Lec. 5 Z - Transform

5.1 Definition of Z.T

The z-transform is a very important tool in describing and analyzing digital systems. It also offers the techniques for digital filter design and frequency analysis of digital signals.

The z-transform of a *causal* sequence x(n), designated by X(z) or Z(x(n)), is defined as:

$$X(z) = Z(x(n)) = \sum_{n=0}^{\infty} x(n)z^{-n}$$
$$= x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} + \ldots \quad (5.1)$$

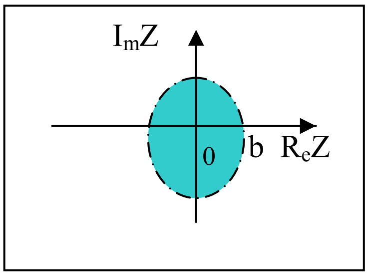
Where, z is the complex variable. Here, the summation taken from n = 0 to n = ∞ is according to the fact that for most situations, the digital signal x(n) is the ***causal*** sequence, that is, x(n) = 0 for n ≤ 0. For non-causal system, the summation starts at n = -∞. Thus, the definition in Equation (5.1) is referred to as a one-sided z-transform or a unilateral transform. The region of convergence is defined based on the particular sequence x(n) being applied. The z-transforms for common sequences are summarized below:

Line No.	$x(n)$, $n \geq 0$	z-Transform $X(z)$	Region of Convergence
1	$x(n)$	$\sum_{n=0}^{\infty} x(n)z^{-n}$	
2	$\delta(n)$	1	$\lvert z\rvert > 0$
3	$au(n)$	$\frac{az}{z-1}$	$\lvert z\rvert > 1$
4	$nu(n)$	$\frac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
5	$n^2u(n)$	$\frac{z(z+1)}{(z-1)^3}$	$\lvert z\rvert > 1$
6	$a^n u(n)$	$\frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
7	$e^{-na}u(n)$	$\frac{z}{(z-e^{-a})}$	$\lvert z\rvert > e^{-a}$
8	$na^n u(n)$	$\frac{az}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$
9	$\sin(an)u(n)$	$\frac{z\sin(a)}{z^2 - 2z\cos(a) + 1}$	$\lvert z\rvert > 1$
10	$\cos(an)u(n)$	$\frac{z[z-\cos(a)]}{z^2 - 2z\cos(a) + 1}$	$\lvert z\rvert > 1$
11	$a^n \sin(bn)u(n)$	$\frac{[a\sin(b)]z}{z^2 - [2a\cos(b)]z + a^2}$	$\lvert z\rvert > \lvert a\rvert$
12	$a^n \cos(bn)u(n)$	$\frac{z[z-a\cos(b)]}{z^2 - [2a\cos(b)]z + a^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
13	$e^{-an}\sin(bn)u(n)$	$\frac{[e^{-a}\sin(b)]z}{z^2 - [2e^{-a}\cos(b)]z + e^{-2a}}$	$\lvert z\rvert > e^{-a}$
14	$e^{-an}\cos(bn)u(n)$	$\frac{z[z-e^{-a}\cos(b)]}{z^2 - [2e^{-a}\cos(b)]z + e^{-2a}}$	$\lvert z\rvert > e^{-a}$
15	$2\lvert A\rvert\lvert P\rvert^n \cos(n\theta + \phi)u(n)$ where P and A are complex constants defined by $P = \lvert P\rvert\angle\theta, A = \lvert A\rvert\angle\phi$	$\frac{Az}{z-P} + \frac{A^*z}{z-P^*}$	

Example(1): Find Z.T including region of convergence of x(n) = - b^n u(- n - 1)

Solution: the system is non- causal

$$X(Z)=\sum_{n=-\infty}^{\infty}-b\,u(-n-1)\,Z^{-n}=-\sum_{n=-\infty}^{-1}(b/Z)^{n}$$



