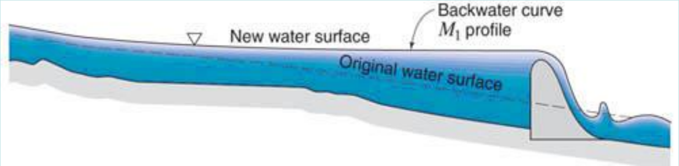
BAE 6333 – Fluvial Hydraulics

Nonuniform Flow

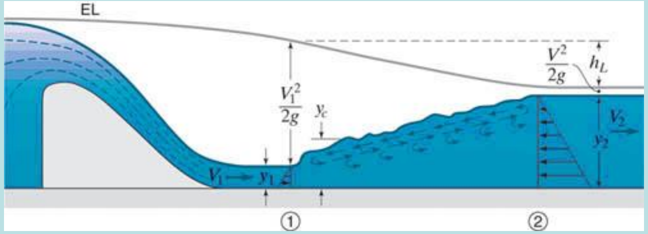
Chapter 4 (Graf, 1998)

Nonuniform Flow

- Two Types of Nonuniform Flow:
  - Gradually Varied Flow – changing conditions extend over a long distance



  - Rapidly Varied Flow – changing flow conditions occur abruptly



Gradually Varied Flow

- Consider prismatic channel with steady, non-uniform flow:

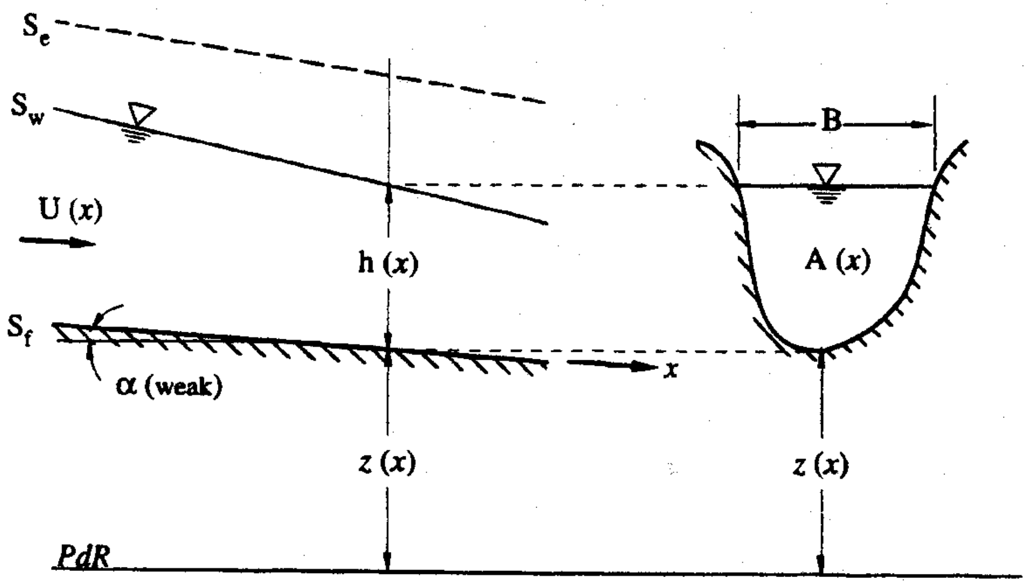


Fig. 4.1 Scheme of steady and non-uniform flow on a fixed bed slope, $z(x)$.

Saint-Venant Equations

- Combined equation of continuity and equation of energy…

$$\frac{\partial Q}{\partial x} + B\frac{\partial h}{\partial t} = 0$$

$$\frac{\partial Q}{\partial x} = A\frac{\partial U}{\partial x} + U\frac{\partial A}{\partial x} = 0$$

$$h\frac{\partial U}{\partial x} + U\frac{\partial h}{\partial x} = 0$$

Saint-Venant Equations

- Combined equation of continuity and equation of energy…

$$H = \frac{U^2}{2g} + h + z$$

$$\frac{d}{dx}\left(\frac{U^2}{2g}\right) + \frac{dh}{dx} + \frac{dz}{dx} = \frac{dH}{dx}$$

$$\frac{dz}{dx} = -S_f \qquad \frac{dH}{dx} = -S_e = f\,\frac{1}{4R_h}\frac{U^2}{2g} = \frac{8g}{C^2}\frac{1}{4R_h}\frac{U^2}{2g}$$

$$\frac{d}{dx}\left(\frac{(Q/A)^2}{2g}\right) + \frac{dh}{dx} - S_f = -S_e = -\frac{(Q/A)^2}{C^2 R_h}$$

Saint-Venant Equations

- Combined equation of continuity and equation of energy…

$$\frac{d}{dx}\left(\frac{(Q/A)^2}{2g}\right)+\frac{dh}{dx}-S_f=-S_e=-\frac{(Q/A)^2}{C^2R_h}$$

$$\begin{aligned}&\mathit{Prismatic:}\\ &A=\mathrm{f}(h)\end{aligned}$$

$$\frac{d}{dx}\left(\frac{(Q/A)^2}{2g}\right)=\frac{d}{dx}\left(\frac{Q^2}{2g\left[A(h)\right]^2}\right)=\frac{Q^2}{2g}\left(\frac{-2}{A^3}\frac{dA}{dx}\right)=-\frac{Q^2}{gA^3}\left(\frac{dA}{dh}\frac{dh}{dx}\right)$$

