Lec. 6 Discrete Fourier Transform and Signal Spectrum

6.1 Discrete Fourier Transform

In time domain, representation of digital signals describes the signal amplitude versus the sampling time instant or the sample number. However, in some applications, signal frequency content is very useful than as digital signal samples.

The algorithm transforming the time domain signal samples to the frequency domain components is known as the discrete Fourier transform, or DFT. The DFT also establishes a relationship between the time domain representation and the frequency domain representation. Therefore, we can apply the DFT to perform frequency analysis of a time domain sequence. In addition, the DFT is widely used in many other areas, including spectral analysis, acoustics, imaging/ video, audio, instrumentation, and communications systems.

6.2 Fourier Series Coefficients of Periodic Digital Signals

To estimate the spectrum of a periodic digital signal x(n), sampled at a rate of f_s Hz with the fundamental period $T_0 = NT$, where there are N samples within the duration of the fundamental period and $T = 1/f_s$ is the sampling period. Fig. 6.1 shows periodic digital signal.

Fourier series expansion of a periodic signal x(t) in a complex form is:

$$c_k = \frac{1}{T_0} \int_{T_0} x(t)\, e^{-jkW_0 t}\, dt, \quad -\infty \le k \le \infty \qquad (6.1)$$

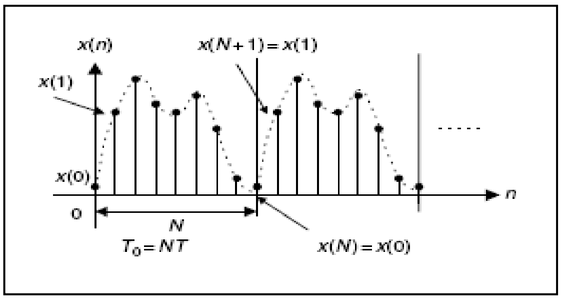


Fig. 6.1 periodic digital signal

Where, k is the number of harmonics corresponding to the harmonic frequency of kf_0 and $W_0 = 2\pi / T_0$ and $f_0 = 1/T_0$ are the fundamental frequency in radians per second and the fundamental frequency in Hz, respectively. To apply Equation (6.1), we substitute $T_0 = NT$, $W_0 = 2\pi / T_0$ and approximate the integration over one period using a summation by substituting $dt = T$ and $t = nT$. We obtain:

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi k n}{N}}, \quad -\infty \le k \le \infty \qquad (6.2)$$

Since the coefficients c_k are obtained from the Fourier series expansion in the complex form, the resultant spectrum c_k will have two sides. Therefore, the two-sided line amplitude spectrum $|c_k|$ is periodic, as shown in Fig. 6.2.

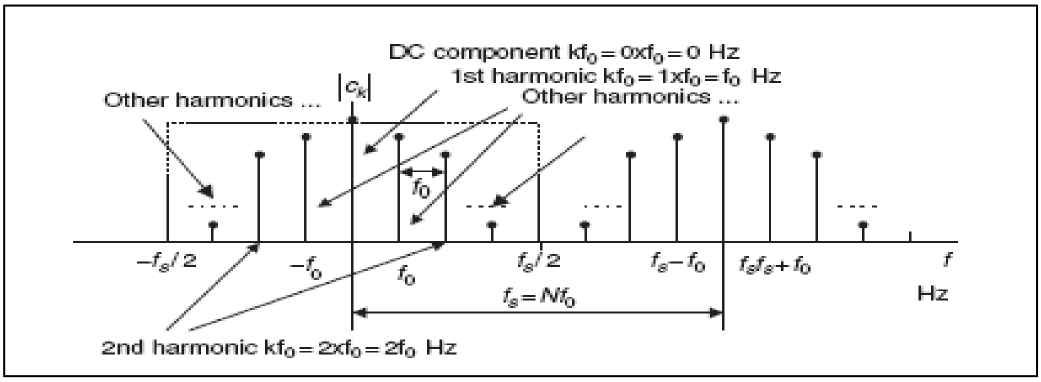


Fig. 6.2 Amplitude Spectrum of periodic Digital signal

As displayed in Figure 6.3 we note the following points:

a. Only the line spectral portion between the frequency $-f_s/2$ and frequency $f_s/2$ (folding frequency) represents the frequency information of the periodic signal.

b. The spectrum is periodic for every Nf_0 Hz.

c. For the kth harmonic, the frequency is $f = kf_0$ Hz. f_0 is called the frequency resolution.

