Lec. 7   Windowing and FFT

7.1 Spectral Estimation Using Window Functions

Consider the pure 1-Hz sine wave with 32 samples shown in Fig. 7.1. As shown in the figure, if we use a window size of N =16 samples, which is a multiple of the two waveform cycles, the second window repeats with continuity. However, when the window size is chosen to be 18 samples, which is not a multiple of the waveform cycles (2.25 cycles), the second window repeats the first window with discontinuity. *It is this discontinuity that produces harmonic frequencies that are not present in the original signal (spectral leakage).* Fig.7.2 shows the spectral plots for both cases using the DFT/FFT directly.
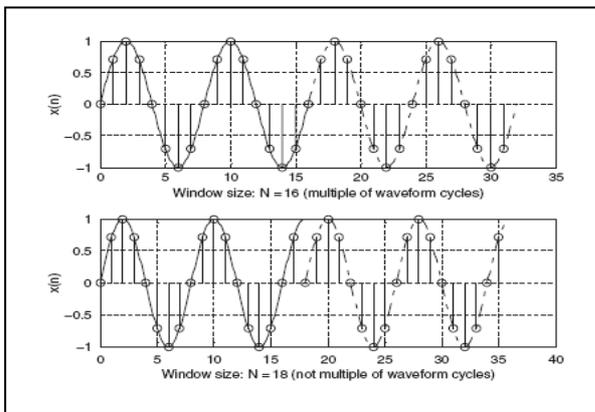
Fig. 7.1 Sampling a 1-Hz sine wave using (top) 16 samples per cycle and (bottom) 18 samples per cycle.
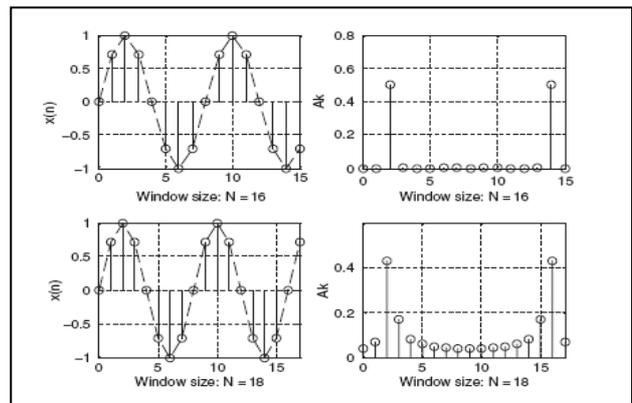
Fig. 7.2 Signal samples and spectra without and with spectral leakage.

The amount of spectral leakage shown in the second plot is due to amplitude discontinuity in time domain. The bigger the discontinuity, the more is the leakage. To reduce the effect of spectral leakage, a window function can be used whose amplitude tapers smoothly and gradually toward zero at both ends. Applying the window function w(n) to a data sequence x(n) to obtain a windowed sequence $x_w(n)$ is better illustrated in Fig. 7.3 using :

$$x_w(n) = x(n)w(n), \text{ for } n = 0, 1, \ldots, N-1.$$

(7.1)

Fig. 7.3 Illustration of the window operation.
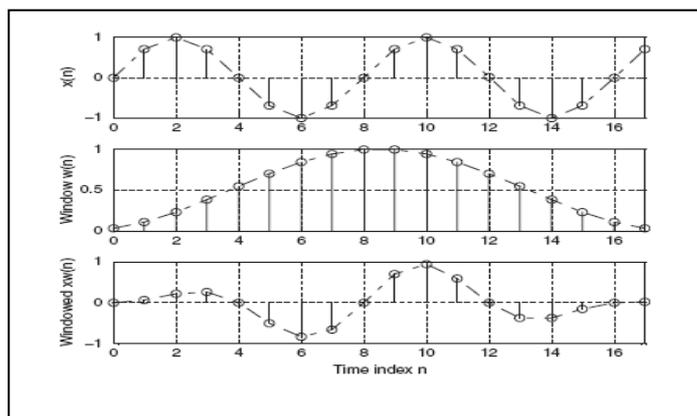
The common window functions are listed as follows:

The rectangular window (no window function):

$$w_R(n) = 1 \qquad 0 \le n \le N - 1 \tag{7.2}$$

The triangular window:

$$w_{tri}(n) = 1 - \frac{|2n - N + 1|}{N - 1}, \ 0 \le n \le N - 1 \tag{7.3}$$

The Hamming window:

$$w_{hm}(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{N - 1}\right), \ 0 \le n \le N - 1 \tag{7.4}$$

The Hanning window:

$$w_{hn}(n) = 0.5 - 0.5\cos\left(\frac{2\pi n}{N - 1}\right), \ 0 \le n \le N - 1 \tag{7.5}$$

Plots for each window function for a size of 20 samples are shown in Figure 7.4.

