Lec. 9 – Part 2

9.6 Finite Impulse Response (FIR) filter

In many cases a linear phase c/cs is required throughout the pass-band of the filter to preserve the shape of a given signal within the pass-band. Assume a LP filter with:

$$H(e^{jW}) = \begin{cases} e^{-jW\alpha} & |W| < W_o \\ 0 & W_o < |W| < \pi \\ periodic & for\ all\ other\ W \end{cases} \qquad (9.32)$$

$$Y(e^{jW}) = X(e^{jW}) \,.\, H(e^{jW}) = X(e^{jW}) \,.\, e^{-jW\alpha} \qquad (9.33\ a)$$

$$Y(Z) = X(Z) \,.\, Z^{-\alpha} \qquad (9.33\ b)$$

$$y(n) = x(n - \alpha) \qquad (9.34)$$

The linear phase filter did not alter the shape of the original signal, simply translated it by an amount α, as shown in Fig. (9.9)

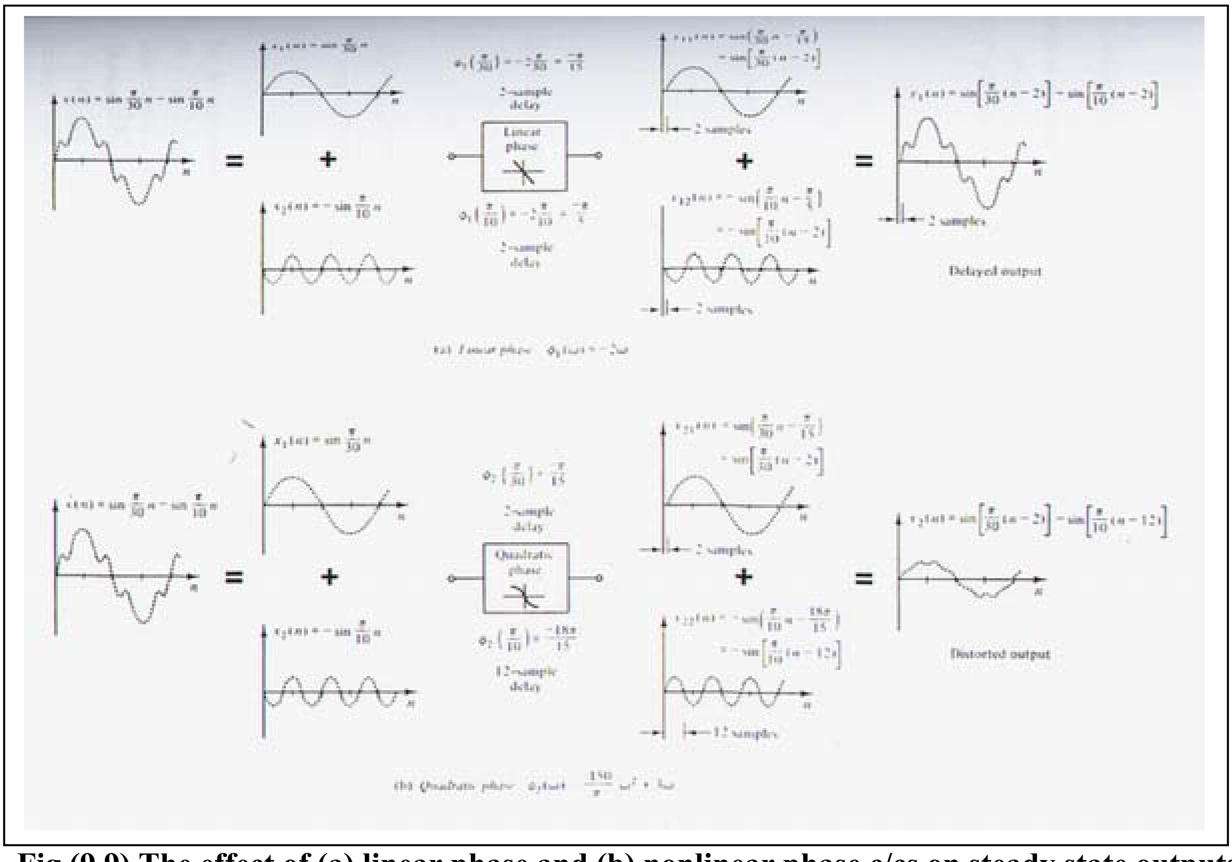


Fig.(9.9) The effect of (a) linear phase and (b) nonlinear phase c/cs on steady state outputs with identical magnitude frequency response curves

A causal IIR filter can not produce a linear phase c/cs and that only special forms of FIR filters can give linear phase.