Let m = - n

$$X(Z)=-\sum_{m=1}^{\infty}(Z/b)^{m}=1-\sum_{m=0}^{\infty}(Z/b)^{m}$$

By using $\sum_{m=0}^{\infty}x^{m}=1+x+x^{2}+x^{3}+....=\frac{1}{x-1}$ $\quad ,|x|\langle 1$

$$X(Z)=1-\frac{1}{1-(Z/b)}=\frac{Z}{Z-b},\quad |(Z/b)|\langle 1 \quad or \quad |Z|\langle |b|$$

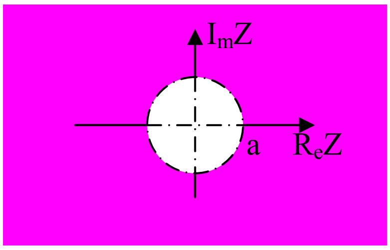
The region of convergence (ROC) is <u>inside</u> the unit circle only.

Example(2): Find Z.T including region of convergence of x(n) = a^n u(n)

$$X(Z)=\sum_{n=0}^{\infty}a^{n}\,Z^{-n}=\sum_{n=0}^{\infty}(a\,Z^{-1})^{n}=\frac{1}{1-aZ^{-1}}=\frac{Z}{Z-a}\quad ,\quad |a\,Z^{-1}|\langle 1$$

Or $|Z| > |a|$

The region of convergence (ROC) is <u>outside</u> the unit circle only.



5.2 Properties of Z.T:

5.2.1 Linearity: The z-transform is a linear transformation, which implies

$$Z\,(a\,x_{1}(n)\pm b\,x_{2}(n)\,)=a\;X_{1}(Z)\pm b\;X_{2}(Z) \qquad (5.2)$$

Where a and b are constants

5.2.2 Shift theorem (without initial conditions): Given X(z), the z-transform of a sequence x(n), the z-transform of x(n - m), the time-shifted sequence, is given by;

$$Z\{x(n-m)\}=Z^{-m}\,X(Z) \qquad (5.3)$$

5.2.3 Convolution: Given two sequences $x_1(n)$ and $x_2(n)$, their convolution can be determined as follows:

$$x(n) = x_1(n) \otimes x_2 = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k) = \sum_{k=-\infty}^{\infty} x_1(n-k)\, x_2(k) \qquad (5.4)$$

Where $\otimes$ designates the linear convolution. In z-transform domain, we have

$$X(Z) = X_1(Z) \,.\, X_2(Z) \qquad (5.5)$$

5.2.4 Multiplication by exponential:

$$Z\{\, a^n\; x(n)\,\} = X(Z)\,\Big|_{Z \to \frac{Z}{a}} \qquad (5.6.a)$$

$$Z\{e^{\pm an}\, x(n)\} = X(Z)\,\Big|_{Z \to e^{\mp a} Z} \qquad (5.6.b)$$

5.2.5 Initial and final value theorems:

$$\lim_{n\to 0} x(n) = \lim_{Z\to\infty} X(Z) = x(0) \quad \textit{initial value theorem} \qquad (5.7.a)$$

$$\lim_{n\to\infty} x(n) = \lim_{Z\to 1} Z^{-1}\;(Z-1)\;\; X(Z) \quad \textit{final value theorem} \qquad (5.7.b)$$

5.2.6 Multiplication by n:

$$Z\{\, n\, x(n)\,\} = -\,Z\,\frac{d}{dZ}\,X(Z) \qquad (5.8)$$

Example(3) : Find $Z\{(n-2)\, a^{(n-2)}\, \cos[\, w(n-2)]\, u(n-2)$.

