and $F_z = 0$ in Eq. (3.19), we obtain

$$\mathbf{M}_O = (xF_y - yF_x)\mathbf{k}$$

We verify that the moment of $\mathbf{F}$ about O is perpendicular to the plane of the figure and that it is completely defined by the scalar

$$M_O = M_z = xF_y - yF_x \tag{3.22}$$

As noted earlier, a positive value for M_O indicates that the vector $\mathbf{M}_O$ points out of the paper (the force $\mathbf{F}$ tends to rotate the body counterclockwise about O), and a negative value indicates that the vector $\mathbf{M}_O$ points into the paper (the force $\mathbf{F}$ tends to rotate the body clockwise about O).

To compute the moment about $B(x_B, y_B)$ of a force lying in the xy plane and applied at $A(x_A, y_A)$ (Fig. 3.18), we set $z_{A/B} = 0$ and $F_z = 0$ in the relations (3.21) and note that the vector $\mathbf{M}_B$ is perpendicular to the xy plane and is defined in magnitude and sense by the scalar

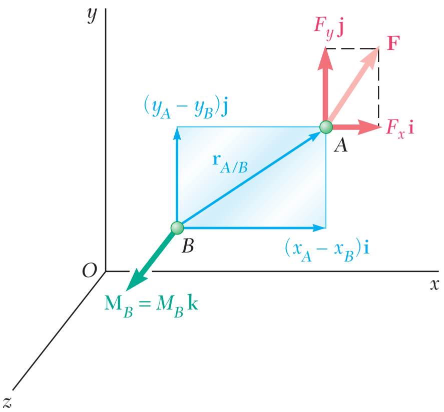
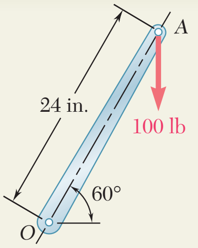


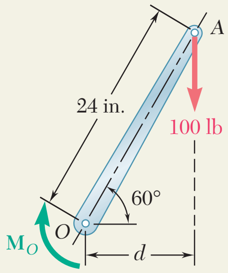
Fig. 3.18

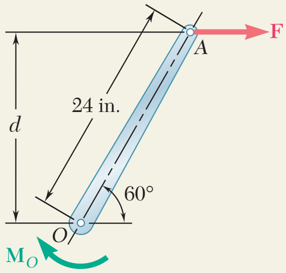
$$M_B = (x_A - x_B)F_y - (y_A - y_B)F_x \tag{3.23}$$

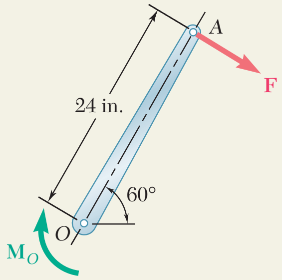


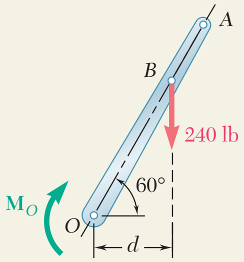
SAMPLE PROBLEM 3.1

A 100-lb vertical force is applied to the end of a lever which is attached to a shaft at O. Determine (*a*) the moment of the 100-lb force about O; (*b*) the horizontal force applied at A which creates the same moment about O; (*c*) the smallest force applied at A which creates the same moment about O; (*d*) how far from the shaft a 240-lb vertical force must act to create the same moment about O; (*e*) whether any one of the forces obtained in parts b, c, and d is equivalent to the original force.









SOLUTION

a. Moment about O. The perpendicular distance from O to the line of action of the 100-lb force is

$$d = (24\text{ in.})\cos 60° = 12\text{ in.}$$

The magnitude of the moment about O of the 100-lb force is

$$M_O = Fd = (100\text{ lb})(12\text{ in.}) = 1200\text{ lb}\cdot\text{in.}$$

Since the force tends to rotate the lever clockwise about O, the moment will be represented by a vector $\mathbf{M}_O$ perpendicular to the plane of the figure and pointing *into* the paper. We express this fact by writing

$$\mathbf{M}_O = 1200\text{ lb}\cdot\text{in. } \circlearrowright \quad ◀$$

b. Horizontal Force. In this case, we have

$$d = (24\text{ in.})\sin 60° = 20.8\text{ in.}$$

Since the moment about O must be 1200 lb · in., we write

$$M_O = Fd$$
$$1200\text{ lb}\cdot\text{in.} = F(20.8\text{ in.})$$
$$F = 57.7\text{ lb} \qquad \mathbf{F} = 57.7\text{ lb} \rightarrow \quad ◀$$

c. Smallest Force. Since $M_O = Fd$, the smallest value of F occurs when d is maximum. We choose the force perpendicular to OA and note that $d = 24$ in.; thus,

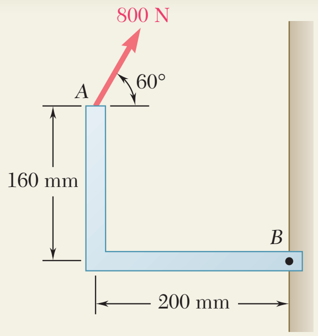
$$M_O = Fd$$
$$1200\text{ lb}\cdot\text{in.} = F(24\text{ in.})$$
$$F = 50\text{ lb} \qquad \mathbf{F} = 50\text{ lb} \searrow 30° \quad ◀$$

d. 240-lb Vertical Force. In this case $M_O = Fd$ yields

$$1200\text{ lb}\cdot\text{in.} = (240\text{ lb})d \qquad d = 5\text{ in.}$$

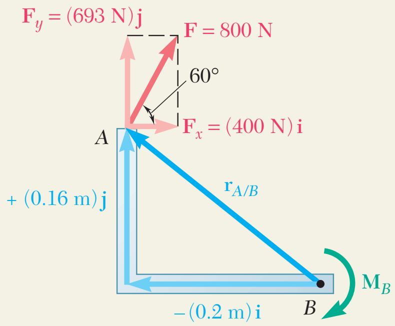
but $\qquad OB\cos 60° = d \qquad OB = 10\text{ in.} \quad ◀$

e. None of the forces considered in parts b, c, and d is equivalent to the original 100-lb force. Although they have the same moment about O, they have different x and y components. In other words, although each force tends to rotate the shaft in the same manner, each causes the lever to pull on the shaft in a different way.



SAMPLE PROBLEM 3.2

A force of 800 N acts on a bracket as shown. Determine the moment of the force about B.



SOLUTION

The moment $\mathbf{M}_B$ of the force $\mathbf{F}$ about B is obtained by forming the vector product

$$\mathbf{M}_B = \mathbf{r}_{A/B} \times \mathbf{F}$$

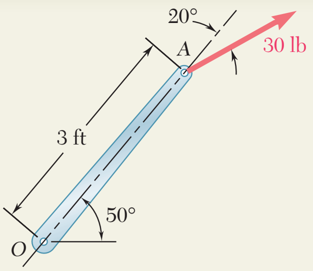
where $\mathbf{r}_{A/B}$ is the vector drawn from B to A. Resolving $\mathbf{r}_{A/B}$ and $\mathbf{F}$ into rectangular components, we have

$$\begin{aligned} \mathbf{r}_{A/B} &= -(0.2\ \text{m})\mathbf{i} + (0.16\ \text{m})\mathbf{j} \\ \mathbf{F} &= (800\ \text{N})\cos 60°\mathbf{i} + (800\ \text{N})\sin 60°\mathbf{j} \\ &= (400\ \text{N})\mathbf{i} + (693\ \text{N})\mathbf{j} \end{aligned}$$

Recalling the relations (3.7) for the cross products of unit vectors (Sec. 3.5), we obtain

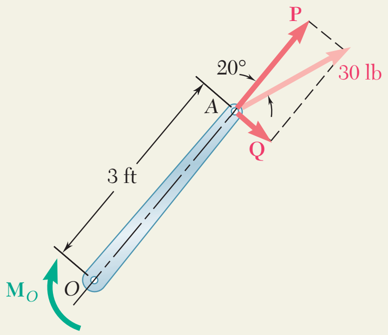
$$\begin{aligned} \mathbf{M}_B = \mathbf{r}_{A/B} \times \mathbf{F} &= [-(0.2\ \text{m})\mathbf{i} + (0.16\ \text{m})\mathbf{j}] \times [(400\ \text{N})\mathbf{i} + (693\ \text{N})\mathbf{j}] \\ &= -(138.6\ \text{N} \cdot \text{m})\mathbf{k} - (64.0\ \text{N} \cdot \text{m})\mathbf{k} \\ &= -(202.6\ \text{N} \cdot \text{m})\mathbf{k} \qquad \mathbf{M}_B = 203\ \text{N} \cdot \text{m} \circlearrowright \blacktriangleleft \end{aligned}$$

The moment $\mathbf{M}_B$ is a vector perpendicular to the plane of the figure and pointing *into* the paper.



