Chapter 3

Block Diagrams and Signal-Flow Graphs

Automatic Control Systems, 9th Edition

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Introduction

- In this chapter, we discuss graphical techniques for modeling control systems and their underlying mathematics.
- We also utilize the block diagram reduction techniques and the Mason's gain formula to find the transfer function of the overall control system.
- Later on in Chapters 4 and 5, we use the material presented in this chapter and Chapter 2 to fully model and study the performance of various control systems.

Objectives of this Chapter

1. To study block diagrams, their components, and their underlying mathematics.
2. To obtain transfer function of systems through block diagram manipulation and reduction.
3. To introduce the signal-flow graphs.
4. To establish a parallel between block diagrams and signal-flow graphs.
5. To use Mason's gain formula for finding transfer function of systems.
6. To introduce state diagrams.
7. To demonstrate the MATLAB tools using case studies.

3-1 BLOCK DIAGRAMS

Block diagrams provide a better understanding of the composition and interconnection of the components of a system. It can be used, together with transfer functions, to describe the cause-and-effect relationships throughout the system.

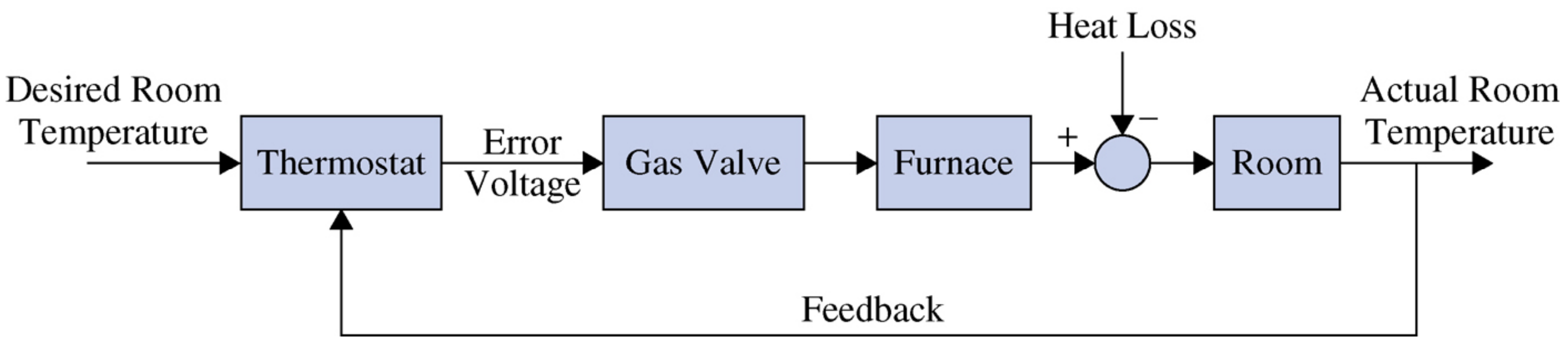


Figure 3-1 A simplified block diagram representation of a heating system.

3-1-1 Typical Elements of Block Diagrams in Control Systems

The common elements in block diagrams of most control systems include:

- Comparators
- Blocks representing individual component transfer functions, including:
  - Reference sensor (or input sensor)
  - Output sensor
- Actuator
- Controller
- Plant (the component whose variables are to be controlled)
- Input or reference signals
- Output signals
- Disturbance signal
- Feedback loops

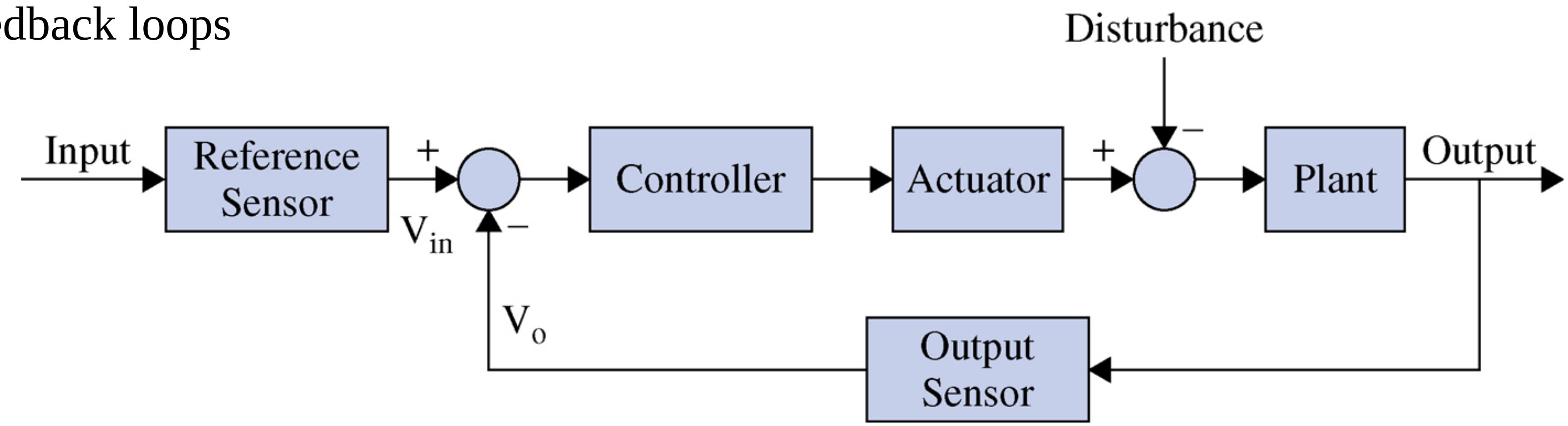
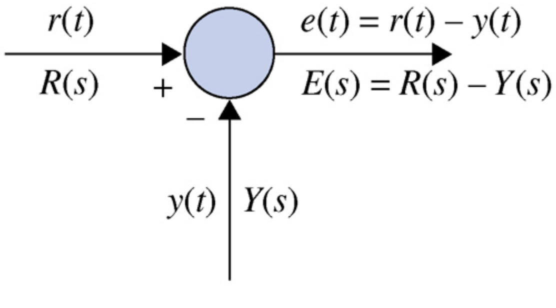
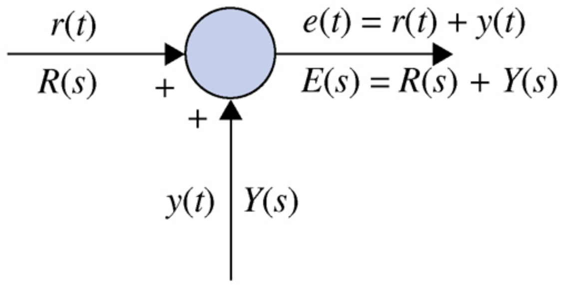


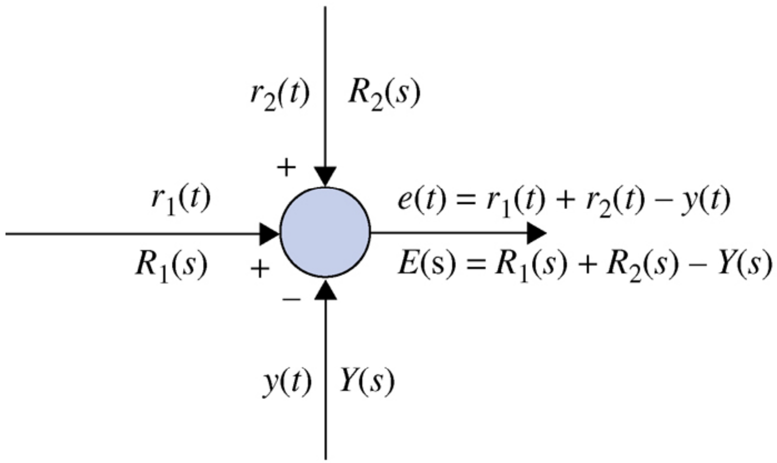
Figure 3-3 Block diagram representation of a general control system.



(a)



(b)



(c)

A **comparator** performs addition and subtraction

Figure 3-4 Block-diagram elements of typical sensing devices of control systems. (a) Subtraction. (b) Addition. (c) Addition and subtraction.

$$X(s) = G(s)U(s) \quad (3\text{-}4)$$

If signal $X(s)$ is the output and signal $U(s)$ denotes the input, the transfer function of the block in Fig. 3-5 is

$$G(s) = \frac{X(s)}{U(s)} \quad (3\text{-}5)$$

Typical block elements that appear in the block diagram representation of most control systems include **plant**, **controller**, **actuator**, and **sensor**.

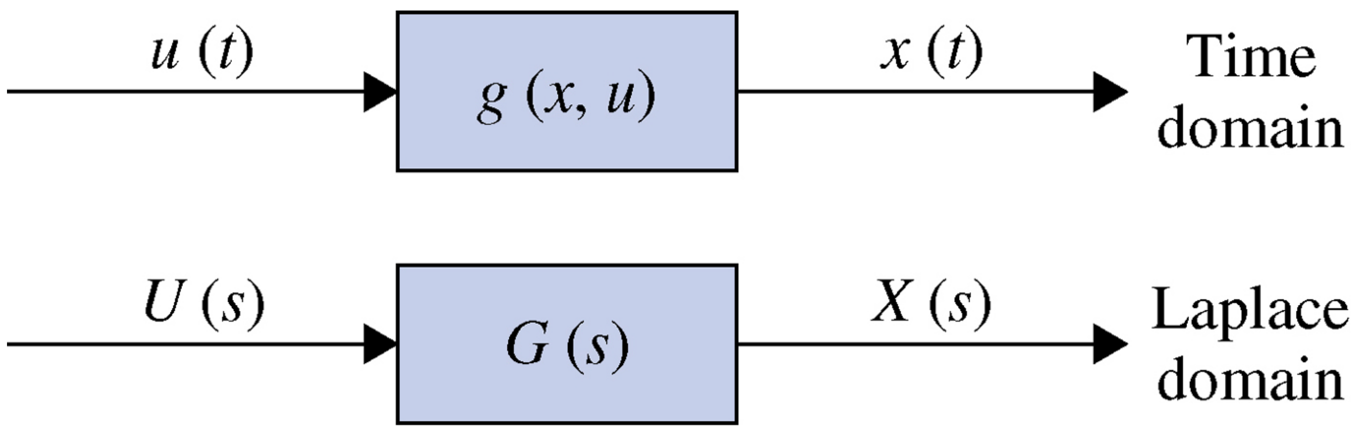


Figure 3-5 Time and Laplace domain block diagrams.

EXAMPLE 3-1-1

$$X(s) = A(s)G_2(s)$$
$$A(s) = U(s)G_1(s)$$
$$X(s) = G_1(s)G_2(s)U(s)$$
$$G(s) = \frac{X(s)}{U(s)}$$

$$G(s) = G_1(s)G_2(s) \qquad (3\text{-}6)$$

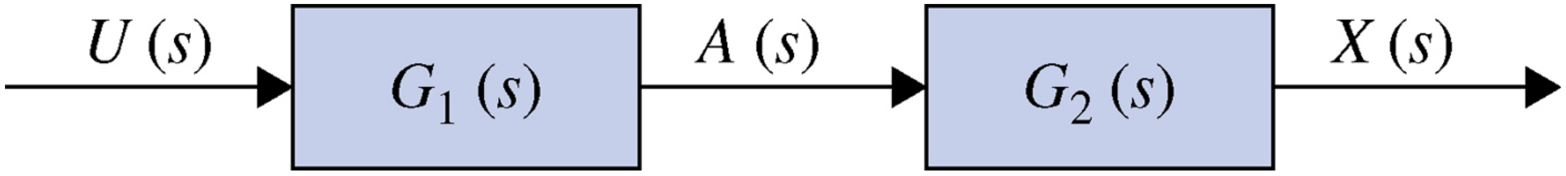


Figure 3-6 Block diagrams $G_1(s)$ and $G_2(s)$ connected in series.

