Lec. 4 Frequency Response and Sinusoidal Steady State Response

4.1 Frequency domain representation

In continuous linear time invariant (CLTI) system, it was important to know the frequency response of a system ($H(j\Omega)$) , which could be used to find the steady-state response of the system . For discrete linear time invariant (DLTI) system, $H(e^{jW})$ will be used to find the frequency response of the system.

$W = \Omega T$	rad/sample	digital frequency.
$\Omega = 2\pi f$	rad/sec.	analog frequency.
$T = 1 / f_s$	sec.	where, sampling rate = 1 / T .

4.1.1 Response to a complex exponential sequence:

If $x(n) = e^{j\,nW}$ (4.1)

And $$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) = \sum_{k=-\infty}^{\infty} h(k)\, e^{jW(n-k)} = e^{jnW} \sum_{k=-\infty}^{\infty} h(k)\, e^{-jWk} \quad (4.2)$$

Let $$H(e^{jW}) = \sum_{k=-\infty}^{\infty} h(k)\, e^{-jWk} \quad (4.3)$$

$$\therefore y(n) = e^{jnW}\, H(e^{jW}) \quad (4.4)$$

$$H(e^{jW}) = H_R(e^{jW}) + j\,H_I(e^{jW}) = |H(e^{jW})|\,\Phi(e^{jW}) \quad (4.5)$$

$$|H(e^{jW})| = [\,\{H_R(e^{jW})\}^2 + \{H_I(e^{jW})\}^2\,]^{1/2} \quad (4.6.a)$$

$$\Phi(e^{jW}) = \tan^{-1}[H_I(e^{jW}) \,/\, H_R(e^{jW})] \quad (4.6.b)$$

4.1.2 Response to a sinusoidal sequence:

If $x(n) = A\cos(W_o n + \theta) = \dfrac{A}{2}(e^{j\theta}\, e^{jW_o n} + e^{-j\theta}\, e^{-jW_o n})$ (4.7)

Substituting equation (4.7) into equation (4.4), and rearrange the terms, then:

$$y(n) = 2\,\mathrm{Re}[0.5\,A\; H(e^{jW_o})\; e^{jnW_o}\; e^{j\theta}]$$

$$y(n) = A\,|H(e^{jW_o})|\,\cos[\,n W_o + \theta + \Phi(e^{jW_o})\,] \quad (4.8)$$

Note: the output to a sinusoid is another sinusoid of the same frequency but with different phase and magnitude.

<u>Example (1):</u> A discrete time system has a unit sample response h(n)

$h(n) = 0.5\ \delta(n) + \delta(n-1) + 0.5\ \delta(n-2)$

a) Find the system frequency response. Plot magnitude and phase.

b) Find the steady-state response of the system to $x(n) = 5 \cos(\pi n/4)$.

c) Find the steady-state response of the system to $x(n) = 5 \cos(3\pi n/4)$.

d) Find the total response to $x(n) = u(n)$ assuming the system is initially at rest.

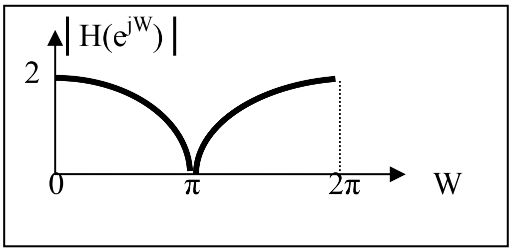
Solution:

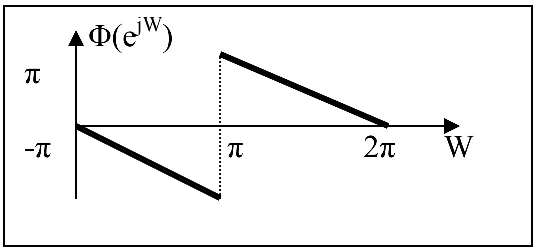
a) $H(e^{jW}) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-jWn} = 0.5\, e^{-0} + e^{-jW} + 0.5\, e^{-j2W}$

