Modeling

M. Sami Fadali
Professor of Electrical Engineering
University of Nevada

Transfer Function

LTI System

$$a_n \frac{d^n c}{dt^n} + a_{n-1} \frac{d^{n-1} c}{dt^{n-1}} + \cdots + a_1 \frac{dc}{dt} + a_0 c$$

$$= b_m \frac{d^m r}{dt^m} + b_{m-1} \frac{d^{m-1} r}{dt^{m-1}} + \cdots + b_1 \frac{dr}{dt} + b_0 r$$

$$G(s) = \left.\frac{C(s)}{R(s)}\right|_{zero\ IC} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

Example $\ddot{c} + 2\dot{c} + 10c = \dot{r} + 4r$

$$G(s) = \frac{s+4}{s^2 + 2s + 10}$$

Response of LTI System

- Convolution Theorem: $g(t)$ = impulse response

$$c(t) = \int_0^t g(t-\tau) r(\tau) d\tau \overset{\mathscr{L}}{\leftrightarrow} C(s) = G(s)R(s)$$

- Impulse Input $r(t) = \delta(t), \overset{\mathscr{L}}{\leftrightarrow} R(s) = 1$

$$g(t) \overset{\mathscr{L}}{\leftrightarrow} G(s)$$

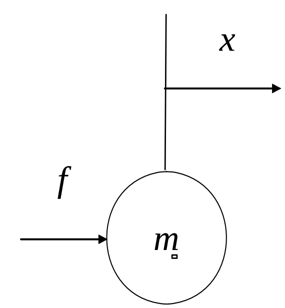
- The transfer function and the impulse response are Laplace transform pairs.

Example: Transfer Function of Point Mass

$f(t) = m\ddot{x}$

$F(s) = ms^2 X(s)$

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2}$$



$F(s) = 1/s$

$$X(s) = \frac{1}{ms^2} \times \frac{1}{s} \Rightarrow x(t) = \frac{t^2}{2m}$$

Analytical Modeling

- Physical systems store and dissipate energy.
- *Lumped* idealized elements: represent energy dissipation and energy storage separately.
- Physical elements may *approximate* the behavior of the idealized elements.
- Physical elements are *not* lumped. They involve both energy dissipation and energy storage.

Relations

- Constitutive (elemental) Relations
  - Govern the behavior of the idealized elements.
  - Hold only approximately for physical elements.
- Connective Relations
  - Govern connections of elements.
  - Often derived from conservation laws.

Electrical Systems

- **Ideal *R*:** energy dissipation.
- **Ideal *L*:** magnetic energy storage.
- **Ideal *C*:** electrostatic energy storage.
- **Connective Relations:** Kirchhoff's Laws
  - *Current Law:* conservation of charge
  - *Voltage Law:* conservation of energy

Constitutive Relations for *R*, *L*, *C*

Table 2.3

				Impedance	**Admittance**
Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)\,d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ = V (volts), $i(t)$ = A (amps), $q(t)$ = Q (coulombs), C = F (farads), R = Ω (ohms), G = ℧ (mhos), L = H (henries).

Mesh Analysis

1. Replace passive elements with $Z(s)$.
2. Define clockwise current for each mesh.
3. Write KVL in matrix form with source voltages that drive clockwise current positive.
4. Use Cramer's rule to solve for the transfer function.

Cramer's Rule

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

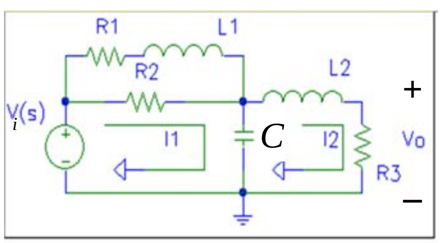
- Provided that a solution exists

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Example: Mesh Analysis

$$Z_1 = (L_1 s + R_1) \| R_2 = \frac{R_2 (L_1 s + R_1)}{L_1 s + R_1 + R_2}$$



$$[\mathbf{Z}(s)]\mathbf{I}(s) = \mathbf{V}_i(s)$$

$$\begin{bmatrix} Z_1 + 1/(sC) & -1/(sC) \\ -1/(sC) & R_3 + L_2 s + 1/(sC) \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} V_i(s) \\ 0 \end{bmatrix}$$

$$V_o(s) = R_3 I_2(s)$$

Cramer's Rule

$$I_2(s) = \frac{\begin{vmatrix} Z_1 + 1/(sC) & V_i \\ -1/(sC) & 0 \end{vmatrix}}{\begin{vmatrix} Z_1 + 1/(sC) & -1/(sC) \\ -1/(sC) & R_3 + L_2 s + 1/(sC) \end{vmatrix}}$$