Saint-Venant Equations

- Combined equation of continuity and equation of energy…

$$-\frac{Q^2}{gA^3}B\frac{dh}{dx}+\frac{dh}{dx}-S_f=-\frac{(Q/A)^2}{C^2R_h}$$

$$Prismatic:$$

$$\frac{dh}{dx}=\frac{S_f-\left(\frac{(Q/A)^2}{C^2R_h}\right)}{1-\frac{BQ^2}{gA^3}}=S_f\,\frac{1-\left(\frac{(Q/A)^2}{C^2R_hS_f}\right)}{1-\frac{BQ^2}{gA^3}}$$

Wide and Rectangular Channel

- Further simplify for wide and rectangular channel…
  - Normal Depth:

$$Q = UA = C(h_n B)\sqrt{h_n S_f}$$

$$q = \frac{Q}{B}$$

$$q = Ch_n\sqrt{h_n S_f}$$

$$q^2 = C^2 h_n{}^3 S_f \rightarrow h_n{}^3 = \frac{q^2}{C^2 S_f}$$

Wide and Rectangular Channel

- Further simplify for wide and rectangular channel…
  - Critical Depth:

$$Fr = 1 = \frac{U}{\sqrt{gD_h}} = \frac{Q}{A\sqrt{gh_c}} = \frac{Q}{Bh_c\sqrt{gh_c}} = \frac{q}{h_c\sqrt{gh_c}}$$

$$q^2 = h_c{}^2 gh_c \rightarrow h_c{}^3 = \frac{q^2}{g}$$

Wide and Rectangular Channel

- Using Chezy Equation…

$$\frac{dh}{dx} = S_f \frac{1-\left(\dfrac{Q^2}{A^2C^2hS_f}\right)}{1-\dfrac{BQ^2}{gA^3}} = S_f \frac{1-\left(\dfrac{Q^2}{B^2h^3C^2S_f}\right)}{1-\dfrac{BQ^2}{g(Bh)^3}}$$

$$\frac{dh}{dx} = S_f \frac{1-\left(\dfrac{q^2}{h^3C^2S_f}\right)}{1-\dfrac{q^2}{gh^3}} = S_f \frac{1-\left(\dfrac{h_n}{h}\right)^3}{1-\left(\dfrac{h_c}{h}\right)^3}$$

Wide and Rectangular Channel

- Using Manning's Equation…

$$\frac{dh}{dx} = S_f \frac{1-\left(\dfrac{q^2}{h^3 C^2 S_f}\right)}{1-\dfrac{q^2}{gh^3}} = S_f \frac{1-\left(\dfrac{h_n}{h}\right)^{10/3}}{1-\left(\dfrac{h_c}{h}\right)^3}$$

Critical Slope

- Bed slope that results in uniform flow at critical depth ($y_n = y_c$) for a given discharge
  - Combine critical flow equation with uniform flow equation:

$$U = C\sqrt{R_h S_f} \leftrightarrow \left(\frac{Q}{A}\right)^2 = \frac{gA}{B}$$

$$C^2 R_h S_c = g\,\frac{A}{B}$$

$$S_c = \frac{gA}{C^2 B R_h}$$

Critical Slope

- If $S_f < S_c$ for a given Q and C, then $h_n > h_c$:
  - Mild Slope
  - Uniform flow corresponding to this normal depth will be subcritical (fluvial)
- If $S_f > S_c$ for a given Q and C, then $h_n < h_c$:
  - Steep Slope
  - Uniform flow corresponding to this normal depth will be supercritical (torrential)

Forms of Water Surface

- Water surface profiles for the possible cases encountered in open-channel flow…
  - First classification based on bed slope, S_f:
    - $S_f = 0$ (Horizontal Slope): H
    - $S_f < 0$ (Adverse Slope): A
    - $S_f > 0$ (Mild Slope, Steep Slope, or Critical Slope): M, S, or C
  - Figure 4.2 in Graf

Conditions Eq. 4.8a	$\frac{h_n}{h}$	Sign num.	$\frac{h_c}{h}$	Sign den.	Sign dh/dx	Change of flow depth	Name
$S_f > 0$							
$S_f < S_c$	< 1	+	< 1	+	+	increase	M1
$h_n > h_c$	< 1	+	> 1	-		not possible	
	> 1	-	< 1	+	-	decrease	M2
	> 1	-	> 1	-	+	increase	M3
$S_f > 0$							
$S_f > S_c$	< 1	+	< 1	+	+	increase	S1
$h_n < h_c$	< 1	+	> 1	-	-	decrease	S2
	> 1	-	> 1	-	+	increase	S3
$S_f > 0$							
$S_f = S_c$	< 1	+	< 1	+	+	increase	C1
$h_n = h_c$	> 1	-	> 1	-	+	increase	C3
$S_f = 0$							
$h_n = \infty$		-	< 1	+	-	decrease	H2
		-	> 1	-	+	increase	H3
$S_f < 0$							
$h_n < 0$	< 1	-	< 1	+	-	decrease	A2
	< 1	-	> 1	-	+	increase	A3