EXAMPLE 3-1-2

$$A_1(s) = U(s)$$

$$A_2(s) = A_1(s)G_1(s)$$

$$A_3(s) = A_1(s)G_2(s)$$

$$X(s) = A_2(s) + A_3(s)$$

$$X(s) = U(s)(G_1(s) + G_2(s))$$

$$G(s) = \frac{X(s)}{U(s)}$$

$$G(s) = G_1(s) + G_2(s) \qquad (3\text{-}7)$$

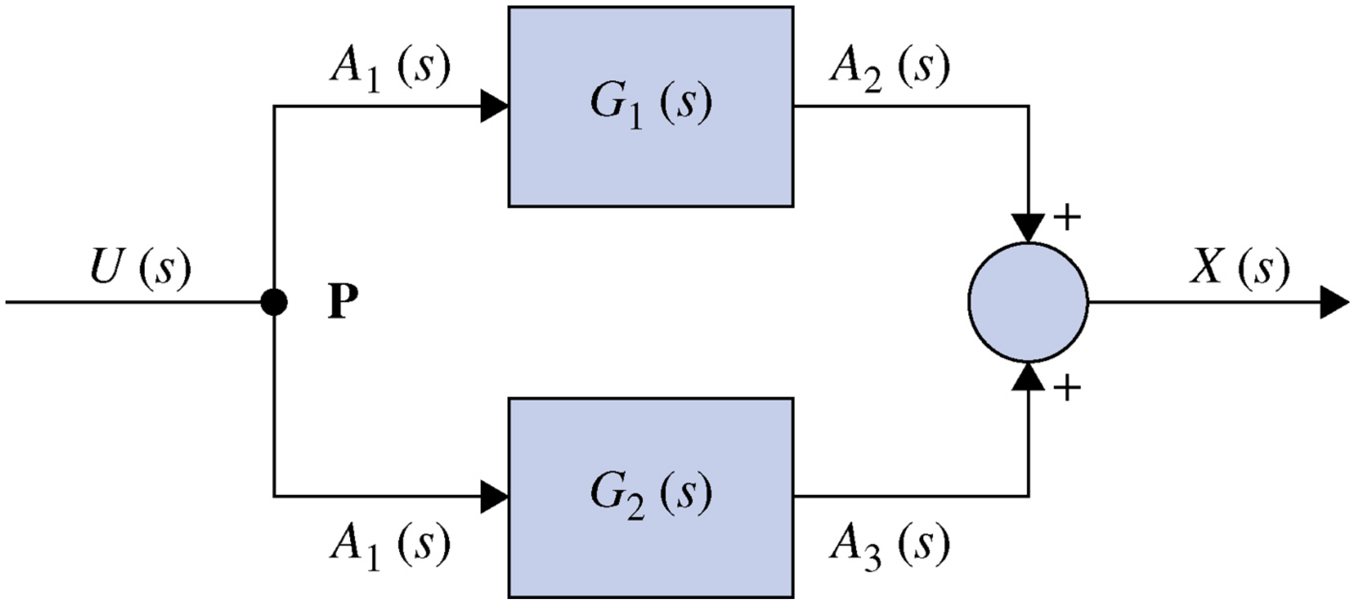


Figure 3-7 Block diagrams $G_1(s)$ and $G_2(s)$ connected in parallel.

Basic block diagram of a feedback control system

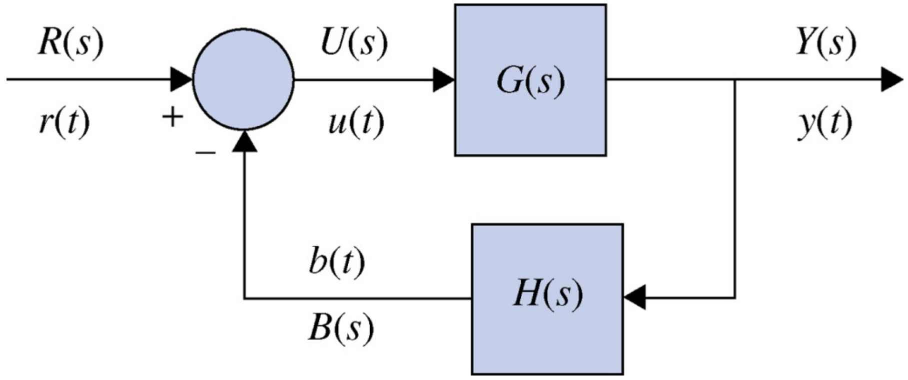


Figure 3-8 Basic block diagram of a feedback control system.

$$Y(s) = G(s)U(s) \quad (3\text{-}8)$$

$$B(s) = H(s)Y(s) \quad (3\text{-}9)$$

$$U(s) = R(s) - B(s) \quad (3\text{-}10)$$

$$Y(s) = G(s)R(s) - G(s)H(s) \quad (3\text{-}11)$$

negative feedback loop

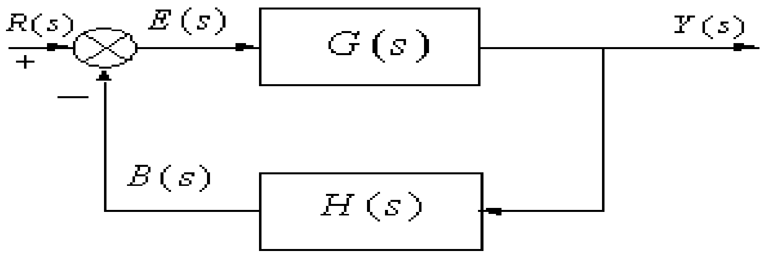
$$M(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \quad (3\text{-}12)$$

positive feedback

$$M(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 - G(s)H(s)} \quad (3\text{-}13)$$

$r(t),\ R(s) =$ reference input (command)
$y(t),\ Y(s) =$ output (controlled variable)
$b(t),\ B(s) =$ feedback signal
$u(t),\ U(s) =$ actuating signal $=$ error signal $e(t), E(s)$, when $H(s) = 1$
$H(s) =$ feedback transfer function
$G(s)H(s) = L(s) =$ loop transfer function
$G(s) =$ forward-path transfer function
$M(s) = Y(s)/R(s) =$ closed-loop transfer function or system transfer function

Feedback Control System



R(s) : 기준입력(reference input), 입력(input), 또는 command

Y(s) : 출력(output, controlled variable), 또는 응답(response)

B(s) : 궤환 신호(feedback signal)

E(s) : 오차신호(error signal) 또는 actuating signal

G(s) : 순방향경로 전달함수(forward-path transfer function)

H(s) : 궤환 전달함수(feedback transfer function, feedback gain)

G(s)H(s) : 루프전달함수(loop transfer function), 개루프전달함수(open-loop transfer function)

M(s) = Y(s)/R(s) : 폐루프전달함수(closed-loop transfer function, system transfer function)

B(s) = H(s)Y(s)

E(s) = R(s) – B(s)

Y(s) = G(s)E(s) = G(s)R(s) – G(s)B(s)

M(s) = Y(s) / R(s) = G(s) / (1 + G(s)H(s))

3-1-2 Relation between Mathematical Equations and Block Diagrams

$$\omega_n^2 U(s) - 2\zeta\omega_n X(s)s - \omega_n^2 X(s) = X(s)s^2 \qquad (3\text{-}16)$$

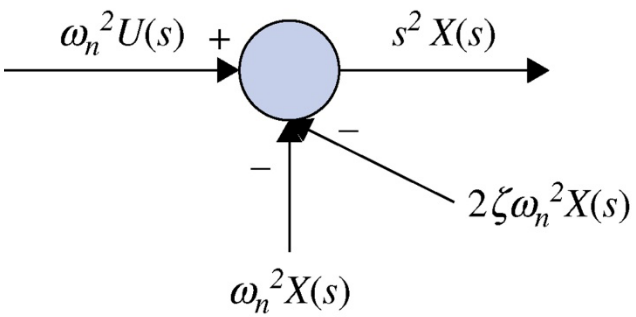


Figure 3-9 Graphical representation of Eq. (3-16) using a comparator.

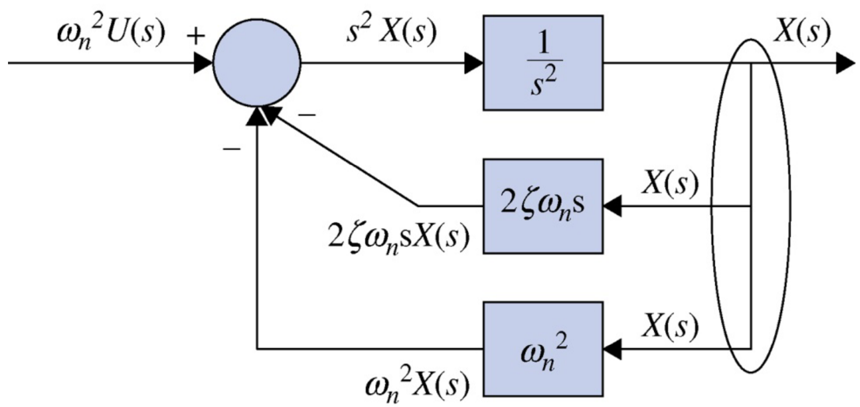


Figure 3-10 Addition of blocks $\frac{1}{s^2}$, $2\zeta\omega_n s$, and ω_n^2 to the graphical representation of Eq. (3-17).

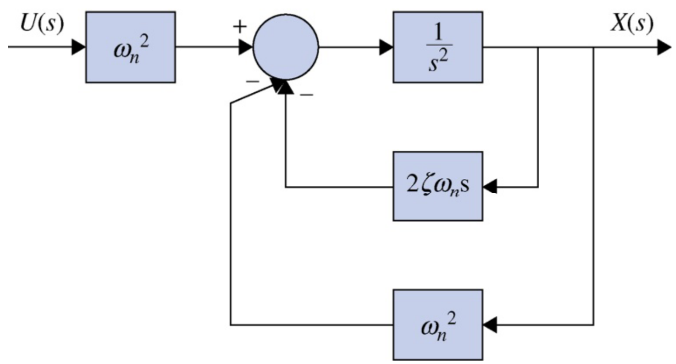


Figure 3-11 Block diagram representation of Eq. (3-17) in Laplace domain.

$$\frac{V(s)}{U(s)} = \frac{s\,\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_2^n} \qquad (3\text{-}20)$$

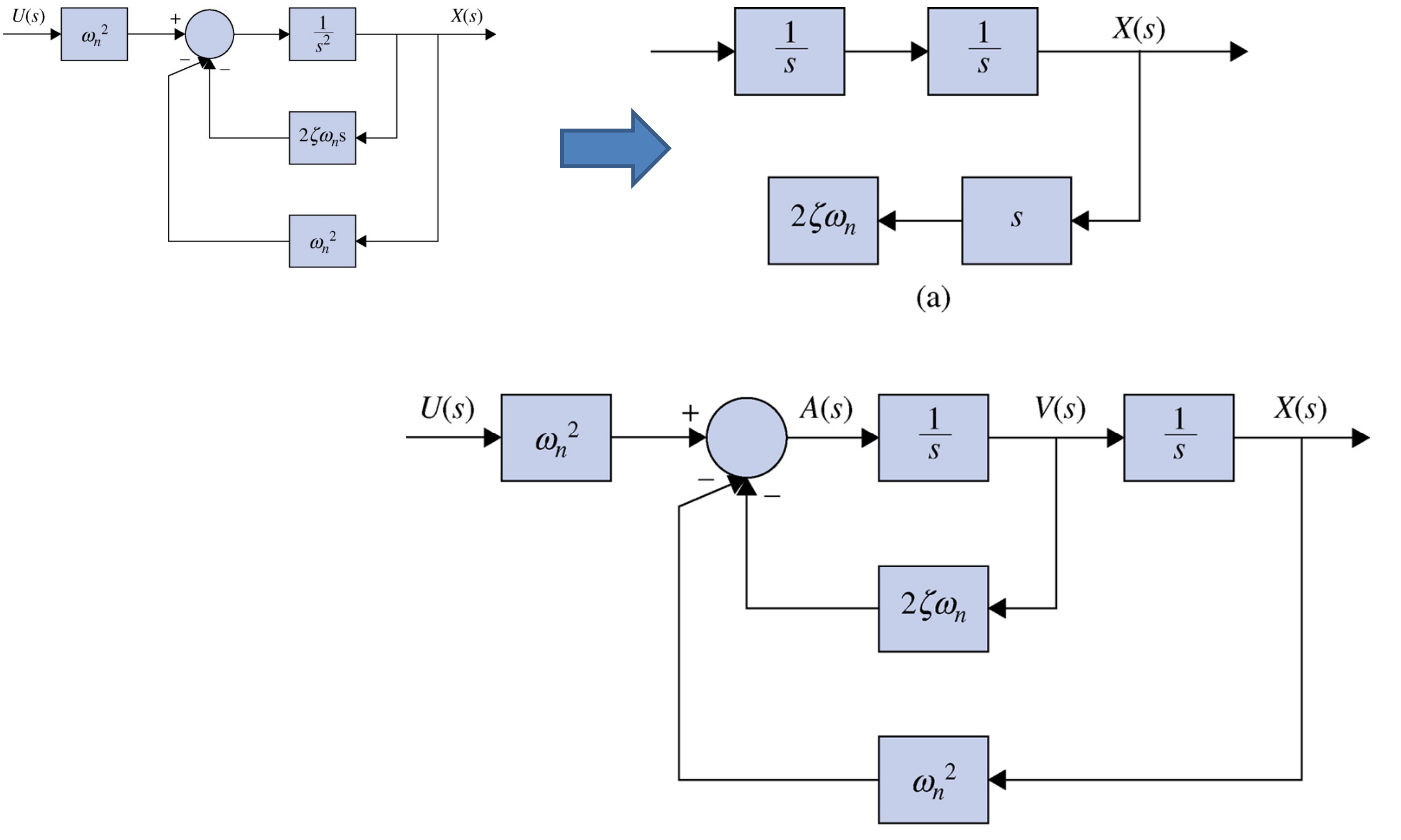


Figure 3-12 (a) Factorization of 1/s term in the internal feedback loop of Fig.3-11.
(b) Final block diagram representation of Eq.(3-17) in Laplace domain .