Example(1):

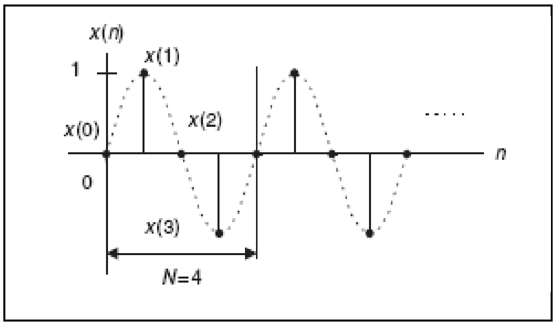
The periodic signal $x(t) = \sin(2\pi t)$ is sampled using the rate $f_s = 4$ Hz.

a. Compute the spectrum c_k using the samples in one period.

b. Plot the two-sided amplitude spectrum $|c_k|$ over the range from −2 to 2 Hz

Solution:

The fundamental frequency $W_0 = 2\pi$ radians per second and $f_0 = 1$, and the fundamental period $T_0 = 1$ second. Since using the sampling interval $T = 1/f_s = 0.25$ second. The sampled signal is $x(n) = \sin(0.5\pi n)$, and the first eight samples of it are plotted as shown



Choosing one period, N = 4, we have x(0) = 0; x(1) = 1; x(2) = 0; and x(3) = −1. Using Eq. (6.2),

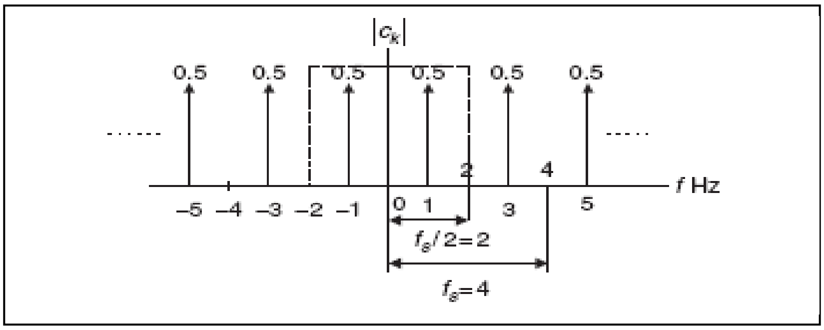
$$c_0 = \frac{1}{4}\sum_{n=0}^{3} x(n) = \frac{1}{4}(x(0) + x(1) + x(2) + x(3)) = \frac{1}{4}(0 + 1 + 0 - 1) = 0$$

$$c_1 = \frac{1}{4}\sum_{n=0}^{3} x(n)e^{-j2\pi \times 1n/4} = \frac{1}{4}\left(x(0) + x(1)e^{-j\pi/2} + x(2)e^{-j\pi} + x(3)e^{-j3\pi/2}\right)$$

$$= \frac{1}{4}(x(0) - jx(1) - x(2) + jx(3) = 0 - j(1) - 0 + j(-1)) = -j0.5.$$

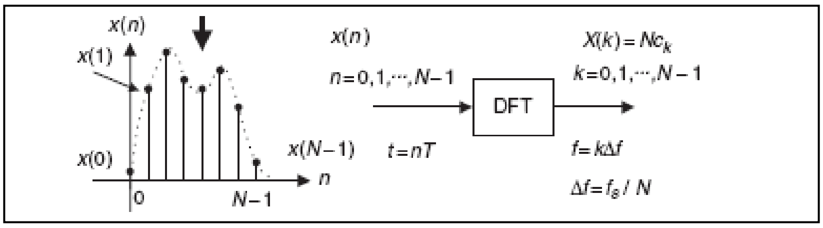
Similarly $c_3 = j\,0.5$. Using periodicity, it follows that $c_{-1} = c_1 = j0{:}5$, and $c_{-2} = c_2 = 0$.