The solution is:

$$= Z^{-2}\; Z\{\, n\, a^n\, \cos wn\, u(n)\,\}$$

$$= Z^{-2}\;(-Z)\,\frac{d}{dZ}\, Z\{\, a^n\, \cos wn\, u(n)\,\}$$

$$= -Z^{-1}\,\frac{d}{dZ}\,\frac{Z^2 - Z\cos w}{Z^2 - 2\,Z\cos w + 1}\;\Big|_{Z\to\frac{Z}{a}}$$

5.3 Inverse of Z.T

$$x(n) = Z^{-1}\,\{X(Z)\} \qquad (5.9)$$

The inverse z-transform may be obtained by the following methods:

1. Using properties.
2. Partial fraction expansion method.

3. Residue method.
4. Power series expansion (the solution is obtained by applying long division because the denominator can't be analyzed. It is not accurate method compared with the above three methods)

Example(4): Find x(n), using properties , if

$$X(z) = \frac{10z}{z^2 - z + 1}$$

Solution:

Since $X(z) = \frac{10z}{z^2 - z + 1} = \left(\frac{10}{\sin(a)}\right)\frac{\sin(a)z}{z^2 - 2z\cos(a) + 1}$,
by coefficient matching, we have

$$-2\cos(a) = -1.$$

Hence, $\cos(a) = 0.5$, and $a = 60°$. Substituting $a = 60°$ into the sine function leads to

$$\sin(a) = \sin(60°) = 0.866.$$

Finally, we have

$$x(n) = \frac{10}{\sin(a)} Z^{-1}\left(\frac{\sin(a)z}{z^2 - 2z\cos(a) + 1}\right) = \frac{10}{0.866}\sin(n \cdot 60^0)$$

$$= 11.547\sin(n \cdot 60^0).$$

Example(5): Find x(n) using partial fraction method , if:

$$X(z) = \frac{1}{(1 - z^{-1})(1 - 0.5z^{-1})}.$$

Solution:

Eliminating the negative power of z by multiplying the numerator and denominator by z^2 yields

$$X(z) = \frac{z^2}{z^2(1 - z^{-1})(1 - 0.5z^{-1})}.$$

$$= \frac{z^2}{(z - 1)(z - 0.5)}$$

Dividing both sides by z leads to

$$\frac{X(z)}{z} = \frac{z}{(z - 1)(z - 0.5)}.$$

Again, we write

$$\frac{X(z)}{z} = \frac{A}{(z - 1)} + \frac{B}{(z - 0.5)}.$$

$$A = (z-1)\frac{X(z)}{z}\Big|_{z=1} = \frac{z}{(z-0.5)}\Big|_{z=1} = 2,$$

$$B = (z-0.5)\frac{X(z)}{z}\Big|_{z=0.5} = \frac{z}{(z-1)}\Big|_{z=0.5} = -1.$$

Thus

$$\frac{X(z)}{z} = \frac{2}{(z-1)} + \frac{-1}{(z-0.5)}.$$

Multiplying z on both sides gives

$$X(z) = \frac{2z}{(z-1)} + \frac{-z}{(z-0.5)}.$$

$$x(n) = 2u(n) - (0.5)^n u(n).$$

Example(6) : Find x(n) using the residue theorem, if

$$X(Z) = \frac{2Z}{(Z-1)^2\ (Z-2)\ (Z-3)}$$

The residue theorem is:

x(n) = ∑ residues of X(Z) Z^{n-1} at the poles of X(Z) Z^{n-1} = $a_{-1} + b_{-1} + c_{-1} + \ldots.$ (5.10)

$$a_{-1} = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dZ^{m-1}}\{(Z-a)^m\ Z^{n-1}\ X(Z)\},\ \ m \textit{ is the order of the pole} \qquad (5.11)$$

Solution:

$$a_{-1} = \frac{1}{1!}\lim_{Z\to 1}\frac{d}{dZ}\frac{2Z\ \ Z^{n-1}}{(z-2)\ (Z-3)} = n + \frac{3}{2}$$

$$b_{-1} = \frac{1}{0!}\lim_{Z\to 2}\frac{2\ Z^n}{(Z-1)^2\ (Z-3)} = -2\ (2)^n$$

$$c_{-1} = \frac{1}{0!}\lim_{Z\to 3}\frac{2\ Z^n}{(Z-1)^2\ (Z-2)} = \frac{1}{2}\ (3)^n$$

$$x(n) = a_{-1} + b_{-1} + c_{-1} = n + \frac{3}{2} - 2\ (2)^n + \frac{1}{2}\ (3)^n$$