Profiles
vertical scale exaggerated

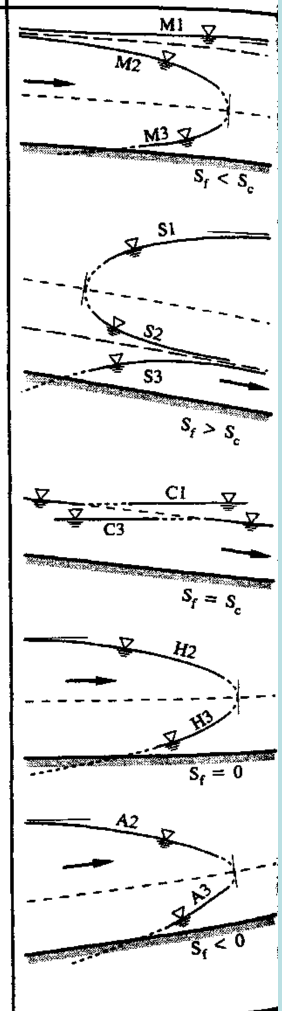


Fig. 4.2 Water-surface profiles for gradually varied flow.
water surface ; — — — normal depth ; - - - - critical depth

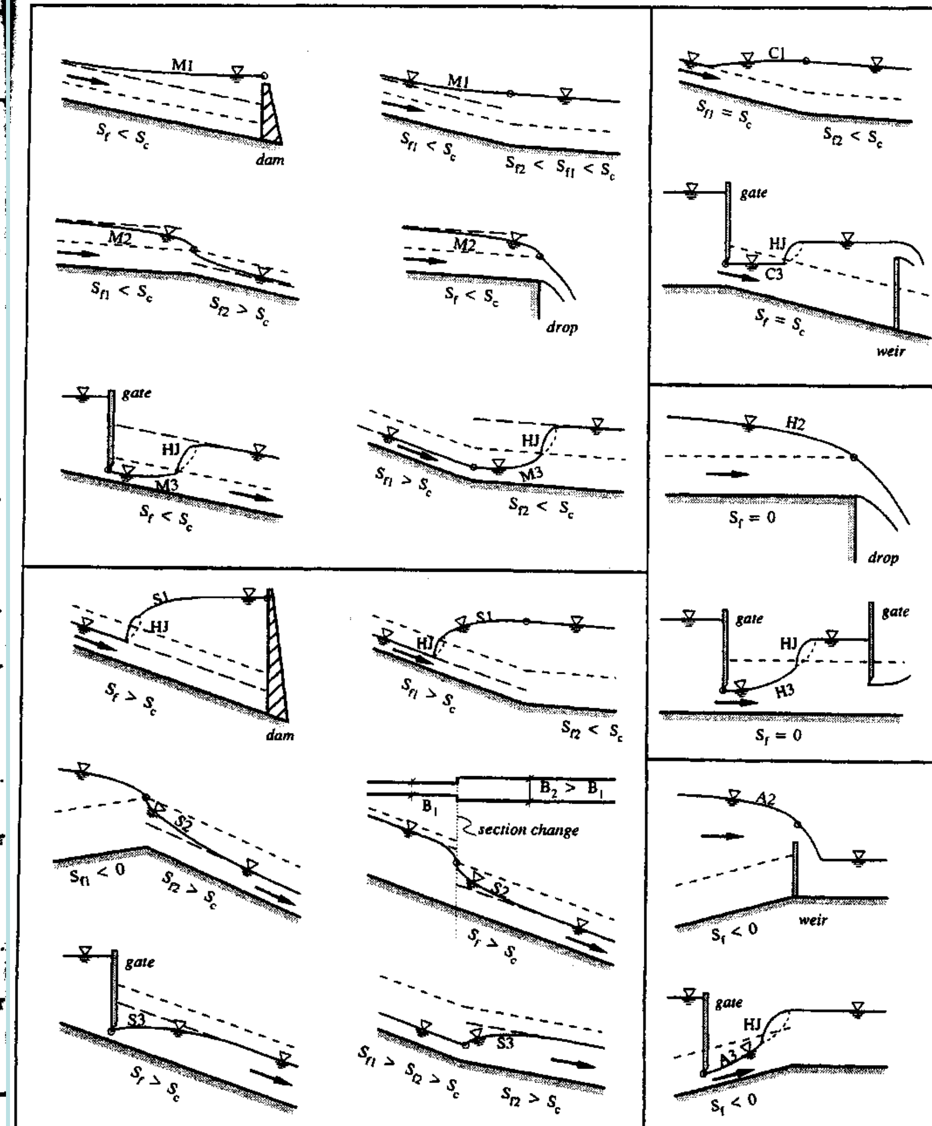
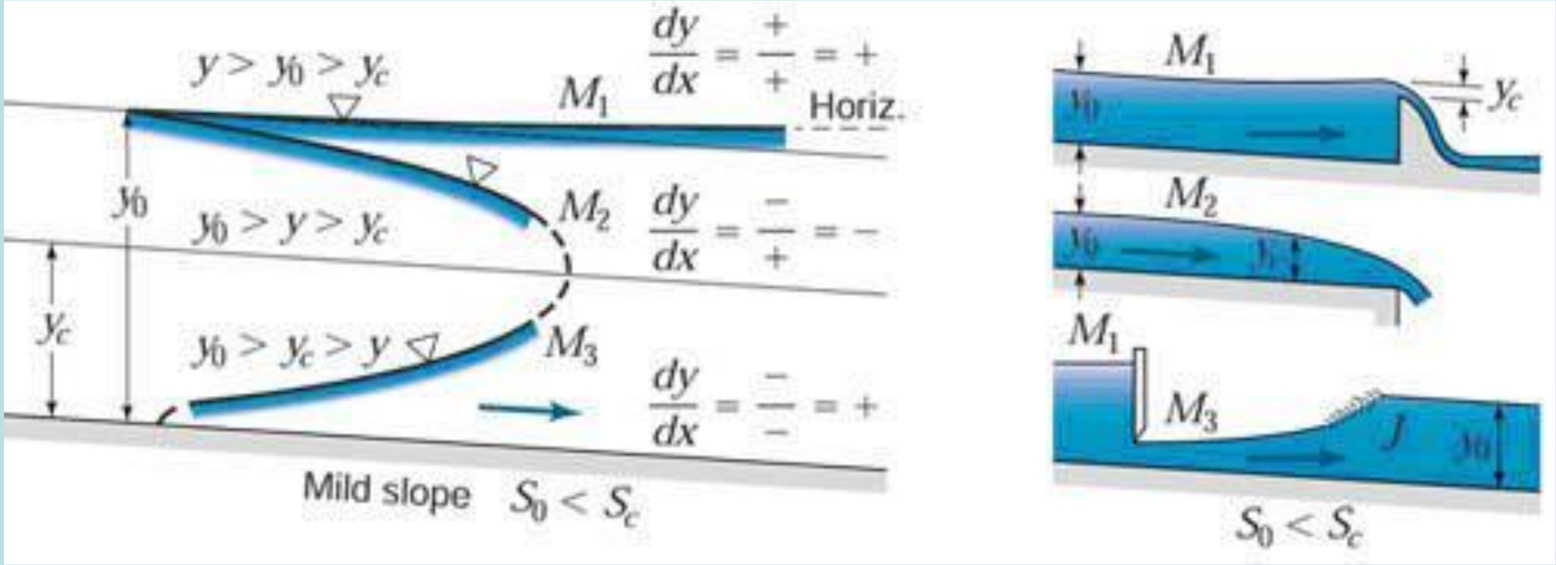