SAMPLE PROBLEM 3.3

A 30-lb force acts on the end of the 3-ft lever as shown. Determine the moment of the force about O.



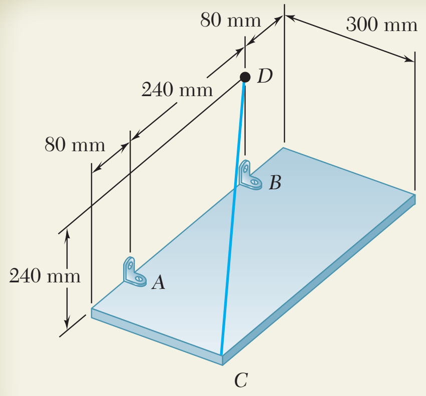
SOLUTION

The force is replaced by two components, one component $\mathbf{P}$ in the direction of OA and one component $\mathbf{Q}$ perpendicular to OA. Since O is on the line of action of $\mathbf{P}$, the moment of $\mathbf{P}$ about O is zero and the moment of the 30-lb force reduces to the moment of $\mathbf{Q}$, which is clockwise and, thus, is represented by a negative scalar.

$$\begin{aligned} Q &= (30\ \text{lb}) \sin 20° = 10.26\ \text{lb} \\ M_O &= -Q(3\ \text{ft}) = -(10.26\ \text{lb})(3\ \text{ft}) = -30.8\ \text{lb} \cdot \text{ft} \end{aligned}$$

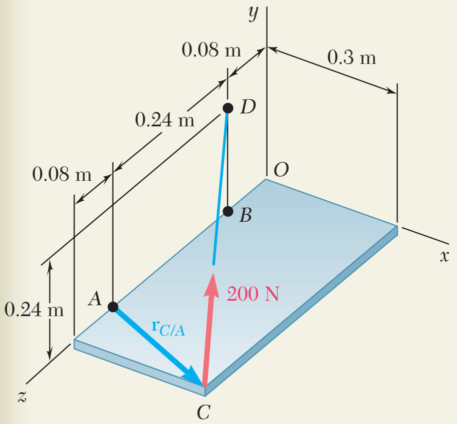
Since the value obtained for the scalar M_O is negative, the moment $\mathbf{M}_O$ points *into* the paper. We write

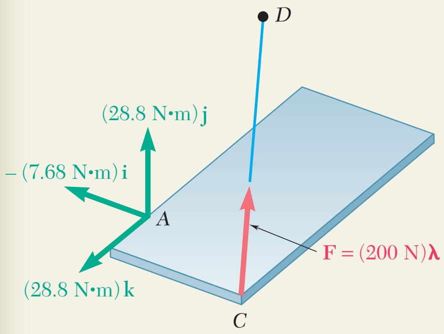
$$\mathbf{M}_O = 30.8\ \text{lb} \cdot \text{ft} \circlearrowright \blacktriangleleft$$



SAMPLE PROBLEM 3.4

A rectangular plate is supported by brackets at A and B and by a wire CD. Knowing that the tension in the wire is 200 N, determine the moment about A of the force exerted by the wire on point C.





SOLUTION

The moment $\mathbf{M}_A$ about A of the force $\mathbf{F}$ exerted by the wire on point C is obtained by forming the vector product

$$\mathbf{M}_A = \mathbf{r}_{C/A} \times \mathbf{F} \tag{1}$$

where $\mathbf{r}_{C/A}$ is the vector drawn from A to C,

$$\mathbf{r}_{C/A} = \overrightarrow{AC} = (0.3 \text{ m})\mathbf{i} + (0.08 \text{ m})\mathbf{k} \tag{2}$$

and $\mathbf{F}$ is the 200-N force directed along CD. Introducing the unit vector $\boldsymbol{\lambda} = \overrightarrow{CD}/CD$, we write

$$\mathbf{F} = F\boldsymbol{\lambda} = (200 \text{ N})\frac{\overrightarrow{CD}}{CD} \tag{3}$$

Resolving the vector $\overrightarrow{CD}$ into rectangular components, we have

$$\overrightarrow{CD} = -(0.3 \text{ m})\mathbf{i} + (0.24 \text{ m})\mathbf{j} - (0.32 \text{ m})\mathbf{k} \qquad CD = 0.50 \text{ m}$$

Substituting into (3), we obtain

$$\mathbf{F} = \frac{200 \text{ N}}{0.50 \text{ m}}[-(0.3 \text{ m})\mathbf{i} + (0.24 \text{ m})\mathbf{j} - (0.32 \text{ m})\mathbf{k}]$$
$$= -(120 \text{ N})\mathbf{i} + (96 \text{ N})\mathbf{j} - (128 \text{ N})\mathbf{k} \tag{4}$$

Substituting for $\mathbf{r}_{C/A}$ and $\mathbf{F}$ from (2) and (4) into (1) and recalling the relations (3.7) of Sec. 3.5, we obtain

$$\mathbf{M}_A = \mathbf{r}_{C/A} \times \mathbf{F} = (0.3\mathbf{i} + 0.08\mathbf{k}) \times (-120\mathbf{i} + 96\mathbf{j} - 128\mathbf{k})$$
$$= (0.3)(96)\mathbf{k} + (0.3)(-128)(-\mathbf{j}) + (0.08)(-120)\mathbf{j} + (0.08)(96)(-\mathbf{i})$$
$$\mathbf{M}_A = -(7.68 \text{ N}\cdot\text{m})\mathbf{i} + (28.8 \text{ N}\cdot\text{m})\mathbf{j} + (28.8 \text{ N}\cdot\text{m})\mathbf{k} \quad ◀$$

Alternative Solution. As indicated in Sec. 3.8, the moment $\mathbf{M}_A$ can be expressed in the form of a determinant:

$$\mathbf{M}_A = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_C - x_A & y_C - y_A & z_C - z_A \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.3 & 0 & 0.08 \\ -120 & 96 & -128 \end{vmatrix}$$

$$\mathbf{M}_A = -(7.68 \text{ N}\cdot\text{m})\mathbf{i} + (28.8 \text{ N}\cdot\text{m})\mathbf{j} + (28.8 \text{ N}\cdot\text{m})\mathbf{k} \quad ◀$$

PROBLEMS

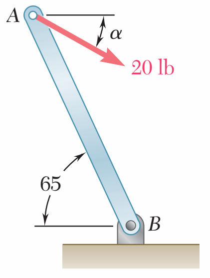


Fig. P3.1 and P3.2

3.1 A 20-lb force is applied to the control rod AB as shown. Knowing that the length of the rod is 9 in. and that $\alpha = 25°$, determine the moment of the force about point B by resolving the force into horizontal and vertical components.

3.2 A 20-lb force is applied to the control rod AB as shown. Knowing that the length of the rod is 9 in. and that the moment of the force about B is 120 lb · in. clockwise, determine the value of α.

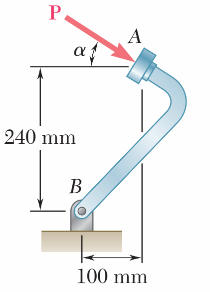


Fig. P3.3 and *P3.4*

3.3 For the brake pedal shown, determine the magnitude and direction of the smallest force **P** that has a 104-N · m clockwise moment about B.

3.4 A force **P** is applied to the brake pedal at A. Knowing that $P = 450$ N and $\alpha = 30°$, determine the moment of **P** about B.

3.5 A 450-N force is applied at A as shown. Determine (*a*) the moment of the 450-N force about D, (*b*) the smallest force applied at B that creates the same moment about D.

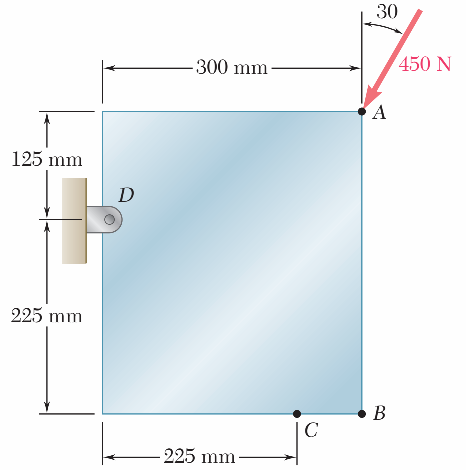


Fig. P3.5 and P3.6

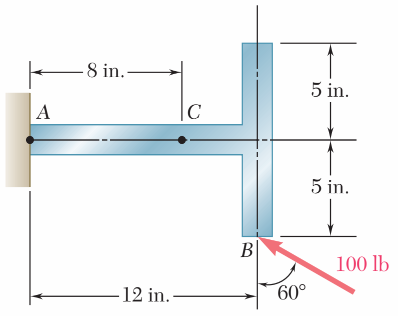


Fig. P3.7 and *P3.8*

3.6 A 450-N force is applied at A as shown. Determine (*a*) the moment of the 450-N force about D, (*b*) the magnitude and sense of the horizontal force applied at C that creates the same moment about D, (*c*) the smallest force applied at C that creates the same moment about D.