Fig. 7.4 Plots of window sequences
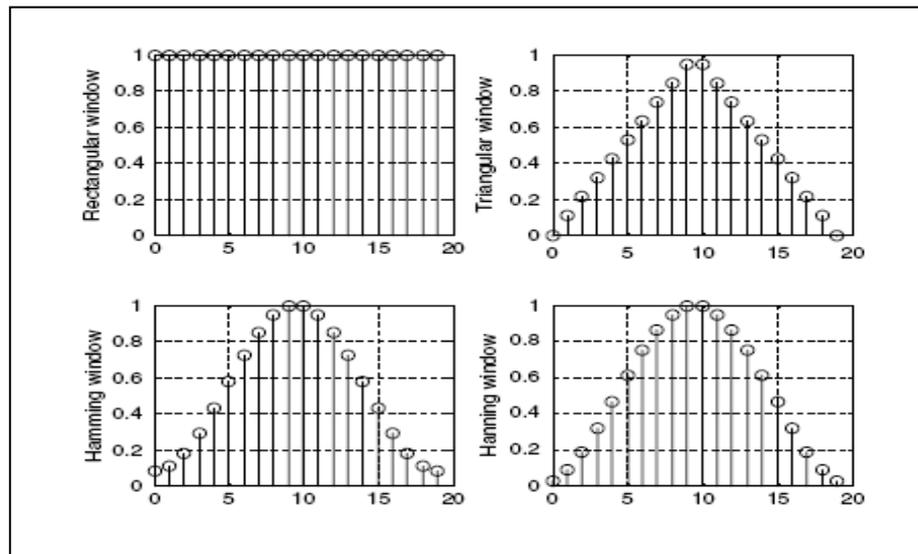
Example (1): Considering the sequence x(0) = 1, x(1) = 2, x(2) = 3, and x(3) = 4, and given f_s = 100 Hz, T = 0.01 seconds, compute the amplitude spectrum, phase spectrum, and power spectrum

a. Using the triangular window function.

b. Using the Hamming window function.

Solution:

  a) Since N = 4, from the triangular window function given in equation (7.3), we have:

$w_{tri}(0) = 0$, $w_{tri}(1) = 0.6667$, $w_{tri}(2) = 0.6667$, and $w_{tri}(3) = 0$.

Now, applying eq. (7.1), we have:

$x_w(0) = x(0) \ w_{tri}(0) = 0$. Similarly $x_w(1) = 1.3334$, $x_w(2) = 2$, and $x_w(3) = 0$

Applying DFT equation (6.3) to $x_w(n)$ for K=0, 1, 2, and 3 , we have:

X(0)= 3.3334, X(1) = −2 − j1.3334 , X(2) = 0.6666, and X(3) = − 2 + j 1.3334

Δf = 1 / NT = 25 Hz

Applying equations (6.11), (6.13), and (6.14):

$$A_0 = \frac{1}{4}|X(0)| = 0.8334, \ \varphi_0 = \tan^{-1}\left(\frac{0}{3.3334}\right) = 0^0,$$

$$P_0 = \frac{1}{4^2}|X(0)|^2 = 0.6954$$

K	A_K	Φ_K in degree	P_K
1	0.6009	− 146.31	0.3611
2	0.1667	0	0.0278
3	0.6009	146.31	0.3611

b. Since N = 4, from the Hamming window function given in eq. (7.4), we have:

$w_{hm}(0)$ = 0.08, $w_{hm}(1)$ = 0.77, $w_{hm}(2)$ = 0.77, and $w_{hm}(3)$ = 0.08. The windowed sequence is computed using eq. (7.1) as:

$x_w(0)$ = x(0) $w_{hm}(0)$ = 0.08, $x_w(1)$ = 1.54, $x_w(2)$ = 2.31, and $x_w(3)$ = 0.32

Applying DFT equation (6.3) to $x_w(n)$ for K=0, 1, 2, and 3 , we have:

X(0)= 4.25, X(1) = −2.23 − j1.22 , X(2) = 0.53, and X(3) = − 2.23 + j 1.22

Δf = 1 / NT = 25 Hz

Applying equations (6.11), (6.13), and (6.14):

$$A_0 = \frac{1}{4}|X(0)| = 1.0625, \ \varphi_0 = \tan^{-1}\left(\frac{0}{4.25}\right) = 0^0,$$

$$P_0 = \frac{1}{4^2}|X(0)|^2 = 1.1289$$

K	A_K	Φ_K in degree	P_K
1	0.6355	−151.32	0.4308
2	0.1325	0	0.0176
3	0.6355	151.32	0.4308