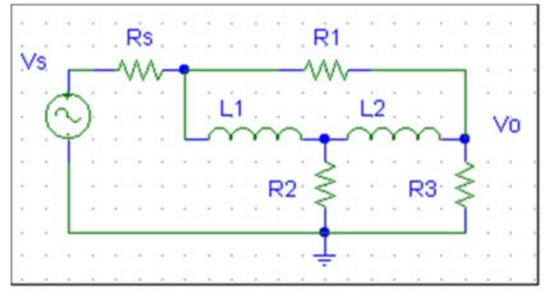
$$G(s) = \frac{V_o}{V_i} = \frac{R_3 I_2(s)}{V_i}$$

$$= \frac{R_3/(sC)}{(Z_1 + 1/(sC))(R_3 + L_2 s + 1/(sC)) - 1/(sC)^2}$$

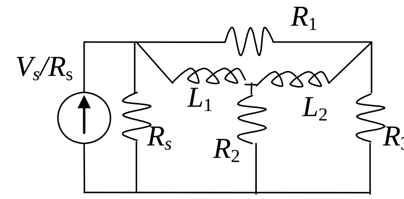
Node Analysis

1. Replace passive elements with $Y(s)$.
2. For each node, define a node voltage relative to a reference node.
3. Write KCL in matrix form with source currents that drive current into a node positive.
4. Solve for the transfer function using Cramer's rule.

Example: Node Analysis



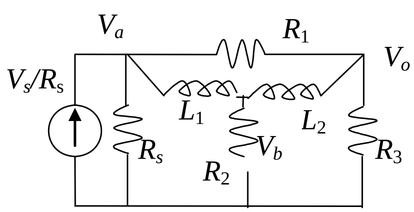
Change voltage source to current source.



Node Equations

$$[\mathbf{Y}(s)]\mathbf{V}(s) = \mathbf{I}(s)$$

$$G = 1/R, V_o = V_c$$



$$\begin{bmatrix} G_s + G_1 + \frac{1}{sL_1} & -\frac{1}{sL_1} & -G_1 \\ -\frac{1}{sL_1} & G_2 + \frac{1}{sL_1} + \frac{1}{sL_2} & -\frac{1}{sL_2} \\ -G_1 & -\frac{1}{sL_2} & G_1 + G_3 + \frac{1}{sL_2} \end{bmatrix} \begin{bmatrix} V_a(s) \\ V_b(s) \\ V_c(s) \end{bmatrix} = \begin{bmatrix} V_s(s)G_s \\ 0 \\ 0 \end{bmatrix}$$

$$G(s) = \frac{V_o}{V_s} = \frac{\left(\frac{1}{V_s}\right)\begin{vmatrix} G_s + G_1 + \frac{1}{sL_1} & -\frac{1}{sL_1} & \frac{V_s}{R_s} \\ -\frac{1}{sL_1} & G_2 + \frac{1}{sL_1} + \frac{1}{sL_2} & 0 \\ -G_1 & -\frac{1}{sL_2} & 0 \end{vmatrix}}{\begin{vmatrix} G_s + G_1 + \frac{1}{sL_1} & -\frac{1}{sL_1} & -G_1 \\ -\frac{1}{sL_1} & G_2 + \frac{1}{sL_1} + \frac{1}{sL_2} & -\frac{1}{sL_2} \\ -G_1 & -\frac{1}{sL_2} & G_1 + G_3 + \frac{1}{sL_2} \end{vmatrix}}$$

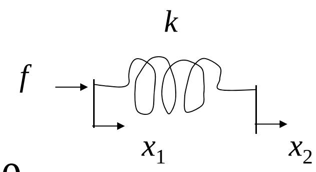
Translational Mechanical Systems

- Ideal *Damper b*: energy dissipation.
- Ideal *Spring k*: potential energy storage.
- Pure *Mass m*: kinetic energy storage.
- Connective Relations: Newton's 2nd Law

$$m\ddot{x} = \sum_i f_i$$

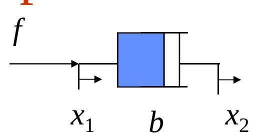
a) Ideal Spring

- Elastic energy (neglect plastic deformation)
- Linear element: force proportional to the deformation $x_1 - x_2$.
- No energy dissipation
- No mass



$$f + f_s = 0$$
$$f_s = -k(x_1 - x_2)$$
$$-f_s = k(x_1 - x_2) = f$$

b) Ideal Viscous Damper

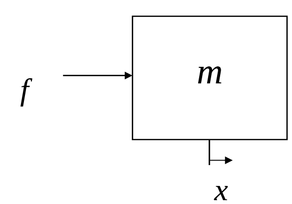


- Energy dissipation
- No mass
- No elastic deformation
- Linear element: force proportional to rate of deformation $\dot{x}_1 - \dot{x}_2$.

$$f + f_d = 0$$
$$f_d = -b(\dot{x}_1 - \dot{x}_2)$$
$$-f_d = b(\dot{x}_1 - \dot{x}_2) = f$$

c) Point Mass

- Perfectly rigid
- No dissipation
- Linear element



$$m\,\ddot{x}(t) = f(t)$$

Constitutive Relations for Spring, Mass, Damper

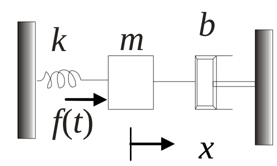
Component	Force-velocity	Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
Spring (K)	$f(t) = K\int_0^t v(\tau)\,d\tau$	$f(t) = Kx(t)$	K
Viscous damper (f_v)	$f(t) = f_v v(t)$	$f(t) = f_v \dfrac{dx(t)}{dt}$	$f_v s$
Mass (M)	$f(t) = M\dfrac{dv(t)}{dt}$	$f(t) = M\dfrac{d^2x(t)}{dt^2}$	Ms^2

Note: The following set of symbols and units is used throughout this book: $f(t)$ = N (newtons), $x(t)$ = m (meters), $v(t)$ = m/s (meters/second), K = N/m (newtons/meter), f_v = N-s/m (newton-seconds/meter), M = kg (kilograms = newton-seconds²/meter).

