General review: -

a) Dot Product

If θ is the angle between the vectors a and b, then

$$a \cdot b = |a||b| \cos \theta$$

NOTE: Two vectors a and b are orthogonal, if and only if $a \cdot b = 0$.

❖ Properties of the Dot Product

If a, b and c are vectors and d is a scalar then

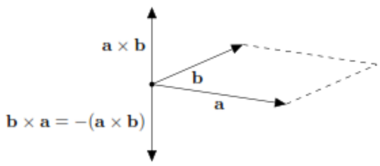
1) $a \cdot a = |a|^2$
2) $a \cdot b = b \cdot a$
3) $a \cdot (b + c) = a \cdot b + a \cdot c$
4) $(d\, a) \cdot \mathrm{b} = d\,(a \cdot b) = a \cdot (d\, b)$
5) $0 \cdot a = 0$

b) Cross Product

Theorem: The vector $a \times b$ is orthogonal to both a and b.

If θ is the angle between a and b, then

$$|a \times b| = |a||b| \sin \theta$$



Two nonzero vectors a and b are parallel if and only if $a \times b = 0$

❖ Properties of the Cross Product

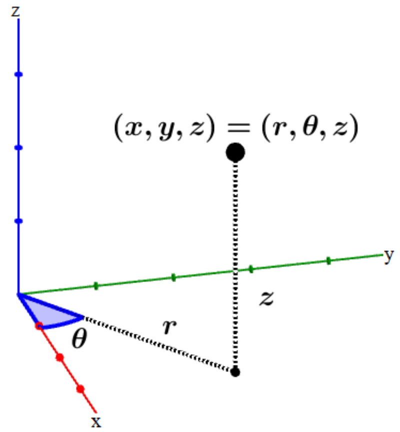
If a and b and c are vectors and d is a scalar, then

1) $a \times b = -b \times a$
2) $(da) \times b = d\,(a \times b) = a \times (d\, b)$
3) $a \times (b + c) = a \times b + a \times c$
4) $(a + b) \times c = a \times c + b \times c$
5) $a \cdot (b \times c) = (a \times b) \cdot c$
6) $a \times (b \times c) = (a \cdot c)\, b - (a \cdot b)\, c$

c) Vector Analysis

1) The cylindrical coordinates

The cylindrical coordinate of a point *P* in three-space is the ordered triple (r; θ; z) where (r; θ) is the polar coordinate of the projection of *P* onto the xy-plane and z is the same as in the Cartesian coordinates as shown in Figure below.



❖ Converting between Cartesian and Cylindrical Coordinates

To convert from cylindrical coordinates to cartesian coordinates:

$$x = r\cos(\theta) \qquad y = r\sin(\theta) \qquad z=z$$

To convert from cartesian coordinates to cylindrical coordinates:

$$r^2 = x^2 + y^2 \qquad \tan(\theta) = \frac{y}{x} \qquad z=z$$

❖ Rectangular to Cylindrical Vectors (and Vice Versa)

In the rectangular coordinate system, we express a vector A as

$$\mathbf{A} = \hat{\mathbf{a}}_x A_x + \hat{\mathbf{a}}_y A_y + \hat{\mathbf{a}}_z A_z$$

Where $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$, are the unit vectors and A_x , A_y , A_z are the components of the vector A in the rectangular coordinate system. We wish to write A as

$$A = \hat{a}_r\, A_r + \hat{a}_\theta\, A_\theta + \hat{a}_z\, A_z$$

where $\hat{a}_r, \hat{a}_\theta, \hat{a}_z$ are the unit vectors and A_r, A_θ, A_z are the vector components in the cylindrical coordinate system. The z-axis is common to both of them. We can write the transformation in matrix form

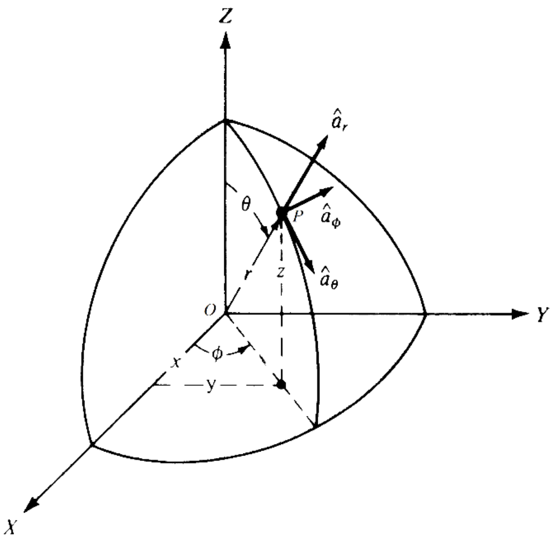
$$\begin{pmatrix} A_r \\ A_\theta \\ A_z \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

we can write the transformation matrix for cylindrical-to-rectangular components as

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_r \\ A_\theta \\ A_z \end{pmatrix}$$

2) The spherical coordinates

The spherical coordinate of a point *P* in three-space is the ordered triple (r; θ; $\emptyset$), where r is the distance from *P* to the origin *O*, θ is the angle from the positive z-axis to the vector *OP*, and $\emptyset$ is the angle from the positive x-axis to the project of vector *OP* as shown in Figure below.