3.7 Compute the moment of the 100-lb force about A, (*a*) by using the definition of the moment of a force, (*b*) by resolving the force into horizontal and vertical components, (*c*) by resolving the force into components along AB and in the direction perpendicular to AB.

3.8 Determine the moment of the 100-lb force about C.

3.9 and 3.10 It is known that the connecting rod *AB* exerts on the crank *BC* a 2.5-kN force directed down to the left along the centerline *AB*. Determine the moment of that force about *C*.

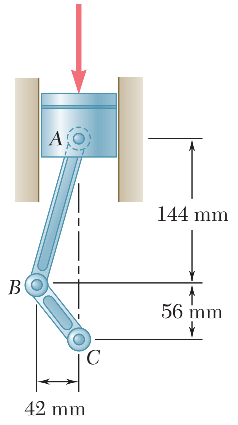


Fig. P3.9

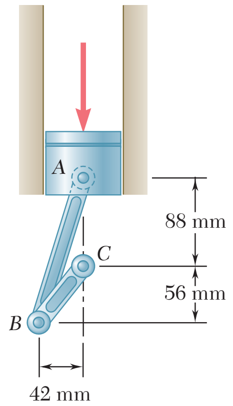


Fig. P3.10

3.11 Rod *AB* is held in place by the cord *AC*. Knowing that the tension in the cord is 300 lb and that $c = 18$ in., determine the moment about *B* of the force exerted by the cord at point *A* by resolving that force into horizontal and vertical components applied (*a*) at point *A*, (*b*) at point *C*.

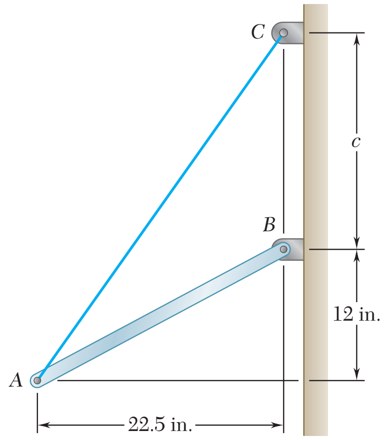


Fig. *P3.11* and P3.12

3.12 Rod *AB* is held in place by the cord *AC*. Knowing that $c = 42$ in. and that the moment about *B* of the force exerted by the cord at point *A* is 700 lb · ft, determine the tension in the cord.

3.13 Determine the moment about the origin of coordinates *O* of the force $\mathbf{F} = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ that acts at a point *A*. Assume that the position of *A* is (*a*) $\mathbf{r} = \mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$, (*b*) $\mathbf{r} = 6\mathbf{i} + \mathbf{j} + 3\mathbf{k}$, (*c*) $\mathbf{r} = 5\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$.

3.14 Determine the moment about the origin of coordinates *O* of the force $\mathbf{F} = -\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ that acts at a point *A*. Assume that the position of *A* is (*a*) $\mathbf{r} = 2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$, (*b*) $\mathbf{r} = 4\mathbf{i} + 6\mathbf{j} + 10\mathbf{k}$, (*c*) $\mathbf{r} = -3\mathbf{i} + 9\mathbf{j} + 15\mathbf{k}$.

3.15 The line of action of the force **P** of magnitude 420 lb passes through the two points A and B as shown. Compute the moment of **P** about O using the position vector (a) of point A, (b) of point B.

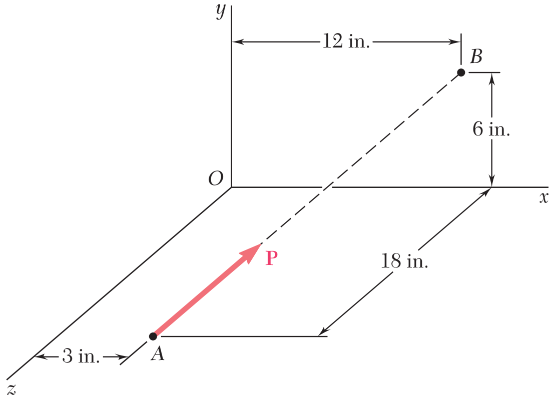


Fig. P3.15

3.16 A force **P** of magnitude 200 N acts along the diagonal BC of the bent plate shown. Determine the moment of **P** about point E.

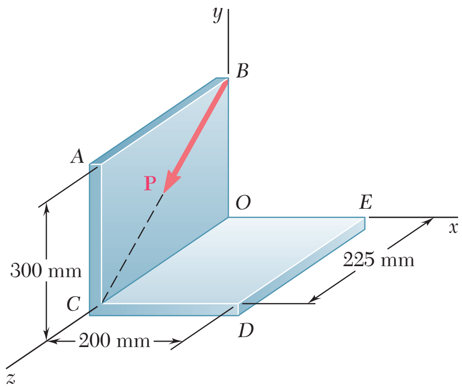


Fig. P3.16

3.17 Knowing that the tension in cable AB is 1800 lb, determine the moment of the force exerted on the plate at A about (a) the origin of coordinates O, (b) corner D.

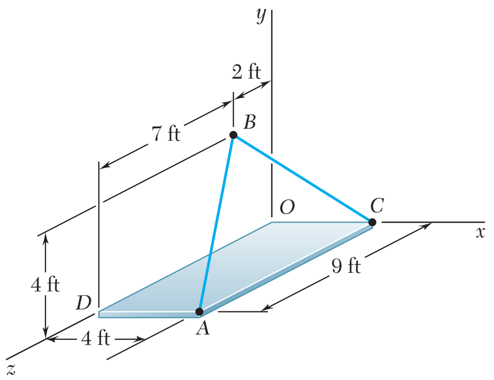


Fig. P3.17 and P3.18

3.18 Knowing that the tension in cable BC is 900 lb, determine the moment of the force exerted on the plate at C about (a) the origin of coordinates O, (b) corner D.

3.19 A 200-N force is applied as shown to the bracket *ABC*. Determine the moment of the force about *A*.

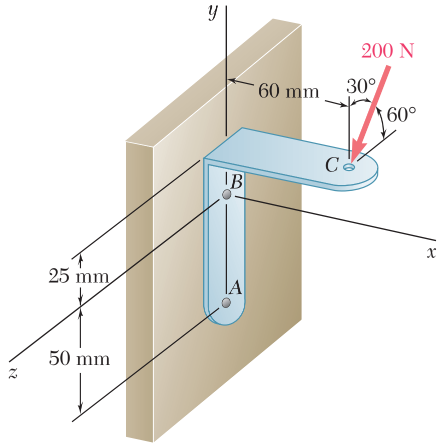


Fig. P3.19

3.20 A small boat hangs from two davits, one of which is shown in the figure. The tension in line *ABAD* is 82 lb. Determine the moment about *C* of the resultant force $\mathbf{R}_A$ exerted on the davit at *A*.

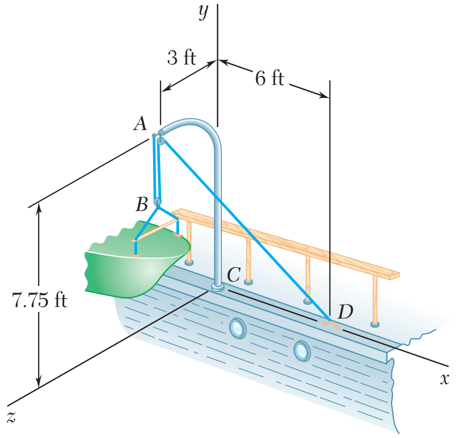


Fig. P3.20

3.21 In Prob. 3.15, determine the perpendicular distance from the line of action of **P** to the origin *O*.

3.22 In Prob. 3.16, determine the perpendicular distance from the line of action of **P** to point *E*.

3.23 In Prob. 3.20, determine the perpendicular distance from the point *C* to the portion *AD* of line *ABAD*.

3.24 In Sample Prob. 3.4, determine the perpendicular distance from point *A* to wire *CD*.

3.9 SCALAR PRODUCT OF TWO VECTORS

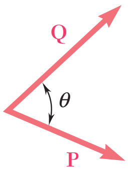


Fig. 3.19

The *scalar product* of two vectors **P** and **Q** is defined as the product of the magnitudes of **P** and **Q** and of the cosine of the angle θ formed by **P** and **Q** (Fig. 3.19). The scalar product of **P** and **Q** is denoted by $\mathbf{P} \cdot \mathbf{Q}$. We write therefore

$$\mathbf{P} \cdot \mathbf{Q} = PQ \cos \theta \tag{3.24}$$

Note that the expression just defined is not a vector but a *scalar*, which explains the name *scalar product*; because of the notation used, $\mathbf{P} \cdot \mathbf{Q}$ is also referred to as the *dot product* of the vectors **P** and **Q**.