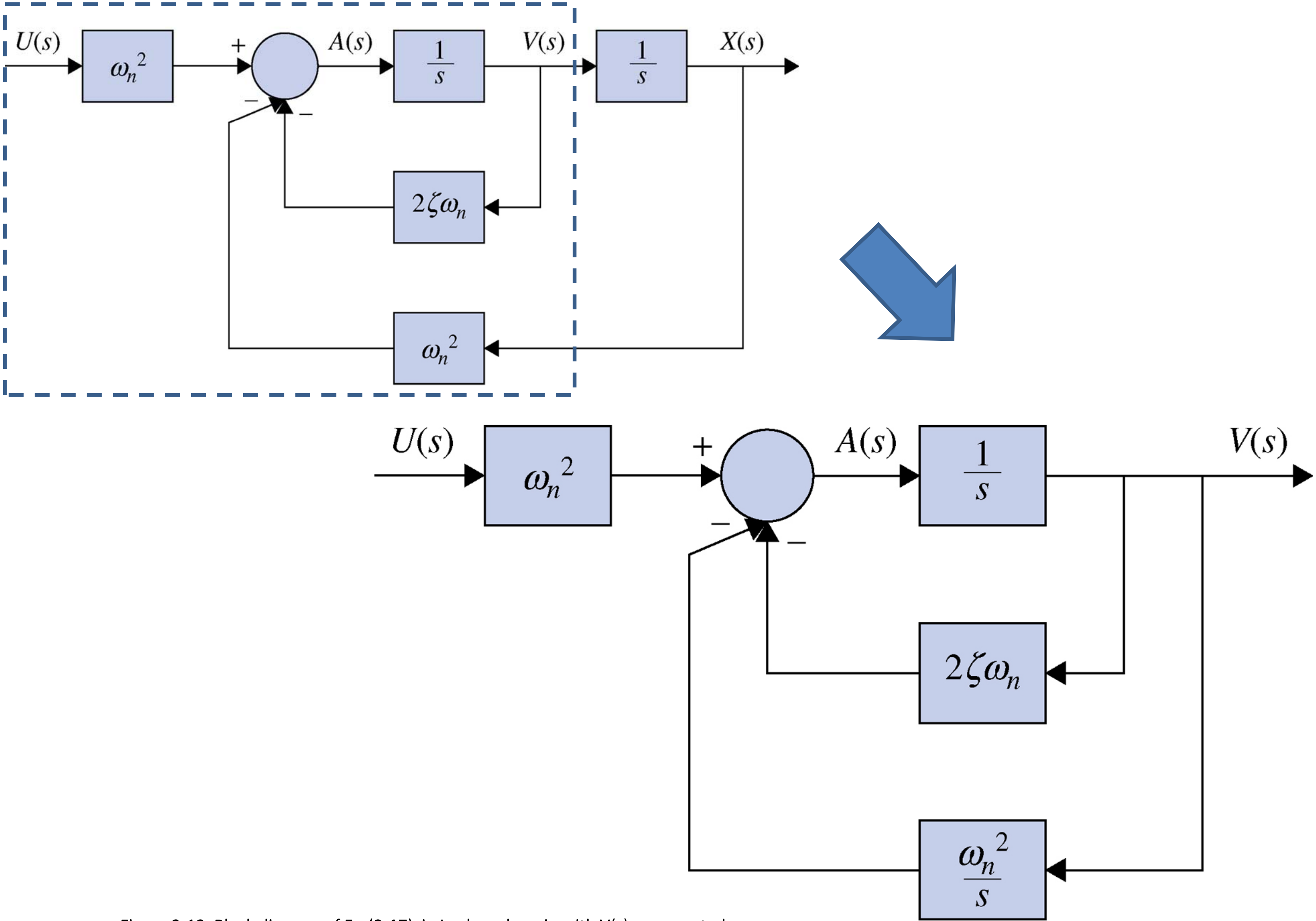


Figure 3-13 Block diagram of Eq.(3-17) in Laplace domain with V(s) represented as the output.

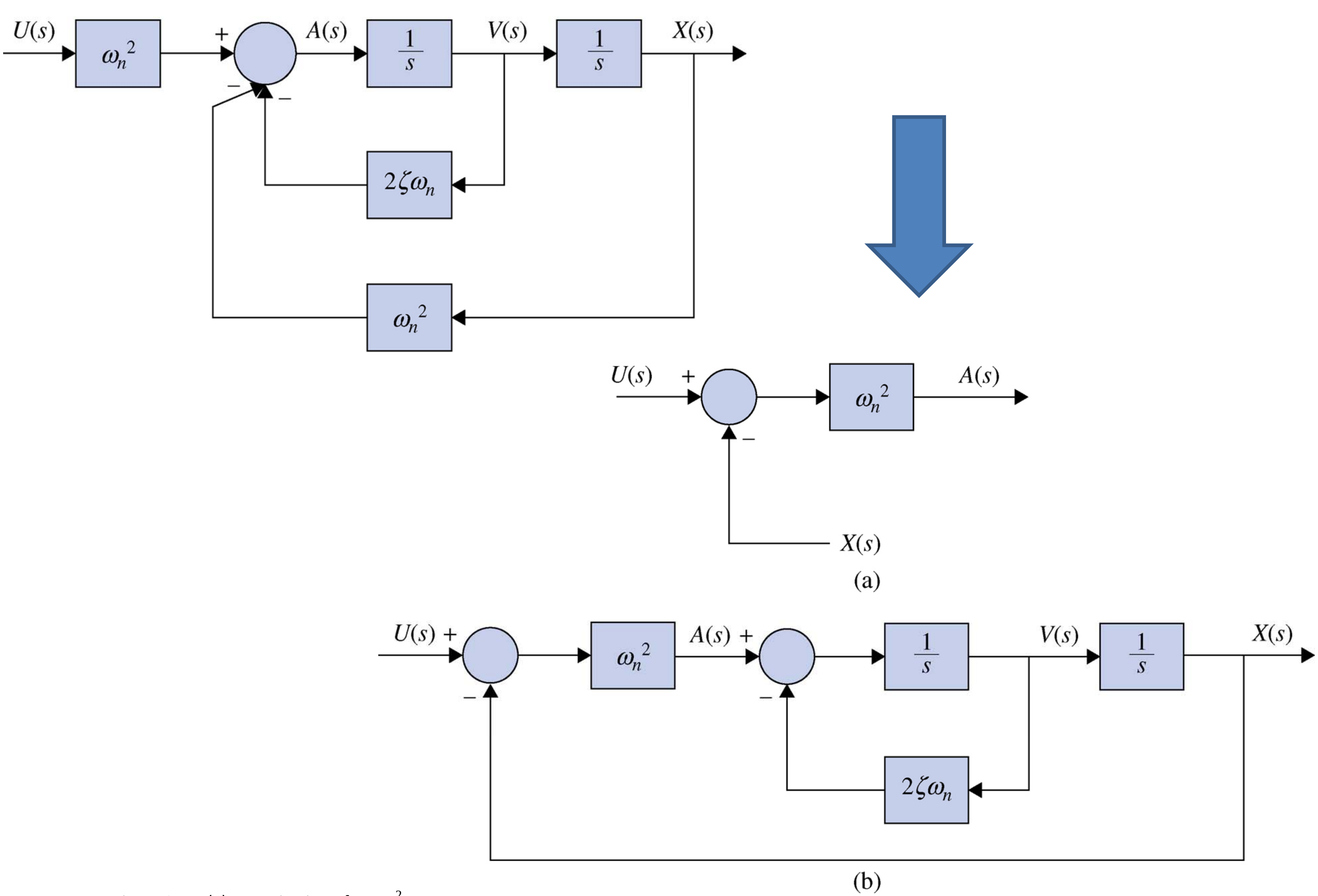


Figure 3-14 (a) Factorization of ω_n^2
(b) Alternative diagram representation of Eq.(3-17) in Laplace domain.

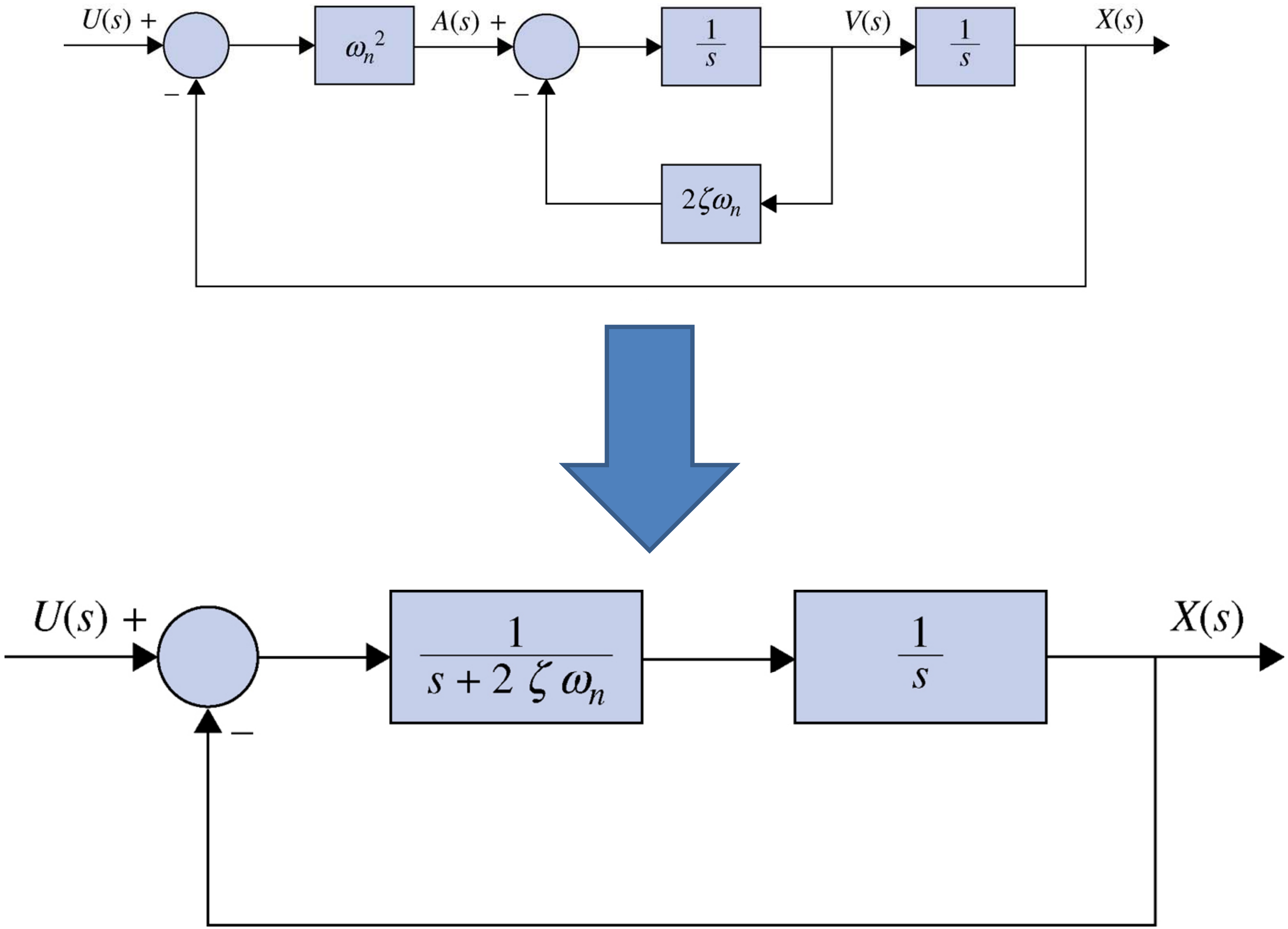
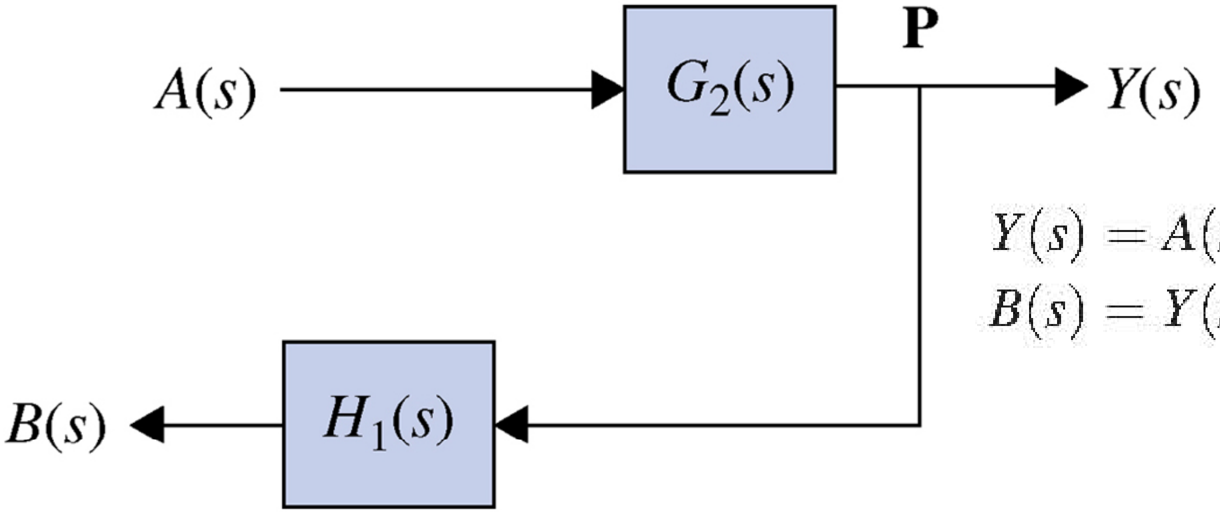


