Lec.8 Analog Filter Design

8.1 Introduction:

Let us review analog filter design using lowpass prototype transformation. This method converts the analog lowpass filter with a cutoff frequency of 1 radian per second, called the lowpass prototype, into practical analog lowpass, highpass, bandpass, and bandstop filters with their frequency specifications.

8.2 Butterworth Filters

8.2.1 Butterworth low-pass filter (LPF)

A typical frequency response for a Butterworth low-pass filter of order n is shown in Fig. 8.1.

$$|H_n(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2n}} \qquad (8.1)$$

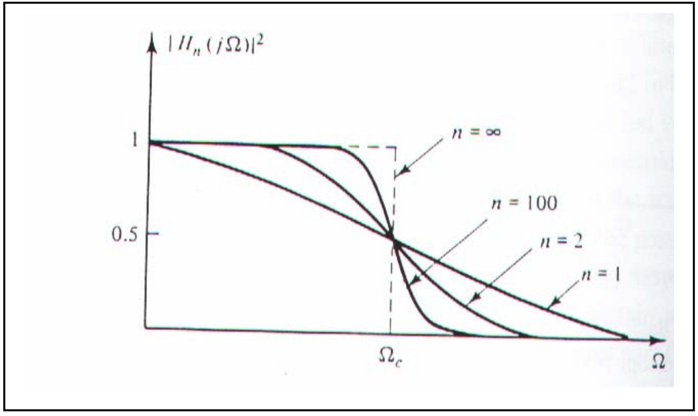


Fig.8.1 Butterworth LPF c/cs

Properties:

$$|H_n(j\Omega)|^2_{\Omega=0} = 1 \quad for \quad all \quad n$$

$$|H_n(j\Omega)|^2_{\Omega=\Omega_c} = \frac{1}{2} \quad for \quad all \quad finite\ n$$

$$|H_n(j\Omega)|_{\Omega=\Omega_c} = 0.707 \quad (-3.0103 \ dB)$$

$|H_n(j\Omega)|^2$ is monotonically decreasing function of Ω, it is also called maximally flat at the origin since all derivatives exist and are zero. As n → ∞ , we get ideal response.

The *normalized* LP Butterworth is obtained when:

Ω_c = 1 rad / sec.

Substituting S = j Ω in eq. (8.1), and rearrange to get the LP Butterworth poles, then:

$S = (-1)^{[(n+1)/2n]}$

For n odd, $S_K = 1\angle k\pi/n \quad , k = 0, 1, 2, ..., 2n-1$ (8.2a)

For n even, $S_K = 1\angle (k\pi/n) + (\pi/2n) \quad , k = 0, 1, 2, ..., 2n-1$ (8.2.b)

For stable and causal filter:

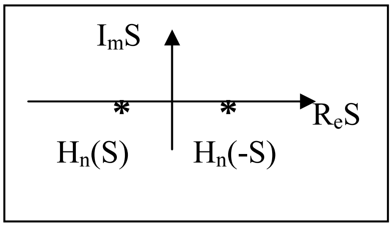
$$H_n(S) = \frac{1}{\prod_{LHP\ poles}(S-S_k)} = \frac{1}{B_n(S)} \qquad (8.3)$$

$B_n(S)$: Butterworth polynomial of order n (see Table (1)).

LHP: Left half plane.

Example(1): Find the transfer function $H_1(S)$ for the normalized Butterworth filter of order one.