It follows from its very definition that the scalar product of two vectors is *commutative*, i.e., that

$$\mathbf{P} \cdot \mathbf{Q} = \mathbf{Q} \cdot \mathbf{P} \tag{3.25}$$

To prove that the scalar product is also *distributive*, we must prove the relation

$$\mathbf{P} \cdot (\mathbf{Q}_1 + \mathbf{Q}_2) = \mathbf{P} \cdot \mathbf{Q}_1 + \mathbf{P} \cdot \mathbf{Q}_2 \tag{3.26}$$

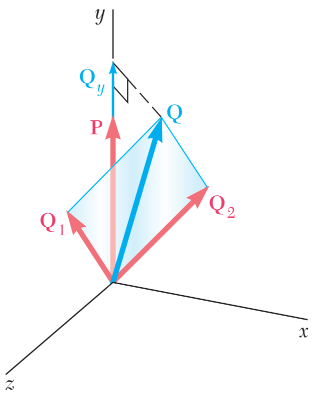


Fig. 3.20

We can, without any loss of generality, assume that **P** is directed along the y axis (Fig. 3.20). Denoting by **Q** the sum of $\mathbf{Q}_1$ and $\mathbf{Q}_2$ and by θ_y the angle **Q** forms with the y axis, we express the left-hand member of (3.26) as follows:

$$\mathbf{P} \cdot (\mathbf{Q}_1 + \mathbf{Q}_2) = \mathbf{P} \cdot \mathbf{Q} = PQ \cos \theta_y = PQ_y \tag{3.27}$$

where Q_y is the y component of **Q**. We can, in a similar way, express the right-hand member of (3.26) as

$$\mathbf{P} \cdot \mathbf{Q}_1 + \mathbf{P} \cdot \mathbf{Q}_2 = P(Q_1)_y + P(Q_2)_y \tag{3.28}$$

Since **Q** is the sum of $\mathbf{Q}_1$ and $\mathbf{Q}_2$, its y component must be equal to the sum of the y components of $\mathbf{Q}_1$ and $\mathbf{Q}_2$. Thus, the expressions obtained in (3.27) and (3.28) are equal, and the relation (3.26) has been proved.

As far as the third property—the associative property—is concerned, we note that this property cannot apply to scalar products. Indeed, $(\mathbf{P} \cdot \mathbf{Q}) \cdot \mathbf{S}$ has no meaning since $\mathbf{P} \cdot \mathbf{Q}$ is not a vector but a scalar.

The scalar product of two vectors **P** and **Q** can be expressed in terms of their rectangular components. Resolving **P** and **Q** into components, we first write

$$\mathbf{P} \cdot \mathbf{Q} = (P_x\mathbf{i} + P_y\mathbf{j} + P_z\mathbf{k}) \cdot (Q_x\mathbf{i} + Q_y\mathbf{j} + Q_z\mathbf{k})$$

Making use of the distributive property, we express $\mathbf{P} \cdot \mathbf{Q}$ as the sum of scalar products, such as $P_x\mathbf{i} \cdot Q_x\mathbf{i}$ and $P_x\mathbf{i} \cdot Q_y\mathbf{j}$. However, from the definition of the scalar product it follows that the scalar products of the unit vectors are either zero or one.

$$\begin{array}{lll} \mathbf{i}\cdot\mathbf{i} = 1 & \mathbf{j}\cdot\mathbf{j} = 1 & \mathbf{k}\cdot\mathbf{k} = 1 \\ \mathbf{i}\cdot\mathbf{j} = 0 & \mathbf{j}\cdot\mathbf{k} = 0 & \mathbf{k}\cdot\mathbf{i} = 0 \end{array} \tag{3.29}$$

Thus, the expression obtained for $\mathbf{P}\cdot\mathbf{Q}$ reduces to

$$\mathbf{P}\cdot\mathbf{Q} = P_xQ_x + P_yQ_y + P_zQ_z \tag{3.30}$$

In the particular case when **P** and **Q** are equal, we note that

$$\mathbf{P}\cdot\mathbf{P} = P_x^2 + P_y^2 + P_z^2 = P^2 \tag{3.31}$$

Applications

1. *Angle formed by two given vectors.* Let two vectors be given in terms of their components:

$$\mathbf{P} = P_x\mathbf{i} + P_y\mathbf{j} + P_z\mathbf{k}$$
$$\mathbf{Q} = Q_x\mathbf{i} + Q_y\mathbf{j} + Q_z\mathbf{k}$$

To determine the angle formed by the two vectors, we equate the expressions obtained in (3.24) and (3.30) for their scalar product and write

$$PQ\cos\theta = P_xQ_x + P_yQ_y + P_zQ_z$$

Solving for $\cos\theta$, we have

$$\cos\theta = \frac{P_xQ_x + P_yQ_y + P_zQ_z}{PQ} \tag{3.32}$$

2. *Projection of a vector on a given axis.* Consider a vector **P** forming an angle θ with an axis, or directed line, OL (Fig. 3.21). The *projection of* **P** *on the axis* OL is defined as the scalar

$$P_{OL} = P\cos\theta \tag{3.33}$$

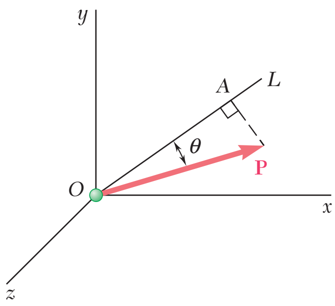


Fig. 3.21

We note that the projection P_{OL} is equal in absolute value to the length of the segment OA; it will be positive if OA has the same sense as the axis OL, that is, if θ is acute, and negative otherwise. If **P** and OL are at a right angle, the projection of **P** on OL is zero.

Consider now a vector **Q** directed along OL and of the same sense as OL (Fig. 3.22). The scalar product of **P** and **Q** can be expressed as

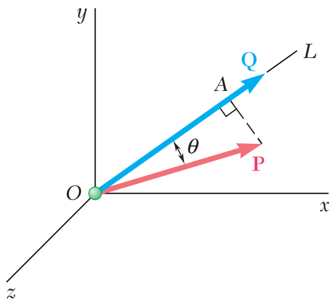


Fig. 3.22

$$\mathbf{P}\cdot\mathbf{Q} = PQ\cos\theta = P_{OL}Q \tag{3.34}$$

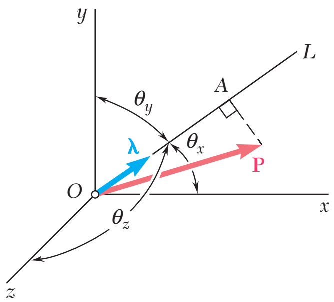


Fig. 3.23

from which it follows that

$$P_{OL} = \frac{\mathbf{P} \cdot \mathbf{Q}}{Q} = \frac{P_xQ_x + P_yQ_y + P_zQ_z}{Q} \tag{3.35}$$

In the particular case when the vector selected along OL is the unit vector $\boldsymbol{\lambda}$ (Fig. 3.23), we write

$$\boxed{P_{OL} = \mathbf{P} \cdot \boldsymbol{\lambda}} \tag{3.36}$$

Resolving $\mathbf{P}$ and $\boldsymbol{\lambda}$ into rectangular components and recalling from Sec. 2.12 that the components of $\boldsymbol{\lambda}$ along the coordinate axes are respectively equal to the direction cosines of OL, we express the projection of $\mathbf{P}$ on OL as

$$P_{OL} = P_x \cos\theta_x + P_y \cos\theta_y + P_z \cos\theta_z \tag{3.37}$$

where θ_x, θ_y, and θ_z denote the angles that the axis OL forms with the coordinate axes.

3.10 MIXED TRIPLE PRODUCT OF THREE VECTORS

We define the *mixed triple product* of the three vectors $\mathbf{S}$, $\mathbf{P}$, and $\mathbf{Q}$ as the scalar expression

$$\boxed{\mathbf{S} \cdot (\mathbf{P} \times \mathbf{Q})} \tag{3.38}$$

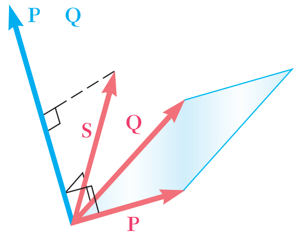


Fig. 3.24

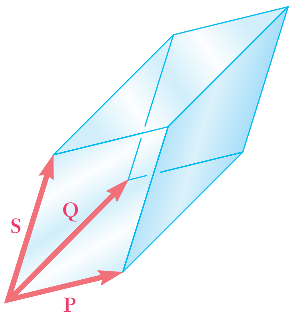


Fig. 3.25

obtained by forming the scalar product of $\mathbf{S}$ with the vector product of $\mathbf{P}$ and $\mathbf{Q}$.