❖ Converting between Cartesian and Spherical Coordinates

To convert from spherical coordinates to cartesian coordinates:

$$x = r\sin(\theta)\cos(\emptyset) \qquad y = r\sin(\theta)\sin(\emptyset) \qquad z = r\cos(\theta)$$

To convert from cartesian coordinates to spherical coordinates

$$r^2 = x^2 + y^2 + z^2 \qquad \cos(\theta) = \frac{z}{\sqrt{x^2+y^2+z^2}} \qquad \tan(\emptyset) = \frac{y}{x}$$

❖ Rectangular to Spherical Vectors (and Vice Versa)

We wish to write A in spherical components as

$$A = \hat{a}_r\, A_r + \hat{a}_\theta\, A_\theta + \hat{a}_\emptyset\, A_\emptyset$$

where $\hat{a}_r, \hat{a}_\theta, \hat{a}_\emptyset$ are the unit vectors and $A_r, A_\theta, A_\emptyset$ are the vector components in the spherical coordinate system. We can write the transformation in matrix form from rectangular to spherical coordinates

$$\begin{pmatrix} A_r \\ A_\theta \\ A_\emptyset \end{pmatrix} = \begin{pmatrix} \sin(\theta)\cos(\emptyset) & \sin(\theta)\ \sin(\emptyset) & \cos(\theta) \\ \cos(\theta)\ \cos(\emptyset) & \cos(\theta)\ \sin(\emptyset) & -\sin(\theta) \\ -\sin(\theta) & \cos(\emptyset) & 0 \end{pmatrix} \begin{pmatrix} \mathrm{A_x} \\ \mathrm{A_y} \\ \mathrm{A_z} \end{pmatrix}$$

And the transformation matrix for spherical-to-rectangular components is

$$\begin{pmatrix} \mathrm{A_x} \\ \mathrm{A_y} \\ \mathrm{A_z} \end{pmatrix} = \begin{pmatrix} \sin(\theta)\ \cos(\emptyset) & \cos(\theta)\ cos(\emptyset) & -\sin(\emptyset) \\ \sin(\theta)\ \sin(\emptyset) & \cos(\theta)\ \sin(\emptyset) & \cos(\emptyset) \\ \cos(\theta) & -\sin(\theta) & 0 \end{pmatrix} \begin{pmatrix} A_r \\ A_\theta \\ A_\emptyset \end{pmatrix}$$

d) VECTOR DIFFERENTIAL OPERATORS

The differential operators of gradient of a scalar (∇ψ), divergence of a vector (∇·A), curl of a vector (∇×A), Laplacian of a scalar ($\nabla^2\psi$), and Laplacian of a vector (∇^2A) frequently encountered in electromagnetic field analysis will be listed in the rectangular and spherical coordinate systems.

❖ Rectangular Coordinates

$$\nabla\psi = \hat{\mathbf{a}}_x \frac{\partial\psi}{\partial x} + \hat{\mathbf{a}}_y \frac{\partial\psi}{\partial y} + \hat{\mathbf{a}}_z \frac{\partial\psi}{\partial z}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\mathbf{A} = \hat{\mathbf{a}}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{a}}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{a}}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla\cdot\nabla\psi = \nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$$

$$\nabla^2\mathbf{A} = \hat{\mathbf{a}}_x \nabla^2 A_x + \hat{\mathbf{a}}_y \nabla^2 A_y + \hat{\mathbf{a}}_z \nabla^2 A_z$$