Solution: applying eq.(8.2a), where n=1, k = 0,1



$S_0 = 1 \angle 0 = H_n(-S)$

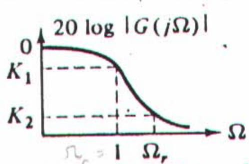
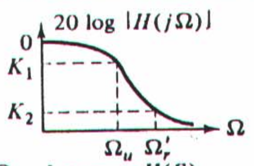
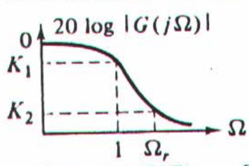
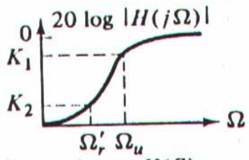
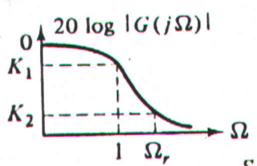
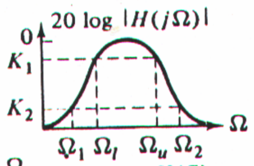
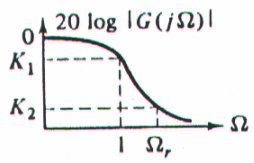
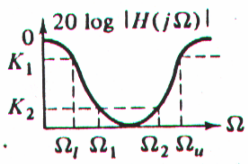
$S_1 = 1 \angle \pi = H_n(S)$. Using eq. (8.3) and taking **LHP poles** S_1:

$$H_1(S) = \frac{1}{S-(-1)} = \frac{1}{S+1}$$

8.2.2 Analog- to analog transformation

To obtain Butterworth filters with cutoff frequencies other than 1 rad /sec. It is convenient to use 1 rad /sec. Butterworth filters as prototypes and apply analog-to-analog transformation (see Table (2)). *The transformational method is not limited in its application to Butterworth filters.*

TABLE 2 ANALOG-TO-ANALOG TRANSFORMATION

Prototype response	Transformed filter response	Design equations
 Low-pass $G(S)$	$S \to S/\Omega_u$  Low-pass $H(S)$	Forward: $\Omega_r' = \Omega_r\Omega_u$ Backward: $\Omega_r = \Omega_r'/\Omega_u$
 Low-pass $G(S)$	$S \to \Omega_u/S$  High-pass $H(S)$	Forward: $\Omega_r' = \Omega_u/\Omega_r$ Backward: $\Omega_r = \Omega_u/\Omega_r'$
 Low-pass $G(S)$	$S \to \frac{S^2+\Omega_l\Omega_u}{S(\Omega_u-\Omega_l)}$  Bandpass $H(S)$	Forward: $\Omega_{av} = (\Omega_u - \Omega_l)/2$ $\Omega_1 = (\Omega_r^2\Omega_{av}^2 + \Omega_l\Omega_u)^{1/2} - \Omega_{av}\Omega_r$ $\Omega_2 = (\Omega_r^2\Omega_{av}^2 + \Omega_l\Omega_u)^{1/2} + \Omega_{av}\Omega_r$ Backward: $\Omega_r = \min\{\lvert A\rvert, \lvert B\rvert\}$ $A = (-\Omega_1^2 + \Omega_l\Omega_u)/[\Omega_1(\Omega_u - \Omega_l)]$ $B = (+\Omega_2^2 - \Omega_l\Omega_u)/[\Omega_2(\Omega_u - \Omega_l)]$
 Low-pass $G(S)$	$S \to \frac{S(\Omega_u-\Omega_l)}{S^2+\Omega_l\Omega_u}$  Bandstop $H(S)$	Forward: $\Omega_{av} = (\Omega_u - \Omega_l)/2$ $\Omega_1 = [(\Omega_{av}/\Omega_r)^2 + \Omega_l\Omega_u]^{1/2} - \Omega_{av}/\Omega_r$ $\Omega_2 = [(\Omega_{av}/\Omega_r)^2 + \Omega_l\Omega_u]^{1/2} + \Omega_{av}/\Omega_r$ Backward: $\Omega_r = \min\{\lvert A\rvert, \lvert B\rvert\}$ $A = \Omega_1(\Omega_u - \Omega_l)/[-\Omega_1^2 + \Omega_l\Omega_u]$ $B = \Omega_2(\Omega_u - \Omega_l)/[-\Omega_2^2 + \Omega_l\Omega_u]$

8.2.3 Design Equations of Butterworth Filters:

A **Butterworth LPF** Filter of order n is given by the following equation:

$$n = \left\lceil \frac{\log_{10}\{(10^{-0.1k_1}-1)/(10^{-0.1k_2}-1)\}}{2\log_{10}(1/\Omega_r)} \right\rceil \qquad (8.4)$$

Here, $1/\Omega_r = \Omega_u / \Omega'_r$, see Table (2).