Ex. Mass-Spring-Damper

$$m\ddot{x} = f + f_s + f_d$$
$$= f - kx - b\dot{x}$$
$$m\ddot{x} + b\dot{x} + kx = f$$

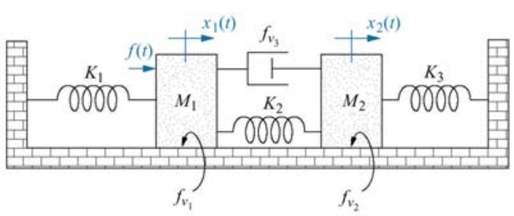
$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$



Example 2.11

$$m_1\ddot{x}_1 + b_1\dot{x}_1 + b_3(\dot{x}_1 - \dot{x}_2) + k_1x_1 + k_2(x_1 - x_2) = f$$
$$m_2\ddot{x}_2 + b_2\dot{x}_2 + b_3(\dot{x}_2 - \dot{x}_1) + k_3x_2 + k_2(x_2 - x_1) = 0$$

$$\begin{bmatrix} m_1s^2 + (b_1+b_3)s + k_1 + k_2 & -(b_3s+k_2) \\ -(b_3s+k_2) & m_2s^2 + (b_2+b_3)s + k_2 + k_3 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}$$



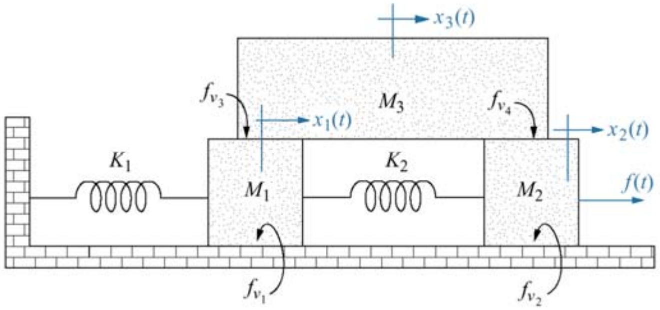
Transfer Function (Cramer's Rule)

$$G(s) = \frac{X_2}{F}$$

$$= \frac{\left(\dfrac{1}{F}\right)\begin{vmatrix} m_1s^2 + (b_1+b_3)s + k_1 + k_2 & F \\ -(b_3s+k_2) & 0 \end{vmatrix}}{\begin{vmatrix} m_1s^2 + (b_1+b_3)s + k_1 + k_2 & -(b_3s+k_2) \\ -(b_3s+k_2) & m_2s^2 + (b_2+b_3)s + k_2 + k_3 \end{vmatrix}}$$

$$= \frac{b_3s + k_2}{\left[m_1s^2 + (b_1+b_3)s + k_1 + k_2\right]\left[m_2s^2 + (b_2+b_3)s + k_2 + k_3\right] - \left[b_3s + k_2\right]^2}$$

3-D.O.F. Translational Mechanical System



Three equations of motion.

Rotational Mechanical Systems

Component	Torque-angular velocity	Torque-angular displacement	Impedance $Z_M(s) = T(s)/\theta(s)$
Spring K ($T(t)$, $\theta(t)$)	$T(t) = K\int_0^t \omega(\tau)d\tau$	$T(t) = K\theta(t)$	K
Viscous damper D ($T(t)$, $\theta(t)$)	$T(t) = D\omega(t)$	$T(t) = D\frac{d\theta(t)}{dt}$	Ds
Inertia J ($T(t)$, $\theta(t)$)	$T(t) = J\frac{d\omega(t)}{dt}$	$T(t) = J\frac{d^2\theta(t)}{dt^2}$	Js^2

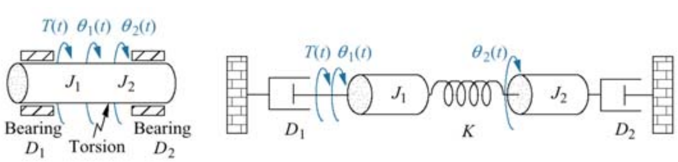
Note: The following set of symbols and units is used throughout this book: $T(t)$ = N-m (newton-meters), $\theta(t)$ = rad (radians), $\omega(t)$ = rad/s (radians/second), K = N-m/rad (newton-meters/radian), D = N-m-s/rad (newton-meters-seconds/radian), J = kg-m² (kilogram-meters² = newton-meters-seconds²/radian).

