Linear Array Theory

11.1 Introduction

Usually the radiation patterns of single-element antennas are relatively wide, i.e., they have relatively low directivity (gain). In long distance communications, antennas with high directivity are often required. Such antennas are possible to construct by enlarging the dimensions of the radiating aperture (maximum size much larger than λ). This approach however may lead to the appearance of multiple side lobes. Besides, the antenna is usually large and difficult to fabricate.

Another way to increase the electrical size of an antenna is to construct it as an assembly of radiating elements in a proper electrical and geometrical configuration – ***antenna array***. Usually, the array elements are identical. This is not necessary but it is practical and simpler for design and fabrication. The individual elements may be of any type (wire dipoles, loops, apertures, etc.)

The total field of an array is a vector superposition of the fields radiated by the individual elements. To provide very directive pattern, it is necessary that the partial fields (generated by the individual elements) interfere constructively in the desired direction and interfere destructively in the remaining space.

There are five basic methods to control the overall antenna pattern:

a. the geometrical configuration of the overall array (linear, circular, spherical, rectangular, etc.).
b. the relative placement of the elements.
c. the excitation amplitude of the individual elements.
d. the excitation phase of each element.
e. the individual pattern of each element.

11.2 Two-element array

Let us represent the electric fields in the far zone of the array elements in the form

$$E_1 = M_1 E_{n1}(\theta_1, \varphi_1) \frac{e^{-j(\beta r_1 - \frac{k}{2})}}{r_1} \widehat{\boldsymbol{\rho}}_{\mathbf{1}}$$

$$E_2 = M_2 E_{n2}(\theta_2, \varphi_2) \frac{e^{-j(\beta r_2 + \frac{k}{2})}}{r_2} \widehat{\boldsymbol{\rho}}_{\mathbf{2}}$$

where:

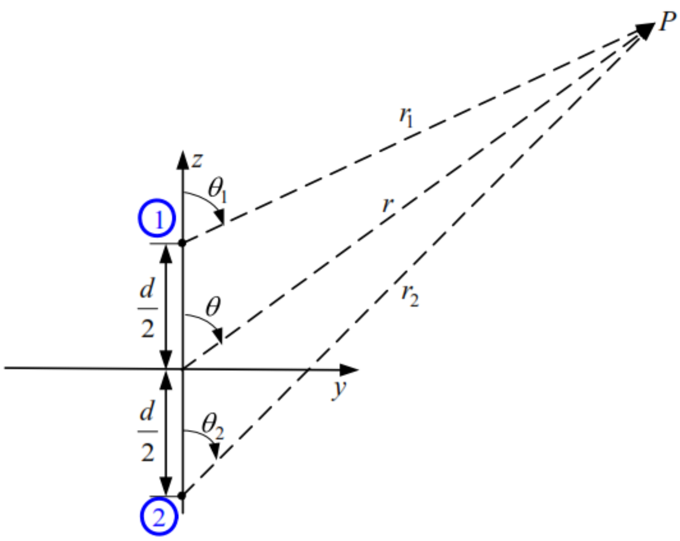
M_1, M_2 field magnitudes (do not include the $1/r$ factor);

E_{n1}, E_{n2} normalized field patterns;

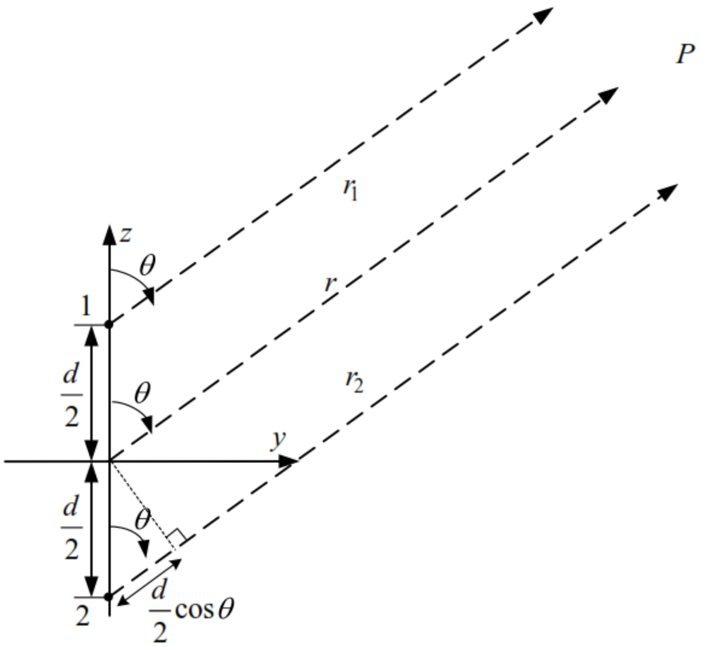
r_1, r_2 distances to the observation point P;