Where, k_1, k_2, Ω_u , and Ω'_r are the pass-band gain and stop-band attenuation with their relative frequencies respectively(see Table (2)).

To satisfy our requirement at Ω_u exactly, then:

$$\Omega_c = \Omega_u / (10^{-0.1k_1}-1)^{1/2n} \qquad (8.5a)$$

To satisfy our requirement at Ω'_r exactly, then:

$$\Omega_c = \Omega'_r / (10^{-0.1k_2}-1)^{1/2n} \qquad (8.5b)$$

Ω_c is the cutoff frequency at – 3dB

Example (2): design an analog Butterworth LPF that has a – 2 dB butter cutoff frequency of 20 rad/sec. and at least 10 dB of attenuation at 30 rad/sec**.**

Solution: Applying eq. (8.4), where k_1= -2 dB, k_2 = -10 dB, Ω_u = 20 rad/sec., and Ω'_r = 30 rad/sec

$$n = \left\lceil \frac{\log_{10}\{(10^{0.2}-1)/(10^{1}-1)\}}{2\log_{10}(20/30)} \right\rceil = \lceil 3.3709 \rceil = 4$$

To satisfy our requirement at Ω_u exactly, then:

$$\Omega_c = 20/(10^{0.2}-1)^{1/8} = 21.3836\ rad/\sec$$

From Table (1) of *normalized* Butterworth LPF (Ω_c =1 rad/ sec) with n = 4 :

$$H_4(S) = \frac{1}{(S^2+0.76536S+1)(S^2+1.84776S+1)}$$

Using Table (2) and applying LP → LP transformation, S→ S / 21.3836, and rearranging:

$$H(S) = \frac{0.20921\times 10^6}{(S^2+16.3686S+457.394)(S^2+39.5176S+457.394)}$$

For **Butterworth HPF**:

1- Put $1/\Omega_r = \Omega'_r / \Omega_u$ in equation (8.4), and find its order n .(see Table(2))

2- Use Table (1) to find the normalized Butterworth LPF equation with order n.

3- Apply LP → HP transformation, $S \rightarrow \Omega_c / S$, and rearrange the equation obtained in step 2.

For **Butterworth BPF**:

1- Calculate $\Omega_r = \min \{ |A|, |B| \}$ using equations given in Table (2). Find the filter order using eq.(8.4)

2- Use Table (1) to find the normalized Butterworth LPF equation with order n.

3- Apply LP → BP transformation, $S \rightarrow \frac{S^2 + \Omega_l \Omega_u}{S(\Omega_u - \Omega_l)}$, and rearrange the equation obtained in step 2

For **Butterworth BSF:**

Refer to Table (2) to see the variables.

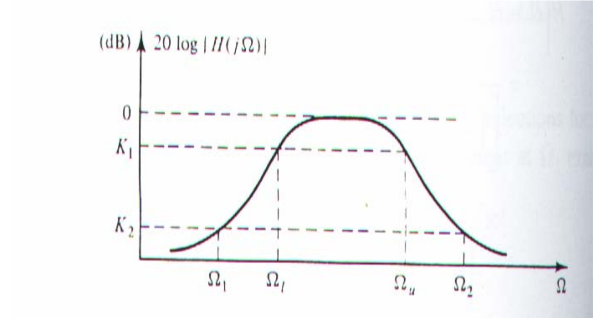


Fig. 8.2 Butterworth BPF

Example (3): Design an analog Butterworth BPF with the following c/cs:

A – 3.0103 dB upper and lower cutoff frequencies of 50 Hz and 20 KHz.

A stop-band attenuation of at least 20 dB at 20 Hz and 45 kHz.

Solution:

$\Omega_1 = 2\pi(20) = 125.663$ rad / sec.

$\Omega_2 = 2\pi(45 \times 10^3) = 2.82743 \times 10^5$ rad / sec.

$\Omega_u = 2\pi(20 \times 10^3) = 1.25663 \times 10^5$ rad / sec.