7.2 Application to Speech Spectral Estimation

The following plots show the comparisons of amplitude spectral estimation for speech data with 2,001 samples and a sampling rate of 8,000 Hz using the rectangular window (no window) function and the Hamming window function. As demonstrated in Fig. 7.5 (two-sided spectrum) and Fig. 7.6 (one-sided spectrum), there is little difference between the amplitude spectrum using the Hamming window function and the spectrum without using the window function. This is due to the fact that when the data length of the sequence (e.g., 2,001 samples) increases, the frequency resolution will be improved and spectral leakage will become less significant. However, when data length is short, reduction of spectral leakage using a window function will come to be prominent.
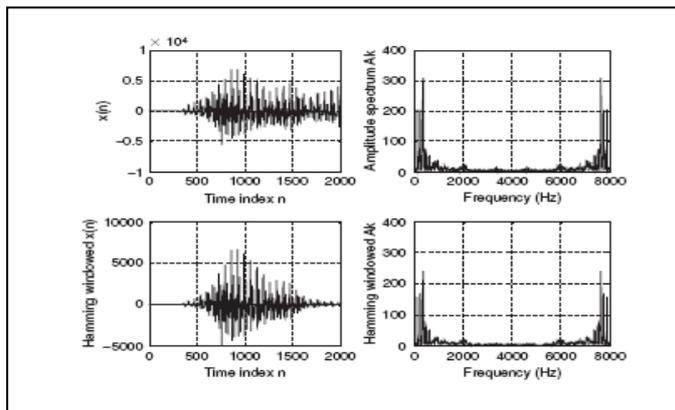
Fig. 7.5 Comparison of a spectrum without using a window function and a spectrum using the Hamming window for speech data.
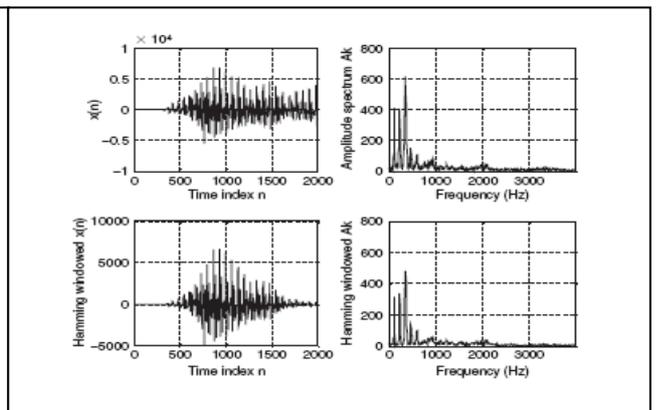
Fig. 7.6 Comparison of a one-sided spectrum without using a window function and a one-sided spectrum using the Hamming window for speech data.

7.3 Fast Fourier Transform

FFT is a very efficient algorithm in computing DFT coefficients and can reduce a very large amount of computational complexity (multiplications).

Consider the digital sequence $x(n)$ consisting of 2^m samples, where m is a positive integer—the number of samples of the digital sequence $x(n)$ is a power of 2, N = 2, 4, 8, 16, etc. If $x(n)$ does not contain 2^m samples, then *we simply append it with zeros* until the number of the appended sequence is equal to an integer of a power of 2 data points.

The number of points $N = 2^m$, where the stages $m = \log_2 N$.

In this section, we focus on two formats. One is called the decimation in- frequency algorithm, while the other is the decimation-in-time algorithm. They are referred to as the radix-2 FFT algorithms.

7.3.1 Method of Decimation-in-Frequency (Reduced DIF FFT)

Beginning with the definition of DFT :

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \text{ for } k = 0, 1, \ldots, N-1,$$

$$(7.6)$$

Where, $W_N = e^{-j2\pi / N}$ is the twiddle factor, and N = 0, 2, 4, 8, 16, …..Equation (7.6) can be expanded as:

$$X(k) = x(0) + x(1)W_N^k + \ldots + x(N-1)W_N^{k(N-1)}.$$

$$(7.7)$$

If we split equation (7.7):

$$X(k) = x(0) + x(1)W_N^k + \ldots + x\left(\frac{N}{2} - 1\right) W_N^{k(N/2-1)}$$
$$+ x\left(\frac{N}{2}\right) W^{kN/2} + \ldots + x(N-1)W_N^{k(N-1)}$$

$$(7.8)$$

Then we can rewrite as a sum of the following two parts:

