Nulls of the *AF*:

To find the nulls, equation of the *AF* is set equal to zero

$$AF_n = \frac{\sin\left(\frac{N}{2}\Psi\right)}{N\cdot\sin\left(\frac{\Psi}{2}\right)}$$

$$\sin\left(\frac{N}{2}\Psi\right) = 0 \Rightarrow \frac{N}{2}\Psi = \pm n\pi \Rightarrow \frac{N}{2}(\beta d\cos\theta_n + k) = \pm n\pi$$

$$\boxed{\theta_n = \cos^{-1}\left[\frac{\lambda}{2\pi d}\left(-k \pm \frac{2n}{N}\pi\right)\right], n = 1,2,3\ldots(n \neq 0, N, 2N, 3N\ldots)}$$

When $n = 0, N, 2N, 3N\ldots$, the *AF* attains its maximum values (see the case below). The values of n determine the order of the nulls. For a null to exist, the argument of the arccosine must be between –1 and +1.

Maxima of the *AF*:

They are studied in order to determine the maximum directivity, the *HPBW*s, the direction of maximum radiation. The maxima of *AF* occur when (see the plot in page 127, note that $x = \psi/2$).

$$\frac{\Psi}{2} = \frac{1}{2}(\beta d\cos\theta_m + k) = \pm m\pi$$

$$\theta_m = \cos^{-1}\left[\frac{\lambda}{2\pi d}(-k \pm 2m\pi)\right], m = 0,1,2,\ldots$$

When $AF_n = 1$, i.e., these are not maxima of minor lobes. The index m shows the maximum's order. It is usually desirable to have a single major lobe, i.e. $m=0$. This can be achieved by choosing d/λ sufficiently small. Then the argument of the $\cos^{-1}$ function in (above equation) becomes greater than unity for $m = 1, 2,\ldots$ and this equation has a single real-valued solution

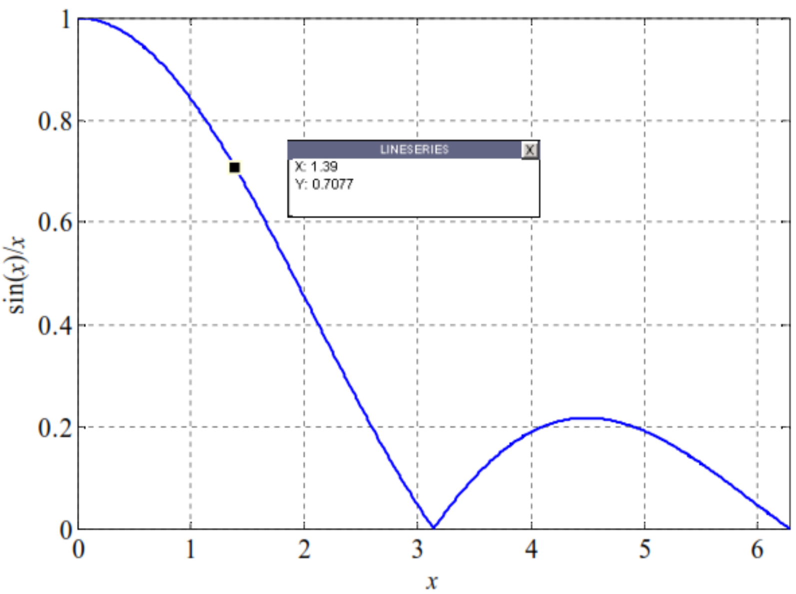
$$\boxed{\theta_0 = \cos^{-1}\left[\frac{-\lambda k}{2\pi d}\right]}$$

The HPBW of a major lobe:

The *HPBW* of a major lobe is calculated by setting the value of AF_n equal to $1/\sqrt{2}$. For the approximate AF_n in $AF_n = \frac{\sin\left(\frac{N}{2}\Psi\right)}{N\cdot\left(\frac{\Psi}{2}\right)}$

$$\frac{N\Psi}{2} = \frac{N}{2}(\beta d \cos\theta_h + k) = \pm 1.4$$

See plot of (sin *x*)/*x* below.



$$\boxed{\theta_h = \cos^{-1}\left[\frac{\lambda}{2\pi d}\left(-k \pm \frac{2.8}{N}\right)\right]}$$