Figure 3-15 A block diagram representation of Eq.(3-19) in Laplace domain.

3-1-3 Block Diagram Reduction: Branch point relocation

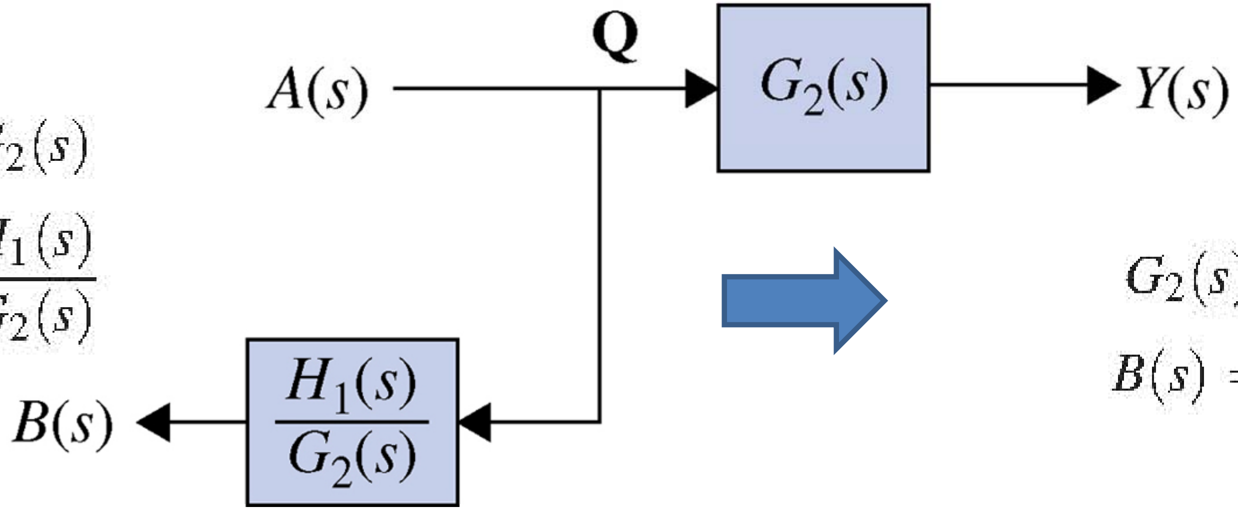
(a)



$$Y(s) = A(s)G_2(s)$$
$$B(s) = Y(s)H_1(s)$$

(b)

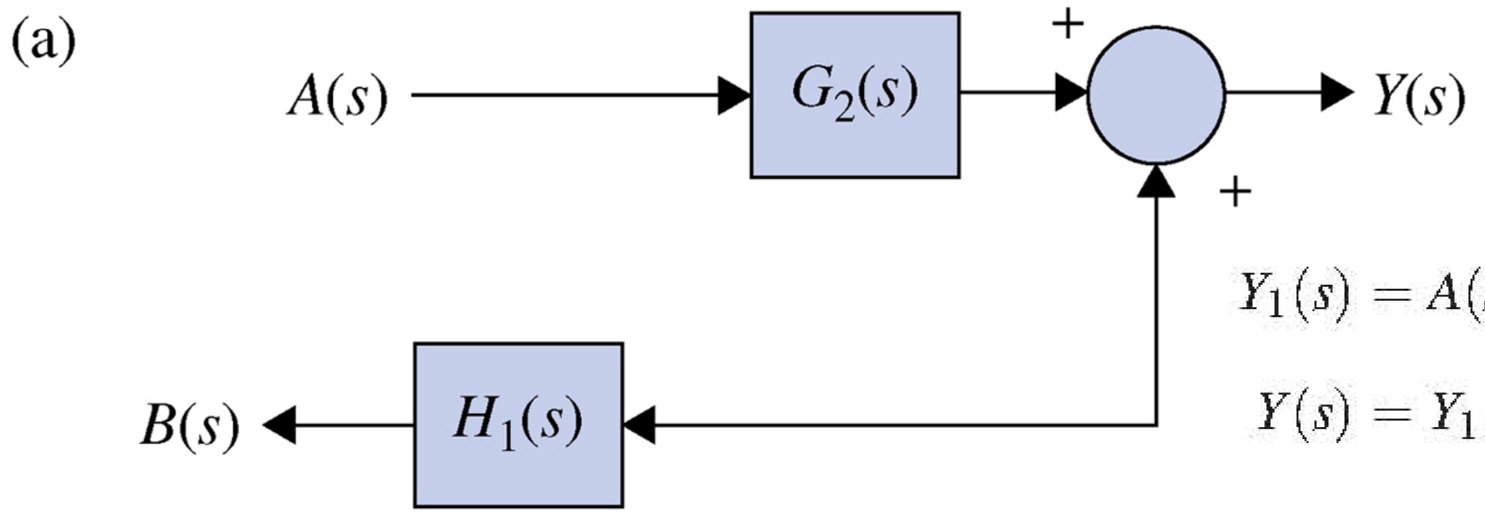
$$Y(s) = A(s)G_2(s)$$
$$B(s) = A(s)\frac{H_1(s)}{G_2(s)}$$



$$G_2(s) = \frac{A(s)}{Y(s)}$$
$$B(s) = Y(s)H_1(s)$$

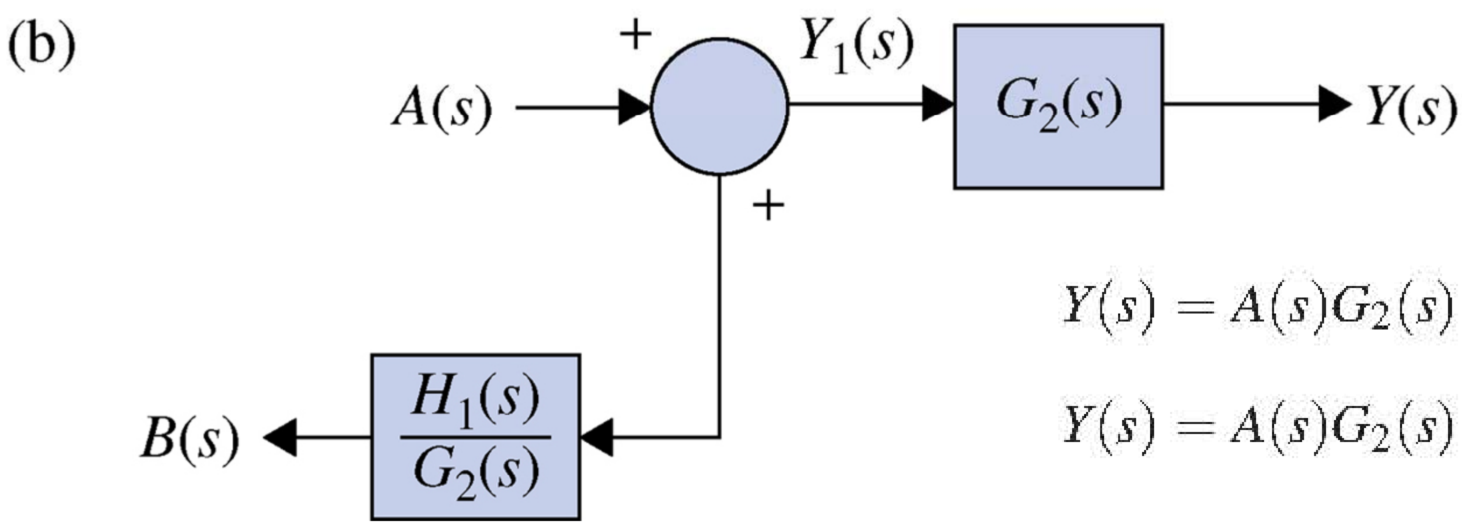
Figure 3-16 (a) Branch point relocation from point **P** to (b) point **Q**.

3-1-3 Block Diagram Reduction: Comparator relocation



$$Y_1(s) = A(s) + B(s)\frac{H_1(s)}{G_2(s)}$$

$$Y(s) = Y_1(s)G_2(s)$$



$$Y(s) = A(s)G_2(s) + B(s)\frac{H_1(s)}{G_2(s)}G_2(s)$$

$$Y(s) = A(s)G_2(s) + B(s)H_1(s)$$

Figure 3-17 (a) Comparator relocation from the right-hand side of block $G_2(s)$ to (b) the left-hand side of block $G_2(s)$.