A simple geometrical interpretation can be given for the mixed triple product of $\mathbf{S}$, $\mathbf{P}$, and $\mathbf{Q}$ (Fig. 3.24). We first recall from Sec. 3.4 that the vector $\mathbf{P} \times \mathbf{Q}$ is perpendicular to the plane containing $\mathbf{P}$ and $\mathbf{Q}$ and that its magnitude is equal to the area of the parallelogram which has $\mathbf{P}$ and $\mathbf{Q}$ for sides. On the other hand, Eq. (3.34) indicates that the scalar product of $\mathbf{S}$ and $\mathbf{P} \times \mathbf{Q}$ can be obtained by multiplying the magnitude of $\mathbf{P} \times \mathbf{Q}$ (i.e., the area of the parallelogram defined by $\mathbf{P}$ and $\mathbf{Q}$) by the projection of $\mathbf{S}$ on the vector $\mathbf{P} \times \mathbf{Q}$ (i.e., by the projection of $\mathbf{S}$ on the normal to the plane containing the parallelogram). The mixed triple product is thus equal, in absolute value, to the volume of the parallelepiped having the vectors $\mathbf{S}$, $\mathbf{P}$, and $\mathbf{Q}$ for sides (Fig. 3.25). We note that the sign of the mixed triple product will be positive if $\mathbf{S}$, $\mathbf{P}$, and $\mathbf{Q}$ form a right-handed triad and negative if they form a left-handed triad [that is, $\mathbf{S} \cdot (\mathbf{P} \times \mathbf{Q})$ will be negative if the rotation which brings $\mathbf{P}$ into line with $\mathbf{Q}$ is observed as clockwise from the tip of $\mathbf{S}$]. The mixed triple product will be zero if $\mathbf{S}$, $\mathbf{P}$, and $\mathbf{Q}$ are coplanar.

Since the parallelepiped defined in the preceding paragraph is independent of the order in which the three vectors are taken, the six mixed triple products which can be formed with **S**, **P**, and **Q** will all have the same absolute value, although not the same sign. It is easily shown that

$$\begin{aligned}\mathbf{S}\cdot(\mathbf{P}\times\mathbf{Q}) &= \mathbf{P}\cdot(\mathbf{Q}\times\mathbf{S}) = \mathbf{Q}\cdot(\mathbf{S}\times\mathbf{P})\\ &= -\mathbf{S}\cdot(\mathbf{Q}\times\mathbf{P}) = -\mathbf{P}\cdot(\mathbf{S}\times\mathbf{Q}) = -\mathbf{Q}\cdot(\mathbf{P}\times\mathbf{S})\end{aligned} \tag{3.39}$$

Arranging in a circle and in counterclockwise order the letters representing the three vectors (Fig. 3.26), we observe that the sign of the mixed triple product remains unchanged if the vectors are permuted in such a way that they are still read in counterclockwise order. Such a permutation is said to be a *circular permutation*. It also follows from Eq. (3.39) and from the commutative property of scalar products that the mixed triple product of **S**, **P**, and **Q** can be defined equally well as $\mathbf{S}\cdot(\mathbf{P}\times\mathbf{Q})$ or $(\mathbf{S}\times\mathbf{P})\cdot\mathbf{Q}$.

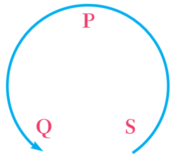


Fig. 3.26

The mixed triple product of the vectors **S**, **P**, and **Q** can be expressed in terms of the rectangular components of these vectors. Denoting $\mathbf{P}\times\mathbf{Q}$ by **V** and using formula (3.30) to express the scalar product of **S** and **V**, we write

$$\mathbf{S}\cdot(\mathbf{P}\times\mathbf{Q}) = \mathbf{S}\cdot\mathbf{V} = S_xV_x + S_yV_y + S_zV_z$$

Substituting from the relations (3.9) for the components of **V**, we obtain

$$\begin{aligned}\mathbf{S}\cdot(\mathbf{P}\times\mathbf{Q}) = S_x(P_yQ_z - P_zQ_y) + S_y(P_zQ_x - P_xQ_z)\\ + S_z(P_xQ_y - P_yQ_x)\end{aligned} \tag{3.40}$$

This expression can be written in a more compact form if we observe that it represents the expansion of a determinant:

$$\mathbf{S}\cdot(\mathbf{P}\times\mathbf{Q}) = \begin{vmatrix} S_x & S_y & S_z \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} \tag{3.41}$$

By applying the rules governing the permutation of rows in a determinant, we could easily verify the relations (3.39) which were derived earlier from geometrical considerations.

3.11 MOMENT OF A FORCE ABOUT A GIVEN AXIS

Now that we have further increased our knowledge of vector algebra, we can introduce a new concept, the concept of *moment of a force about an axis*. Consider again a force **F** acting on a rigid body and the moment $\mathbf{M}_O$ of that force about *O* (Fig. 3.27). Let *OL* be

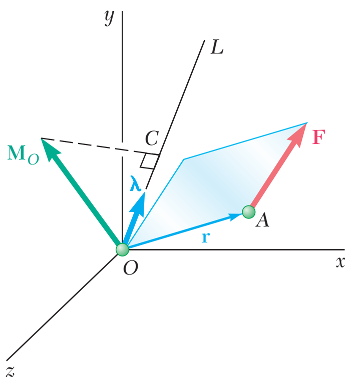


Fig. 3.27

an axis through O; *we define the moment* M_{OL} *of* **F** *about* OL *as the projection* OC *of the moment* $\mathbf{M}_O$ *onto the axis* OL. Denoting by $\boldsymbol{\lambda}$ the unit vector along OL and recalling from Secs. 3.9 and 3.6, respectively, the expressions (3.36) and (3.11) obtained for the projection of a vector on a given axis and for the moment $\mathbf{M}_O$ of a force **F**, we write

$$M_{OL} = \boldsymbol{\lambda} \cdot \mathbf{M}_O = \boldsymbol{\lambda} \cdot (\mathbf{r} \times \mathbf{F}) \tag{3.42}$$

which shows that the moment M_{OL} of **F** about the axis OL is the scalar obtained by forming the mixed triple product of $\boldsymbol{\lambda}$, **r**, and **F**. Expressing M_{OL} in the form of a determinant, we write

$$M_{OL} = \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} \tag{3.43}$$

where $\lambda x, \lambda y, \lambda z$ = direction cosines of axis OL
x, y, z = coordinates of point of application of **F**
F_x, F_y, F_z = components of force **F**

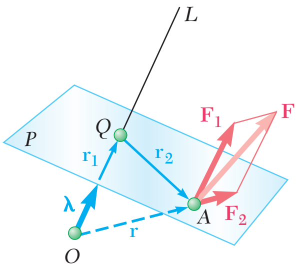


Fig. 3.28

The physical significance of the moment M_{OL} of a force **F** about a fixed axis OL becomes more apparent if we resolve **F** into two rectangular components $\mathbf{F}_1$ and $\mathbf{F}_2$, with $\mathbf{F}_1$ parallel to OL and $\mathbf{F}_2$ lying in a plane P perpendicular to OL (Fig. 3.28). Resolving **r** similarly into two components $\mathbf{r}_1$ and $\mathbf{r}_2$ and substituting for **F** and **r** into (3.42), we write

$$\begin{aligned} M_{OL} &= \boldsymbol{\lambda} \cdot [(\mathbf{r}_1 + \mathbf{r}_2) \times (\mathbf{F}_1 + \mathbf{F}_2)] \\ &= \boldsymbol{\lambda} \cdot (\mathbf{r}_1 \times \mathbf{F}_1) + \boldsymbol{\lambda} \cdot (\mathbf{r}_1 \times \mathbf{F}_2) + \boldsymbol{\lambda} \cdot (\mathbf{r}_2 \times \mathbf{F}_1) + \boldsymbol{\lambda} \cdot (\mathbf{r}_2 \times \mathbf{F}_2) \end{aligned}$$

Noting that all of the mixed triple products except the last one are equal to zero, since they involve vectors which are coplanar when drawn from a common origin (Sec. 3.10), we have

$$M_{OL} = \boldsymbol{\lambda} \cdot (\mathbf{r}_2 \times \mathbf{F}_2) \tag{3.44}$$

The vector product $\mathbf{r}_2 \times \mathbf{F}_2$ is perpendicular to the plane P and represents the moment of the component $\mathbf{F}_2$ of **F** about the point Q where OL intersects P. Therefore, the scalar M_{OL}, which will be positive if $\mathbf{r}_2 \times \mathbf{F}_2$ and OL have the same sense and negative otherwise, measures the tendency of $\mathbf{F}_2$ to make the rigid body rotate about the fixed axis OL. Since the other component $\mathbf{F}_1$ of **F** does not tend to make the body rotate about OL, we conclude that *the moment* M_{OL} *of* **F** *about* OL *measures the tendency of the force* **F** *to impart to the rigid body a motion of rotation about the fixed axis* OL.