k phase difference between the feed of the two array elements;

$\widehat{\boldsymbol{\rho}}_{\mathbf{1}}, \widehat{\boldsymbol{\rho}}_{\mathbf{2}}$ polarization vectors of the far-zone **E** fields.



The far-field approximation of the two-element array problem:



Let us assume that:

1. the array elements are identical, i.e.,

   $E_{n1}(\theta_1, \varphi_1) = E_{n2}(\theta_1, \varphi_1) = E_n(\theta, \varphi)$

2. they are oriented in the same way in space (they have identical polarizations), i.e., $\hat{\boldsymbol{\rho}}_1 = \hat{\boldsymbol{\rho}}_2 = \hat{\boldsymbol{\rho}}$

3. their excitation is of the same amplitude, i.e., $M_1 = M_2 = M$

Then, the total field can be derived as

$$E = E_1 + E_2,$$

$$E = \hat{\boldsymbol{\rho}} M E_n(\theta, \varphi) \frac{1}{r} \left[e^{-j\beta\left(r - \frac{d}{2}\cos\theta\right) + j\frac{k}{2}} + e^{-j\beta\left(r + \frac{d}{2}\cos\theta\right) - j\frac{k}{2}} \right]$$

$$E = \hat{\boldsymbol{\rho}} \frac{M}{r} e^{-j\beta r} E_n(\theta, \varphi) \left[e^{j\left(\frac{\beta d}{2}\cos\theta + \frac{k}{2}\right)} + e^{-j\left(\frac{\beta d}{2}\cos\theta + \frac{k}{2}\right)} \right]$$

$$\boxed{E = \hat{\boldsymbol{\rho}} M \frac{e^{-j\beta r}}{r} E_n(\theta, \varphi) \times \underbrace{2\cos\left(\frac{\beta d \cos\theta + k}{2}\right)}_{AF}}$$

The total field of the array is equal to the product of the field created by a single element located at the origin and the ***Array Factor***, *AF*:

$$AF = 2\cos\left(\frac{\beta d\cos\theta + k}{2}\right)$$

Using the normalized field pattern of a single element, $E_n(\theta,\varphi)$, and the normalized *AF*,

$$AF_n = \cos\left(\frac{\beta d\cos\theta + k}{2}\right),$$

the normalized field pattern of the array is expressed as their product:

$$f_n(\theta,\varphi) = E_n(\theta,\varphi) \times AF_n(\theta,\varphi)$$

The concept expressed by above equation is the so-called ***pattern multiplication*** rule valid for arrays of identical elements. This rule holds for any array consisting of decoupled identical elements, where the excitation magnitudes, the phase shift between the elements and the displacement between them are not necessarily the same. The total pattern, therefore, can be controlled via the single–element pattern, $E_n(\theta,\varphi)$ or via the *AF*.

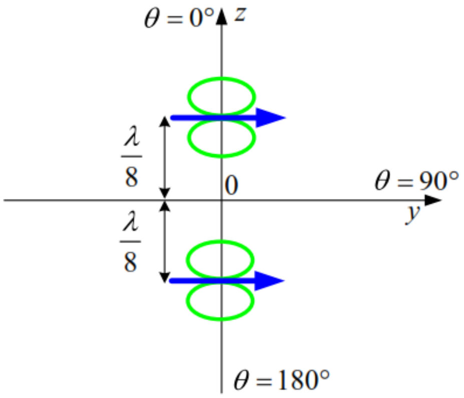
The *AF*, in general, depends on:

- number of elements,
- mutual placement,
- relative excitation magnitudes and phases.