EXAMPLE 3-1-5 Find the input–output transfer function of the system

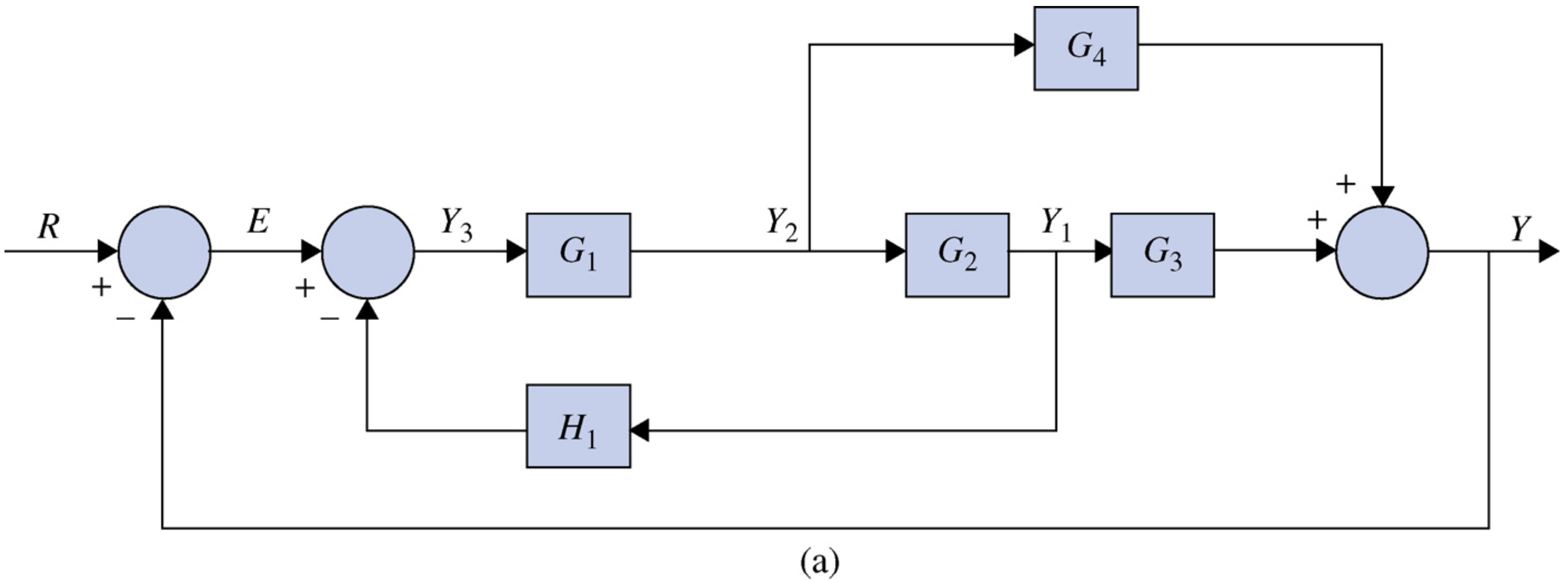
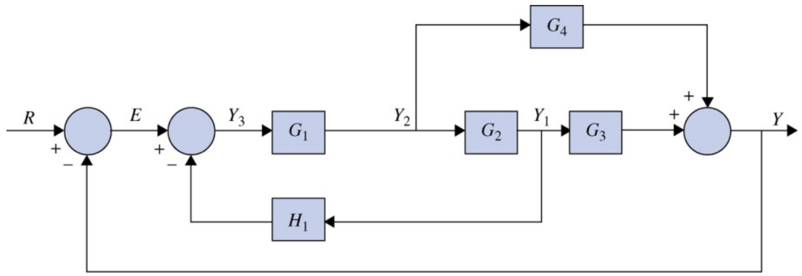
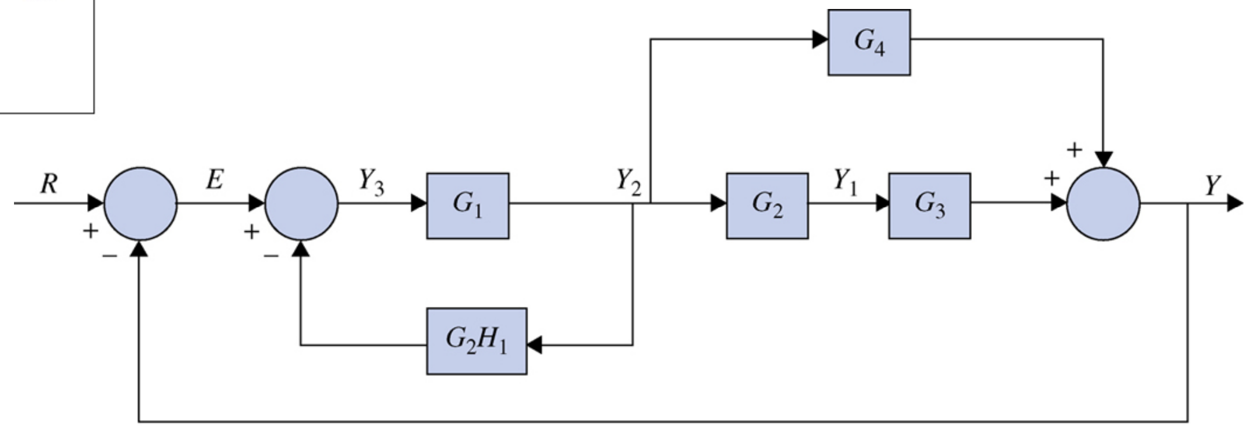


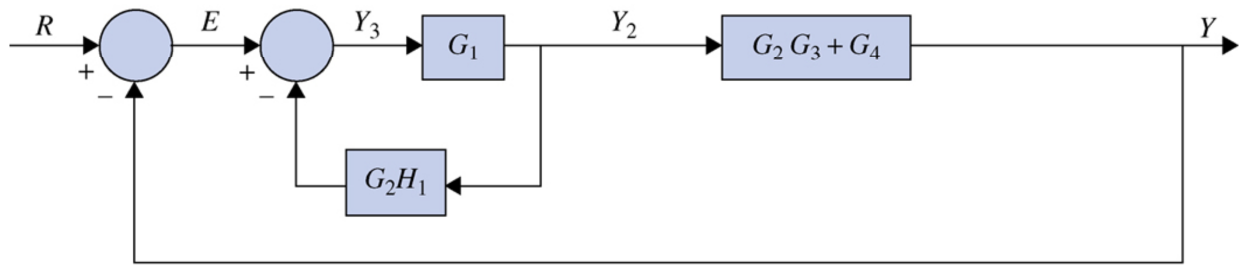
Figure 3-18 (a) Original block diagram.
(b) Moving the branch point at Y_1 to the left of block G_2.
(c) Combining the blocks G_1, G_2, and G_3.
(d) Eliminating the inner feedback loop.



(a)

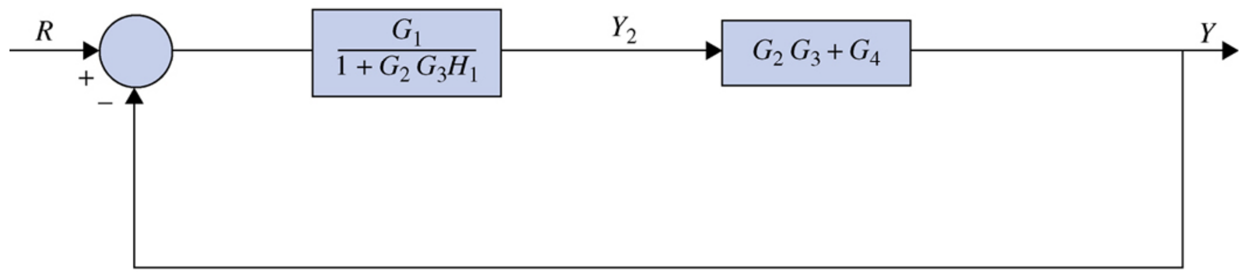


(b)



(c)

$$\frac{Y(s)}{E(s)} = \frac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2H_1 + G_1G_2G_3 + G_1G_4}$$



(d)

Figure 3-18 (Continued)

3-1-4 Block Diagram of Multi-Input Systems—Special Case: Systems with a Disturbance

Super Position: For linear systems, the overall response of the system under multi-inputs is the summation of the responses due to the individual inputs, i.e., in this case,

$$Y_{total} = Y_R|_{D=0} + Y_D|_{R=0} \tag{3-28}$$

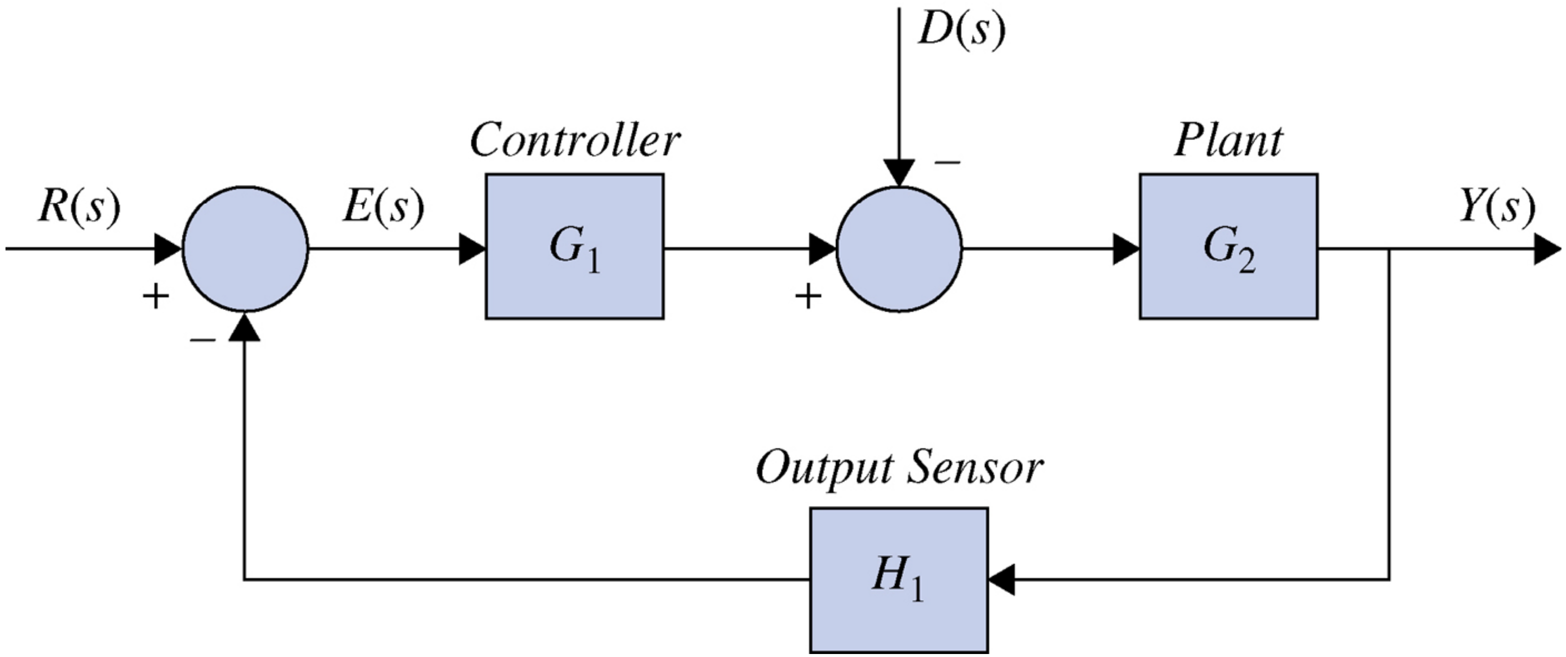


Figure 3-19 Block diagram of a system undergoing disturbance.

When $D(s) = 0$, the block diagram is simplified (Fig. 3-20) to give the transfer function

$$\frac{Y(s)}{R(s)} = \frac{G_1(s)\,G_2(s)}{1 + G_1(s)\,G_2\,H_1(s)} \tag{3-29}$$

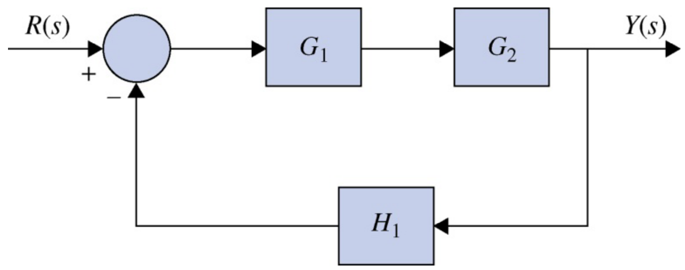


Figure 3-20 Block diagram of the system in Fig. 3-19 when D(s) = 0.

When $R(s) = 0$, the block diagram is rearranged to give (Fig. 3-21):

$$\frac{Y(s)}{D(s)} = \frac{-G_2(s)}{1 + G_1(s)\,G_2(s)\,H_1(s)} \tag{3-30}$$

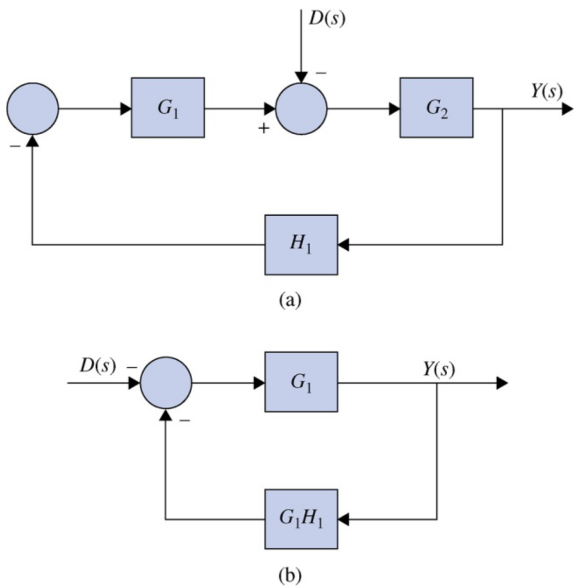


Figure 3-21 Block diagram of the system in Fig. 3-19 when R(s) = 0.