Example 2.19

$$J_1\ddot{\theta}_1 + D_1\dot{\theta}_1 + K(\theta_1 - \theta_2) = \tau$$

$$J_2\ddot{\theta}_2 + D_2\dot{\theta}_2 + K(\theta_2 - \theta_1) = 0$$

$$\begin{bmatrix} J_1s^2+D_1s+K & -K \\ -K & J_2s^2+D_2s+K \end{bmatrix}\begin{bmatrix}\Theta_1 \\ \Theta_2\end{bmatrix} = \begin{bmatrix}\mathrm{T} \\ 0\end{bmatrix}$$



(a) (b)

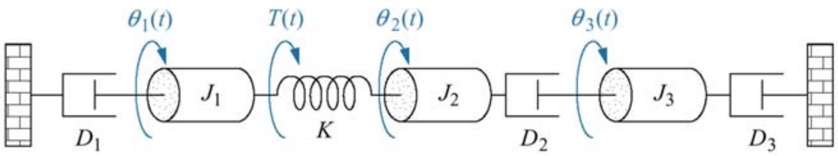
Transfer Function (Cramer's Rule)

$$G(s) = \frac{\Theta_2}{\mathrm{T}} = \frac{\left(\frac{1}{\mathrm{T}}\right)\begin{vmatrix} J_1s^2 + D_1s + K & \mathrm{T} \\ -K & 0 \end{vmatrix}}{\begin{vmatrix} J_1s^2 + D_1s + K & -K \\ -K & J_2s^2 + D_2s + K \end{vmatrix}}$$

$$= \frac{K}{\Delta}$$

$$\Delta = (J_1s^2 + D_1s + K)(J_2s^2 + D_2s + K) - K^2$$

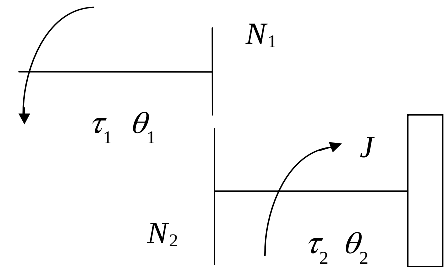
Example 2.20



$$J_1\ddot{\theta}_1 + D_1\dot{\theta}_1 + K(\theta_1 - \theta_2) = \tau$$
$$J_2\ddot{\theta}_2 + D_2(\dot{\theta}_2 - \dot{\theta}_3) + K(\theta_2 - \theta_1) = 0$$
$$J_3\ddot{\theta}_3 + D_2(\dot{\theta}_3 - \dot{\theta}_2) + D_3\dot{\theta}_3 = 0$$

$$\begin{bmatrix} J_1s^2 + D_1s + K & -K & 0 \\ -K & J_2s^2 + D_2s + K & -D_2s \\ 0 & -D_2s & J_3s^2 + (D_2 + D_3)s \end{bmatrix} \begin{bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{bmatrix} = \begin{bmatrix} T \\ 0 \\ 0 \end{bmatrix}$$

Gears



Assume:

1- No losses.

2- No inertia.

3- Perfectly rigid.

Single velocity at point of contact $r_1\dot{\theta}_1 = r_2\dot{\theta}_2$

Equal arc length $r_1\theta_1 = r_2\theta_2$

$$\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2} = \frac{\dot{\theta}_2}{\dot{\theta}_1} = \frac{\ddot{\theta}_2}{\ddot{\theta}_1}$$

Energy Balance

- Assume no losses $\tau_1\theta_1 = \tau_2\theta_2$

$$\frac{\tau_2}{\tau_1} = \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1} = \frac{\dot{\theta}_1}{\dot{\theta}_2} = \frac{\ddot{\theta}_1}{\ddot{\theta}_2}$$

- Trade speed for torque

$N_2 > N_1$: output side slower but delivers more torque

$N_2 < N_1$: output side faster but delivers less torque

Energy Storage

- Translation $E = \int_0^x f\,dx$
- Mass $E = \int_0^x m\left(\frac{dv}{dt}\right) dx = \int_0^t m\left(\frac{dv}{dt}\right)\left(\frac{dx}{dt}\right) dt$
  $= \int_0^v mv\,dv = \frac{1}{2}mv^2$
- Spring $E = \int_0^x kx\,dx = \frac{1}{2}kx^2$
- Rotation
- Inertia $E = \frac{1}{2}J\omega^2$
- Spring $E = \frac{1}{2}K\theta^2$

Energy Dissipation

- Power dissipated
- Translation

$$P = f \times v = bv \times v = bv^2$$

- Rotation

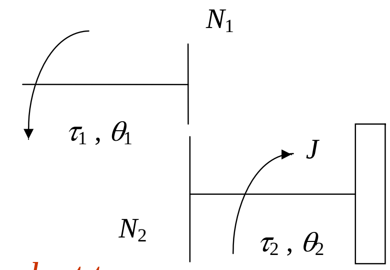
$$P = \tau \times \omega = B\omega \times \omega = B\omega^2$$

Equivalent Inertia

$$E = \frac{1}{2} J_1 \omega_1^2 = \frac{1}{2} J_2 \omega_2^2$$

$$\frac{J_1}{J_2} = \left(\frac{\omega_2}{\omega_1}\right)^2 = \left(\frac{N_1}{N_2}\right)^2$$

$$J_e = J_1 = J_2 \left(\frac{N_1}{N_2}\right)^2$$

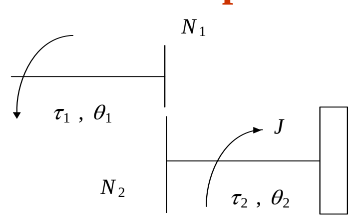


Equivalent to inertia on output side

$$J_e = J \left(\frac{\text{no. teeth of destination}}{\text{no. teeth of source}}\right)^2$$