❖ Spherical Coordinates

$$\nabla\psi = \hat{\mathbf{a}}_r \frac{\partial\psi}{\partial r} + \hat{\mathbf{a}}_\theta \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\mathbf{a}}_\phi \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \frac{\hat{\mathbf{a}}_r}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right] + \frac{\hat{\mathbf{a}}_\theta}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right] + \frac{\hat{\mathbf{a}}_\phi}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\nabla^2\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla\times\nabla\times\mathbf{A}$$

or in an expanded form

$$\begin{aligned}\nabla^2\mathbf{A} = \hat{\mathbf{a}}_r &\left(\frac{\partial^2 A_r}{\partial r^2} + \frac{2}{r}\frac{\partial A_r}{\partial r} - \frac{2}{r^2}A_r + \frac{1}{r^2}\frac{\partial^2 A_r}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial A_r}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_r}{\partial\phi^2}\right.\\
&\left. -\frac{2}{r^2}\frac{\partial A_\theta}{\partial\theta} - \frac{2\cot\theta}{r^2}A_\theta - \frac{2}{r^2\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right)\\
+\hat{\mathbf{a}}_\theta &\left(\frac{\partial^2 A_\theta}{\partial r^2} + \frac{2}{r}\frac{\partial A_\theta}{\partial r} - \frac{A_\theta}{r^2\sin^2\theta} + \frac{1}{r^2}\frac{\partial^2 A_\theta}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial A_\theta}{\partial\theta}\right.\\
&\left. +\frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_\theta}{\partial\phi^2} + \frac{2}{r^2}\frac{\partial A_r}{\partial\theta} - \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right)\\
+\hat{\mathbf{a}}_\phi &\left(\frac{\partial^2 A_\phi}{\partial r^2} + \frac{2}{r}\frac{\partial A_\phi}{\partial r} - \frac{1}{r^2\sin^2\theta}A_\phi + \frac{1}{r^2}\frac{\partial^2 A_\phi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial A_\phi}{\partial\theta}\right.\\
&\left. +\frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_\phi}{\partial\phi^2} + \frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial\phi} + \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial A_\theta}{\partial\phi}\right)\end{aligned}$$

e) Overview of Electromagnetic fields

Mainly, electromagnetic fields can be classified into two classes.

1- *Static fields*: -

As the name indicates, these are time-invariant fields. This means that all the field quantities are not function of time, i.e., they do not change their state with time. They are also called the DC fields since their frequency is zero Hertz. Static fields have negligible radiation and hence they do not support wave propagation. Therefore, static fields cannot be used for communications. It is interesting to note here that the static electric field and the static magnetic field are not coupled; therefore, they can be analyzed separately. The static electric field (also called electrostatic field) can be generated from time-invariant charge at rest. On the other hand, the static magnetic field (also called magnetostatic field) can be generated from steady electric current.

2- *Dynamic fields:*

These fields are time-dependent. With the high frequency band, these fields support propagating waves and hence they represent the communication waves. Dynamic electric field and dynamic magnetic field are coupled. This means that they cannot be analyzed separately. Here, the electric field and magnetic field generate each other by the mechanism of mutual induction. According to the operating frequency, dynamic field can be divided into two types.

i. Slowly time-varying electromagnetic field (also called *quasi-static* electromagnetic field). This is the low-frequency side of the dynamic electromagnetic field. With such a low frequency, this type is not significant when characterizing propagating electromagnetic fields. In the quasi-static electromagnetic fields, the displacement current is negligible compared to the conduction current.

ii. *Rapidly time-varying* electromagnetic field. This is the high-frequency side of the dynamic electromagnetic field. Having an effective radiation, high-frequency electromagnetic field support propagating waves. Here, the displacement current cannot be negligible compared to conduction current. Therefore, high-frequency electromagnetic field represents the most general field, i.e., all the field parameters are presence in the relevant Maxwell's equations. The other types of electromagnetic fields (Static fields and Quasi-static field) can be regarded as special cases of this general case.

f) Which Dynamic Field: Low- or High-Frequency?

In this section, we will learn how to differentiate between the low- frequency (Quasi-static) and the high-frequency dynamic fields. To set the distinction rule, it is necessary to be introduced to the *retardation effect.*

❖ The Concept of Retardation Effect

The mutual induction of time-varying electric and magnetic fields is the basis of time retardation in electromagnetic systems. The time- retardation concept states that there is a time lag between a change of the field sources, i.e., of time-varying charges and currents, and the associated change of the fields, so that the values of field intensities at a distance from the sources depend on the values of charge and current densities at an earlier time. This means, it takes some time for the effect of a change of charges and currents to be "sensed" at distant field points.