As a result, from Eq. (3-28) to Eq. (3-32), we ultimately get

$$\begin{aligned} Y_{total} &= \left.\frac{Y(s)}{R(s)}\right|_{D=0} R(s) + \left.\frac{Y(s)}{D(s)}\right|_{R=0} D(s) \\ Y(s) &= \frac{G_1 G_2}{1 + G_1 G_2 H_1} R(s) + \frac{-G_2}{1 + G_1 G_2 H_1} D(s) \end{aligned} \tag{3-31}$$

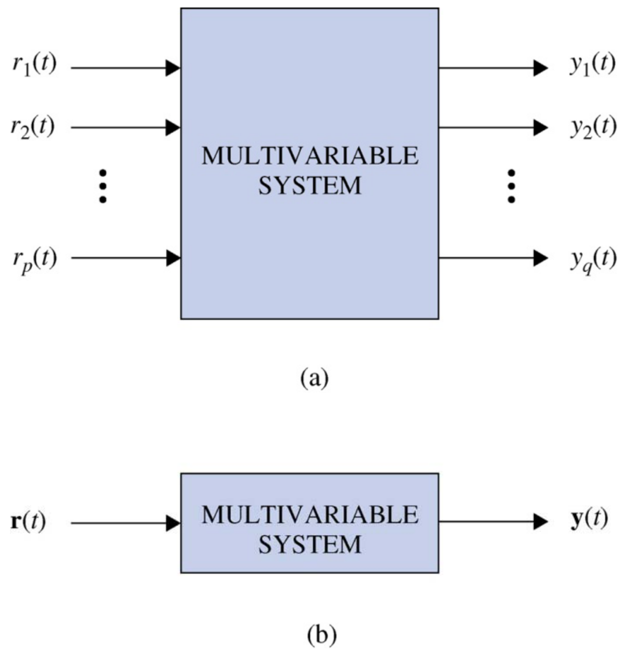


Figure 3-22 Block diagram representations of a multivariable system.

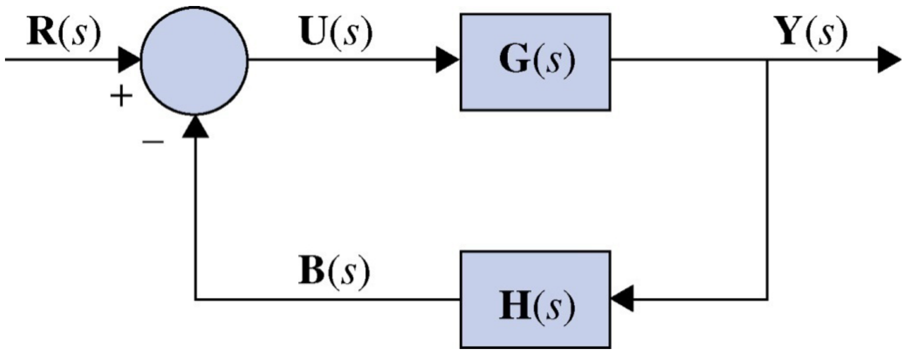


Figure 3-22 Block diagram representations of a multivariable feedback control system.

$$\mathbf{M}(s) = [\mathbf{I} + \mathbf{G}(s)\mathbf{H}(s)]^{-1}\mathbf{G}(s) \qquad (3\text{-}37)$$

$$\mathbf{Y}(s) = \mathbf{M}(s)\mathbf{R}(s) \qquad (3\text{-}38)$$

EXAMPLE 3-1-6 Consider that the forward-path transfer function matrix and the feedback-path transfer function matrix of the system shown in Fig. 3-23 are

$$\mathbf{G}(s) = \begin{bmatrix} \frac{1}{s+1} & -\frac{1}{s} \\ 2 & \frac{1}{s+2} \end{bmatrix} \qquad \mathbf{H}(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3-39}$$

respectively. The closed-loop transfer function matrix of the system is given by Eq. (3-15), and is evaluated as follows:

$$\mathbf{I} + \mathbf{G}(s)\mathbf{H}(s) = \begin{bmatrix} 1 + \frac{1}{s+1} & -\frac{1}{s} \\ 2 & 1 + \frac{1}{s+2} \end{bmatrix} = \begin{bmatrix} \frac{s+2}{s+1} & -\frac{1}{s} \\ 2 & \frac{s+3}{s+2} \end{bmatrix} \tag{3-40}$$

The closed-loop transfer function matrix is

$$\mathbf{M}(s) = [\mathbf{I} + \mathbf{G}(s)\mathbf{H}(s)]^{-1}\mathbf{G}(s) = \frac{1}{\Delta} \begin{bmatrix} \frac{s+3}{s+2} & \frac{1}{s} \\ -2 & \frac{s+2}{s+1} \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} & -\frac{1}{s} \\ 2 & \frac{1}{s+2} \end{bmatrix} \tag{3-41}$$

where

$$\Delta = \frac{s+2}{s+1}\frac{s+3}{s+2} + \frac{2}{s} = \frac{s^2+5s+2}{s(s+1)} \tag{3-42}$$

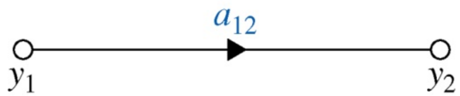
Thus,

$$\mathbf{M}(s) = \frac{s(s+1)}{s^2+5s+2} \begin{bmatrix} \frac{3s^2+9s+4}{s(s+1)(s+2)} & -\frac{1}{s} \\ 2 & \frac{3s+2}{s(s+1)} \end{bmatrix} \tag{3-43}$$

◀

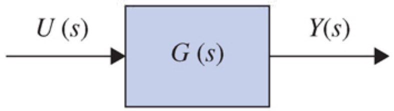
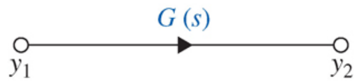
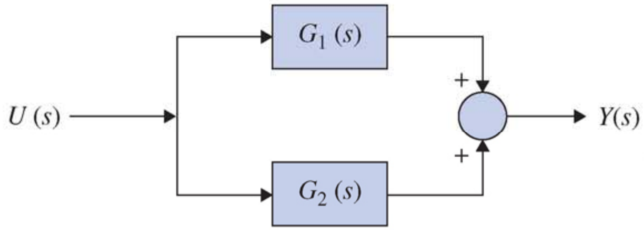
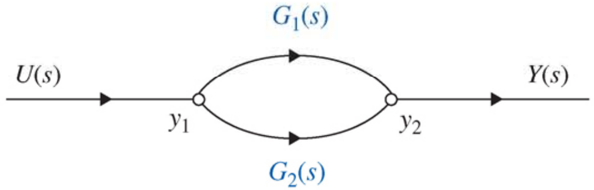
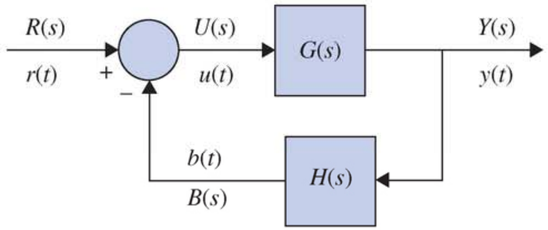
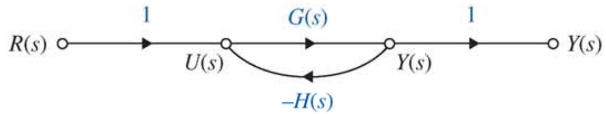
3-2 SIGNAL-FLOW GRAPHS (SFGs)

Output Node (Sink): *An* output node *is a node that has only incoming branches:*



Input Node (Source): *An* input node *is a node that has only outgoing branches*

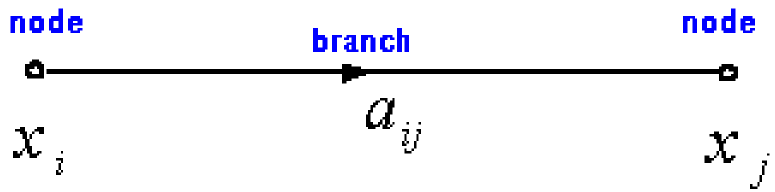
TABLE 3-1 Block diagrams and their SFG equivalent representations

	Block Diagram	Signal Flow Diagram
Simple Transfer Function $\frac{Y(s)}{U(s)} = G(s)$		
Parallel Feedback		
$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$		

1. SFG applies only to linear systems.
2. The equations for which an SFG is drawn must be algebraic equations in the form of cause-and-effect.
3. Nodes are used to represent variables. Normally, the nodes are arranged from left to right, from the input to the output, following a succession of cause-and-effect relations through the system.
4. Signals travel along branches only in the direction described by the arrows of the branches.
5. The branch directing from node y_k to y_j represents the dependence of y_j upon y_k but not the reverse.
6. A signal y_k traveling along a branch between y_k and y_j is multiplied by the gain of the branch a_{kj}, so a signal $a_{kj}y_k$ is delivered at y_j.

Signal-Flow Graphs(SFG, 신호흐름선도)

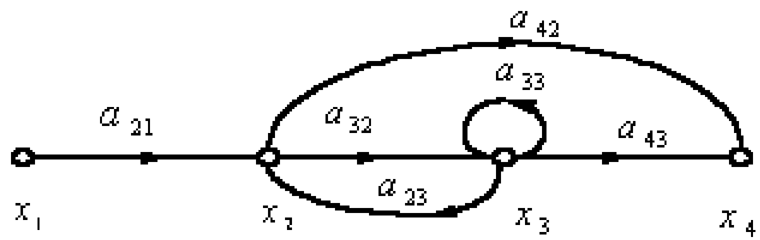
신호흐름도는 신호의 입·출력관계를 cause-and-effect 원리에 따라 대수적으로 나타낸 흐름도로서 절점(node)과 가지(branch)로 구성되며, 아래그림과 같이 각 node는 변수(variable)를 나타내고 branch는 전달되는 변수의 이득(gain)과 방향을 나타낸다.



[$x_j=a_{ij}x_i$를 나타낸 node와 branch]

output = ∑ gain x input 즉, ***j*** **th output = ∑ (gain from *k* to *j*) x (*k*th cause)**

$Y_j(s) = \sum G_{kj}(s)\, Y_k(s)$



SFG Terms의 정의

- **입력 노드(Input node, Source)**
  나가는 방향의 branch만 연결되어 있는 node
  예] 위의 그림에서 x_1
- **출력 노드(Output node, Sink)**
  들어오는 방향의 branch만 연결되어 있는 node
  예] 위의 그림에서 x_4

- **이득(Gain)**

  branch로 연결되어 있는 변수간의 비율
  예) x_1과 x_2를 연결하는 branch의 이득은 a_{21}이며,
  $x_2 = a_{21}x_1$+(다른 입력에 의한 항들)의 관계를 나타냄.
  (주의 : $x_2/x_1 = a_{21}$ 이라는 것은 아님)

- **경로(Path)**

  지정된 방향으로 연결된 branch의 집합으로 어떤 한 변수에서 출발하여, 지정된 어떤 변수에 이르는 경로를 이룬다. 단, 경로가 되기 위한 조건으로, 경로를 따라 신호가 전달될 때 어떤 경우에도 같은 node를 두 번 지나서는 안된다.
  예) x_1에서 x_3로 가는 path는 다음과 같이 두 개의 경로가 있다.

$$x_1 \xrightarrow{a_{21}} x_2 \xrightarrow{a_{32}} x_3$$

$$x_1 \xrightarrow{a_{21}} x_2 \xrightarrow{a_{32}} x_3 \xrightarrow{a_{33}} x_3$$

- **전방향 경로(Forward path)**

  입력 node에서 출력 node에 전방향으로 전달하는 path
  예) $x_1 \rightarrow x_4$의 forward path는 아래와 같이 2개의 경로가 있다.

i) $x_1 \xrightarrow{a_{21}} x_2 \xrightarrow{a_{32}} x_3 \xrightarrow{a_{43}} x_4$

ii) $x_1 \xrightarrow{a_{21}} x_2 \xrightarrow{a_{42}} x_4$

- **궤환 경로(Feedback path)**

  입출력 node간을 역방향으로 되돌아 진행하는 path.

- **Loop, Self loop**

  경로 중에서 출발 노드와 도착 노드가 동일한 경로를 루프(loop)라고 하고, 그 경로내부에 다른 node가 없으면 (또는 한개의 branch로 구성된 loop) self-loop 라고 함.

  예)

  i) loop: $x_2 \xrightarrow{a_{32}} x_3 \xrightarrow{a_{23}} x_2$

  ii) self-loop: $x_3 \xrightarrow{a_{33}} x_3$

- **Nontouching loops**

  Loop중에서 공통인 node가 없는 loop

- **경로 이득(Path gain)**

  정해진 path를 이루는 각 branch gain의 곱.