Effect of Loading Output Side on Input Side

$$\frac{\tau_2}{\tau_1} = \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1}$$



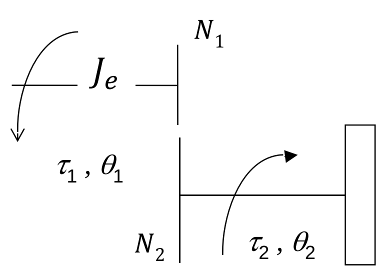
Inertia

$$\tau_2 = J\ddot{\theta}_2$$

$$\frac{N_2}{N_1}\tau_1 = J\left(\frac{N_1}{N_2}\ddot{\theta}_1\right)$$

$$J_e = \frac{\tau_1}{\ddot{\theta}_1} = J\left(\frac{N_1}{N_2}\right)^2$$

$$= J\left(\frac{\text{no. teeth of destination}}{\text{no. teeth of source}}\right)^2$$



Damper and Spring

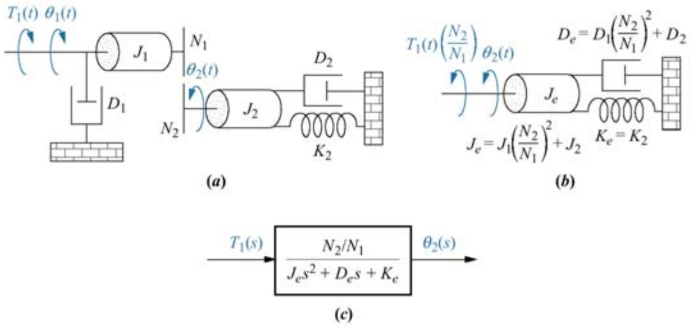
Damper

$$D_e = \frac{\tau_1}{\dot{\theta}_1} = D\left(\frac{N_1}{N_2}\right)^2 = D\left(\frac{\text{no. teeth of destination}}{\text{no. teeth of source}}\right)^2$$

Spring

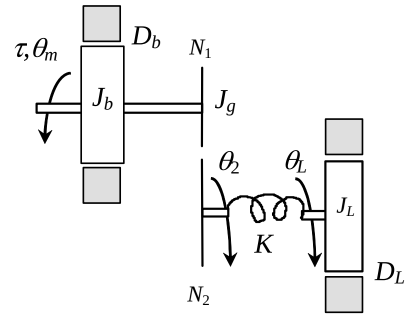
$$K_e = \frac{\tau_1}{\theta_1} = K\left(\frac{N_1}{N_2}\right)^2 = K\left(\frac{\text{no. teeth of destination}}{\text{no. teeth of source}}\right)^2$$

Example (Special Case)



$$\left(J_e s^2 + D_e s + K\right)\theta_2(s) = T_2(s) = \left(\frac{N_2}{N_1}\right)T_1(s)$$

Example

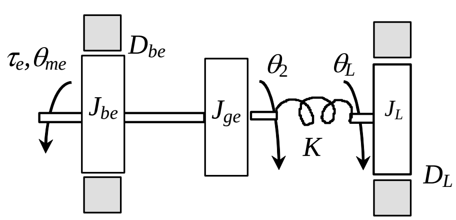


- Two equations of motion.
- Cannot simply add all rotational masses!

Solution

Redraw the schematic with

(i) added "*e*" for elements (and variables) moved, and (ii) gears removed.



Gear Train

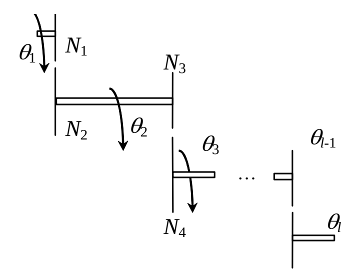
$$\frac{\theta_l}{\theta_1} = \left(\frac{\theta_2}{\theta_1}\right)\left(\frac{\theta_3}{\theta_2}\right)\cdots\left(\frac{\theta_l}{\theta_{l-1}}\right)$$

$$= \left(\frac{N_1}{N_2}\right)\left(\frac{N_3}{N_4}\right)\cdots\left(\frac{N_{2l-3}}{N_{2l-2}}\right)$$

$$= n_e$$

$$\frac{\tau_l}{\tau_1} = \frac{1}{n_e}$$

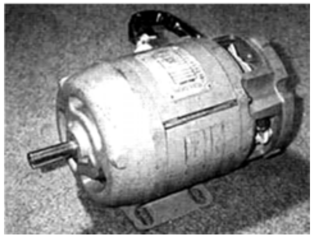
$$J_l = n_e^2 J_1 \qquad B_l = n_e^2 B_1 \qquad K_l = n_e^2 K_1$$



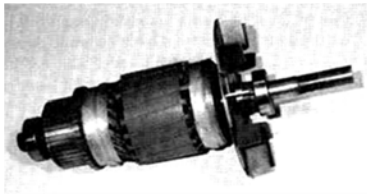
TFs of Electromechanical Systems

- Electrical Subsystem
  - Varies with motor type
  - *armature* (rotor) conductors current i_a
  - *field* (stator) conductors or permanent magnet
- Mechanical Subsystem
  - Varies with load.
  - Write equations of motion.