$$X(k) = \sum_{n=0}^{(N/2)-1} x(n) W_N^{kn} + \sum_{n=N/2}^{N-1} x(n) W_N^{kn}.$$

$$(7.9)$$

Modifying the second term in Equation (7.9) yields:

$$X(k) = \sum_{n=0}^{(N/2)-1} x(n) W_N^{kn} + W_N^{(N/2)k} \sum_{n=0}^{(N/2)-1} x\left(n + \frac{N}{2}\right) W_N^{kn}.$$

$$(7.10)$$

Recall $W_N^{N/2} = e^{-j\frac{2\pi(N/2)}{N}} = e^{-j\pi} = -1$; then we have

$$X(k) = \sum_{n=0}^{(N/2)-1} \left(x(n) + (-1)^k x\left(n + \frac{N}{2}\right) \right) W_N^{kn}.$$

$$(7.11)$$

Now letting k = 2m as an even number achieves:

$$X(2m) = \sum_{n=0}^{(N/2)-1} \left(x(n) + x\left(n + \frac{N}{2}\right) \right) W_N^{2mn},$$

$$(7.12)$$

While substituting k = 2m + 1 as an odd number yields:

$$X(2m+1) = \sum_{n=0}^{(N/2)-1} \left(x(n) - x\left(n+\frac{N}{2}\right) \right) W_N^n W_N^{2mn}.$$

(7.13)

Using the fact that $W_N^2 = e^{-j\frac{2\pi \times 2}{N}} = e^{-j\frac{2\pi}{(N/2)}} = W_{N/2}$, it follows that

$$X(2m) = \sum_{n=0}^{(N/2)-1} a(n) W_{N/2}^{mn} = DFT\{a(n) \text{ with } (N/2) \text{ points}\}$$

(7.14)

$$X(2m+1) = \sum_{n=0}^{(N/2)-1} b(n) W_N^n W_{N/2}^{mn} = DFT\{b(n) W_N^n \text{ with } (N/2) \text{ points}\},$$

(7.15)

Where, a(n) and b(n) are introduced and expressed as:

$$a(n) = x(n) + x\left(n+\frac{N}{2}\right), \text{ for } n = 0,1 \ldots, \frac{N}{2} - 1$$

$$b(n) = x(n) - x\left(n+\frac{N}{2}\right), \text{ for } n = 0,1,\ldots, \frac{N}{2} - 1.$$

(7.16)

$$DFT\{x(n) \text{ with } N \text{ points}\} = \begin{cases} DFT\{a(n) \text{ with } (N/2) \text{ points}\} \\ DFT\{b(n) W_N^n \text{ with } (N/2) \text{ points}\} \end{cases}$$

(7.17)

Figure 7.7(a) illustrates the block diagram of N-point DIF FFT. Fig. 7.7(b) illustrates **reduced** DIF FFT computation for the eight-point DFT, where there are 12 complex multiplications as compared with the eight-point DFT with 64 complex multiplications. For a data length of N, the number of complex multiplications for DFT and FFT, respectively, are determined by:

Complex multiplications of DFT = N^2, and   (7.18a)

Complex multiplications of FFT (***With Reduction***) = $(N / 2) \log_2 (N)$ (7.18b)
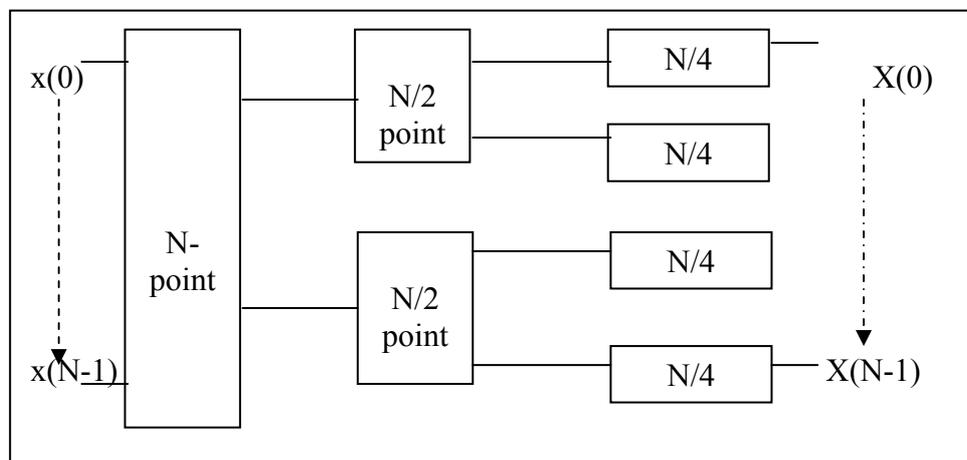
Fig. 7.7(a) Block diagram of DIF FFT
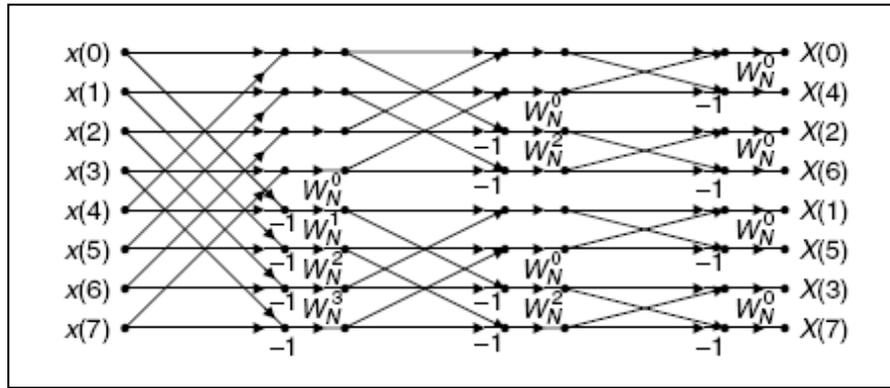
Fig. 7.7(b) The eight-point FFT (total twelve multiplications).