Fig. 4.2a Some examples of flow.
o control section ; HJ hydraulic jump

Water-Surface Profiles

- M1 –
  - Curves goes downstream towards a horizontal tangent
  - Upstream of a dam or weir, pier, at junctures of certain bed slopes

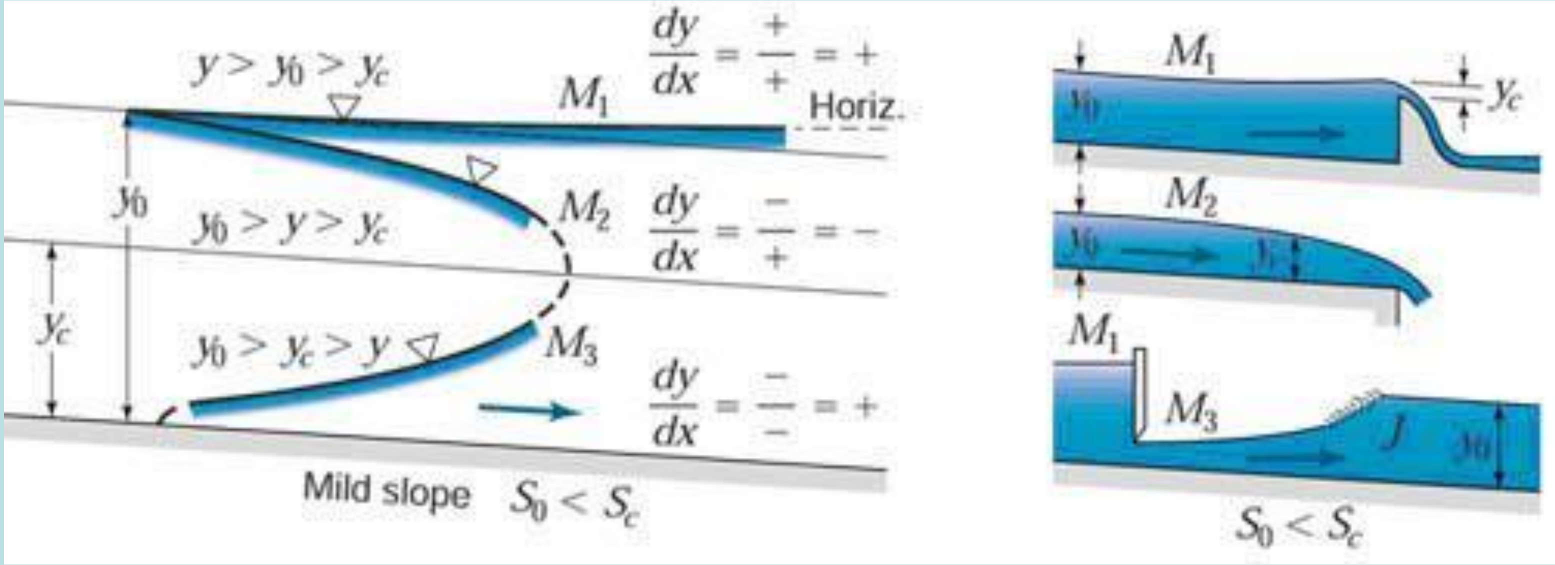


Finnemore and Franzini, Fluid Mechanics

Note: $y = h$, $y_o = h_n$

Water-Surface Profiles

- M2 –
  - Curves goes downstream towards the critical depth
  - Upstream of an increase in bed slope and upstream or a hydraulic drop