DC Motor



DC motor.



DC motor armature (rotor)

National Instruments:
http://zone.ni.com/devzone/cda/ph/p/id/52

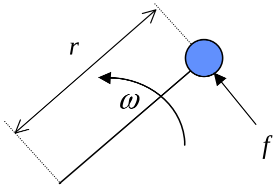
Torque Equation

- Magnetic flux $\phi(i_f)$ Wb
- Force $f = B\ l\ i_a$

l = conductor length

B = magnetic flux density

i_a = armature (rotor) current



$$T = f \times r \approx K\phi(i_f)i_a$$

Control

- Vary torque by changing i_a or i_f

Field Control

- Changing i_f and fixing i_a
- *Back EMF* (Faraday's Law)
- Voltage induced in moving coil proportional to the rate of cutting of lines of magnetic flux.

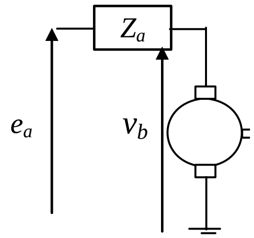
$$v_b = Blv = Blr\omega_m$$

- i_a is only approximately constant through the use of high resistance (inefficient)

$$i_a = \frac{e_a - v_b}{R_a} \approx \frac{e_a}{R_a}$$

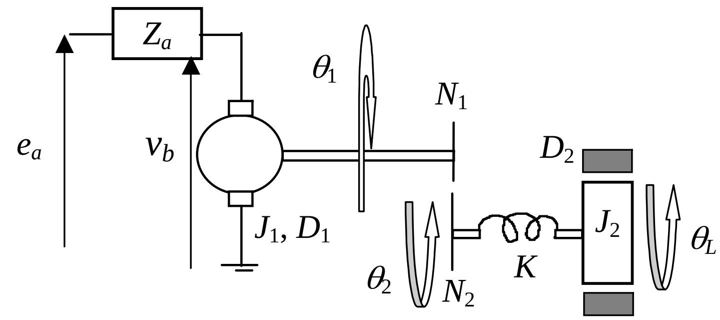
Armature Control

- Changing i_a and fixing i_f
- Used in practice.
- KVL



$$[L_a s + R_a]I_a + V_b = E_a$$
$$V_b = K_b \Omega(s) = K_b s\theta(s)$$
$$[L_a s + R_a]I_a + K_b s\theta(s) = E_a$$
$$T(s) = K_t I_a(s)$$

Schematic

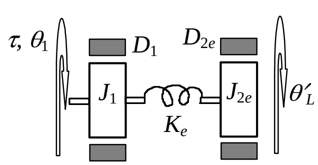


Mechanical Subsystem

Equations of motion for rotational system

$$K_e = K\left(\frac{N_1}{N_2}\right)^2, D_{2e} = D_2\left(\frac{N_1}{N_2}\right)^2, J_{2e} = J_2\left(\frac{N_1}{N_2}\right)^2$$

$$J_1\ddot{\theta}_1 + D_1\dot{\theta}_1 + K_e\left(\theta_1 - \theta_L'\right) = \tau = K_t i_a$$
$$J_{2e}\ddot{\theta}_L' + D_{2e}\dot{\theta}_L' + K_e\left(\theta_L' - \theta_1\right) = 0$$



Matrix Form

$$[L_a s + R_a]I_a + K_b s\theta(s) = E_a$$
$$J_1\ddot{\theta}_1 + D_1\dot{\theta}_1 + K_e\left(\theta_1 - \theta_L'\right) = \tau = K_t i_a$$
$$J_{2e}\ddot{\theta}_L' + D_{2e}\dot{\theta}_L' + K_e\left(\theta_L' - \theta_1\right) = 0$$

$$\begin{bmatrix} K_b s & 0 & L_a s + R_a \\ J_1 s^2 + D_1 s + K_e & -K_e & -K_t \\ -K_e & J_{2e}s^2 + D_{2e}s + K_e & 0 \end{bmatrix}\begin{bmatrix} \Theta_1 \\ \Theta_L' \\ I_a \end{bmatrix} = \begin{bmatrix} E_a \\ 0 \\ 0 \end{bmatrix}$$

Transfer Function

$$G(s)=\frac{\Theta_L}{E_a}=\frac{\Theta_L'(N_1/N_2)}{E_a}$$

$$=\frac{(N_1/N_2)\left(\frac{1}{E_a}\right)\begin{vmatrix} K_b s & E_a & L_a s+R_a \\ J_1 s^2+D_1 s+K_e & 0 & -K_t \\ -K_e & 0 & 0 \end{vmatrix}}{\begin{vmatrix} K_b s & 0 & L_a s+R_a \\ J_1 s^2+D_1 s+K_e & -K_e & -K_t \\ -K_e & J_{2e}s^2+D_{2e}s+K_e & 0 \end{vmatrix}}$$