Reduced DIF FFT

Note: The input sequence is in normal order index and the output frequency bin number is in reversal bits order. The *Butterfly structure* for DIF FFT and DIT FFT is shown below:

$C = A+B$, $D = (A-B)$ W_N^r *(7.19a)* , $F= E +W_N^r G$, $H= E -W_N^r G$ *(7.19b)*
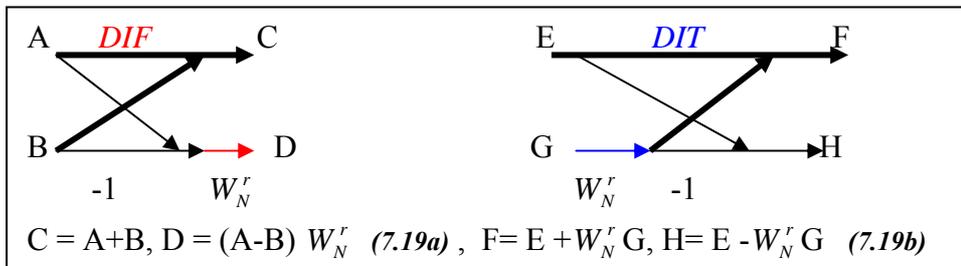
The inverse FFT is defined as:

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn} = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \tilde{W}_N^{kn}, \text{ for } k = 0, 1, \ldots, N-1.$$

$$(7.20)$$
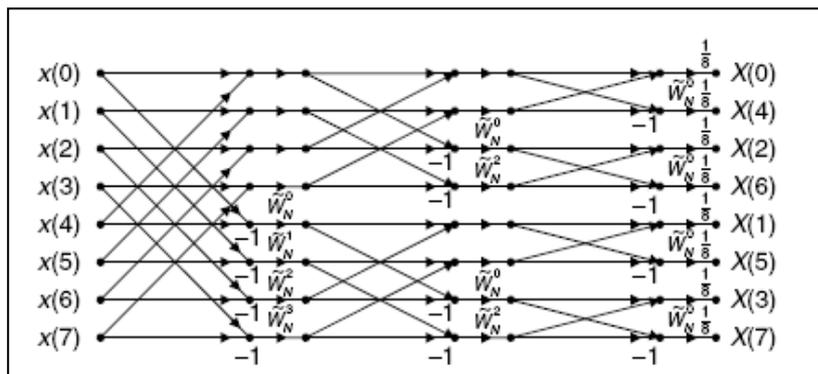
Fig. 7.8 Block diagram for the inverse of eight-point FFT.

Reduced DIF IFFT

The twiddle factor W_N is changed to be $\tilde{W}_N = W_N^{-1}$, and the sum is multiplied by a factor of $1/N$. Hence, the inverse FFT block diagram is achieved as shown in Fig. 7.8

Example (2): Given a sequence x(n) for $0 \leq n \leq 3$, where x(0) = 1, x(1) = 2, x(2) = 3, and x(3) = 4,

a. Evaluate its DFT X(k) using the decimation-in-frequency FFT method.

b. Determine the number of complex multiplications.