Example: -

An array consists of two horizontal infinitesimal dipoles (their normalized field patterns is $E_n(\theta,\varphi) = \sqrt{1-\sin^2\theta\sin^2\varphi}$) located at a distance $d = \lambda/4$ from each other. Find the nulls of the total field in the elevation plane $\varphi = \pm 90^o$, if the excitation magnitudes are the same and the phase difference is:

a) $k = 0$
b) $k = \pi/2$
c) $k = -\pi/2$



Solution:

the element factor $E_n(\theta,\varphi) = \sqrt{1-\sin^2\theta\sin^2\varphi}$ does not depend on k, and it produces in all three cases the same null. For $\varphi = \pm 90^o$, $E_n(\theta, \varphi) = |\cos\theta|$ and the null is at $\theta = \pi/2$.

a) at $k = 0$

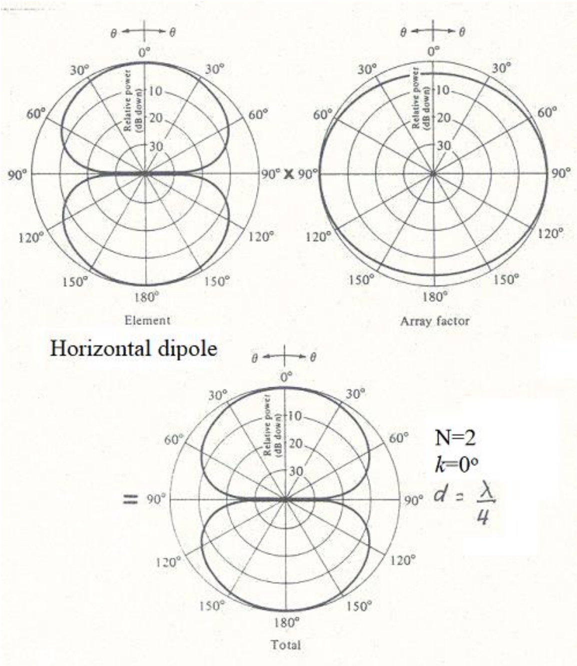
$$AF_n = \cos\left(\frac{\beta d\cos\theta + k}{2}\right) = 0$$

$$AF_n = \cos\left(\frac{\frac{\pi}{2}\cos\theta + 0}{2}\right) = 0$$

$$AF_n = \cos\left(\frac{\pi}{4}\cos\theta\right) = 0$$

$$\frac{\pi}{4}\cos\theta_n = (2n+1)\frac{\pi}{2} \Rightarrow \cos\theta_n = 2\cdot(2n+1), n = 0, \pm 1, \pm 2, \dots$$

A solution with a real-valued angle does not exist. In this case, the total field pattern has only one null at $\theta = 90°$