$\Omega_l = 2\pi(50) = 314.159$ rad / sec

Calculate $\Omega_r = \min \{ |A|, |B| \} = \min(|2.5053|, |2.2545|) = 2.2545$ by using equations given in Table (2) . Apply eq. (8.4) to find:

$n = \lceil 2.829 \rceil = 3$

From Table (1) of *normalized* Butterworth LPF (Ω_c =1 rad/ sec) with n = 3:

$$H_3(S) = \frac{1}{S^3 + 2S^2 + 2S + 1}$$

Apply LP → BP transformation by substituting $S \rightarrow \frac{S^2 + \Omega_l \Omega_u}{S(\Omega_u - \Omega_l)} = \frac{S^2 + 3.94784 \times 10^7}{S(1.25349 \times 10^5)}$, in the above equation and rearrange it to obtain H_{BPF} (as H.W)

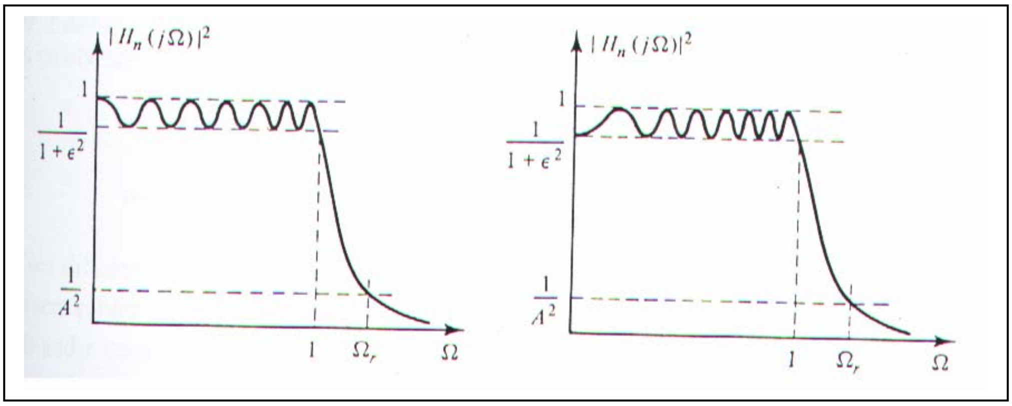
8.3 Chebyshev Filters:

There are two types of Chebyshev Filters:

1- One containing a ripple in the pass-band (type 1).
2- One containing a ripple in the stop-band (type 2).

$$|H_n(j\,\Omega)|^2 = \frac{1}{1+\varepsilon^2\, T_n^2(\Omega)} \qquad (8.6)$$

$T_n(\Omega)$ is the nth order Chebyshev polynomial where $T_0(x) = 1$, and $T_1(x) = x$ as listed in Table (3). ε^2 is a parameter chosen to provide the proper pass-band ripple. Fig. (8.3) shows *normalized* Chebyshev Filters of both types.



n odd ***n even***

Fig.(8.3) Normalized Chebyshev filters of type 1 for (n odd), and (n even)

8.3.1 Design Equations of Chebyshev Filters:

$$n = \left\lceil \frac{\log_{10}[g+\sqrt{g^2-1}\;]}{\log_{10}[\Omega_r+\sqrt{\Omega_r^{\,2}-1}\;]} \right\rceil \qquad (8.7)$$

$$20\log_{10}[1/A^2]^{1/2} = stopband\, attenuation\,(dB) \qquad (8.8a)$$

$$g = [(A^2-1)/\varepsilon^2\;]^{1/2} \qquad (8.8b)$$

$$|H_n(S)| = \frac{K}{\prod_{\substack{LPF\\poles}}(S-S_K)} = \frac{K}{V_n(S)}$$

$$K = V_n(0) = b_0 \qquad n\ odd \qquad (8.9)$$

$$K = V_n(0)\,/\,\sqrt{(1+\varepsilon^2} \qquad n\ even$$

Table (4) gives V_n(S) for n =1 to n =10 and ε corresponding to 0.5, 1, 2, and 3 dB ripples. Table (5) gives the zeros {poles of H_n(S) } for the same n and ε.

8.3.2 Design steps of Chebeshev LPF, HPF, BPF, and BSF :

1- Use the *backward design equations* from Table (2) to obtain normalized LPF requirements (Ω_r).