  예) path : $x_1 \xrightarrow{a_{21}} x_2 \xrightarrow{a_{42}} x_4$에 대한 path gain은 $a_{21}a_{42}$

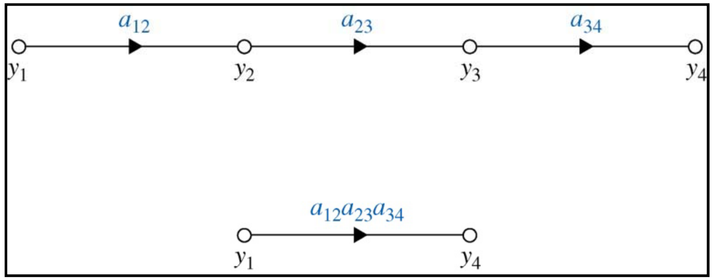
  (주의 : 이 예에서 path gain이 $a_{21}a_{42}$ 라고해서 $x_4/x_1=a_{21}a_{42}$라는 뜻은 아님)

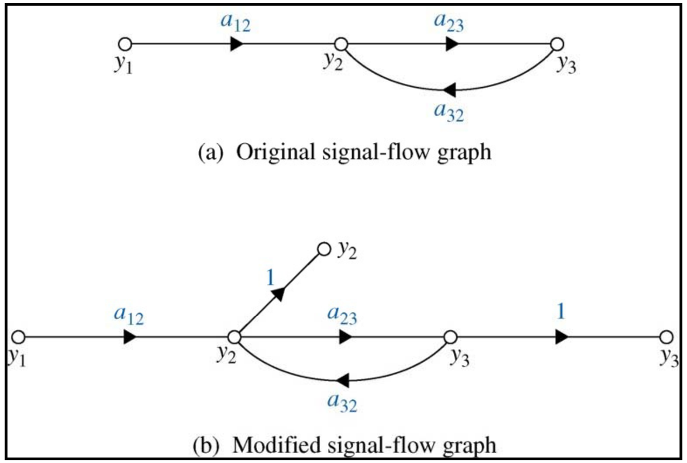
- **Loop gain**

  지정된 loop을 형성하는 각 branch gain의 곱 (loop의 path gain)

  예) loop: $x_2 \xrightarrow{a_{32}} x_3 \xrightarrow{a_{23}} x_2$ loop gain은 $a_{23}a_{32}$

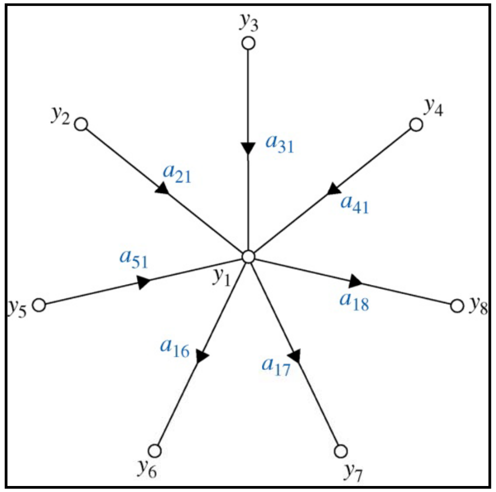
3-2-4 SFG Algebra

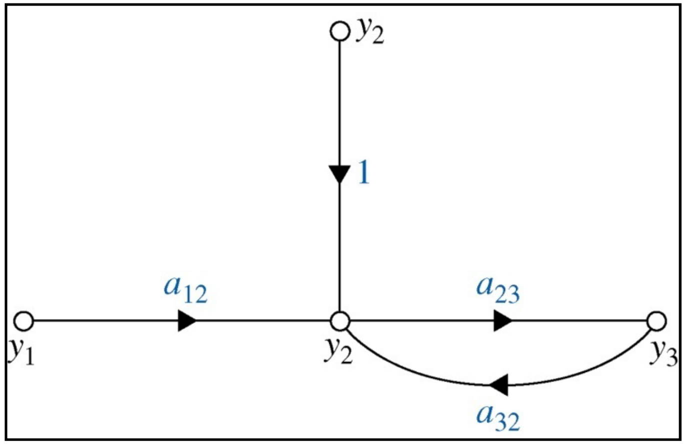




(a) Original signal-flow graph

(b) Modified signal-flow graph





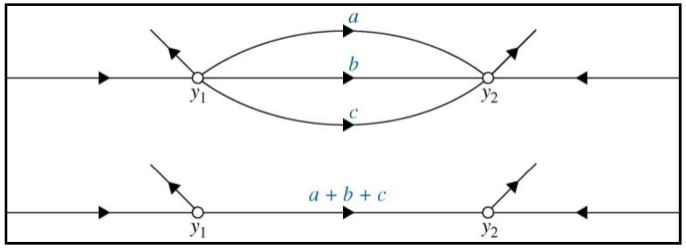


Figure 3-29~31 Signal-flow graph.

3-2-7 Gain Formula for SFG

Given an SFG with N forward paths and K loops, the gain between the input node y_{in} and output node y_{out} is [3]

$$M = \frac{y_{\text{out}}}{y_{\text{in}}} = \sum_{k=1}^{N} \frac{M_k \Delta_k}{\Delta} \tag{3-54}$$

where

y_{in} = input-node variable

y_{out} = output-node variable

M = gain between y_{in} and y_{out}

N = total number of forward paths between y_{in} and y_{out}

M_k = gain of the kth forward paths between y_{in} and y_{out}

$$\Delta = 1 - \sum_i L_{i1} + \sum_j L_{j2} - \sum_k L_{k3} + \ldots \tag{3-55}$$

L_{mr} = gain product of the mth $(m = i, j, k, \ldots)$ possible combination of r non-touching loops $(1 \le r \le K)$.

or

Δ = 1− (sum of the gains of **all individual** loops) + (sum of products of gains of all possible combinations of **two** nontouching loops) − (sum of products of gains of all possible combinations of **three** nontouching loops) + ⋯

Δ_k = the Δ for that part of the SFG that is nontouching with the kth forward path.

The gain formula in Eq. (3-54) may seem formidable to use at first glance. However, Δ and Δ_k are the only terms in the formula that could be complicated if the SFG has a large number of loops and nontouching loops.

Care must be taken when applying the gain formula to ensure that it is applied between an **input node** and an **output node**.

Gain Formula for SFG **(Mason's gain rule)**

M : The gain between input node y_{in} and output node y_{out}

$$M = y_{out} / y_{in} = \sum M_k \Delta_k \ / \Delta \ , \quad k = 1, \dots, N$$

여기서,
N : Total number of forward path
M_k : k번째 forward path의 gain
Δ : signal flow graph determinant 또는 characteristic function

$$\Delta = 1 - \sum L_{i1} + L_{j2} - L_{k3} + \dots\dots$$

L_{mr} = r nontouching loops 의 m^{th} possible combination의 gain product ($1 \le r \le L$)

Δ = 1 - (모든 각각의 loop 이득의 합)
+ (2개의 비접 loop의 가능한 모든 조합의 이득곱의 합)
- (3개의 비접 loop의 가능한 모든 조합의 이득곱의 합)
+ .....

L = loops의 수

Δ_k : k^{th} forward path와 nontouching하는 Δ part

Δ_k = k번째의 전향경로와 접하지 않는 graph의 부분에 대한 Δ 의 값

k번째 경로의 모든 branch를 제거한 신호흐름도에서 구한 Δ

$$\Delta_i = \Delta - \sum \textit{loop gain touching the i-th forward path}$$

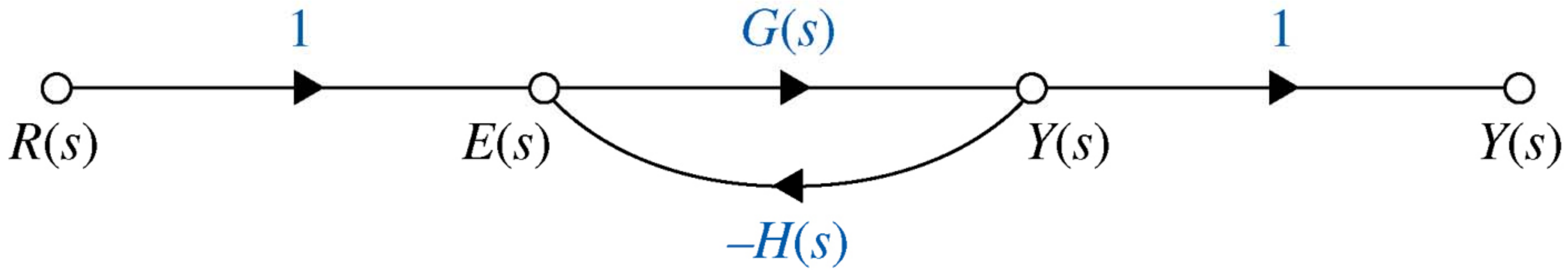


Figure 3-32 Signal-flow graph of the feedback control system shown in Fig. 3-8.

EXAMPLE 3-2-2 Consider that the closed-loop transfer function $Y(s)/R(s)$ of the SFG in Fig. 3-32 is to be determined by use of the gain formula, Eq. (3-54). The following results are obtained by inspection of the SFG:

1. There is only one forward path between $R(s)$ and $Y(s)$, and the forward-path gain is

$$M_1 = G(s) \tag{3-56}$$

2. There is only one loop; the loop gain is

$$L_{11} = -G(s)H(s) \tag{3-57}$$

3. There are no nontouching loops since there is only one loop. Furthermore, the forward path is in touch with the only loop. Thus, $\Delta_1 = 1$, and

$$\Delta = 1 - L_{11} = 1 + G(s)H(s) \tag{3-58}$$

Using Eq. (3-54), the closed-loop transfer function is written

$$\frac{Y(s)}{R(s)} = \frac{M_1\Delta_1}{\Delta} = \frac{G(s)}{1 + G(s)H(s)} \tag{3-59}$$

which agrees with Eq. (3-12).

EXAMPLE 3-2-3

Consider the SFG shown in Fig. 3-25(d). Let us first determine the gain between y_1 and y_5 using the gain formula.

The three forward paths between y_1 and y_5 and the forward-path gains are

$$
\begin{aligned}
M_1 &= a_{12}a_{23}a_{34}a_{45} \quad \text{Forward path:} \quad y_1 - y_2 - y_3 - y_4 - y_5 \\
M_2 &= a_{12}a_{25} \quad \text{Forward path:} \quad y_1 - y_2 - y_5 \\
M_3 &= a_{12}a_{24}a_{45} \quad \text{Forward path:} \quad y_1 - y_2 - y_4 - y_5
\end{aligned}
$$

The four loops of the SFG are shown in Fig. 3-28. The loop gains are

$$L_{11} = a_{23}a_{32} \quad L_{21} = a_{34}a_{43} \quad L_{31} = a_{24}a_{43}a_{32} \quad L_{41} = a_{44}$$

There is only one pair of nontouching loops; that is, the two loops are

$$y_2 - y_3 - y_2 \quad \text{and} \quad y_4 - y_4$$

Thus, the product of the gains of the two nontouching loops is

$$L_{12} = a_{23}a_{32}a_{44} \tag{3-60}$$

All the loops are in touch with forward paths M_1 and M_3. Thus, $\Delta_1 = \Delta_3 = 1$. Two of the loops are not in touch with forward path M_2. These loops are $y_3 - y_4 - y_3$ and $y_4 - y_4$. Thus,

$$\Delta_2 = 1 - a_{34}a_{43} - a_{44} \tag{3-61}$$

Substituting these quantities into Eq. (3-54), we have

$$
\begin{aligned}
\frac{y_5}{y_1} &= \frac{M_1\Delta_1 + M_2\Delta_2 + M_3\Delta_3}{\Delta} \\
&= \frac{(a_{12}a_{23}a_{34}a_{45}) + (a_{12}a_{25})(1 - a_{34}a_{43} - a_{44}) + a_{12}a_{24}a_{45}}{1 - (a_{23}a_{32} + a_{34}a_{43} + a_{24}a_{32}a_{43} + a_{44}) + a_{23}a_{32}a_{44}}
\end{aligned} \tag{3-62}
$$

where

$$
\begin{aligned}
\Delta &= 1 - (L_{11} + L_{21} + L_{31} + L_{41}) + L_{12} \\
&= 1 - (a_{23}a_{32} + a_{34}a_{43} + a_{24}a_{32}a_{43} + a_{44}) + a_{23}a_{32}a_{44}
\end{aligned} \tag{3-63}
$$

The reader should verify that choosing y_2 as the output,

$$\frac{y_2}{y_1} = \frac{a_{12}(1 - a_{34}a_{43} - a_{44})}{\Delta} \tag{3-64}$$

where Δ is given in Eq. (3-63).