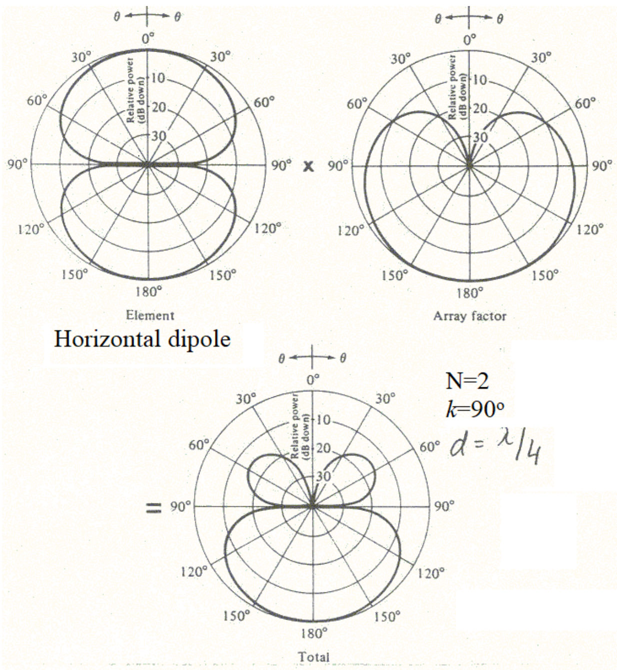
b) $k = \pi/2$

$$AF_n = \cos\left(\frac{\pi}{4}\cos\theta + \frac{\pi}{4}\right) = 0$$

$$\frac{\pi}{4}(\cos\theta_n + 1) = (2n+1)\frac{\pi}{2} \Longrightarrow \cos\theta_n + 1 = 2\cdot(2n+1)\,,when\ n = 0$$

$$\Longrightarrow \cos\theta = 1 \Longrightarrow \theta = 0$$

The solution for *n*=0 is the only real-valued solution. Thus, the total field pattern has two nulls at θ =90° and at θ =0°:

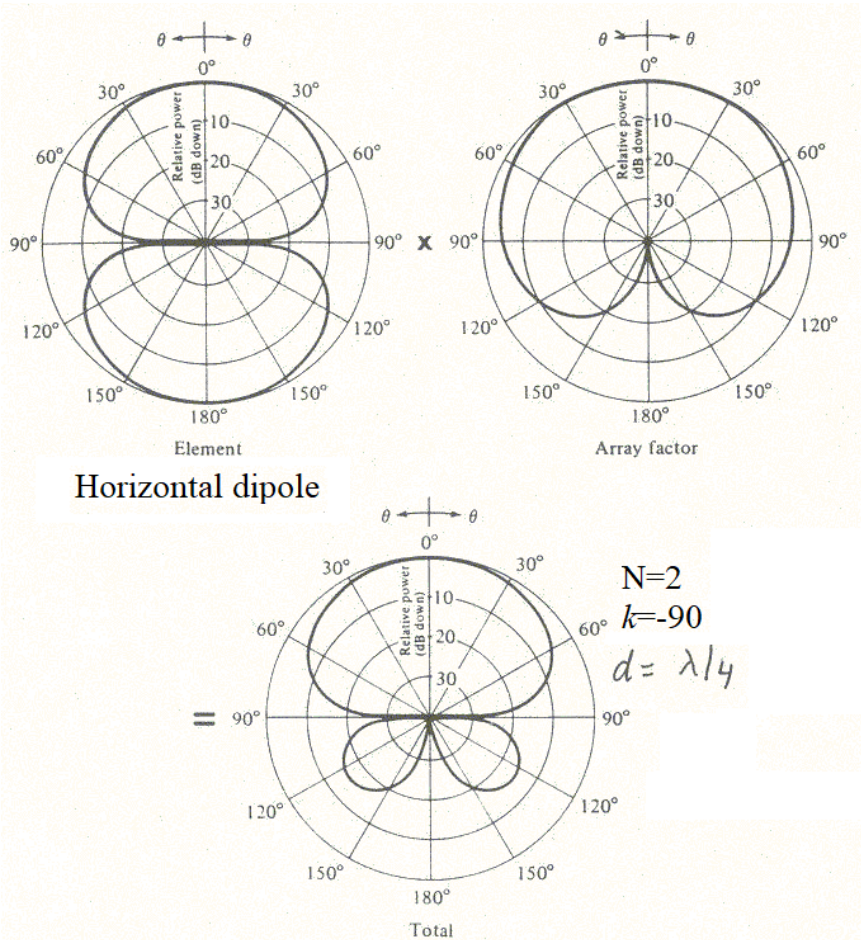


c) $k = -\pi/2$

$$AF_n = \cos\left(\frac{\pi}{4}\cos\theta - \frac{\pi}{4}\right) = 0$$

$$\frac{\pi}{4}(\cos\theta_n - 1) = (2n+1)\frac{\pi}{2} \Rightarrow \cos\theta_n - 1 = 2\cdot(2n+1)\,, when\; n = -1 \;\Rightarrow \cos\theta = -1 \Rightarrow \theta = \pi$$

The total field pattern has two nulls at θ =90° and at θ =180°:



Example:

Consider a 2-element array of identical (infinitesimal) dipoles oriented along the *y*-axis. Find the angles of observation in the plane φ = ± 90°, where the nulls of the pattern occur, as a function of the distance between the dipoles, *d*, and the phase difference, *k*.