2-Calculate A using eq. (8.8a)

3-Calculate g from eq. (8.8b), then apply eq.(8.7) to find the order n.

4- Use Table (4) and Table (5) to find the Chebeshev Filter equation with order n.

5- Apply LP → LP or HP or BP or BS transformation (Table (2)) and rearrange the equation obtained in step 4.

Example (4): Design a Chebshev filter to satisfy the following specifications:

1-Acceptable pass-band ripple of 2dB

2-Cutoff frequency of 40 rad/sec.

3- stop-band attenuation of 20 dB or more at 52 rad/sec.

Solution: From Table (2)

$\Omega_r = \Omega'_r / \Omega_u$ = 52/ 40 = 1.3 rad/sec.

$20\log_{10}[1/A^2]^{1/2} = -20$, A = 10, using ε = 2 dB = 0.76478 (see Table (4) and Table (5))

Applying eq. (8.8b), then g = 13.01

$$n = \left\lceil \frac{\log_{10}[13.01 + \sqrt{(13.01)^2 - 1}\ ]}{\log_{10}[1.3 + \sqrt{(1.3)^2 - 1}\ ]} \right\rceil = \lceil 4.3 \rceil = 5 \quad \text{, n odd}$$

From Table (4) with n = 5 and ε = 2 dB = 0.76478

$$H_5(S) = \frac{0.08172}{S^5 + 0.70646S^4 + 1.499S^3 + 0.6934S^2 + 0.459349S + 0.08172}$$

Using poles from Table (5):

$$H_5(S) = \frac{0.08172}{(S + 0.218303)(S^2 + 0.134922S + 0.95215)(S^2 + 0.35323S + 0.393115)}$$

Using Table (2) and applying LP → LP transformation, S→ S / 40, and rearranging the above equation:

$$H_{LPF}(S) = \frac{8.366 \times 10^6}{(S + 8.73212)(S^2 + 5.3969S + 1523.44)(S^2 + 14.1292S + 628.984)}$$

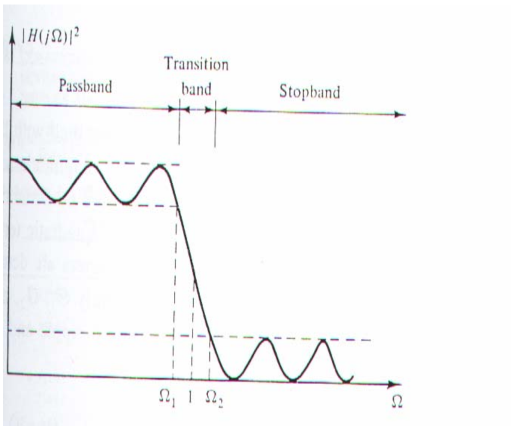
Notes:

1. Butterworth or maximally flat amplitude; as the order (n) is increased the response becomes flatter in the pass-band and the attenuation is greater in the stop-band.

2. Chebshev Filter has a sharper cutoff; i.e., a narrower transition band (best amplitude response) than a Butterworth filter of the same order (n)
3. Chebshev Filter provides poorest phase response (most nonlinear). The Butterworth filter compromise between amplitude and phase (this is one of the reasons for its widespread popularity).

8.4 Elliptic Filters:

A LP elliptic filter provides a smaller transition width and is optimum in the sense that no other filter of the same order has a narrower transition width for a given pass-band ripple and stop-band attenuation.



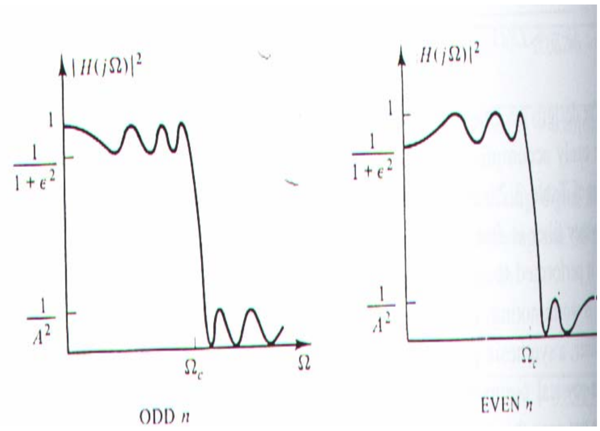


Fig. 8.4(a) normalized elliptic LPF **Fig. 8.4(b) elliptic LP filters types**

Fig. 8.4 (a) shows a normalized elliptic LPF and Fig. 8.4 (b) shows elliptic LP filters of type 1 (n odd), and type 2 (n even).