It follows from the definition of the moment of a force about an axis that the moment of **F** about a coordinate axis is equal to the component of $\mathbf{M}_O$ along that axis. Substituting successively each of the unit vectors **i**, **j**, and **k** for $\boldsymbol{\lambda}$ in (3.42), we observe that the expressions thus obtained for the *moments of* **F** *about the coordinate axes* are respectively equal to the expressions obtained in Sec. 3.8 for the components of the moment $\mathbf{M}_O$ of **F** about O:

$$\begin{aligned} M_x &= yF_z - zF_y \\ M_y &= zF_x - xF_z \\ M_z &= xF_y - yF_x \end{aligned} \qquad (3.18)$$

We observe that just as the components F_x, F_y, and F_z of a force **F** acting on a rigid body measure, respectively, the tendency of **F** to move the rigid body in the x, y, and z directions, the moments M_x, M_y, and M_z of **F** about the coordinate axes measure the tendency of **F** to impart to the rigid body a motion of rotation about the x, y, and z axes, respectively.

More generally, the moment of a force **F** applied at A about an axis which does not pass through the origin is obtained by choosing an arbitrary point B on the axis (Fig. 3.29) and determining the projection on the axis BL of the moment $\mathbf{M}_B$ of **F** about B. We write

$$M_{BL} = \boldsymbol{\lambda} \cdot \mathbf{M}_B = \boldsymbol{\lambda} \cdot (\mathbf{r}_{A/B} \times \mathbf{F}) \qquad (3.45)$$

where $\mathbf{r}_{A/B} = \mathbf{r}_A - \mathbf{r}_B$ represents the vector drawn from B to A. Expressing M_{BL} in the form of a determinant, we have

$$M_{BL} = \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ x_{A/B} & y_{A/B} & z_{A/B} \\ F_x & F_y & F_z \end{vmatrix} \qquad (3.46)$$

where $\lambda_x, \lambda_y, \lambda_z$ = direction cosines of axis BL
$x_{A/B} = x_A - x_B \qquad y_{A/B} = y_A - y_B \qquad z_{A/B} = z_A - z_B$
F_x, F_y, F_z = components of force **F**

It should be noted that the result obtained is independent of the choice of the point B on the given axis. Indeed, denoting by M_{CL} the result obtained with a different point C, we have

$$\begin{aligned} M_{CL} &= \boldsymbol{\lambda} \cdot [(\mathbf{r}_A - \mathbf{r}_C) \times \mathbf{F}] \\ &= \boldsymbol{\lambda} \cdot [(\mathbf{r}_A - \mathbf{r}_B) \times \mathbf{F}] + \boldsymbol{\lambda} \cdot [(\mathbf{r}_B - \mathbf{r}_C) \times \mathbf{F}] \end{aligned}$$

But, since the vectors $\boldsymbol{\lambda}$ and $\mathbf{r}_B - \mathbf{r}_C$ lie in the same line, the volume of the parallelepiped having the vectors $\boldsymbol{\lambda}$, $\mathbf{r}_B - \mathbf{r}_C$, and **F** for its sides is zero, as is the mixed triple product of these three vectors (Sec. 3.10). The expression obtained for M_{CL} thus reduces to its first term, which is the expression used earlier to define M_{BL}. In addition, it follows from Sec. 3.6 that, when computing the moment of **F** about the given axis, A can be any point on the line of action of **F**.

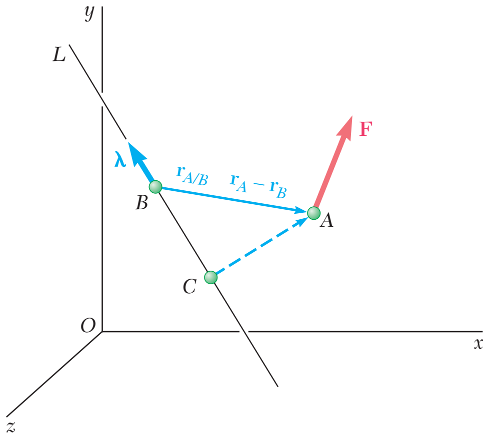
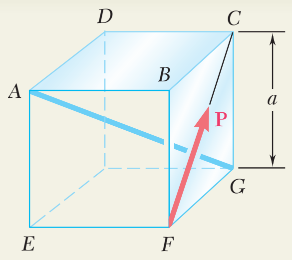
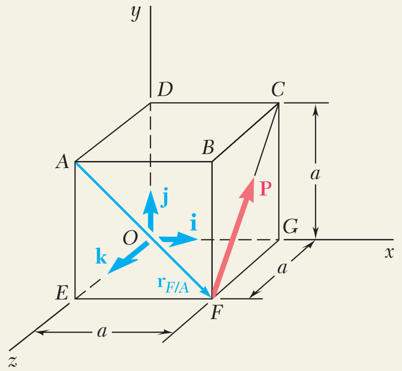


Fig. 3.29



SAMPLE PROBLEM 3.5

A cube of side a is acted upon by a force $\mathbf{P}$ as shown. Determine the moment of $\mathbf{P}$ (*a*) about A, (*b*) about the edge AB, (*c*) about the diagonal AG of the cube. (*d*) Using the result of part *c*, determine the perpendicular distance between AG and FC.



SOLUTION

a. Moment about *A*. Choosing x, y, and z axes as shown, we resolve into rectangular components the force $\mathbf{P}$ and the vector $\mathbf{r}_{F/A} = \overrightarrow{AF}$ drawn from A to the point of application F of $\mathbf{P}$.

$$\mathbf{r}_{F/A} = a\mathbf{i} - a\mathbf{j} = a(\mathbf{i} - \mathbf{j})$$
$$\mathbf{P} = (P/\sqrt{2})\mathbf{j} - (P/\sqrt{2})\mathbf{k} = (P/\sqrt{2})(\mathbf{j} - \mathbf{k})$$

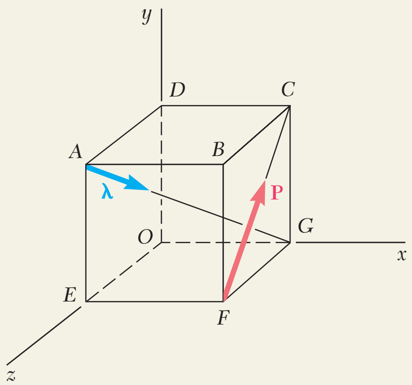
The moment of $\mathbf{P}$ about A is

$$\mathbf{M}_A = \mathbf{r}_{F/A} \times \mathbf{P} = a(\mathbf{i} - \mathbf{j}) \times (P/\sqrt{2})(\mathbf{j} - \mathbf{k})$$
$$\mathbf{M}_A = (aP/\sqrt{2})(\mathbf{i} + \mathbf{j} + \mathbf{k}) \quad ◀$$

b. Moment about *AB*. Projecting $\mathbf{M}_A$ on AB, we write

$$M_{AB} = \mathbf{i} \cdot \mathbf{M}_A = \mathbf{i} \cdot (aP/\sqrt{2})(\mathbf{i} + \mathbf{j} + \mathbf{k})$$
$$M_{AB} = aP/\sqrt{2} \quad ◀$$

We verify that, since AB is parallel to the x axis, M_{AB} is also the x component of the moment $\mathbf{M}_A$.

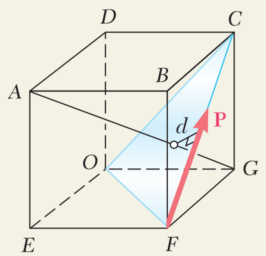


c. Moment about Diagonal *AG*. The moment of $\mathbf{P}$ about AG is obtained by projecting $\mathbf{M}_A$ on AG. Denoting by $\boldsymbol{\lambda}$ the unit vector along AG, we have

$$\boldsymbol{\lambda} = \frac{\overrightarrow{AG}}{AG} = \frac{a\mathbf{i} - a\mathbf{j} - a\mathbf{k}}{a\sqrt{3}} = (1/\sqrt{3})(\mathbf{i} - \mathbf{j} - \mathbf{k})$$
$$M_{AG} = \boldsymbol{\lambda} \cdot \mathbf{M}_A = (1/\sqrt{3})(\mathbf{i} - \mathbf{j} - \mathbf{k}) \cdot (aP/\sqrt{2})(\mathbf{i} + \mathbf{j} + \mathbf{k})$$
$$M_{AG} = (aP/\sqrt{6})(1 - 1 - 1) \qquad M_{AG} = -aP/\sqrt{6} \quad ◀$$

Alternative Method. The moment of $\mathbf{P}$ about AG can also be expressed in the form of a determinant:

$$M_{AG} = \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ x_{F/A} & y_{F/A} & z_{F/A} \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} 1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} \\ a & -a & 0 \\ 0 & P/\sqrt{2} & -P/\sqrt{2} \end{vmatrix} = -aP/\sqrt{6}$$



d. Perpendicular Distance between *AG* and *FC*. We first observe that $\mathbf{P}$ is perpendicular to the diagonal AG. This can be checked by forming the scalar product $\mathbf{P} \cdot \boldsymbol{\lambda}$ and verifying that it is zero:

$$\mathbf{P} \cdot \boldsymbol{\lambda} = (P/\sqrt{2})(\mathbf{j} - \mathbf{k}) \cdot (1/\sqrt{3})(\mathbf{i} - \mathbf{j} - \mathbf{k}) = (P\sqrt{6})(0 - 1 + 1) = 0$$

The moment M_{AG} can then be expressed as $-Pd$, where d is the perpendicular distance from AG to FC. (The negative sign is used since the rotation imparted to the cube by $\mathbf{P}$ appears as clockwise to an observer at G.) Recalling the value found for M_{AG} in part *c*,

$$M_{AG} = -Pd = -aP/\sqrt{6} \qquad d = a/\sqrt{6} \quad ◀$$

PROBLEMS

3.25 Given the vectors $\mathbf{P} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{Q} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$, and $\mathbf{S} = -4\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, compute the scalar products $\mathbf{P} \cdot \mathbf{Q}$, $\mathbf{P} \cdot \mathbf{S}$, and $\mathbf{Q} \cdot \mathbf{S}$.