Solution:

The normalized total field pattern is

$f_n(\theta,\varphi) = E_n(\theta,\varphi) \times AF_n(\theta,\varphi)$, $E_n(\theta,\varphi) = |\cos\theta|$, $AF_n = \cos\left(\frac{\beta d\cos\theta + k}{2}\right)$

$$f_n(\theta,\varphi) = |\cos\theta| \times \cos\left(\frac{\beta d\cos\theta + k}{2}\right)$$

In order to find the nulls, the above equation must be equal to zero

$$|\cos\theta| \times \cos\left(\frac{\beta d\cos\theta + k}{2}\right) = 0$$

The element factor, | cos |, produces one null at

$$\theta = \pi/2$$

The array factor leads to the following solution:

$$\cos\left(\frac{\beta d\cos\theta + k}{2}\right) = 0 \Rightarrow \frac{\beta d\cos\theta + k}{2} = (2n+1)\frac{\pi}{2}, n = 0, \pm 1, \pm 2, \dots$$

$$\theta = \cos^{-1}\left[\frac{\lambda}{2\pi d}(-k \pm (2n+1)\pi)\right]$$

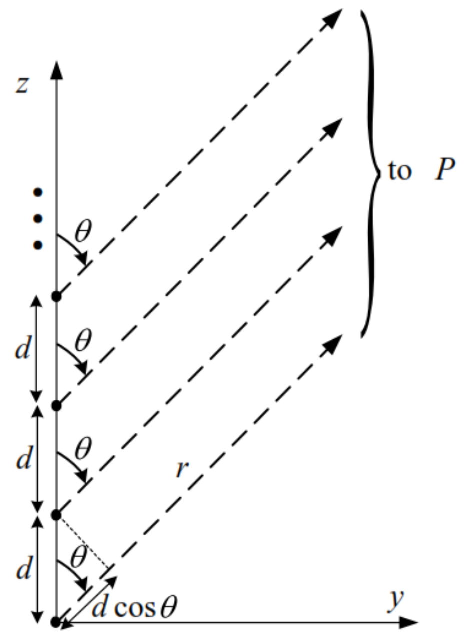
When there is no phase difference between the two element feeds (*k*= 0), the separation *d* must satisfy $d \geq \frac{\lambda}{2}$ in order at least one null to occur.

11.3 *N*-element linear array with uniform amplitude and spacing

We assume that each succeeding element has a k progressive phase lead current excitation relative to the preceding one. An array of identical elements with identical magnitudes and with a progressive phase is called a ***uniform array***. The *AF* of the uniform array can be obtained by considering the individual elements as point (isotropic) sources. Then, the total field pattern can be obtained by simply multiplying the *AF* by the normalized field pattern of the individual element (provided the elements are not coupled).

The *AF* of an *N*-element linear array of isotropic sources is

$$AF = 1 + e^{j(\beta d\cos\theta + k)} + e^{j2(\beta d\cos\theta + k)} + \cdots + e^{j(N-1)(\beta d\cos\theta + k)}$$