The criterion that can distinguish between low- and high-frequency dynamic fields can be stated in either of two senses:

- In time sense, the criterion states that the dynamic field is a low-frequency (Quasi-static) dynamic field if

$$\tau \ll T \qquad \ldots (1)$$

where τ is the time lag in free space given by $\tau = \frac{D_{max}}{c}$

and D_{max} is the maximum dimension of the related domain. The domain can be the charge distribution region or the transmitter-to-receiver region.

c is the speed of light in free space, which can be approximated to $3x10^8$ m/s

T is the time of change of source, which is the source wave period given by

$$T = \frac{1}{f}$$

- In space sense, the criterion depends on the source wavelength. The criterion will be derived from equation (1) mentioned above.

$$\tau \ll T$$

$$\frac{D_{max}}{c} \ll \frac{1}{f}$$

$$D_{max} \ll \frac{c}{f}$$

$$D_{max} \ll \lambda \qquad \ldots (2)$$

If equation (1) or equation (2) does not apply, then the dynamic field is a high-frequency (rapidly time-varying) field.

Example:

Check whether the following two fields are Quasi-static or rapidly time-varying fields.

1- Distance between transmitter and receiver is 10km and the signal carrier frequency is 30kHz
2- The time lag is 17ps and the frequency is 3GHz

Solution:

1- Here the distance is given therefore it is more convenient to use equation (2) where we compare D_{max} with λ. D_{max} is given as 10km = 10^4m. The frequency is 30kHz = $3 \text{x} 10^4$ so the wavelength can be found as

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{3 \times 10^4} = 10km$$

This means that D_{max} is NOT much smaller than the wavelength. Hence, equation (2) does NOT apply, which means that this field is a high-frequency dynamic field.

2- The time is given; therefore, it is easier to use equation (1) where τ = $17 \text{x} 10^{-12}$ s and frequency is $3 \text{x} 10^9$ Hz. The period is

$$T = \frac{1}{f} = \frac{1}{3 \times 10^9} = 333.3\, ps$$

Here equation (1) is verified since the period is much greater than the time lag. This means that this is a low-frequency (Quasi-static) field.

Notice that this example has shown that the low- and high- frequency fields do not mean that the source frequency is low or high. The frequency should be compared according to any of equations (1) and (2).

g) Review of Electromagnetic Fields

The Table shown below introduces the first two constitutive equations.

Electric field	Magnetic field
$\bar{D} = \varepsilon\bar{E}$	$\bar{B} = \mu\bar{H}$
$\bar{E}$:electric field intensity (V/m)	$\bar{H}$:magnetic field intensity (A/m)
$\bar{D}$: electric flux density (C/m^2)	$\bar{B}$: magnetic flux density ($T=Wb/m^2$)
ε : permittivity of medium (F/m) where $\varepsilon = \varepsilon_r\varepsilon_o$	μ: permeability of medium (H/m) where $\mu = \mu_r\mu_o$
ε_r: medium relative permittivity	μ_r: medium relative permeability
ε_o: permittivity of free space $\varepsilon_o = 8.854 \times 10^{-12}$ F/m	μ_o: permeability of free space $\mu_o = 4\pi \times 10^{-7}$ H/m

Table 1. Comparison between the first two constitutive equations

i. Volume Charge Density

The charge density per unit volume (ρ_v measured in C/m^3) is related to the electric flux density by

$$\rho_v = \bar{\nabla} \cdot \bar{D}$$

Thus, the total charge (Q measured in C) can be found by

$$Q = \int_v \rho_v \, dv$$

ii.Electric Current Types

Notice that there is no magnetic current in practical physics; therefore, when using the word "current" it means electric current. Electric current density *(J* measured in A/m^2) can be seen in different forms as stated below.

1) *Impressed current density* or called the *source current density* (J_S). This is the density of the current that causes the field.

2) *Convection current density* (J_v). This is the density of the current that is generated due to motion of charges in space which is a function of the drift velocity (V_d) as given by

$$\overline{J_v} = \rho_v \overline{V_d}$$

3) *Induced current density*, which is the density of the current that is caused by the electric field. There are three types of induced current.

a) *Conduction current density* (J_c), which is the density of the current that is generated due to the motion of charges in conductors.

$$\overline{J_c} = \sigma \bar{E}$$

where σ is the conductivity of the conductor measured in S/m. Therefore, conduction current exists in conducting materials, i.e., when $\sigma > 0$.

b) *Displacement current density* (J_d), which is the density of the current that is generated due to the motion of charges in dielectrics. Displacement current is taken to be negligible (compared to the conduction current) if $\sigma >> \omega\varepsilon$ otherwise, displacement current should be taken into account. Notice that the above rule depends on the radian frequency ω, which is measured in radians per second. The displacement current density is given by

$$\overline{J_d} = \frac{\partial \bar{D}}{\partial t}$$

This means that the displacement current density per unit area is the rate of change of the electric flux density with respect to time.

c) *Polarization current density* (J_p), which is part of the displacement current density that is caused by the non-free space effect as shown below.