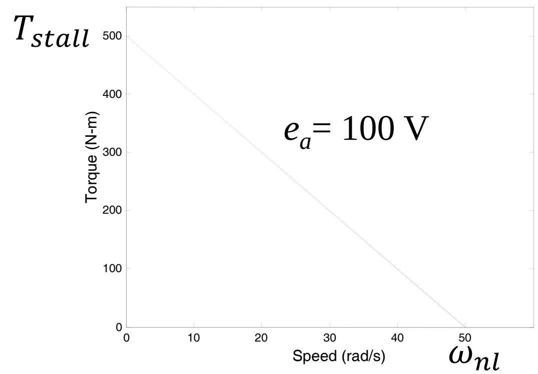
$$=\frac{(N_1/N_2)K_e K_t}{(L_a s+R_a)\left[(J_1 s^2+D_1 s+K_e)(J_{2e}s^2+D_{2e}s+K_e)-K_e^2\right]+K_t K_b s(J_{2e}s^2+D_{2e}s+K_e)}$$

Evaluation of Motor Parameters

Dynamometer measure speed & torque for constant e_a

Dynamometer Test gives speed-torque curves

Assume J_m, D_m supplied by manufacturer



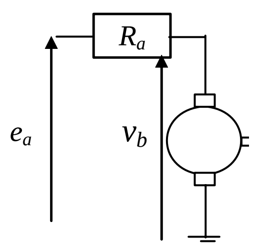
T_{stall}=stall torque

ω_{nl}=no-load speed

Solve for Parameters

$$R_a i_a + K_b \omega_m = e_a$$

$$T = K_t i_a = -\left(\frac{K_b K_t}{R_a}\right)\omega_m + \left(\frac{K_t}{R_a}\right)e_a$$



- Stall torque: $\omega_m = 0$

$$T_{stall} = \left(\frac{K_t}{R_a}\right)e_a \Rightarrow \frac{K_t}{R_a} = \frac{T_{stall}}{e_a}$$

- No-load speed: $T = 0$

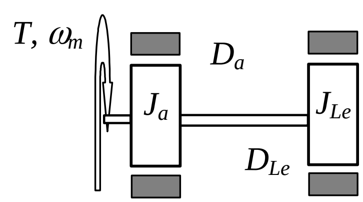
$$\omega_{nl} = \frac{e_a}{K_b} \Rightarrow K_b = \frac{e_a}{\omega_{nl}}$$

Transfer Function

$$I_a(s) = \frac{E_a(s) - K_b\Omega_m(s)}{R_a}, \qquad T(s) = K_t I_a(s)$$

$$(J_e s + D_e)\Omega_m(s) = K_t I_a(s) = K_t\frac{E_a(s)-K_b\Omega_m(s)}{R_a}$$

$$G(s)=\frac{\Omega_m(s)}{E_a(s)}=\frac{(K_t/R_a)}{J_e s + D_e + (K_t/R_a)K_b}$$



$$J_e = J_a + J_{Le}$$
$$D_e = D_a + D_{Le}$$

Linearity

(i) Homogeneity $r \to c \Rightarrow \alpha r \to \alpha c$

(ii) Additivity

$$r_i \to c_i, i = 1{,}2 \Rightarrow r_1 + r_2 \to c_1 + c_2$$

- Affine $\quad y(x) = ax + b$

$$y(\alpha x) = \alpha a x + b \neq \alpha y(x)$$

$$y(x_1 + x_2) = a(x_1 + x_2) + b \neq y(x_1) + y(x_2)$$

Nonlinearities

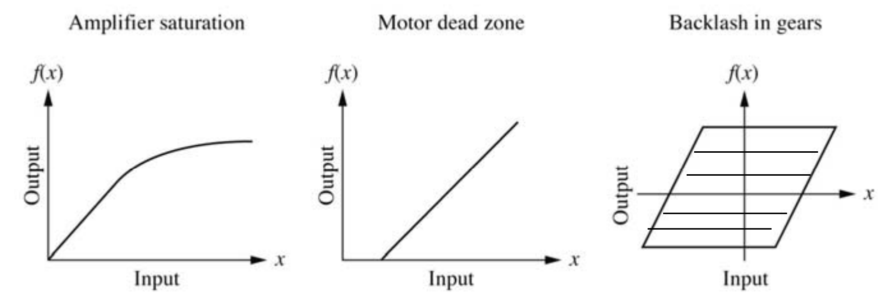


Figure 2.46

Linearization

1st order approximation (in the vicinity of x_0)

$$f(x) = f(x_0) + \left.\frac{df}{dx}\right]_{x=x_0} \Delta x + O(\Delta x^2)$$

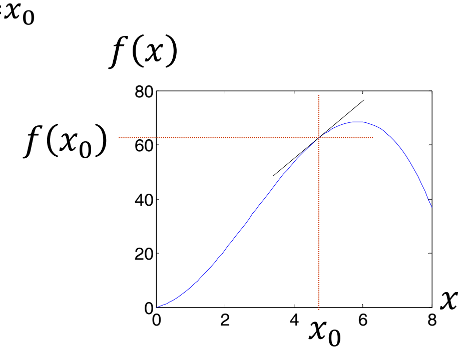
$$\Delta x = x - x_0$$

$$\Delta f = f(x) - f(x_0)$$

$$\approx \left.\frac{df}{dx}\right]_{x=x_0} \Delta x$$

for small Δx

$$\Delta f = m \Delta x$$



Equilibrium Point

- System at an equilibrium stays there unless perturbed.
- Set all derivatives equal to zero for equilibrium.