Solution:

$$W_4^0 = e^{-j\frac{2\pi}{4}(0)} = 1 \ and \ W_4^1 = e^{-j\frac{2\pi}{4}(1)} = -j$$
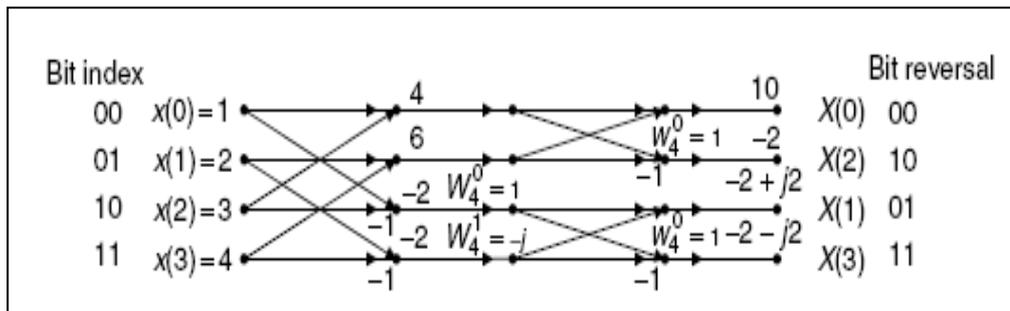
b) The number of complex multiplications is four, which can also be determined from eq. (7.18b), where N=4

7.3.2 Method of Decimation-in-Time (Reduced DIT FFT):

In this method, we split the input sequence x(n) into the even indexed x(2m) and x(2m + 1), each with N data points. Then Equation (7.6) becomes:

$$X(k) = \sum_{m=0}^{(N/2)-1} x(2m) W_N^{2mk} + \sum_{m=0}^{(N/2)-1} x(2m + 1) W_N^k W_N^{2mk},$$

$$\text{for } k = 0, 1, \ldots, N - 1. \tag{7.21}$$

Using $W_N^2 = W_{N/2}$, it follows that:

$$X(k) = \sum_{m=0}^{(N/2)-1} x(2m) W_{N/2}^{mk} + W_N^k \sum_{m=0}^{(N/2)-1} x(2m + 1) W_{N/2}^{mk},$$

$$\text{for } k = 0, 1, \ldots, N - 1. \tag{7.22}$$

Define new functions as:

$$G(k) = \sum_{m=0}^{(N/2)-1} x(2m) W_{N/2}^{mk} = DFT\{x(2m) \ with \ (N/2) \ points\}$$

$$H(k) = \sum_{m=0}^{(N/2)-1} x(2m + 1) W_{N/2}^{mk} = DFT\{x(2m + 1) \ with \ (N/2) \ points\}.$$

$$\tag{7.23}$$

Note that:

$$G(k) = G\left(k + \frac{N}{2}\right), \text{ for } k = 0, 1, \ldots, \frac{N}{2} - 1$$

$$H(k) = H\left(k + \frac{N}{2}\right), \text{ for } k = 0, 1, \ldots, \frac{N}{2} - 1.$$

$$(7.24)$$

Substituting Equations (7.24) into Equation (7.22) yields the first half frequency bins

$$X(k) = G(k) + W_N^k H(k), \text{ for } k = 0, 1, \ldots, \frac{N}{2} - 1.$$

$$(7.25)$$

Considering the following fact and using Equations (7.24):

$$W_N^{(N/2+k)} = -W_N^k.$$

$$(7.26)$$

Then the second half of frequency bins can be computed as follows:

$$X\left(\frac{N}{2} + k\right) = G(k) - W_N^k H(k), \text{ for } k = 0, 1, \ldots, \frac{N}{2} - 1.$$

$$(7.27)$$

The block diagram for the eight-point DIT FFT algorithm is illustrated in Fig.. 7.9
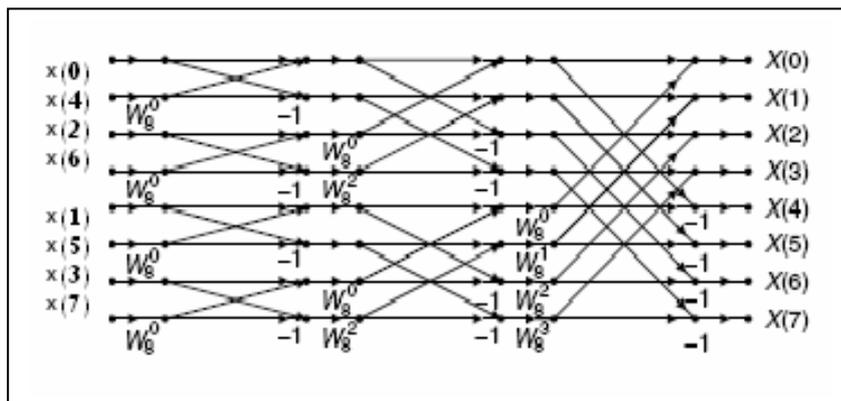
Fig.7.9 The eight-point FFT algorithm using decimation-in-time (twelve complex multiplications). Reduced DIT FFT