$= e^{-jW}\,[\,0.5\, e^{jW} + 1 + 0.5\, e^{-jW}\,] = e^{-jW}\,(1 + \cos W)$

$|H(e^{jW})| = |e^{-jW}| \cdot |(1 + \cos W)| = 1 + \cos W$

$\Phi(e^{jW}) = \tan^{-1}(e^{-jW}) + \tan^{-1}(1 + \cos W) = -W$





b) Applying equation (4.8), where, $W_0 = \pi / 4$

$|H(e^{jW_0})| = |H(e^{j\pi/4})| = 1 + \cos(\pi/4) = 1.707$

$\Phi(e^{jW_0}) = -\pi/4$

Then $y(n) = 5\,(1.707) \cos[\,(n\pi/4) - (\pi/4)\,] = 8.535 \cos[\pi(n-1)/4]$

c) $|H(e^{j3\pi/4})| = 1 + \cos(3\pi/4) = 0.2928$

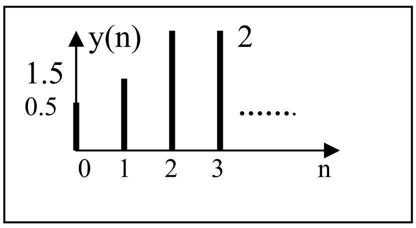
$\Phi(e^{jW_0}) = -3\pi/4$

$y(n) = 5\,(0.2928) \cos[\,(n\pi/4) - (3\pi/4)\,] = 1.4644 \cos[3\pi(n-1)/4]$

In part (b) the input signal is amplified, while in part (c) the input signal is attenuated.

d) $y(n) = x(n) \otimes h(n)$

$= 0.5\, x(n) + x(n-1) + 0.5\, x(n-2)$



= 0.5 u(n) + u(n – 1) + 0.5 u(n – 2)

Note: $\delta(t - t_o) \otimes f(t) = f(t - t_o)$

Properties of frequency response:

1- $H(e^{jW})$ is a continuous function in W.

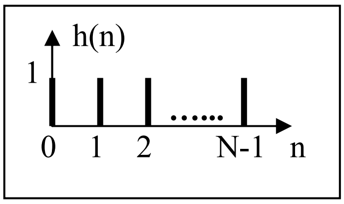
2- $H(e^{jW})$ is periodic in W with period 2π.

3- $|H(e^{jW})|$ is an even function of W and symmetrical about π.

4- $\Phi(e^{jW})$ is an odd function of W and unti-symmetrical about π.

Example (2): Find and plot the frequency response of a rectangular window filter if :

$$h(n) = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{elsewhere} \end{cases}$$



Solution:

$$H(e^{jW}) = \sum_{k=-\infty}^{\infty} h(k)\, e^{-jWk} = \sum_{k=0}^{N-1} e^{-jWk} = \frac{1 - e^{-jWN}}{1 - e^{-jW}}$$

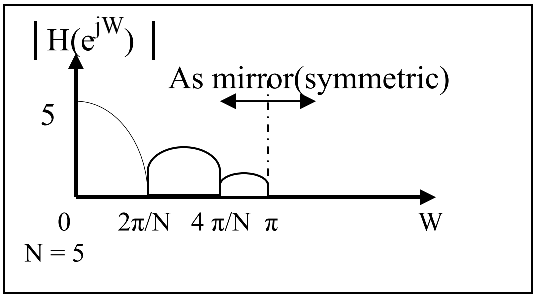
By using $\sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a}, \quad a \ne 1$

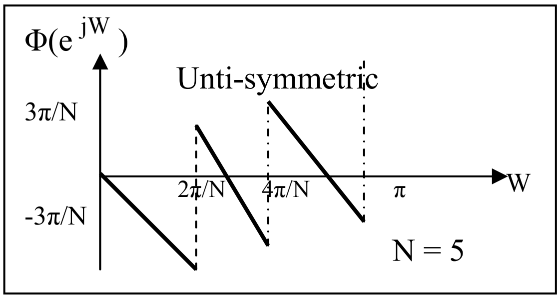
$$H(e^{jW}) \frac{e^{-jWN/2}\,(e^{jWN/2} - e^{-jWN/2})}{e^{-jW/2}\,(e^{jW/2} - e^{-jW/2})} = e^{-jW(N-1)/2}\, \frac{\sin(WN/2)}{\sin(W/2)}$$