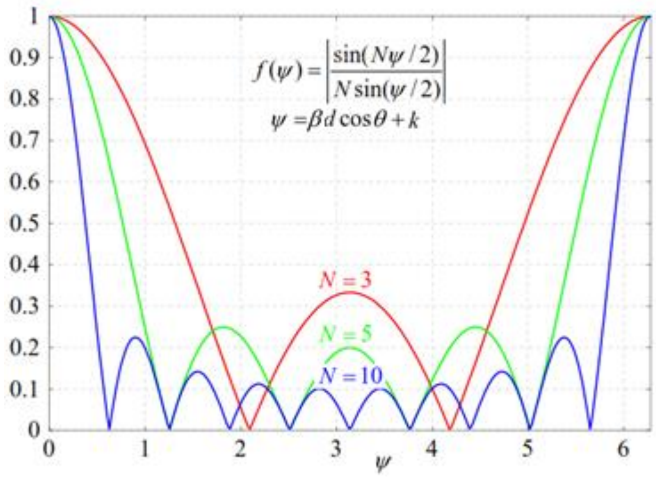
For a symmetrical pattern around θ_m (the angle at which maximum radiation occurs), the *HPBW* is calculated as

$$\boxed{HPBW = 2|\theta_m - \theta_h|}$$

Maxima of minor lobes (secondary maxima)

They are the maxima of AF_n, where $AF_n \leq 1$. This is clearly seen in the plot of the array factors as a function of ($\psi = \beta d \cos\theta + k$) for a uniform equally spaced linear array(N=3,5,10). See the plot below.

The secondary maxima occur approximately where the numerator attains a maximum and the AF is beyond its 1st null:



$$\sin\left(\frac{N}{2}\Psi\right) = \pm 1 \Rightarrow \frac{N}{2}(\beta d \cos\theta + k) = \pm(2s+1)\frac{\pi}{2}$$

$$\boxed{\Rightarrow \theta_s = \cos^{-1}\left\{\frac{\lambda}{2\pi d}\left(-k \pm \left(\frac{2s+1}{N}\right)\pi\right)\right\}}$$

Broadside array

A broadside array is an array, which has maximum radiation at θ =90° (normal to the axis of the array). For optimal solution, both the element factor and the *AF*, should have their maxima at θ =90°.

It follows that the maximum of the *AF* occurs when

$$\Psi = (\beta d \cos\theta + k) = 0$$

Equation

$$\frac{\Psi}{2} = \frac{1}{2}(\beta d \cos\theta_m + k) = \pm m\pi$$

is valid for the 0^{th} order maximum, m =0. If θ_m =π/2, then

$$\Psi = k = 0$$

The uniform linear array has its maximum radiation at θ =90°, if all array elements have their excitation with the same phase.

To ensure that there are no maxima in the other directions (grating lobes), the separation between the elements should not be equal to multiples of a wavelength:

$$d \neq n\lambda, n = 1,2,3, \dots$$

Otherwise, additional maxima, AF_n =1, appear. Assume that d=$n\lambda$. Then,

$$\Psi = \beta d \cos\theta = \frac{2\pi}{\lambda} \cdot n\lambda \cdot \cos\theta = 2n\pi \cos\theta$$

The maximum condition for

$$\frac{\Psi}{2} = \frac{1}{2}(\beta d \cos\theta_m + k) = \pm m\pi \Rightarrow \Psi_m = 2\pi m, m = 0, \pm 1, \pm 2 \dots$$

is fulfilled not only for $\theta_0 = \pi/2$ but also for

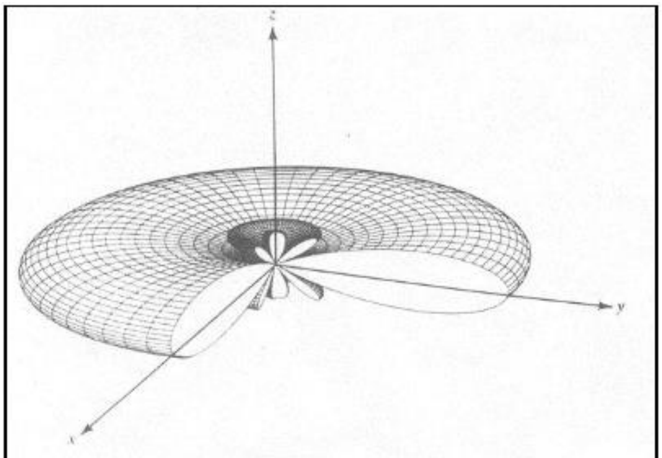
$$\theta_g = \cos^{-1}\left(\frac{m}{n}\right), m = \pm 1, \pm 2 \dots$$

If, for example, d =λ (n=1), above equation results in two additional major lobes at

$$\theta_g = \cos^{-1}(\pm 1) \Rightarrow \theta_{g_{1,2}} = 0^\circ, 180^\circ$$

If, d =2λ (n=2), this equation results in four additional major lobes at

$$\theta_{\mathrm{g}} = \cos^{-1}\left(\pm\frac{1}{2}, \pm 1\right) \Rightarrow \theta_{\mathrm{g}_{1,2,3,4}} = 0^{\circ}, 60^{\circ}, 120^{\circ}, 180^{\circ}$$



(a) Broadside

The best way to ensure the existence of only one maximum is to choose $\boldsymbol{d_{max} < \lambda}$.