8.4.1 Design steps of Elliptic LPF, HPF, BPF, and BSF : using Table (6)

1. Locate k_1= acceptable pass-band ripple (dB) , and k_2 = stop-band attenuation (dB).
2. Calculate Ω_r using Table (2), pp.55.
3. At Ω_r column, take a value *less than* Ω_r .
4. The filter order (n) is the far left of that row, and the coefficients for the filter are found in all rows corresponding to that (n).
5. According to (n), the normalized elliptic LPF equations are:

$$H_n(S) = \frac{H_o}{(S+S_o)} \prod_{i=1}^{(n-1)/2} \frac{S^2 + A_{0i}}{S^2 + B_{1i}\, S + B_{oi}} \quad \text{, n odd} \qquad (8.10\text{ a})$$

$$H_n(S) = H_0(S) \prod_{i=1}^{n/2} \frac{S^2 + A_{0i}}{S^2 + B_{1i}\, S + B_{oi}} \qquad \text{, n even} \qquad (8.10\ b)$$

6- Apply LP → LP or HP or BP or BS transformation (Table (2)) and rearrange the equation obtained in step 5.

Notes:

For *n**ormalized*** elliptic filter**,** $\Omega_0 = (\Omega_2\ \Omega_1)^{0.5} = 1$ = geometric mean**,** and $\Omega_r = \Omega_2 / \Omega_1$,then $\Omega_1 = (\Omega_r)^{-0.5}$, and $\Omega_2 = (\Omega_r)^{0.5}$

For ***not normalized*** elliptic filter**,** $\Omega_0 = (\Omega_2'\ \Omega_1')^{0.5}$, where $\Omega_1 = \Omega_1' / \Omega_0$ and $\Omega_2 = \Omega_2' / \Omega_0$

Then $\Omega_r = \Omega_2' / \Omega_1' = \Omega_2 / \Omega_1$.

n (elliptic) $\leq$ n (chebeshev) $\leq$ n (Butterworth)

Example (5) : Find the transfer function for an elliptic LPF with − 2 dB cutoff value at 10000 rad/sec., and a stop-band attenuation of 40 dB for all Ω past 14400 rad/sec.

Solution:

$\Omega_0 = (\Omega_2'\ \Omega_1')^{0.5} = \{(14400)\ (10000)\}^{0.5} = 12000$

$\Omega_1 = \Omega_1' / \Omega_0 = 10000/12000 = 5/6$ and $\Omega_2 = \Omega_2' / \Omega_0 = 14400/12000 = 6/5$

$\Omega_r = \Omega_2' / \Omega_1' = \Omega_2 / \Omega_1 = 1.44$, $k_1 = -2$ dB, and $k_2 = -40$ dB. From Table (6), n =4

Applying eq. (8.10 b), Where:

$H_0 = 0.01$, $A_{01} = 7.25202$, $B_{01} = 0.212344$, and $B_{11} = 0.467290$, i =1

$A_{02} = 1.57676$, $B_{02} = 0.677934$, and $B_{12} = 0.127954$ i = 2

$$H_4(S) = \frac{0.01(S^2 + 7.25202)\,(S^2 + 1.57676)}{(S^2 + 0.467290S + 0.212344)\,(S^2 + 0.127954S + 0.677934)}$$

Apply LP → LP transformation (Table (2)), where Ω_0 = geometric mean = 12000. Substituting S → S / 12000 in the above equation:

$$H_{LPF}(S) = \frac{0.01(S^2 + 1.04429\times10^9)\,(S^2 + 2.27053\times10^8)}{(S^2 + 5607.48S + 30577536)\,(S^2 + 1535.448S + 97622497)}$$