b. The amplitude spectrum for the digital signal is sketched below:



6.3 Discrete Fourier Transform Formulas

Given a sequence x(n), $0 \le n \le N - 1$, its **DFT** is defined as:



$$X(k)=\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi k n}{N}} = \sum_{n=0}^{N-1} x(n)\, W_N^{k n}, \; for\; k=0,1,..N-1 \qquad (6.3)$$

Where the factor W_N (called the twiddle factor in some textbooks) is defined as

$$W_N = e^{-j\frac{2\pi}{N}} = \cos\left(\frac{2\pi}{N}\right) - j\sin\left(\frac{2\pi}{N}\right) \qquad (6.4)$$

The **inverse DFT** is given by:

$$x(n)=\frac{1}{N}\sum_{n=0}^{N-1} X(k)\, e^{j\frac{2\pi k n}{N}} = \frac{1}{N}\sum_{n=0}^{N-1} X(k)\, W_N^{-k n}, \; for\; n=0,1,..N-1 \qquad (6.5)$$

We can use MATLAB functions **fft()** and **ifft()** to compute the DFT coefficients and the inverse DFT.

<u>Example (2):</u> Given a sequence x(n) for $0 \le n \le 3$, where x(0) = 1, x(1) = 2, x(2) = 3, and x(3) = 4. Evaluate its DFT X(k).

Solution:

Since N = 4, $W_4 = e^{-j\pi/2}$, then using:

$$X(k) = \sum_{n=0}^{3} x(n) W_4^{kn} = \sum_{n=0}^{3} x(n) e^{-j\frac{\pi k n}{2}}.$$

For K= 0, X(0) = 10. Similarly, X(1) = −2 + j 2 , X(2) = −2, X(3) = −2 −j 2

Let us verify the result using the MATLAB function **fft():**

X = fft([1 2 3 4])

X = 10.0000 −2.0000+ 2.0000i − 2.0000 − 2.0000 − 2.0000i

Mapping the frequency bin k to its corresponding frequency is as follows:

$$\omega = \frac{k\omega_s}{N} \text{ (radians per second)}, \quad (6.6)$$

Since $w_s = 2\pi f_s$, then:

$$f = \frac{kf_s}{N} \text{ (Hz)}, \quad (6.7)$$

We can define the frequency resolution as the frequency step between two consecutive DFT coefficients to measure how fine the frequency domain presentation is and achieve

$$\Delta\omega = \frac{\omega_s}{N} \text{ (radians per second)}, \quad (6.8)$$

$$\Delta f = \frac{f_s}{N} \text{ (Hz)}. \quad (6.9)$$

Example (3): In example (2), If the sampling rate is 10 Hz,

a. Determine the sampling period, time index, and sampling time instant for a digital sample x(3) in time domain.

b. Determine the frequency resolution, frequency bin number, and mapped frequency for each of the DFT coefficients X(1) and X(3) in frequency domain.