Then, in the case of ***the broadside array*** **(**k =0**)**, equation

$$\theta_m = \cos^{-1}\left[\frac{\lambda}{2\pi d}\,(-k \pm 2m\pi)\right], m = 0,1,2,\ldots$$

produces no solution for $m \geq 1$

Ordinary end-fire array

An end-fire array is an array, which has its maximum radiation along the axis of the array (θ =0°,180°). It may be required that the array radiates only in one direction – either θ =0° or θ =180°. For an *AF* maximum at θ =0°,

$$\Psi = \beta d \cos\theta + k = 0$$

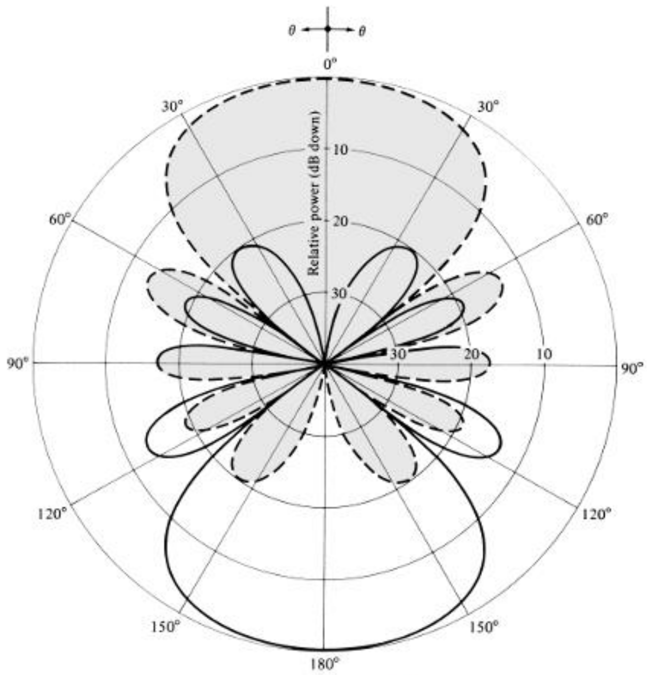
$$at\ \theta = 0^{\circ} \Rightarrow \beta d + k = 0 \Rightarrow -\beta d = k, for\ \theta_{max} = 0^{\circ}$$

For an *AF* maximum at θ =180°,

$$\Psi = \beta d \cos\theta + k = 0$$

$$at\ \theta = 180^{\circ} \Rightarrow -\beta d + k = 0 \Rightarrow \beta d = k, for\ \theta_{max} = 180^{\circ}$$

If the element separation is multiple of a wavelength, ***d*=*n*λ**, then in addition to the end-fire maxima there also exist maxima in the broadside directions. As with the broadside array, in order to avoid grating lobes, the maximum spacing between the element should be less than λ: that mean $\boldsymbol{d_{max}<\lambda}$.



$k=+\beta d$
$k=-\beta d$

Phased (scanning) arrays

It was already shown that the 0th order maximum (*m*=0) of AF_n occurs when

$$\Psi = \beta d \cos\theta_0 + k = 0$$

This gives the relation between the direction of the main beam θ_0 and the phase difference k. The direction of the main beam can be controlled by the phase shift $\boldsymbol{k}$. This is the basic principle of ***electronic scanning*** for phased arrays.

The scanning must be continuous. That is why the feeding system should be capable of continuously varying the progressive phase $\boldsymbol{k}$ between the elements.

Example:

Derive the values of the progressive phase shift $\underline{k}$ as dependent on the direction of the main beam θ_0 for a uniform linear array with $d = \lambda/4$.