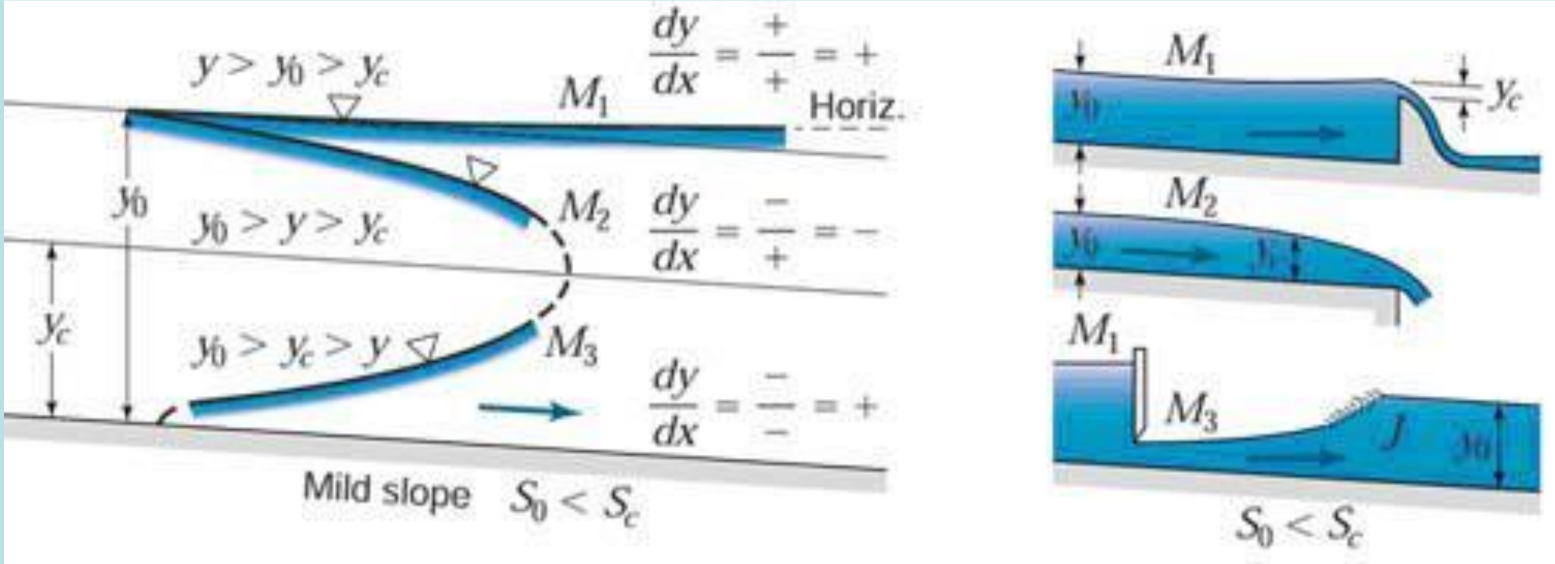


Finnemore and Franzini, Fluid Mechanics

Note: $y = h$, $y_o = h_n$

Water-Surface Profiles

- M3 –
  - Curves goes downstream towards the critical depth where it terminates at a hydraulic jump
  - Occurs when supercritical flow enters a mild channel and after a change in bed slope from steep to mild

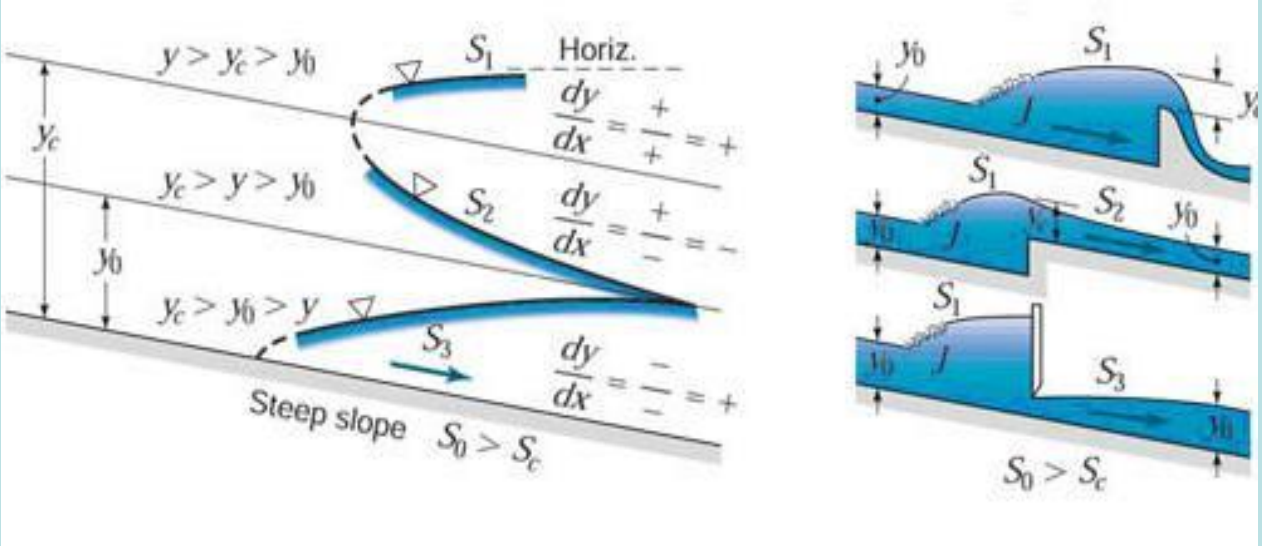


Finnemore and Franzini, Fluid Mechanics

Note: $y = h$, $y_o = h_n$

Water-Surface Profiles

- S1 –
  - Curves begins at critical depth of a hydraulic jump and terminates as a tangent to a horizontal line
  - Upstream of a dam or weir and at a juncture of certain bed slopes