Solution:

a. In time domain, we have the sampling period calculated as

$$T = 1/f_s = 1/10 = 0.1 \text{ second.}$$

For data $x(3)$, the time index is $n = 3$ and the sampling time instant is determined by

$$t = nT = 3 \cdot 0.1 = 0.3 \text{ second.}$$

b. In frequency domain, since the total number of DFT coefficients is four, the frequency resolution is determined by

$$\Delta f = \frac{f_s}{N} = \frac{10}{4} = 2.5 \text{ Hz.}$$

The frequency bin number for $X(1)$ should be $k = 1$ and its corresponding frequency is determined by

$$f = \frac{kf_s}{N} = \frac{1 \times 10}{4} = 2.5 \text{ Hz.}$$

Similarly, for $X(3)$ and $k = 3$,

$$f = \frac{kf_s}{N} = \frac{3 \times 10}{4} = 7.5 \text{ Hz.}$$

6.4 Amplitude Spectrum and Power Spectrum

One of the DFT applications is transformation of a finite-length digital signal x(n) into the spectrum in frequency domain. Fig. 6.3 demonstrates such an application, where A_k and P_k are the computed amplitude spectrum and the power spectrum, respectively, using the DFT coefficients X(k).

First, we achieve the digital sequence x(n) by sampling the analog signal x(t) and truncating the sampled signal with a data window with a length T_0 = NT, where T is the sampling period and N the number of data points. The time for data window is T_0 = NT**.**

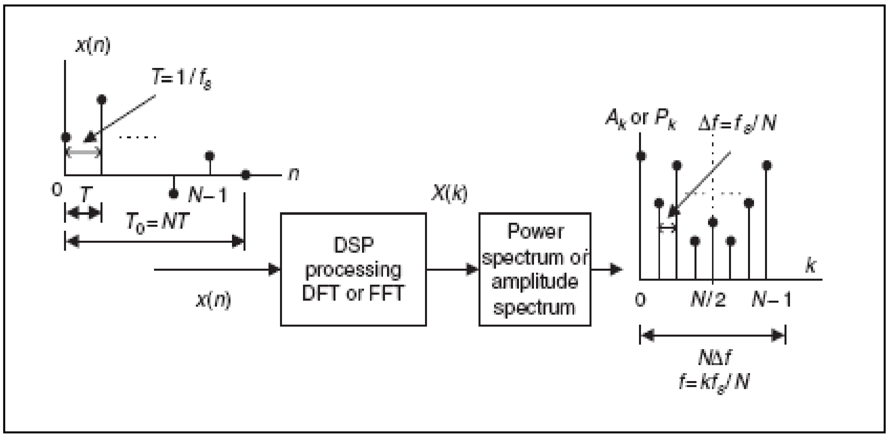


Fig. 6.3 Applications of DFT/ FFT

Next, we apply the DFT to the obtained sequence, x(n), to get the N DFT coefficients

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}, \text{ for } k = 0, 1, 2, \ldots, N-1. \quad (6.10)$$

We define the amplitude spectrum as:

$$A_k = \frac{1}{N}|X(k)| = \frac{1}{N}\sqrt{(\text{Real}[X(k)])^2 + (\text{Imag}[X(k)])^2}, \quad k = 0, 1, 2, \ldots, N-1. \quad (6.11)$$

Keeping original DC term at k = 0, a one-sided amplitude spectrum for equation (6.11) is:

$$\bar{A}_k = \begin{cases} \frac{1}{N}|X(0)|, & k = 0 \\ \frac{2}{N}|X(k)|, & k = 1, \ldots, N/2 \end{cases} \quad (6.12)$$

Correspondingly, the phase spectrum is given by:

$$\varphi_k = \tan^{-1}\left(\frac{\text{Imag}[X(k)]}{\text{Real}[X(k)]}\right), k = 0, 1, 2, \ldots, N-1. \quad (6.13)$$

Besides the amplitude spectrum, the power spectrum is also used. The DFT power spectrum is defined as:

$$P_k = \frac{1}{N^2}|X(k)|^2 = \frac{1}{N^2}\left\{(\text{Real}[X(k)])^2 + (\text{Imag}[X(k)])^2\right\},$$
$$k = 0, 1, 2, \ldots, N-1. \qquad (6.14)$$