From equation $\Psi = \beta d \cos\theta_0 + k = 0$

$$-\beta d \cos\theta_0 = k = -\frac{2\pi}{\lambda}\frac{\lambda}{4}\cos\theta_0 = -\frac{\pi}{2}\cos\theta_0$$

θ_0	k
0°	-90°
60°	-45°
120°	+45°
180°	+90°

The *HPBW* of a scanning array is obtained using

$$\theta_{h_{1,2}} = \cos^{-1}\left[\frac{\lambda}{2\pi d}\left(-k \pm \frac{2.8}{N}\right)\right]$$

With $\beta d \cos\theta_0 + k = 0$

The total beamwidth is $HPBW = \theta_{h1} - \theta_{h2}$

$$HPBW = \cos^{-1}\left[\frac{\lambda}{2\pi d}\left(\beta d \cos\theta_0 - \frac{2.8}{N}\right)\right] - \cos^{-1}\left[\frac{\lambda}{2\pi d}\left(\beta d \cos\theta_0 + \frac{2.8}{N}\right)\right]$$

Since $\beta = 2\pi/\lambda$,

$$HPBW = \cos^{-1}\left[\cos\theta_0 - \frac{2.8}{N\beta d}\right] - \cos^{-1}\left[\cos\theta_0 + \frac{2.8}{N\beta d}\right]$$

A new term L can be defined here as the length of the array,

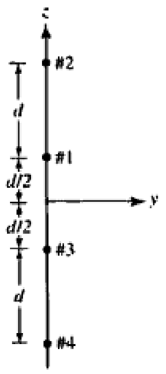
Where $L = Nd - d \Longrightarrow N = (L + d)/d$

$$\boxed{HPBW = \cos^{-1}\left[\cos\theta_0 - 0.44\left(\frac{\lambda}{L+d}\right)\right] - \cos^{-1}\left[\cos\theta_0 + 0.44\left(\frac{\lambda}{L+d}\right)\right]}$$

<u>Example:</u>

Four isotropic sources are placed along the z-axis as shown below. Assuming that the amplitudes of elements number one and number two are +1 and the amplitudes of elements number three and number four are -1 (or 180 degrees out phase with number one and number two), find

a) The array factor in simplified form
b) All the nulls when $d = \lambda/2$



Solution:

a) $E = \frac{e^{-j\beta r_1}}{r_1} + \frac{e^{-j\beta r_2}}{r_2} - \frac{e^{-j\beta r_3}}{r_3} - \frac{e^{-j\beta r_4}}{r_4}$

$$r_1 = r - \frac{d}{2}\cos\theta, r_2 = r - \frac{3d}{2}\cos\theta, r_3 = r + \frac{d}{2}\cos\theta, r_4$$

$$= r + \frac{3d}{2}\cos\theta$$

$$\approx \frac{e^{-j\beta r}}{r}(e^{j\beta\frac{d}{2}\cos\theta} + e^{j\beta\frac{3d}{2}\cos\theta} - e^{-j\beta\frac{d}{2}\cos\theta} - e^{-j\beta\frac{3d}{2}\cos\theta})$$

$$AF = 2j\left[\sin\left(\beta\frac{3d}{2}\cos\theta\right) + \sin\left(\beta\frac{d}{2}\cos\theta\right)\right]$$

$$= 4j\left[\sin(\beta d\cos\theta)\cos\left(\frac{\beta d}{2}\cos\theta\right)\right]$$

b) For $d = \lambda/2$

$$AF = 4j\left[\sin\left(\frac{2\pi}{\lambda}\frac{\lambda}{2}\cos\theta\right)\cos\left(\frac{2\pi}{\lambda}\frac{\lambda}{4}\cos\theta\right)\right]$$

$$AF = 4j\left[\sin(\pi\cos\theta)\cos\left(\frac{\pi}{2}\cos\theta\right)\right]$$

Either $\sin(\pi\cos\theta) = 0$, or $\cos\left(\frac{\pi}{2}\cos\theta\right) = 0$

$$\pi\cos\theta = \sin^{-1}(0) \Rightarrow \pi\cos\theta = \pm n\pi \Rightarrow \cos\theta = \pm n$$

$$\Rightarrow \theta_n = 0^\circ, 90^\circ, 180^\circ$$

$$or\ \cos\left(\frac{\pi}{2}\cos\theta\right) = 0 \Rightarrow \frac{\pi}{2}\cos\theta = \cos^{-1}0 \Rightarrow \frac{\pi}{2}\cos\theta = \pm(2n+1)\frac{\pi}{2}$$

$$\Rightarrow \cos\theta = \pm(2n+1) for\ n = 0 \Rightarrow \cos\theta = \pm 1$$

$$\Rightarrow \theta_n = 0^\circ, 180^\circ$$