The index for each input sequence element can be achieved by bit reversal of the frequency index in a sequential order. Similar to the method of decimation-in-frequency, after we change W_N to $\tilde{W}_N$ in Fig. 7.9 and multiply the output sequence by a factor of 1/N, we derive the inverse FFT block diagram for the eight-point inverse FFT in Fig. 7.10.
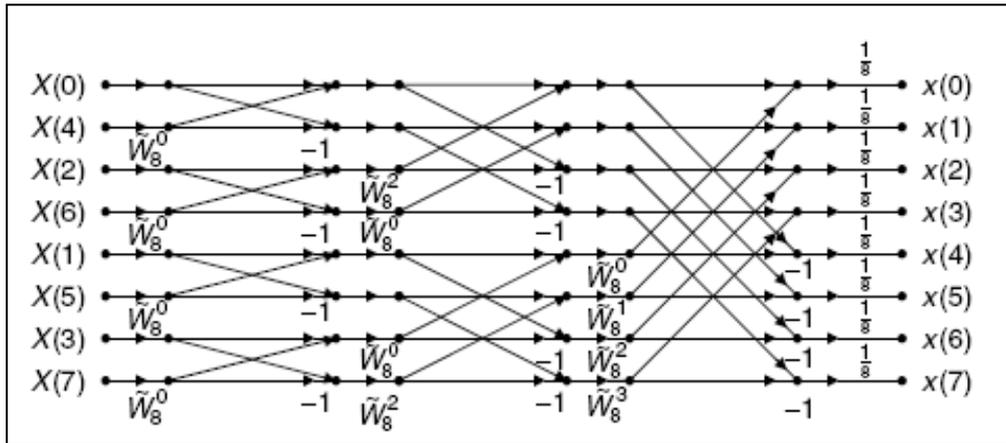
Fig. 7.10 The eight-point IFFT using decimation-in-time (Reduced method).

Example(3): Given a sequence x(n) for $0 \leq n \leq 3$, where x(0) = 1, x(1) = 2, x(2) = 3, and x(3) = 4. Evaluate its DFT X(k) using the decimation-in-time FFT method.

Solution:
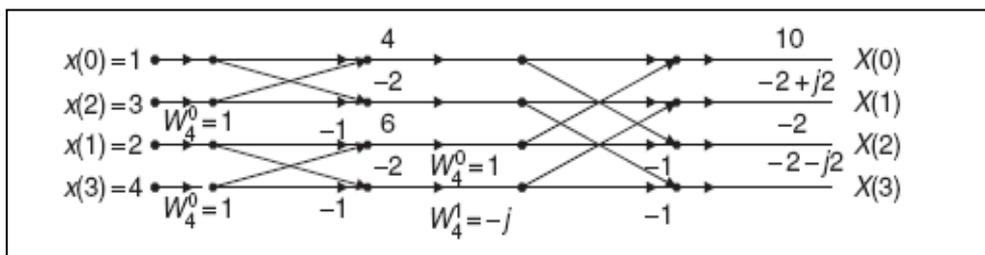
Bit reversal   bit indexed

H.W Find DFT of the following sequence [1 -1 -1 -1 1 1 1 -1], using:

a) Reduced DIT FFT

b) Reduced DIF FFT

$$Ans: [0 \quad -\sqrt{2} + j(\sqrt{2} + 2) \quad 2 - j2 \quad \sqrt{2} + j(\sqrt{2} - 2) \quad 4 \quad \sqrt{2} - j(\sqrt{2} - 2)$$
$$2 + j2 \quad -\sqrt{2} - j(\sqrt{2} + 2) \quad]$$

7.4 Properties of DFT for real x(n):

$$X(K) = X^*(N - K)$$
$$\text{Re}\{X(K)\} = \text{Re}\{X(N - K)\}$$
$$\text{Im}\{X(K)\} = -\text{Im}\{X(N - K)\}$$
$$x(n) = \frac{1}{N} FFT \ [X^*(K)]^*$$

$$K \rightarrow \qquad W_K = K.2\pi/N \rightarrow \quad \Omega_K = K.2\pi/NT$$

$$\textit{frequency} \qquad \textit{digital frequency} \qquad \textit{ana}\log \textit{frequency}$$
$$\textit{index} \qquad\qquad (\textit{rad}) \qquad\qquad (\textit{rad}/\sec)$$