The necessary conditions for linear phase:

1. h(n) have finite duration (for causal FIR filter, h(n) begins at zero and ends at N-1)

h(n) = h(N-1-n)   , n = 0, 1, …., N-1   (9.35)

2. Symmetric about its mid-point (see Fig. (9.10))

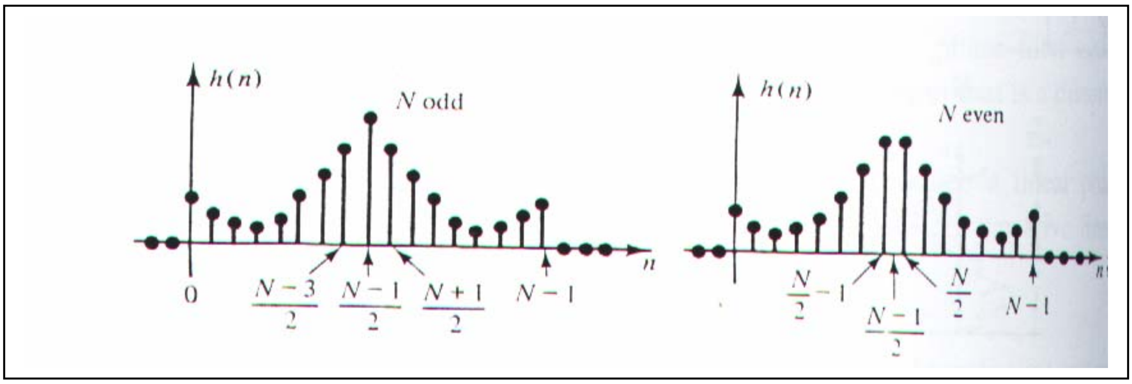


Fig. (9.10) General shapes of h(n) that give linear phase for odd and even N.

If h(n) is as given in the above conditions, we now show that $H(e^{jW})$ has linear phase. For N even:

$$H(e^{jW}) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-jWn} = \sum_{n=0}^{N-1} h(n)\, e^{-jWn} \quad \text{(Finite duration)} \qquad (9.36)$$

$$H(e^{jW}) = \sum_{n=0}^{(N/2)-1} h(n)\, e^{-jWn} + \sum_{n=N/2}^{N-1} h(n)\, e^{-jWn} = H_1(e^{jW}) + H_2(e^{jW}) \qquad (9.37)$$

Let m = N-1-n

$$H_2(e^{jW}) = \sum_{m=(\frac{N}{2})-1}^{0} h(N-1-m)\, e^{-jW(N-1-m)} = \sum_{m=0}^{(\frac{N}{2})-1} h(m)\, e^{-jW(N-1-m)} \qquad (9.38)$$

$$\therefore H(e^{jW}) = \sum_{n=0}^{(\frac{N}{2})-1} h(n)\, e^{-jWn} + \sum_{m=0}^{(\frac{N}{2})-1} h(m)\, e^{-jW(N-1-m)} \qquad (9.39)$$

$$H(e^{jW}) = \sum_{n=0}^{(\frac{N}{2})-1} h(n)\, e^{-jW(\frac{N-1}{2})} \left\{ e^{-jW(n-\frac{N-1}{2})} + e^{-jW(N-1-n-\frac{N-1}{2})} \right\} \qquad (9.40)$$

$$H(e^{jW}) = \sum_{n=0}^{(\frac{N}{2})-1} 2\, h(n)\, e^{-jW(\frac{N-1}{2})} \left\{ \cos\left[W\left(n - \frac{N-1}{2}\right) \right] \right\} \qquad (9.41)$$

For N even:

$$H(e^{jW}) = e^{-jW(\frac{N-1}{2})} \sum_{n=0}^{(\frac{N}{2})-1} 2\,h(n) \left\{ \cos\left[W\left(n - \frac{N-1}{2}\right)\right] \right\} \qquad (9.42)$$

Linear phase *magnitude*

For N odd:

$$H(e^{jW}) = e^{-jW(\frac{N-1}{2})} \{h(\frac{N-1}{2}) + \sum_{n=0}^{(N-3)/2} 2\,h(n) \left\{ \cos\left[W\left(n - \frac{N-1}{2}\right)\right] \right\} \qquad (9.43)$$

For N odd, the slope of – α = – (N–1) /2 causes a delay in the output of (N–1)/2 , which is an integer number of samples, whereas for N even, the slope causes a non-integer delay. The non-integer delay will cause the values of the sequence to be changed, which, in some cases, may be undesirable.