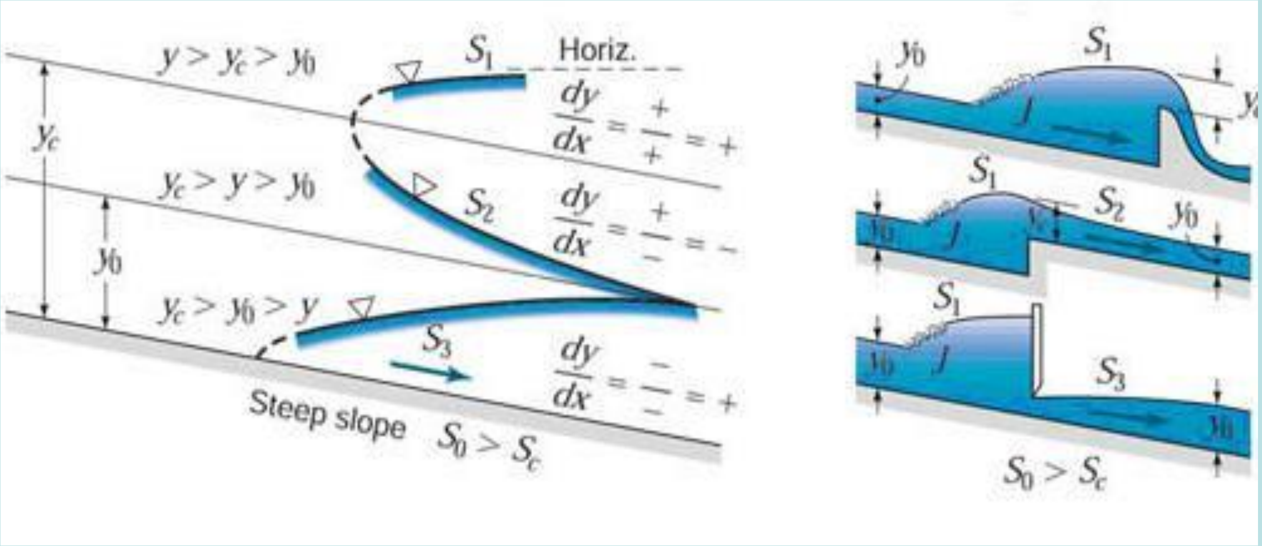


Finnemore and Franzini, Fluid Mechanics

Note: $y = h$, $y_o = h_n$

Water-Surface Profiles

- S2 –
  - Takes place in transition between critical depth and uniform flow (usually very short)
  - Occurs downstream of a sudden increase in bed slope and downstream of an enlargement

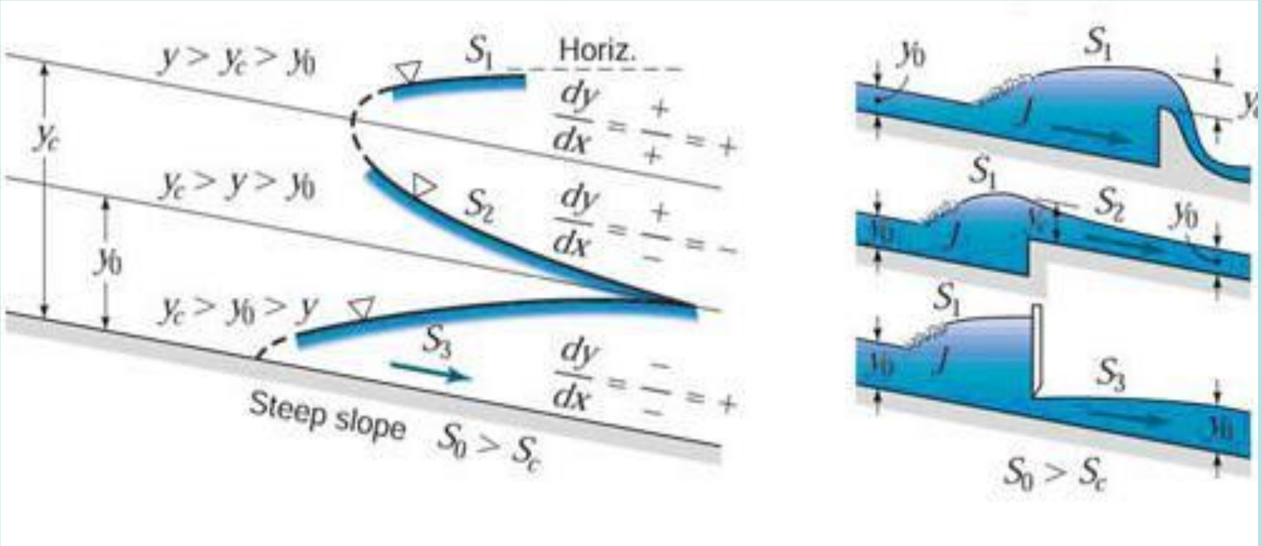


Finnemore and Franzini, Fluid Mechanics

Note: $y = h$, $y_o = h_n$

Water-Surface Profiles

- S3 –
  - Takes place in transition between supercritical flow and uniform flow and approaches a tangent
  - Occurs downstream of a gate, when the flow is below the normal depth and when the bed slope is reduced