3.26 Form the scalar product $\mathbf{P}_1 \cdot \mathbf{P}_2$, and use the result obtained to prove the identity $\cos(\theta_1 - \theta_2) = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2$.

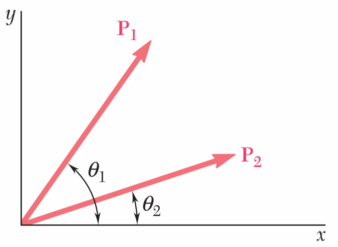


Fig. P3.26

3.27 Knowing that the tension in cable *BC* is 1400 N, determine (*a*) the angle between cable *BC* and the boom *AB*, (*b*) the projection on *AB* of the force exerted by cable *BC* at point *B*.

3.28 Knowing that the tension in cable *BD* is 900 N, determine (*a*) the angle between cable *BD* and the boom *AB*, (*b*) the projection on *AB* of the force exerted by cable *BD* at point *B*.

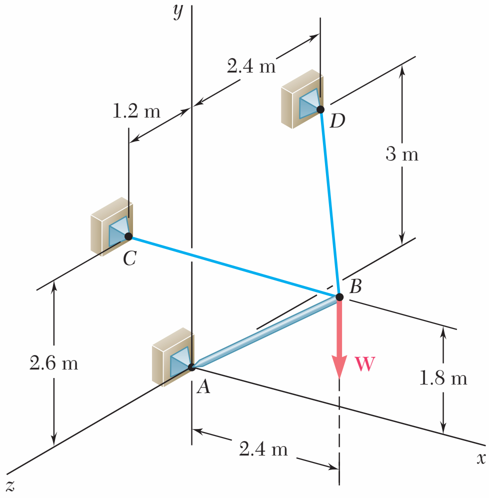


Fig. P3.27 and P3.28

3.29 Three cables are used to support a container as shown. Determine the angle formed by cables *AB* and *AD*.

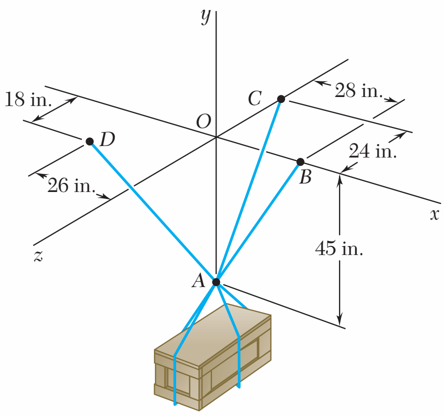


Fig. P3.29 and *P3.30*

3.30 Three cables are used to support a container as shown. Determine the angle formed by cables *AC* and *AD*.

3.31 The 500-mm tube *AB* can slide along a horizontal rod. The ends *A* and *B* of the tube are connected by elastic cords to the fixed point *C*. For the position corresponding to $x = 275$ mm, determine the angle formed by the two cords, (*a*) using Eq. (3.32), (*b*) applying the law of cosines to triangle *ABC*.

3.32 Solve Prob. 3.31 for the position corresponding to $x = 100$ mm.

3.33 Given the vectors $\mathbf{P} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{Q} = 2\mathbf{i} + \mathbf{j}$, and $\mathbf{S} = \mathbf{i}$, compute $\mathbf{P} \cdot (\mathbf{Q} \times \mathbf{S})$, $(\mathbf{P} \times \mathbf{Q}) \cdot \mathbf{S}$, and $(\mathbf{S} \times \mathbf{Q}) \cdot \mathbf{P}$.

3.34 Given the vectors $\mathbf{P} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, $\mathbf{Q} = -\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, and $\mathbf{S} = -3\mathbf{i} - \mathbf{j} + S_z\mathbf{k}$, determine the value of S_z for which the three vectors are coplanar.

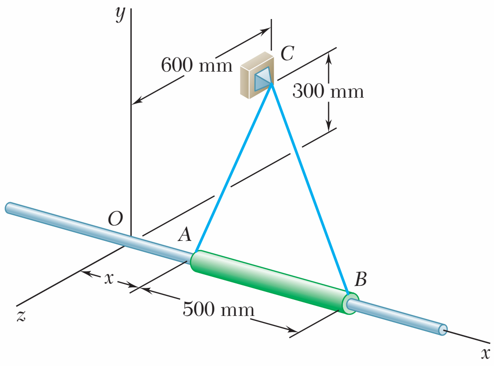


Fig. P3.31

3.35 The jib crane is oriented so that the boom DA is parallel to the x axis. At the instant shown, the tension in cable AB is 13 kN. Determine the moment about each of the coordinate axes of the force exerted on A by the cable AB.

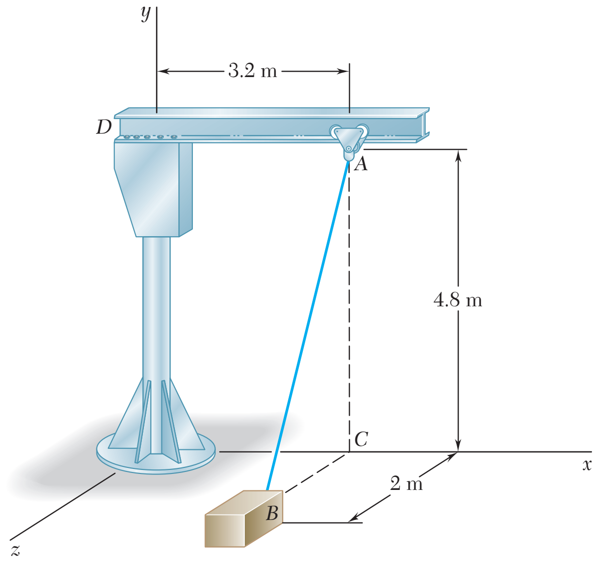


Fig. P3.35 and *P3.36*

3.36 The jib crane is oriented so that the boom DA is parallel to the x axis. Determine the maximum permissible tension in the cable AB if the absolute values of the moments about the coordinate axes of the force exerted on A must be as follows: $|M_x| \leq 10$ kN · m, $|M_y| \leq 6$ kN · m, and $|M_z| \leq 16$ kN · m.

3.37 The primary purpose of the crank shown is to produce a moment about the x axis. Show that a single force acting at A and having moment M_x different from zero about the x axis must also have a moment different from zero about at least one of the other coordinate axes.

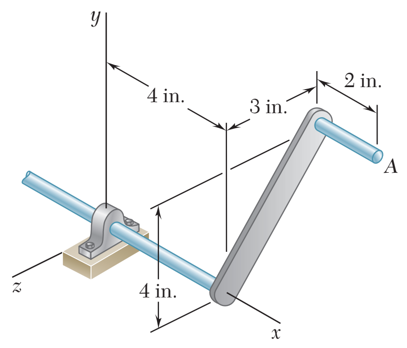


Fig. P3.37 and *P3.38*

3.38 A single force **F** of unknown magnitude and direction acts at point A of the crank shown. Determine the moment M_x of **F** about the x axis knowing that $M_y = +180$ lb · in. and $M_z = -320$ lb · in.

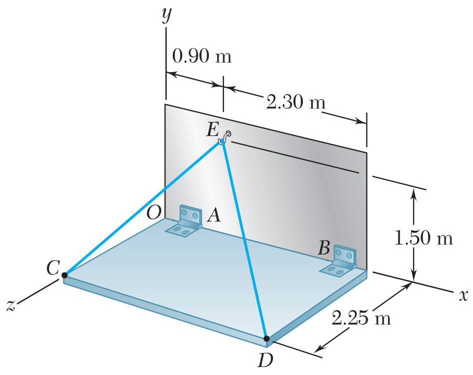


Fig. P3.39

3.39 The rectangular platform is hinged at A and B and supported by a cable that passes over a frictionless hook at E. Knowing that the tension in the cable is 1349 N, determine the moment about each of the coordinate axes of the force exerted by the cable at C.