5.4 Solution of linear constant coefficient difference equation using Z.T

$$Z\{x(n-m)\} = Z^{-m}\,\{X(Z) + \sum_{k=-m}^{-1} x(k)\,Z^{-k}\} \qquad (5.12)$$

Example(7) : Solve y(n) – (3/2) y(n – 1) + (1/2) y(n – 2) = $(1/4)^n$, y(-1) = 4, y(-2) = 10 for n ≥ 0

Solution:

$$Y(Z) - \frac{3}{2}\{Y(Z)\,.\,Z^{-1} + y(-1)\} + \frac{1}{2}\{Z^{-2}\,Y(Z) + Z^{-1}\,y(-1) + y(-2)\} = \frac{Z}{Z-\frac{1}{4}}$$

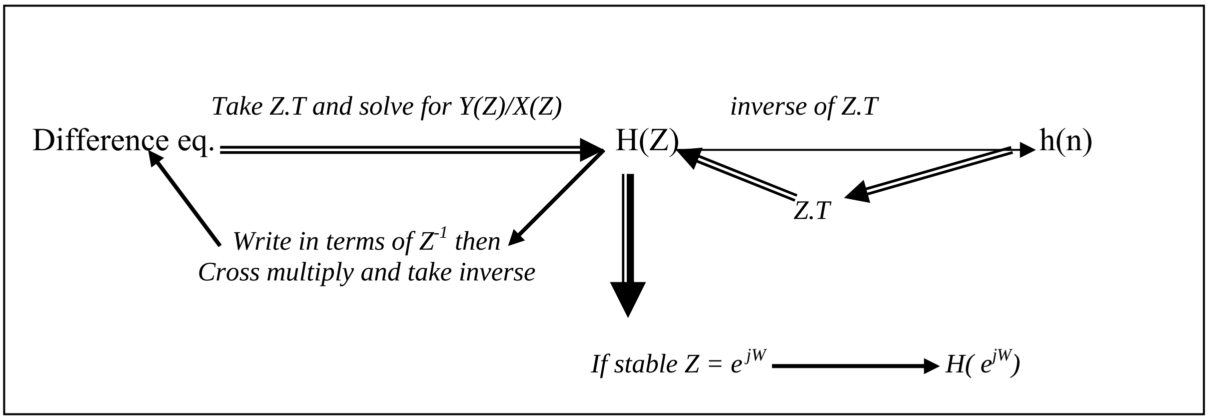
$$Y(Z)\{1 - \frac{3}{2}Z^{-1} + \frac{1}{2}Z^{-2}\} = \frac{Z}{Z-\frac{1}{4}} + 1 - 2Z^{-1}$$

$$Y(Z) = \frac{Z\,(2Z^2 - \frac{9}{4}Z + \frac{1}{2})}{(Z-\frac{1}{4})\,(Z-\frac{1}{2})\,(Z-1)}$$

$$Y(Z) = \frac{(1/3)\,Z}{(Z-\frac{1}{4})} + \frac{Z}{Z-\frac{1}{2}} + \frac{(2/3)\,Z}{Z-1}$$

$$y(n) = \{\frac{1}{3}(\frac{1}{4})^n + (\frac{1}{2})^n + \frac{2}{3}\}\,u(n)$$

5.5 Relations between system representations:



Continuous time system	Discrete time system
Differential equation	Difference equation
H_a (S)	H(Z)
H(jΩ)	$H(e^{jW})$
$H_a(t) = L^{-1}$ { H_a (S)}	$h(n) = Z^{-1}$ { H(Z) }