$$|H(e^{jW})| = \frac{\sin(WN/2)}{\sin(W/2)}$$

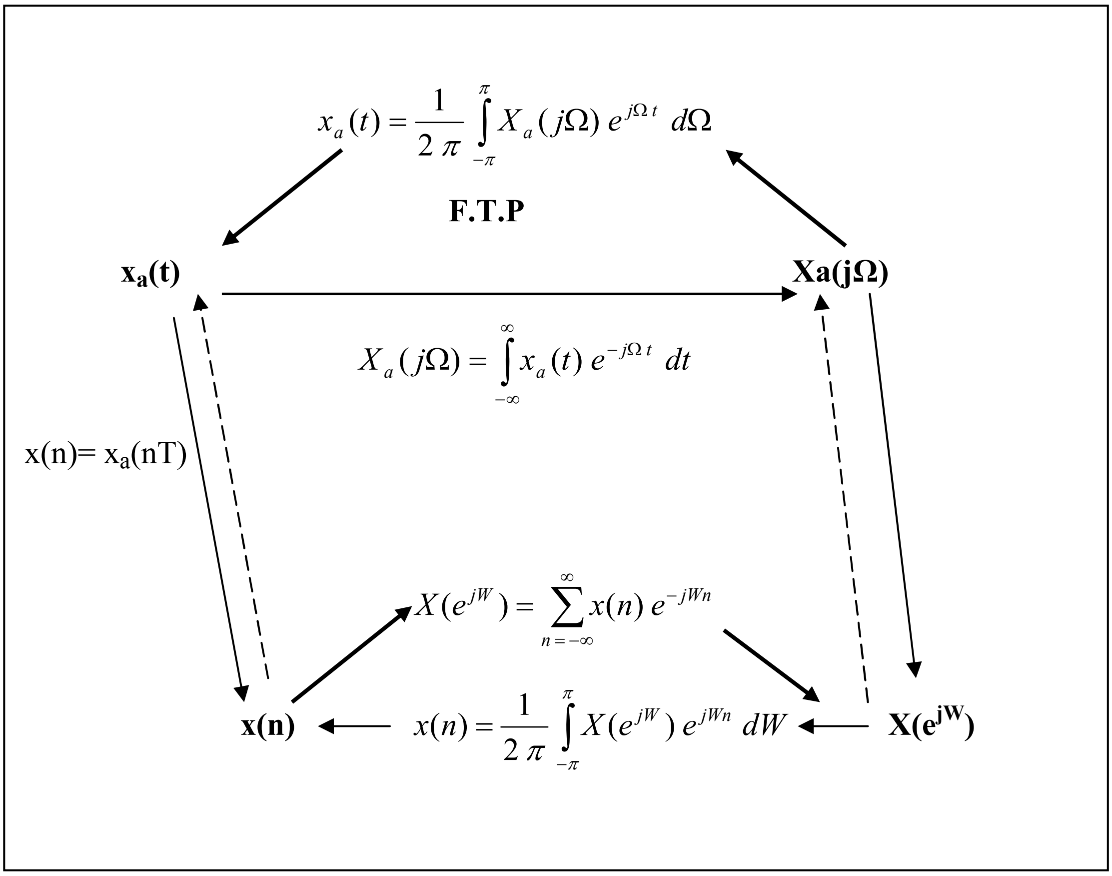
$$\Phi(e^{jW}) = -W(N-1)/2 + \tan^{-1}\left\{ \frac{\sin(WN/2)}{\sin(W/2)} \right\}$$

Assuming **N = 5** , then





4.3 Theorems:



The dotted lines do not hold if:

1- $x_a(t)$ is not band limited

2- $x_a(t)$ is band limited but the sampling rate is less than Nyquest rate.

4.3.1 Fourier Transform of a sequence:

$$\Im\{x(n)\} = X(e^{jW}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jnW} \qquad (4.10)$$

$$\Im^{-1}\{X(e^{jW})\} = x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jW})\, e^{jnW}\, dW \qquad (4.11)$$

$$\text{Energy} = E = \sum_{n=-\infty}^{\infty} x(n)\, x^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jW})\;\; X^*(e^{jW})\, dW \qquad (4.12)$$

* = complex conjugate

$$Y(e^{jW}) = H(e^{jW})\, .\;\; X(e^{jW}) \qquad (4.13)$$

$$y(n) = \Im^{-1}\{\, H(e^{jW})\, .\;\; X(e^{jW})\} \qquad (4.14)$$

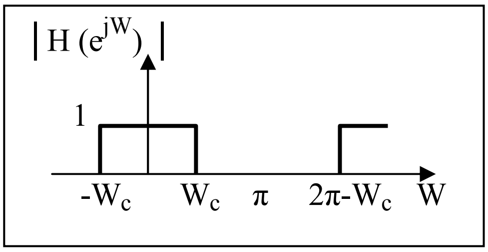
4.3.2 Sampling a continuous signal:

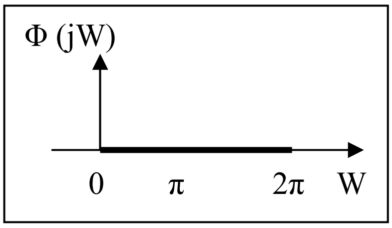
$$x_a(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_a(j\Omega)\, e^{j\Omega t}\, d\Omega \qquad (4.15)$$

$$X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t)\, e^{-j\Omega t}\, dt \qquad (4.16)$$

$$X(e^{jW}) = \frac{1}{T} \sum_{r=-\infty}^{\infty} X_a\{\, j\,(\Omega + \frac{2\pi\, r}{T})\,\} \qquad (4.17)$$

Example(3): The frequency response of an ideal L.P.F. is given below. Find and plot h(n), if $W_c = \pi/2$.





Solution:

$$h(n) = \frac{1}{2\pi} \int_{-W_c}^{W_c} e^{jWn}\, dW = \frac{\sin W_c\, n}{\pi\, n}$$

$$h(0) = W_c / \pi$$