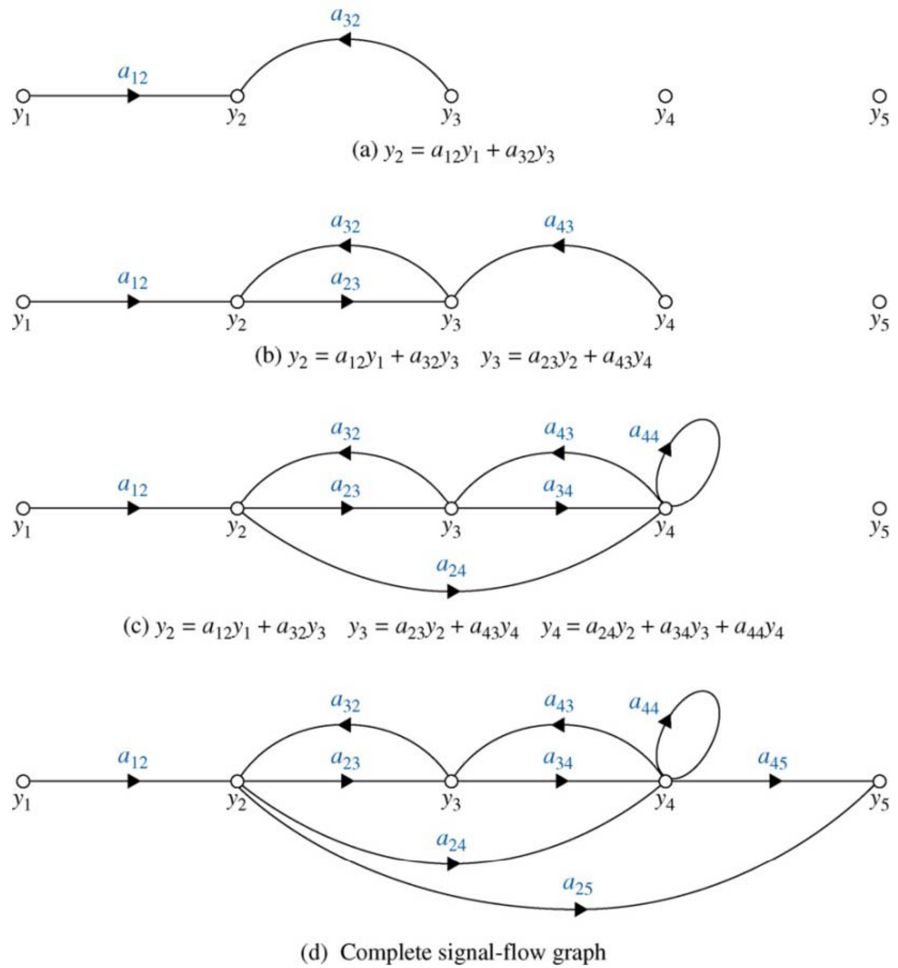


Figure 3-33 Signal-flow graph for Example 3-2-3.

EXAMPLE 3-2-4 Consider the SFG in Fig. 3-33. The following input–output relations are obtained by use of the gain formula:

$$\frac{y_2}{y_1} = \frac{1 + G_3H_2 + H_4 + G_3H_2H_4}{\Delta} \tag{3-65}$$

$$\frac{y_4}{y_1} = \frac{G_1G_2(1 + H_4)}{\Delta} \tag{3-66}$$

$$\frac{y_6}{y_1} = \frac{y_7}{y_1} = \frac{G_1G_2G_3G_4 + G_1G_5(1 + G_3H_2)}{\Delta} \tag{3-67}$$

where

$$\begin{aligned}\Delta = 1 &+ G_1H_1 + G_3H_2 + G_1G_2G_3H_3 + H_4 + G_1G_3H_1H_2 \\ &+ G_1H_1H_4 + G_3H_2H_4 + G_1G_2G_3H_3H_4 + G_1G_3H_1H_2H_4\end{aligned} \tag{3-68}$$

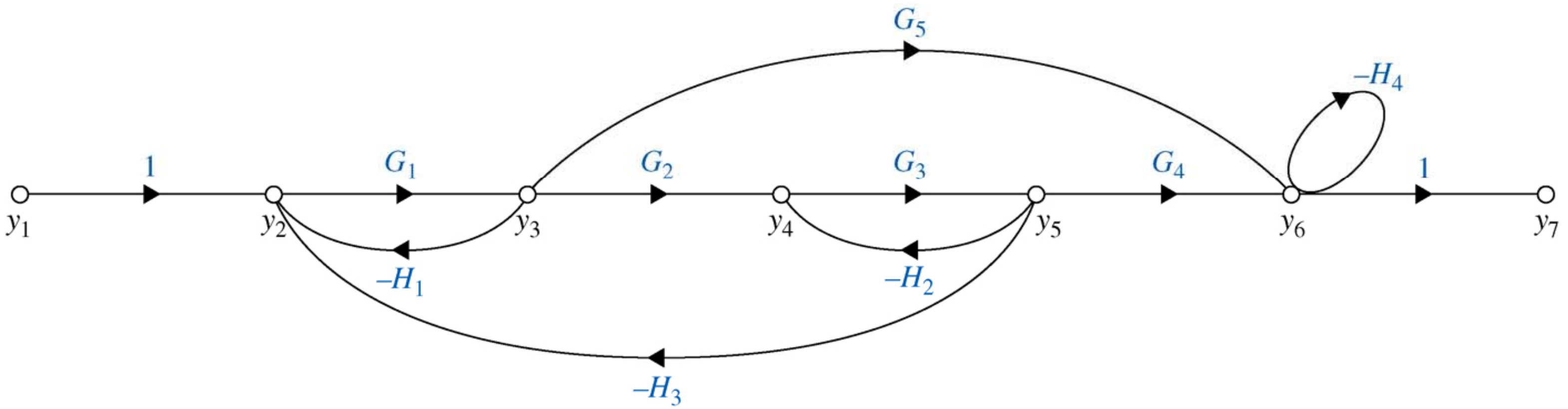
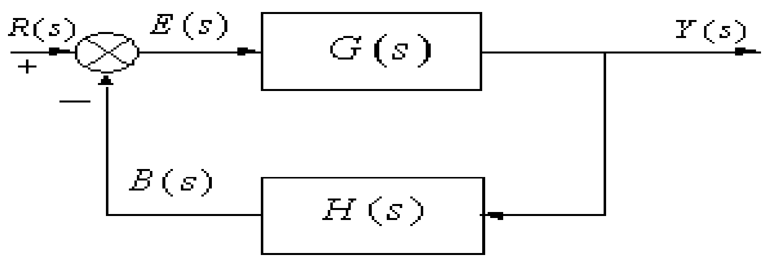


Figure 3-33 Signal-flow graph for Example 3-2-4.

- **Ex. 3-2-2**



M_1 = G(s)

L_{11} = -G(s)H(s)

Δ_1 = 1

Δ = 1 + G(s)H(s)

Closed -loop transfer function

$M = Y(s) / R(s) = M_1 \Delta_1 / \Delta = G(s) / (1 + G(s)H(s))$

- **Ex. 3-2-4**

  $y_2 / y_1 =$

  $y_4 / y_1 =$

  *** Δ는 chosen output에 관계없이 same**

- **Noninput node와 output node사이의 gain**

  $y_{out} / y_2 = (y_{out} / y_{in}) / (y_2 / y_{in}) = (\sum M_k\Delta_k \mid \text{from } y_{in} \text{ to } y_{out} / \Delta) / (\sum M_k\Delta_k \mid \text{from } y_{in} \text{ to } y_2 / \Delta)$

  $= (\sum M_k\Delta_k \mid \text{from } y_{in} \text{ to } y_{out}) / (\sum M_k\Delta_k \mid \text{from } y_{in} \text{ to } y_2)$

- **Ex. 3-2-5 & 3-2-6**

3-2-9 Application of the Gain Formula to Block Diagrams

EXAMPLE 3-2-6

Forward Path Gains: 1. $G_1G_2G_3$; 2. G_1G_4

Loop Gains: 1. $-G_1G_2H_1$; 2. $-G_2G_3H_2$; 3. $-G_1G_2G_3$; 4. $-G_4H_2$; 5. $-G_1G_4$

$$\frac{Y(s)}{R(s)} = \frac{G_1G_2G_3 + G_1G_4}{\Delta} \tag{3-72}$$

$$\Delta = 1 + G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3 + G_4H_2 + G_1G_4 \tag{3-73}$$

$$\frac{E(s)}{R(s)} = \frac{1 + G_1G_2H_1 + G_2G_3H_2 + G_4H_2}{\Delta} \tag{3-74}$$

$$\frac{Y(s)}{E(s)} = \frac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2H_1 + G_2G_3H_2 + G_4H_2} \tag{3-75}$$

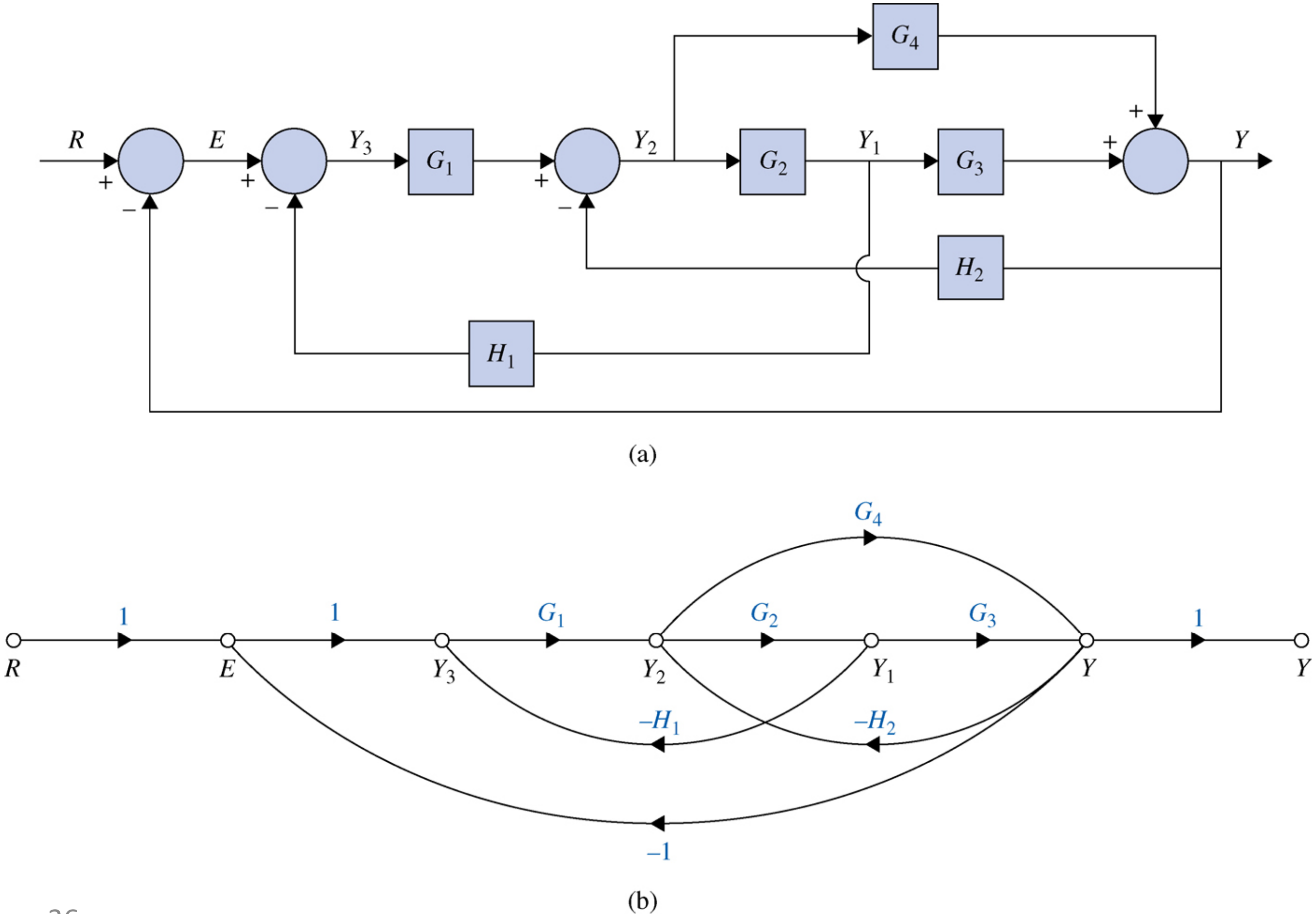


Figure 3-34 (a) Block diagram of a control system. (b) Equivalent signal-flow graph.

3-2-10 Simplified Gain Formula

From Example 3-2-6, we can see that *all loops and forward paths are touching* in this case. As a general rule, if there are no nontouching loops and forward paths (e.g., $y_2 - y_3 - y_2$ and $y_4 - y_4$ in Example 3-2-3) in the block diagram or SFG of the system, then Eq. (3-54) takes a far simpler look, as shown next.

$$M = \frac{y_{out}}{y_{in}} = \sum \frac{Forward\ Path\ Gains}{1 - Loop\ Gains} \qquad (3\text{-}76)$$

Redo Examples 3-2-2 through 3-2-6 to confirm the validity of Eq. (3-76).