$$\frac{d^n c}{dt^n} + a_{n-1}\frac{d^{n-1}c}{dt^{n-1}} + \cdots + a_1 \frac{dc}{dt} + f(c) = r$$

$$\xrightarrow{equilibrium} f(c_0) = r_0$$

- r_0= value of forcing function at equilibrium c_0
- Cancel constants $f(c_0)$ and r_0

$$\frac{d^n c}{dt^n} + a_{n-1}\frac{d^{n-1}c}{dt^{n-1}} + \cdots + a_1 \frac{dc}{dt} + f(c) - f(c_0) = \Delta r$$

$$f(c) - f(c_0) \approx a_0 \Delta c, \qquad \Delta r = r - r_0$$

In the Vicinity of the Equilibrium

Special case: nonlinearity in output c only

$$\frac{d\Delta c}{dt} = \frac{d(c-c_0)}{dt} = \frac{dc}{dt} \quad \textit{Similarly} \quad \frac{d^i\Delta c}{dt^i} = \frac{d^i c}{dt^i}, i = 1, 2, \cdots, n$$

$$\frac{d^n\Delta c}{dt^n} + a_{n-1}\frac{d^{n-1}\Delta c}{dt^{n-1}} + \cdots + a_1\frac{d\Delta c}{dt} + f(c) - f(c_0) = r - r_0 = \Delta r$$

1st order approximation

$$\frac{d^n\Delta c}{dt^n} + a_{n-1}\frac{d^{n-1}\Delta c}{dt^{n-1}} + \cdots + a_1\frac{d\Delta c}{dt} + \left.\frac{df}{dc}\right|_{c_0}\Delta c \approx \Delta r$$

Linearized Differential Equation

$$\frac{d^n c}{dt^n} + a_{n-1}\frac{d^{n-1}c}{dt^{n-1}} + \cdots + a_1\frac{dc}{dt} + f(c) = r$$

$$\xrightarrow{equilibrium} f(c_0) = r_0$$

$$\frac{d^n\Delta c}{dt^n} + a_{n-1}\frac{d^{n-1}\Delta c}{dt^{n-1}} + \cdots + a_1\frac{d\Delta c}{dt} + a_0\Delta c \approx \Delta r, \quad a_0 = \left.\frac{df}{dc}\right|_{c_0}$$

Linear: can Laplace transform to get the TF

$$G(s) = \frac{\Delta C}{\Delta R} = \frac{1}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + a_1 s + a_0}$$

Procedure

1. Determine the equilibrium point(s).
2. Find the first order approximation of all nonlinear functions.
3. Rewrite the system differential equation in terms of perturbations canceling the constants using Step 1.

Example: Pendulum

Moment of inertia $J = ml^2$

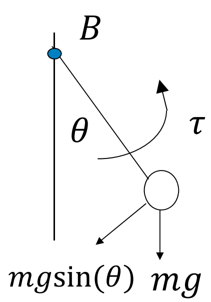
Equation of Motion

$$J\ddot{\theta} + B\dot{\theta} + mgl\sin\theta = \tau$$

- Linearize about $\theta = 30^o$
- Equilibrium at $\theta = 30^o$

$$mgl\sin 30° = \tau_0 \Rightarrow \tau_0 = mgl/2$$

$$J\Delta\ddot{\theta} + B\Delta\dot{\theta} + mgl\sin(30° + \Delta\theta) = \tau_0 + \Delta\tau$$



Linearization

- Using Trigonometric Identity

$$\sin(30^\circ + \Delta\theta) = \sin(30^\circ)\cos(\Delta\theta) + \cos(30^\circ)\sin(\Delta\theta)$$
$$\approx 1/2 + (\sqrt{3}/2)\Delta\theta$$
$$\cos(\Delta\theta) \approx 1 \qquad \sin(\Delta\theta) \approx \Delta\theta$$

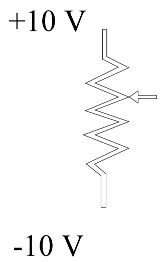
- Using 1st order approximation formula

$$\sin\theta \approx \sin(30^\circ) + \left.\frac{d\sin\theta}{d\theta}\right|_{30^\circ} \Delta\theta = \sin(30^\circ) + \cos(30^\circ)\Delta\theta$$

$$J\Delta\ddot{\theta} + B\Delta\dot{\theta} + mgl\left[1/2 + (\sqrt{3}/2)\Delta\theta\right] = \tau_0 + \Delta\tau$$
$$J\Delta\ddot{\theta} + B\Delta\dot{\theta} + mgl(\sqrt{3}/2)\Delta\theta = \Delta\tau$$

Potentiometer

- 10 turns
- 1 turn = 2 π rad
- 20 V
- Pot Gain = 20/(10 X 2 π)
  = (1/π) V/rad



Fluid Systems

Linearized Model

$h = R\,q$

Conservation of Mass

$$\frac{dCh}{dt} = Q + q_{in} - Q - q_o$$

$C = Area$

$$C\frac{dh}{dt} = q_{in} - \frac{h}{R} \Rightarrow \tau\frac{dh}{dt} + h = Rq_{in}$$

$$\frac{h(s)}{q_{in}(s)} = \frac{R}{\tau s + 1}$$