9.7 Design of FIR filters using Windows

If $h_d(n)$ represents the impulse response of a desired IIR filter, then an FIR filter with impulse response h(n) can be obtained as follows:

$$h(n) = h_d(n) \,.\, w(n)$$

$$h(n) = \begin{cases} h_d(n) & N_1 \le n \le N_2 \\ 0 & otherwise \end{cases}$$

$$w(n) = \begin{cases} 1 & N_1 \le n \le N_2 \\ 0 & otherwise \end{cases} \text{, window function} \qquad (9.44)$$

$$H(e^{jW}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\theta})\, W(e^{j(W-\theta)})\, d\theta = H_d(e^{j\theta}) \otimes W(e^{j\theta})$$

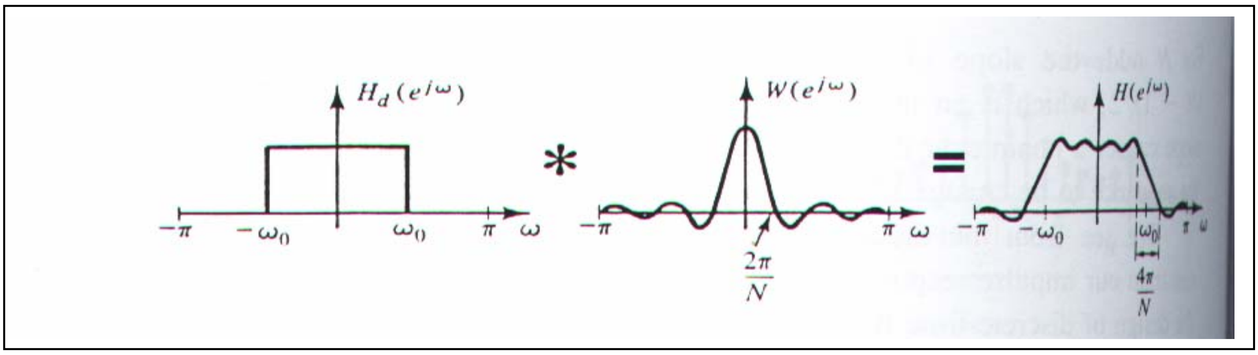


Fig. (9.11) Frequency response obtained by rectangularly windowing ideal LP impulse response.

As shown in Fig.(9.11), the convolution produces a smeared version of ideal LP frequency response $H_d(e^{jW})$. In general, the wider the main lobe of $W(e^{jW})$, the more spreading, whereas the narrower the main lobe (larger N), the closer $|H(e^{jW})|$ comes to $|H_d(e^{jW})|$.

Some of the most commonly used windows are:

1. Rectangular:

$$w_R(n) = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & otherwise \end{cases} \tag{9.45}$$

2. Bartlett:

$$w_B(n) = \begin{cases} \dfrac{2n}{(N-1)} & 0 \le n \le (N-1)/2 \\ \dfrac{2-2n}{(N-1)} & (N-1)/2 \le n \le (N-1) \\ 0 & elsewhere \end{cases} \tag{9.46}$$

3. Hanning:

$$w_{Han}(n) = \begin{cases} 0.5\ [1-\cos(\dfrac{2\pi n}{(N-1)})], & 0 \le n \le N-1 \\ 0 & elsewhere \end{cases} \tag{9.47}$$

4. Hamming:

$$w_{Ham}(n) = \begin{cases} 0.54 - 0.46\cos(\dfrac{2\pi n}{(N-1)}) , & 0 \le n \le N-1 \\ 0 & elsewhere \end{cases} \tag{9.48}$$

5. Blackman:

$$w_{Bl}(n) = \begin{cases} 0.42 - 0.5\cos(\dfrac{2\pi n}{(N-1)}) + 0.08\cos(\dfrac{4\pi n}{(N-1)}), & 0 \le n \le N-1 \\ 0 & elsewhere \end{cases} \tag{9.49}$$

An ideal LP filter with linear phase of slope $-\alpha$ and cutoff w_c can be characterized in frequency domain by:

$$H_d(e^{jW}) = \begin{cases} e^{-jW\alpha} & |W| < W_c \\ 0 & W_c < |W| < \pi \end{cases} \tag{9.50}$$