Phase terms of the partial fields:

$$1^{\text{st}} \longrightarrow e^{-j\beta r}$$

$$2^{\text{nd}} \longrightarrow e^{-j\beta(r - d\cos\theta)}$$

$$3^{\text{rd}} \longrightarrow e^{-j\beta(r - 2d\cos\theta)}$$

$$\cdots$$

$$N^{\text{th}} \longrightarrow e^{-j\beta(r-(N-1)d\cos\theta)}$$

AF can be re-written as

$$AF = \sum_{n=1}^{N} e^{j(n-1)(\beta d \cos\theta + k)}$$

$$AF = \sum_{n=1}^{N} e^{j(n-1)\Psi}$$

where $\Psi = \beta d \cos\theta + k$

Tt is obvious that the *AF*s of uniform linear arrays can be controlled by the relative phase *k* between the elements. The *AF* can be expressed in a closed form, which is more convenient for pattern analysis

$$AF \cdot e^{j\Psi} = \sum_{n=1}^{N} e^{jn\Psi}$$

$$AF \cdot e^{j\Psi} - AF = e^{jN\Psi} - 1$$

$$AF = \frac{e^{jN\Psi} - 1}{e^{j\Psi} - 1} = \frac{e^{j\frac{N}{2}\Psi}\left(e^{j\frac{N}{2}\Psi} - e^{-j\frac{N}{2}\Psi}\right)}{e^{j\frac{\Psi}{2}}\left(e^{j\frac{\Psi}{2}} - e^{-j\frac{\Psi}{2}}\right)} = e^{j\left(\frac{N-1}{2}\right)\Psi} \cdot \frac{\sin\left(\frac{N}{2}\Psi\right)}{\sin\left(\frac{\Psi}{2}\right)}$$

Here, *N* shows the location of the last element with respect to the reference point in steps with length *d*. The phase factor $e^{j\left(\frac{N-1}{2}\right)\Psi}$ is not important unless the array output signal is further combined with the output signal of another antenna. It represents the phase shift of the array's phase center relative to the origin, and it would be identically equal to one if the origin were to coincide with the array center. Neglecting the phase factor gives

$$\boxed{AF = \frac{\sin\left(\frac{N}{2}\Psi\right)}{\sin\left(\frac{\Psi}{2}\right)}}$$

For small values of ψ,

$$AF = \frac{\sin\left(\frac{N}{2}\Psi\right)}{\left(\frac{\Psi}{2}\right)}$$

To normalize *AF*, we need the maximum of the *AF*. We re-write the above equation as

$$AF = N \cdot \frac{\sin\left(\frac{N}{2}\Psi\right)}{N \cdot \sin\left(\frac{\Psi}{2}\right)}$$

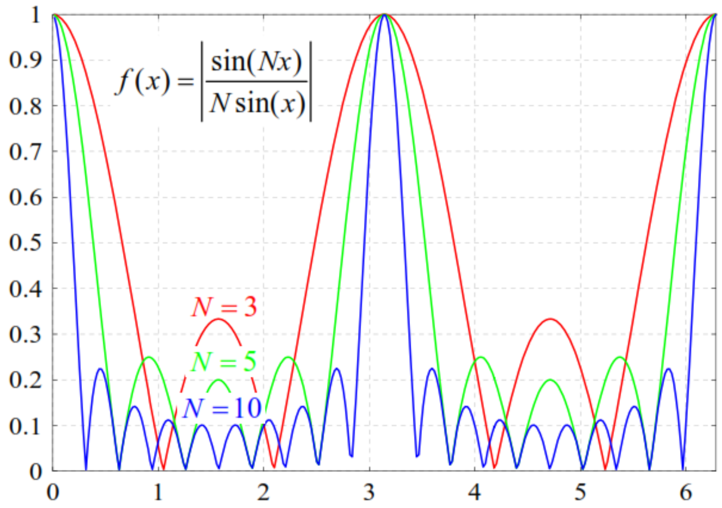
The function

$$f(x) = \frac{\sin(Nx)}{N \cdot \sin(x)}$$

has its maximum at $x = 0, \pi, \ldots$, and the value of this maximum is $f_{max} = 1$. Therefore, $AF_{max} = N$. The normalized *AF* is obtained as

$$AF_n = \frac{\sin\left(\frac{N}{2}\Psi\right)}{N \cdot \sin\left(\frac{\Psi}{2}\right)}$$

The above function is plotted below.



For small values of ψ,

$$AF_n = \frac{\sin\left(\frac{N}{2}\Psi\right)}{N \cdot \left(\frac{\Psi}{2}\right)}$$