History of Routh's Stability Criterion

The first systematic study of the stability of feedback control was apparently given in the paper "On Governors" by J. C. Maxwell (1868).[7] In this paper, Maxwell developed the differential equations of the governor, linearized them about equilibrium, and stated that stability depends on the roots of a certain (characteristic) equation having negative real parts. Maxwell attempted to derive conditions on the coefficients of a polynomial that would hold if all the roots had negative real parts. He was successful only for second- and third-order cases. Determining criteria for stability was the problem for the Adams Prize of 1877, which was won by E. J. Routh.[8] His criterion, developed in his essay, remains of sufficient interest that control engineers are still learning how to apply his simple technique. Analysis of the characteristic equation remained the foundation of control theory until the invention of the electronic feedback amplifier by H. S. Black in 1927 at Bell Telephone Laboratories.

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7 An exposition of Maxwell's contribution is given in Fuller (1976).

8 E. J. Routh was first academically in his class at Cambridge University in 1854, while J. C. Maxwell was second. In 1877 Maxwell was on the Adams Prize Committee that chose the problem of stability as the topic for the year.

Source: "Feedback Control of Dynamic Systems," 5th ed. Franklin, et al., Pg 12 Chapt 1

1

STABILITY VIA ROUTH HURWITZ

$$H(s) = \frac{1}{(s-1+j\sqrt{7})(s-1-j\sqrt{7})(s+3)}$$ UNSTable w 2-RHP Poles

$$= \frac{1}{s^3 + s^2 + 2s + 24}$$

s^3	1	2
s^2	1	24
s^1	$\dfrac{-\begin{vmatrix}1 & 2\\ 1 & 24\end{vmatrix}}{1} = -22$	$\dfrac{-\begin{vmatrix}1 & 0\\ 1 & 0\end{vmatrix}}{1} = 0$
s^0	$\dfrac{-\begin{vmatrix}1 & 24\\ -22 & 0\end{vmatrix}}{-22} = a_0 = 24$	0

Since There are 2 sign changes $\Rightarrow$ $\exists$ 2 RHP Poles. We Know that they are $P_1 = (1+j\sqrt{7})$ & $P_2 = (1-j\sqrt{7})$ from above

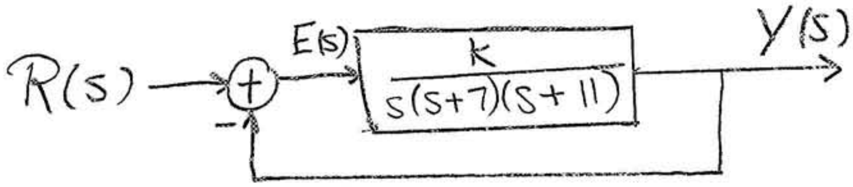
$\boxed{2}$ $H(s) = \dfrac{1}{s^3 + 11s^2 + 10s + K}$ find $K \ni H(s)$ is stable

s^3	1	10
s^2	11	K
s^1	$\dfrac{-\begin{vmatrix} 1 & 10 \\ 11 & K \end{vmatrix}}{11} = \boxed{10 - \frac{K}{11}}$	0
s^0	$\dfrac{-\begin{vmatrix} 11 & K \\ (10-\frac{K}{11}) & 0 \end{vmatrix}}{(10 - K/11)} = a_0 = \boxed{K}$	0

- Since 1st + 2nd elements of column 1 are positive, for $H(s)$ to be stable 3rd + 4th elements must also be positive $\Rightarrow \boxed{0 < K < 110}$
- NOTE if $K > 110 \Rightarrow H(s)$ is unstable with 2 poles in RHP since 3rd element is negative.
- It can also be shown that $H(s)$ is marginally stable for $K = 110$.

STABILITY VIA ROUTH-HURWITZ

3. Find the Range of Gain K for which The system is a) Stable b) unstable and c) (marginally) neutrally stable



$$H(s) = \frac{Y(s)}{R(s)} = \frac{\frac{K}{s(s+7)(s+11)}}{1 + \frac{K}{s(s+7)(s+11)}} = \frac{K}{s(s^2+18s+77)+K}$$

$$H(s) = \frac{K}{s^3 + 18s^2 + 77s + K}$$

s^3	1	77
s^2	18	K
s^1	$b_1 = \dfrac{-\begin{vmatrix} 1 & 77 \\ 18 & K \end{vmatrix}}{18}$	0
s^0	$\dfrac{-\begin{vmatrix} 18 & K \\ b_1 & 0 \end{vmatrix}}{b_1}$	TBD

s^3	1	77
s^2	18	K
s^1	$\dfrac{77(18)-K}{18} = 77 - K/18$	0
s^0	K	0

a) First Column is has no sign changes (ie. system is stable) if $K>0$ + $77 - K/18 > 0 \Rightarrow 0 < K < 1386$

b) System is unstable if $K > 1386 \Rightarrow 3^{rd}$ element of First column is negative $\Rightarrow$ 2 RHP poles (+1 in LHP)

c) System is NEUTRALLY stable on the boundary between stability + Instability $K = 1386$ Note (3^{rd} element $= 0$)

c) (continued) System has 2 roots on $j\omega$ axis and 1 root in LHP (IT can be shown)
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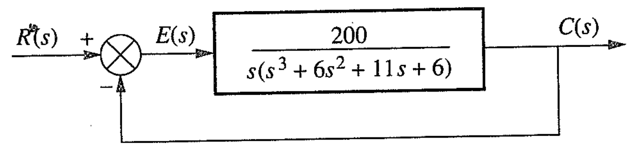
Standard Routh-Hurwitz

Problem Find the number of poles in the left half-plane, the right half-plane, and on the $j\omega$-axis for the system of Figure 6.6.



Solution First find the closed-loop transfer function as

$$T(s) = \frac{200}{s^4 + 6s^3 + 11s^2 + 6s + 200} \tag{6.14}$$

Column 1 Column 2

$$s^2: \quad -\frac{\begin{vmatrix}1 & 11\\ 1 & 1\end{vmatrix}}{1} = \cancel{10}\ \ 1 \qquad -\frac{\begin{vmatrix}1 & 200\\ 1 & 0\end{vmatrix}}{1} = \cancel{200}\ \ 20$$

$$s^1: \quad -\frac{\begin{vmatrix}1 & 1\\ 1 & 20\end{vmatrix}}{1} = -19 \qquad 0$$

Table 6.10 Routh table for Example 6.6

s^4	1	11	200
s^3	$\cancel{6}$ 1	$\cancel{6}$ 1	
s^2	$\cancel{10}$ 1	$\cancel{200}$ 20	
s^1	−19		
s^0	20		

The Routh table for the denominator of Eq. (6.14) is shown as Table 6.10. For clarity we leave most zero cells blank. At the s^1 row there is a negative coefficient; thus, there are two sign changes. The system is unstable, since it has two right-half-plane poles and two left-half-plane poles. The system cannot have $j\omega$ poles since a row of zeros did not appear in the Routh table.

NOTE:
IN some cases the coefficients in the first column of the Routh Table are zero. These cases will not be considered in this Class, but can also be solved via the Routh Table approach