Using inverse F.T (eq. (4.11), PP. 28):

$$h_d(n) = \frac{\sin[\,w_c\,(n-\alpha)\,]}{\pi\,(n-\alpha)} \tag{9.51}$$

For a causal FIR filter, and using :

$$h(n) = h_d(n) \,.\, w(n) \qquad (9.52)$$

Substituting eq.(9.51) into eq.(9.52), yield:

$$h(n) = \frac{\sin[\, w_c\ (n-\alpha)\,]}{\pi\ (n-\alpha)} \,.\, w(n) \qquad (9.53)$$

For h(n) to be a linear phase filter, $\alpha = (N-1)/2$.

Table (3) shows $h_d(n)$ for LPF, HPF, BPF, and BSF:

Table (3) $h_d(n)$ and $h_d(\alpha)$ for LPF, HPF, BPF, and BS

Filter Type	$h_d(n)$	$h_d(\alpha)$
LPF	$h_d(n) = \frac{\sin[\, w_c\ (n-\alpha)\,]}{\pi\ (n-\alpha)}$	$h_d(\alpha) = w_c/\pi$
HPF	$h_d(n) = -\frac{\sin[\, w_c\ (n-\alpha)\,]}{\pi\ (n-\alpha)}$	$h_d(\alpha) = 1-(w_c/\pi)$
BPF	$N_1 = \frac{2\pi k}{w_l - w_1}, \qquad N_2 = \frac{2\pi k}{w_2 - w_u}$ N = max (N_1 , N_2) $h_d(n) = \frac{\sin\{w_u(n-\alpha)\} - \sin\{w_l(n-\alpha)\}}{\pi(n-\alpha)}$	$h_d(\alpha) = (w_u - w_l)/\pi$
BSF	$N_1 = \frac{2\pi k}{w_1 - w_l}, \qquad N_2 = \frac{2\pi k}{w_u - w_2}$ N = max (N_1 , N_2) $h_d(n) = \frac{\sin\{w_l(n-\alpha)\} - \sin\{w_u(n-\alpha)\}}{\pi(n-\alpha)}$	$h_d(\alpha) = (\pi - w_u - w_l)/\pi$

In general, for all the above filters with N odd:

$$h(n) = h_d(n) \,.\, w(n)$$

$$H(e^{jW}) = e^{-jW(\frac{N-1}{2})}\ \{h(\frac{N-1}{2}) + \sum_{n=0}^{(N-3)/2} 2\,h(n)\ \left\{\cos[\,W(n-\frac{N-1}{2})\,]\right\}$$

$\Phi(W) = -W\alpha$, with $\alpha = (N-1)/2$

Notes:

- The stop-band gain for the LPF designed is relatively insensitive to the size of the window and the selection of w_c depending mainly on the type of window.

- The transition width of the designed LPF is approximately equal to the main lobe of the window used. See Table (4)

Table (4) Design table for FIR LPF

Window	**Transition Width (w_t)**	**Minimum stop-band attenuation**
Rectangular	$4\pi / N$	− 21 dB
Bartlett	$8\pi / N$	− 25 dB
Hanning	$8\pi / N$	− 44 dB
Hamming	$8\pi / N$	− 53 dB
Blackman	$12\pi / N$	− 74 dB

9.8 Design procedure for an FIR filter

Requirements: k_1, w_1, k_2, and w_2 represents the cutoff and stop-band requirements for digital filters.

1. From Table (4), select the window type such that the stop-band gain exceeds k_2
2 Selects the number of points in the window,

$w_t = w_2 - w_1 \geq k(2\pi / N)$,

$N \geq k(2\pi) / (w_2 - w_1)$, *N is preferred odd*

3. Select α and w_c , where :

$w_c = w_1$, and $\alpha = (N - 1)/2$

4. Find h(n) from eq. (9.52) using the specified window type and Table (3) .
5. Use eq. (9.42) or eq.(9.43) to plot the frequency response $H(e^{jW})$, and check to see if the given specifications are satisfied.
6. If the attenuation requirement at w_1 is not satisfied, increase w_c and return to step 4, and 5 .
7. If the frequency response requirements are satisfied, check to see if a further reduction of N might be possible. If a further reduction in N is not possible, then h(n) found is the desired design, otherwise, reduce N and return to step 3.
8. If the filter is to be used in A/D- H(Z) – D/A structure, the equivalent analog specifications must be converted to digital specifications. For analog critical frequencies, Ω_i , the corresponding digital specifications using a sampling rate of 1 / T samples /sec. ;

$w_i = \Omega_i T$

Example (9): Design a LP digital filter to be used in A/D- H(Z) – D/A structure that will have a − 3 dB cutoff of 30 π rad / sec. and an attenuation of 50 dB at 45 π rad/sec. *The filter is required to have linear phase*. The system will use a sampling rate of 100 samples/sec.