$$\overline{J_d} = \frac{\partial \overline{D}}{\partial t} = \varepsilon \frac{\partial \overline{E}}{\partial t}$$

The permittivity is given as $\varepsilon = \varepsilon_o \varepsilon_r$ where ε_o is the permittivity of free space. If we subtract the free space part from ε_r, then we can write the permittivity as $\varepsilon = \varepsilon_o + \varepsilon - \varepsilon_o$. Apply the expression of ε in the displacement current density equation to get

$$\overline{J_d} = [\varepsilon_o + \varepsilon - \varepsilon_o] \frac{\partial \overline{E}}{\partial t} = \varepsilon_o \frac{\partial \overline{E}}{\partial t} + (\varepsilon - \varepsilon_o) \frac{\partial \overline{E}}{\partial t}$$

$$= \varepsilon_o \frac{\partial \overline{E}}{\partial t} + \frac{\partial \overline{P}}{\partial t}$$

$$= \overline{J_{do}} + \overline{J_P}$$

where P is the dielectric polarization vector measured in C/m^2 and J_{do} is the displacement current density in a vacuum, which is measured in A/m^2

iii.Power Computations

Poynting theorem states that for a volume (v) bounded by a closed surface (s), the complex power (P_s) delivered by the source in volume v is given by

$$-\frac{1}{2}\left(H^* \cdot M^i + E \cdot J^{*i}\right) = \frac{1}{2}\nabla \cdot (E \times H^*) + \frac{1}{2}\sigma|E|^2 + 2j\omega\left(\frac{1}{4}\mu|H|^2 - \frac{1}{4}\varepsilon|E|^2\right)$$

where M (V/m^2) magnetic current density, and J (A/m^2) electric current density

or

$$P_s = P_{rad} + P_{loss} + j2\omega(W_m - W_e)$$

where $P_s = -\left(H^* \cdot M^i + E \cdot J^{*i}\right)/2$ is the source complex power density (w/m^3),

$P_{rad} = P_f = \nabla \cdot (E \times H^*)/2$ is the power flowing out through closed surface s (or complex power density entering or leaving the point) (w/m^3), given by

$$P_{rad} = P_f = \frac{1}{2}\oiint_s \bar{E} \times \overline{H^*} \cdot \overline{ds}, W$$

Note that $\overline{ds} = d_s\hat{a}_n$, where s is a unit area and $\hat{a}_n$ is a unit vector normal to s,

$P_{loss} = P_d = \sigma|E|^2/2$ is the time-averaged dissipated power in volume v bounded by the closed surface s (the loss power density (real only)) (w/m^3), given by

$$P_d = P_{loss} = \frac{1}{2}\iiint_v \sigma\,|\bar{E}|^2 \cdot d_v, W$$

$Wm = \mu|H|^2/4$ is the time-averaged stored magnetic energy density (J/m^3), given by

$$W_m = \frac{1}{2}\iiint_v \frac{1}{2}\mu\,|\bar{H}|^2 \cdot d_v$$

$W_e = \varepsilon|E|^2/4$ is the time-averaged stored electric energy density (J/m^3), given by

$$W_e = \frac{1}{2}\iiint_v \frac{1}{2}\varepsilon\,|\bar{E}|^2 \cdot d_v$$

In order to compute the complex impedance of an antenna, we need to consider the integral form of Poynting's theorem.

$$-\iiint_v (\mathbf{H}^* \cdot \mathbf{M}^i + \mathbf{E} \cdot \mathbf{J}^{*i}) dv =$$

$$\oiint_{S[v]} (\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{s} + \iiint_v \sigma |\mathbf{E}|^2 \, dv + j\omega \iiint_v (\mu |\mathbf{H}|^2 - \varepsilon |\mathbf{E}|^2) dv$$

Taking a sufficiently large sphere ($r \to \infty$),

$$\Rightarrow \frac{1}{2} V I^* = \frac{1}{2} Z_A I^2 = \Pi_{rad} + \Pi_{loss} + j2\omega(W_m - W_e)$$

or

$$Z_A = \frac{\Pi_{rad} + \Pi_{loss} + j2\omega(W_m - W_e)}{0.5 I^2}.$$

The vector $E \times H$ has the dimension of power density per unit area (expressed in W/m^2). It is called the *Poynting vector* and is designated by S. Thus, the instantaneous Poynting vector is expressed as

$$\bar{S} = \bar{E} \times \bar{H}$$

And the average Poynting vector is given by

$$\overline{S_{av}} = \frac{1}{2} Re[\bar{E} \times \overline{H^*}]$$