3.40 For the platform of Prob. 3.39, determine the moment about each of the coordinate axes of the force exerted by the cable at D.

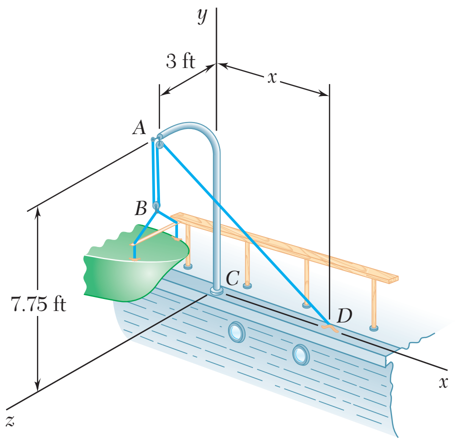


Fig. P3.41

3.41 A small boat hangs from two davits, one of which is shown in the figure. It is known that the moment about the z axis of the resultant force $\mathbf{R}_A$ exerted on the davit at A must not exceed 279 lb · ft in absolute value. Determine the largest allowable tension in the line $ABAD$ when $x = 6$ ft.

3.42 For the davit of Prob. 3.41, determine the largest allowable distance x when the tension in the line $ABAD$ is 60 lb.

3.43 A force **P** of magnitude 25 lb acts on a bent rod as shown. Determine the moment of **P** about (*a*) a line joining points *C* and *F*, (*b*) a line joining points *O* and *C*.

3.44 A force **P** of magnitude 25 lb acts on a bent rod as shown. Determine the moment of **P** about (*a*) a line joining points *A* and *C*, (*b*) a line joining points *A* and *D*.

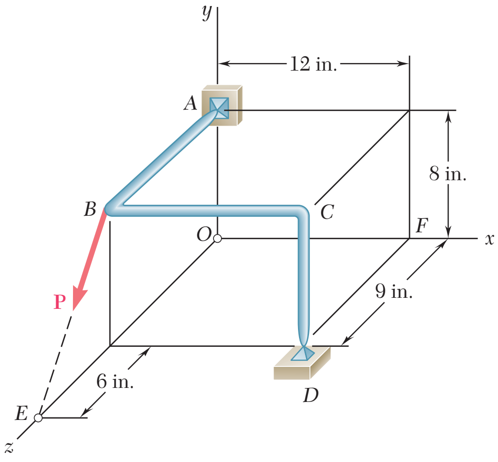


Fig. P3.43 and P3.44

3.45 Two rods are welded together to form a T-shaped lever that is acted upon by a 650-N force as shown. Determine the moment of the force about rod *AB*.

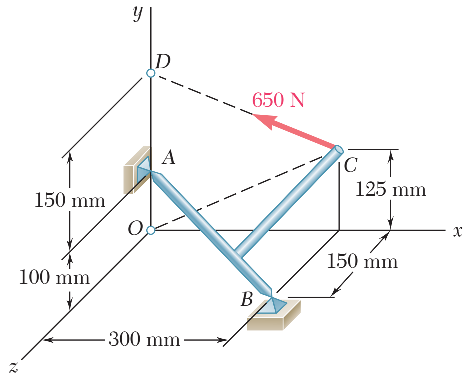


Fig. *P3.45*

3.46 The rectangular plate *ABCD* is held by hinges along its edge *AD* and by the wire *BE*. Knowing that the tension in the wire is 546 N, determine the moment about *AD* of the force exerted by the wire at point *B*.

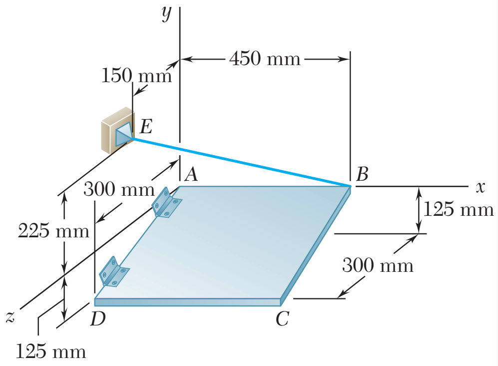


Fig. P3.46

3.47 The 23-in. vertical rod *CD* is welded to the midpoint *C* of the 50-in. rod *AB*. Determine the moment about *AB* of the 235-lb force **P**.

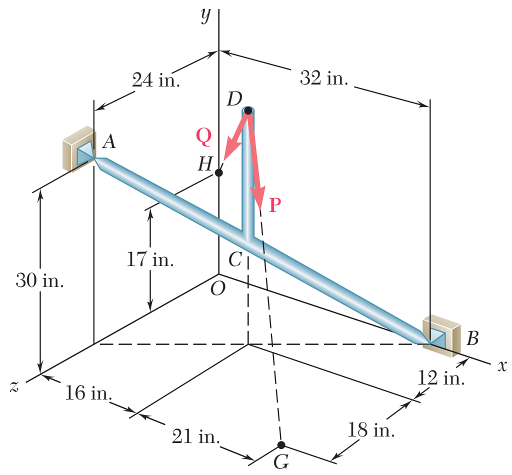


Fig. *P3.47* and P3.48

3.48 The 23-in. vertical rod *CD* is welded to the midpoint *C* of the 50-in. rod *AB*. Determine the moment about *AB* of the 174-lb force **Q**.

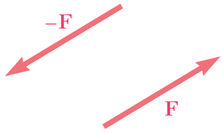


Fig. 3.30

3.12 MOMENT OF A COUPLE

Two forces **F** *and* −**F** *having the same magnitude, parallel lines of action, and opposite sense are said to form a couple* (Fig. 3.30). Clearly, the sum of the components of the two forces in any direction is zero. The sum of the moments of the two forces about a given point, however, is not zero. While the two forces will not translate the body on which they act, they will tend to make it rotate.

Denoting by $\mathbf{r}_A$ and $\mathbf{r}_B$, respectively, the position vectors of the points of application of **F** and −**F** (Fig. 3.31), we find that the sum of the moments of the two forces about O is

$$\mathbf{r}_A \times \mathbf{F} + \mathbf{r}_B \times (-\mathbf{F}) = (\mathbf{r}_A - \mathbf{r}_B) \times \mathbf{F}$$

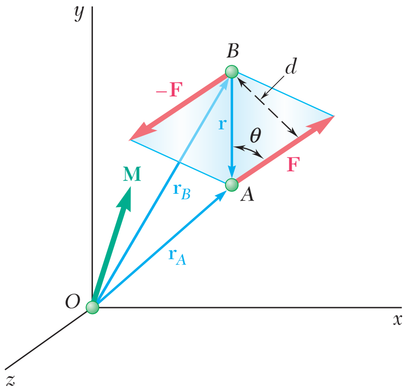


Fig. 3.31

Setting $\mathbf{r}_A - \mathbf{r}_B = \mathbf{r}$, where $\mathbf{r}$ is the vector joining the points of application of the two forces, we conclude that the sum of the moments of **F** and −**F** about O is represented by the vector

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} \tag{3.47}$$

The vector **M** is called the *moment of the couple;* it is a vector perpendicular to the plane containing the two forces, and its magnitude is

$$M = rF \sin\theta = Fd \tag{3.48}$$

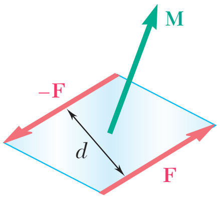


Fig. 3.32

where d is the perpendicular distance between the lines of action of **F** and −**F**. The sense of **M** is defined by the right-hand rule.

Since the vector **r** in (3.47) is independent of the choice of the origin O of the coordinate axes, we note that the same result would have been obtained if the moments of **F** and −**F** had been computed about a different point O'. Thus, the moment **M** of a couple is a *free vector* (Sec. 2.3) which can be applied at any point (Fig. 3.32).

From the definition of the moment of a couple, it also follows that two couples, one consisting of the forces $\mathbf{F}_1$ and $-\mathbf{F}_1$, the other of the forces $\mathbf{F}_2$ and $-\mathbf{F}_2$ (Fig. 3.33), will have equal moments if

$$F_1 d_1 = F_2 d_2 \tag{3.49}$$

and if the two couples lie in parallel planes (or in the same plane) and have the same sense.

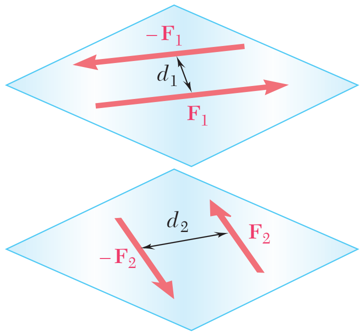


Fig. 3.33