Solution:

$w_c = w_1 = \Omega_u T = 30\,\pi\,(1/100) = 0.3\,\pi$ rad

$w_2 = w_r = \Omega_r T = 45\,\pi\,(1/100) = 0.45\,\pi$ rad

1. Hamming window is chosen.
2. From step (2):

   $(8\,\pi / N) = k\,(2\,\pi / N)$, Then $k = 4$

   $N \geq 4\,(2\,\pi) / (0.45 - 0.3)\,\pi = 53.3 = 55$

3. $w_c = w_u = 0.3\,\pi$ rad , and $\alpha = (N-1)/2 = 27$
4. Using eq. (9.48) for w_{Ham} and the value of $h_d(n)$ from Table (3) to find h(n):

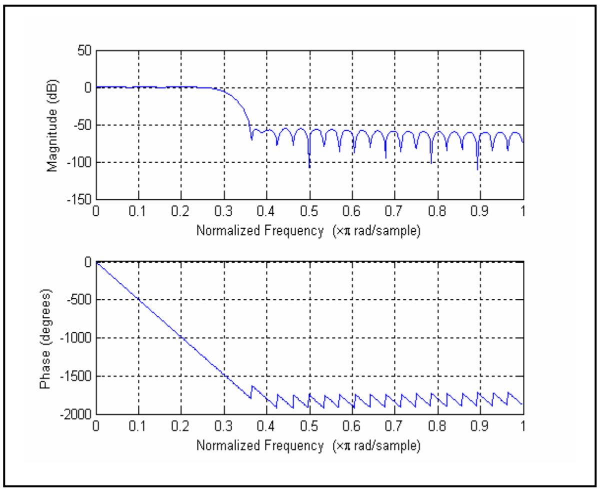
$$h(n) = \frac{\sin[\,0.3\pi\,(n-27)\,]}{\pi\,(n-27)} \,.\,\{0.54 - 0.46\cos(2\pi n / 54)\}\,,\; 0 \le n \le 54$$

$$H(e^{jW}) = e^{-jW(27)}\,\{h(27) + \sum_{n=0}^{26} 2\,h(n)\,\{\cos[\,W(n-27)\,]\}$$

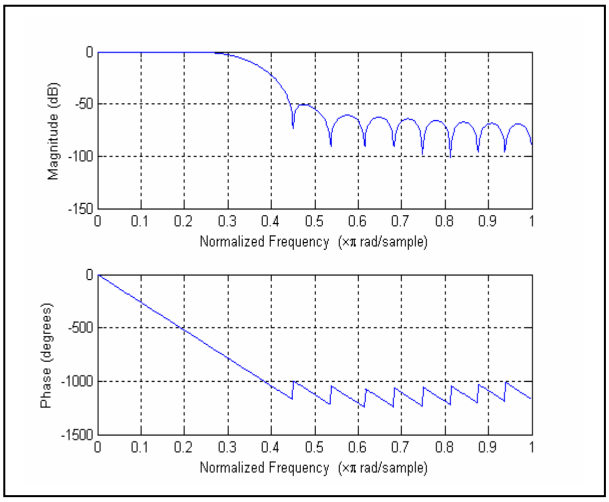
From the results obtained from MATLAB program, the attenuation is seen to be too much at $w_c = w_1$. The design is improved by making $w_c = 0.33$ rad / sec, then N = 29 , α = 14 and

$$h(n) = \frac{\sin[\,0.33\,\pi\,(n-14)\,]}{\pi\,(n-14)} \,.\,\{0.54 - 0.46\cos(2\pi n / 28)\}\,,\; 0 \le n \le 28$$

$$H(e^{jW}) = e^{-jW(14)}\,\{h(14) + \sum_{n=0}^{13} 2\,h(n)\,\{\cos[\,W(n-14)\,]\}$$



N= 55, $w_c = w_u = 0.3\,\pi$ rad



N= 29, $w_c = w_u = 0.33\,\pi$ rad