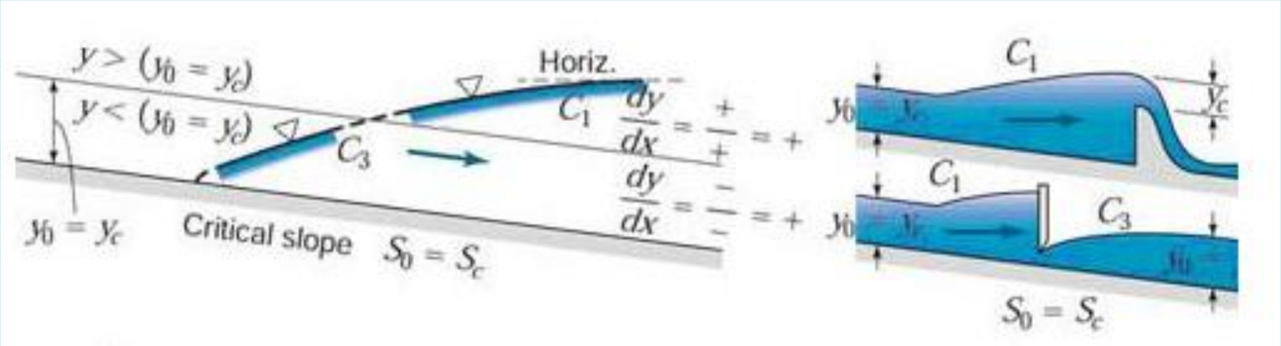


Finnemore and Franzini, Fluid Mechanics

Note: $y = h$, $y_o = h_n$

Water-Surface Profiles

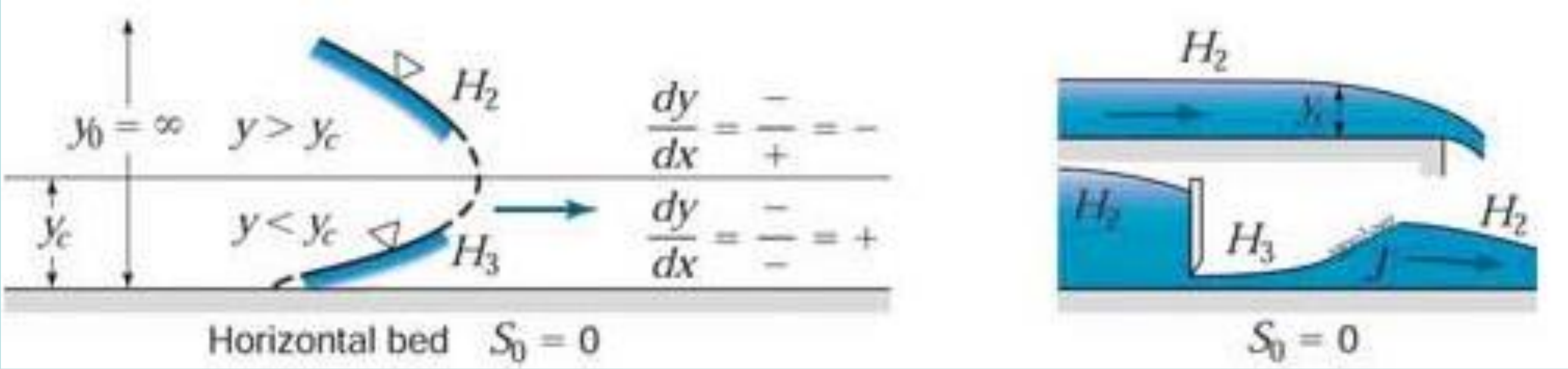
- Critical Slope – C
  - C1 – Curve is horizontal and occurs at a juncture of certain bed slopes and upstream of a dam (weir)
  - C2 - ?????
  - C3 – Curve is horizontal and occurs when the bed slope is reduced to critical slope and downstream of a sluice gate when the flow is below normal depth



$$\frac{dh}{dx} = S_f \frac{1 - \left(\dfrac{q^2}{h^3 C^2 S_f} \right)}{1 - \dfrac{q^2}{g h^3}} = S_f \frac{1 - \left(\dfrac{h_n}{h} \right)^{10/3}}{1 - \left(\dfrac{h_c}{h} \right)^3}$$