(7.26)

* means complex conjugate

For N even:

$$x(n) = \frac{X(0)}{N} + \sum_{K=1}^{(N/2)-1} \frac{2}{N} \{X_R(K)\cos(\frac{2\pi}{N}nK) - X_I(K)\sin(\frac{2\pi}{N}nK)\} + \frac{X(N/2)}{N}\cos\pi n$$

(7.27a)

For N odd:

$$x(n) = \frac{X(0)}{N} + \sum_{K=1}^{(N-1)/2} \frac{2}{N} \{X_R(K)\cos(\frac{2\pi}{N}nK) - X_I(K)\sin(\frac{2\pi}{N}nK)\}$$

(7.27b)

x(n)	X(K)
Real	Real part is even, imaginary part is odd
Real and even	Real and even
Real and odd	Imaginary and odd

Example (4): Find x(n) for $X_R(K)$ and $X_I(K)$, then find $x_a(t)$ if T = 0.1 sec.
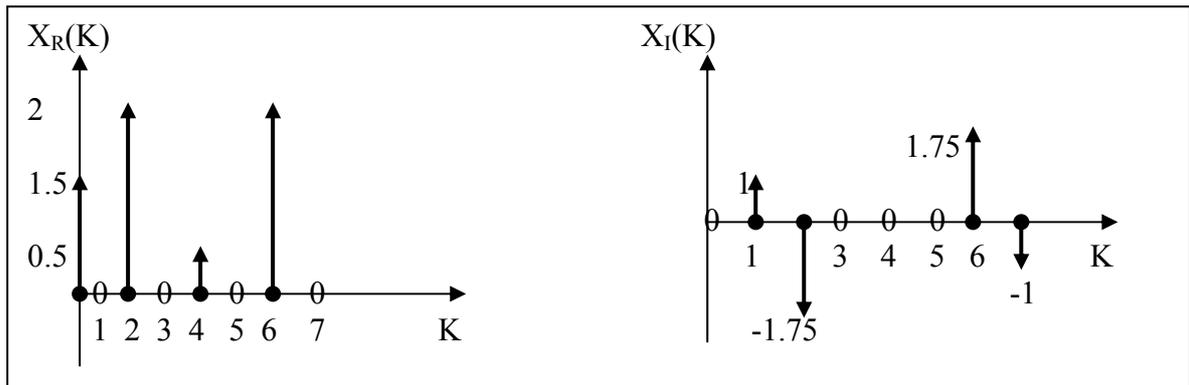
N=8 , then using eq.(7.27a):

$$x(n) = \frac{1.5}{8} + \frac{2}{8}\{-(1)\sin(\frac{2\pi}{8}1n) + (2)\cos(\frac{2\pi}{8}2n) - (-1.75)\sin(\frac{2\pi}{8}2n)\} + \frac{0.5}{8}\cos\pi n$$

$$for\ K = 1, 2, ...\frac{N}{2} - 1 = 3$$

$$\Omega_K = W_k / T = 2\pi K / NT, \qquad t = nT, \quad n = \frac{t}{T}$$

$$x_a(t) = \frac{1.5}{8} + \frac{2}{8}\{-(1)\sin(\frac{2\pi}{8}\frac{t}{0.1}) + (2)\cos(\frac{2\pi}{8}2\frac{t}{0.1}) - (-1.75)\sin(\frac{2\pi}{8}2\frac{t}{0.1})\} + \frac{0.5}{8}\cos\pi n$$

$$for\ T = 0.1\,sec.$$

$$x_a(t) = 0.1875 - 0.25\sin 2.5\pi t + 0.5\cos 5\pi t + 0.4375\sin 5\pi t + 0.0625\cos 10\pi t$$

7.5 DFT and Fourier transform relations:

The Fourier transform $X(e^{jW})$ of an x(n) is given for all W:

$$X(e^{jW}) = \sum_{-\infty}^{\infty} x(n)\, e^{-jWn} \;=\; \sum_{n=0}^{N-1} x(n)\, e^{-jWn} \;,\; n = 0,1,2,..N-1 \tag{7.28}$$

From eq. (7.28), $X(e^{jW})$ is a *continuous* function of W.

The DFT (N-point) of an x(n) is given by:

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\,2\pi Kn/N} \quad,\quad K = 0,1,2...N-1 \tag{7.29}$$

Comparing eq.(7.28) and eq.(7.29), the DFT of x(n) is the sampled version of the Fourier transform sequence as shown below

$$X(K) = X(e^{jW})\, \Big|_{W=(2\pi K/N)} \quad, K = 0,1,2,..N-1 \tag{7.30}$$