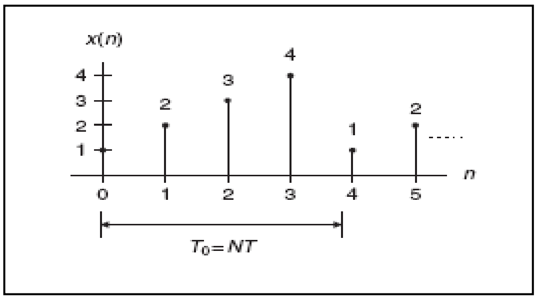
Similarly, for a one-sided power spectrum, we get:

$$\bar{P}_k = \begin{cases} \frac{1}{N^2}|X(0)|^2 & k = 0 \\ \frac{2}{N^2}|X(k)|^2 & k = 0, 1, \ldots, N/2 \end{cases}$$

and $$f = \frac{kf_s}{N}. \qquad (6.15)$$

The frequency resolution is defined in equation (6.9). It follows that better frequency resolution can be achieved by using a longer data sequence.

Example (4) : Consider the sequence:



Assuming that f_s = 100 Hz. Compute the amplitude spectrum, phase spectrum, and power spectrum.

Solution:

Since N = 4, DFT coefficients are: X(0) = 10, X(1) = −2 + j 2 , X(2) = −2, X(3) = −2 −j 2

For $k = 0$, $f = k \cdot f_s/N = 0 \times 100/4 = 0\,\text{Hz}$,

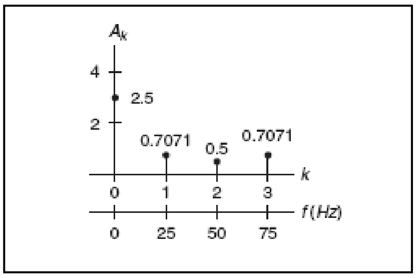
$$A_0 = \frac{1}{4}|X(0)| = 2.5, \; \varphi_0 = \tan^{-1}\left(\frac{\text{Imag}[X(0)]}{\text{Real}([X(0)]}\right) = 0^0,$$

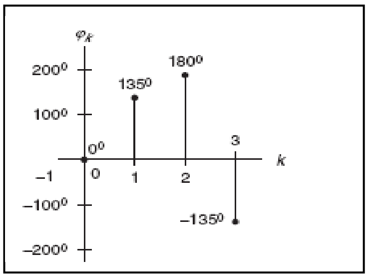
$$P_0 = \frac{1}{4^2}|X(0)|^2 = 6.25.$$

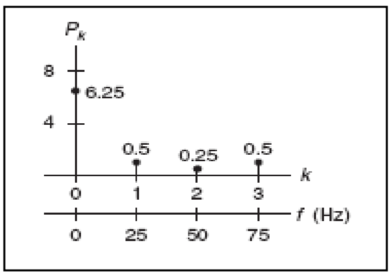
Similarly:

K	f	A_K	Φ_K in degree	P_K
1	25	0.7071	135	0.5
2	50	0.5	180	0.25
3	75	0.7071	-135	0.5

Thus, the sketches for the amplitude spectrum, phase spectrum, and power spectrum are given in the below Figures:



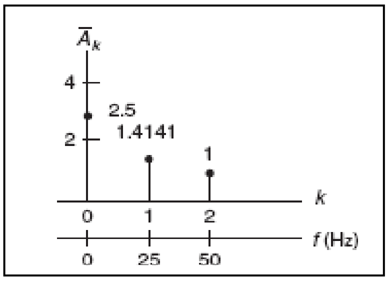




We can easily find the one-sided amplitude spectrum and one-sided power spectrum as:

$\bar{A}_0 = 2.5,\ \bar{A}_1 = 1.4141,\ \bar{A}_2 = 1$ and
$\bar{P}_0 = 6.25,\ \bar{P}_1 = 2,\ \bar{P}_2 = 1.$

We plot the one-sided amplitude spectrum for comparison:



Note that in the one-sided amplitude spectrum, the negative-indexed frequency components are added back to the corresponding positive-indexed frequency components; thus each amplitude value other than the DC term is doubled. It represents the frequency components up to the folding frequency.