Water-Surface Profiles

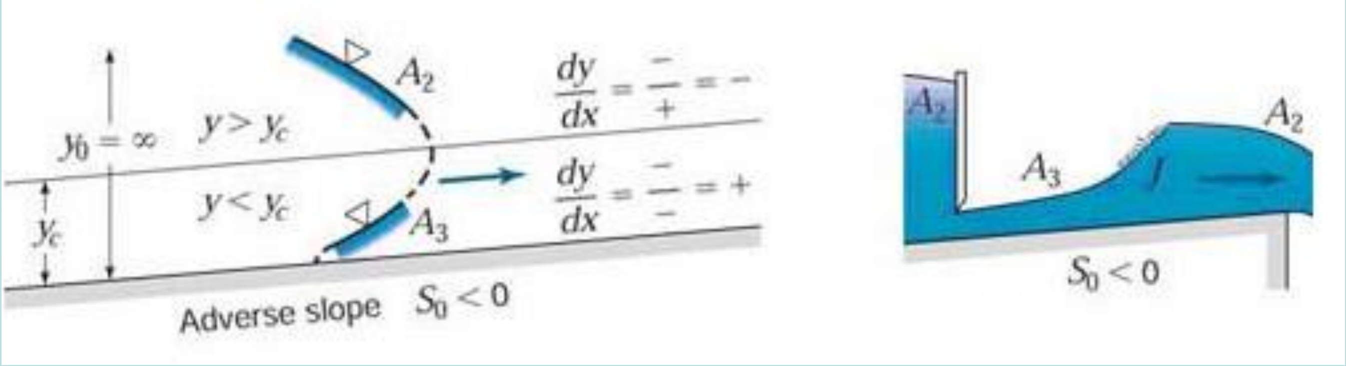
- Horizontal Slope – H (S_f=0, h_n is infinite)
  - H1 – not established because h_n is infinite
  - H2 and H3 correspond to M2 and M3 when the channel bed becomes horizontal
  - H2 is encountered at a hydraulic drop
  - H3 is encountered when supercritical flow enters into a horizontal channel



$$\frac{dh}{dx} = S_f \frac{1-\left(\dfrac{q^2}{h^3C^2S_f}\right)}{1-\dfrac{q^2}{gh^3}} = S_f \frac{1-\left(\dfrac{h_n}{h}\right)^{10/3}}{1-\left(\dfrac{h_c}{h}\right)^3}$$

Water-Surface Profiles

- Adverse Slope – A (S_f<0, h_n does not exist)
  - A1 – not established because h_n does not exist
  - A2 and A3 correspond to H2 and H3
  - A2 is encountered at a juncture of certain bed slopes
  - H3 is encountered when supercritical flow enters into an adverse channel



$$\frac{dh}{dx} = S_f \frac{1 - \left(\dfrac{q^2}{h^3 C^2 S_f}\right)}{1 - \dfrac{q^2}{gh^3}} = S_f \frac{1 - \left(\dfrac{h_n}{h}\right)^{10/3}}{1 - \left(\dfrac{h_c}{h}\right)^3}$$

Example Problem 4.21 Flow profile

A wide, rectangular channel is carrying 10 cfs/ft down a 0.5% slope. The channel has a Manning's n of 0.025. A 2.5-ft barrier in the channel causes flow to pass over the barrier at critical depth. Compute the flow profile upstream from the barrier to a point where the depth is within 10% of normal depth. Figure 4.25 illustrates the physical situation.

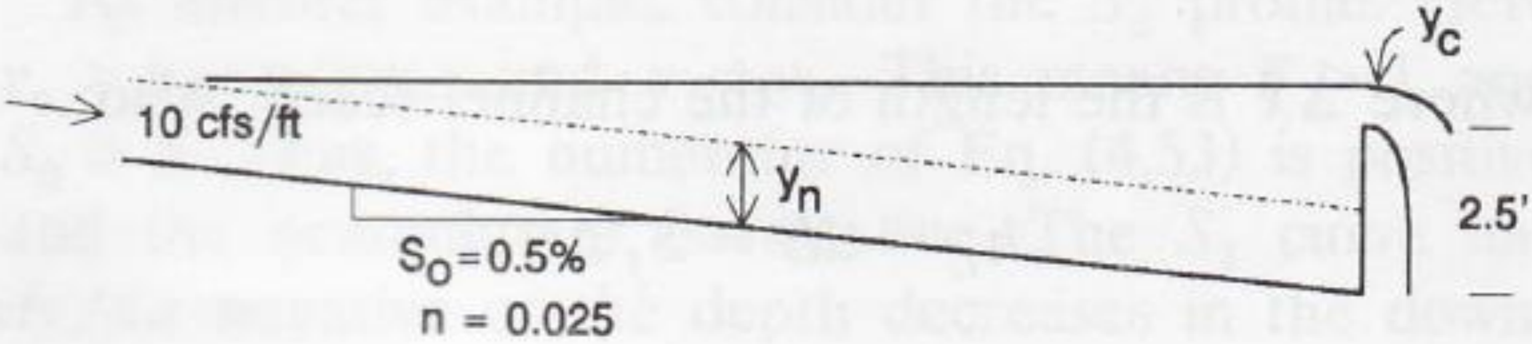


Table 4.15 Profile Calculations for Example Problem 4.21

y (ft)	v[a] (fps)	$V^2/2g$ (ft)	E[b] (ft)	y_m (ft)[c]	S_f[d]	dx (ft)[e]	x[f] (ft)
3.96	2.525	0.099	4.059				0
3.50	2.857	0.127	3.627	3.730	0.00035	–93	–93
3.25	3.077	0.147	3.397	3.375	0.00048	–51	–143
3.00	3.333	0.173	3.173	3.125	0.00063	–51	–195
2.75	3.636	0.205	2.955	2.875	0.00083	–52	–247
2.50	4.000	0.248	2.748	2.625	0.00112	–53	–300
2.25	4.444	0.307	2.557	2.375	0.00157	–56	–356
2.00	5.000	0.388	2.388	2.125	0.00227	–62	–418
1.75	5.714	0.507	2.257	1.875	0.00345	–85	–503
1.70	5.882	0.537	2.237	1.725	0.00456	–45	–548

[a] $v = q/y$.
[b] $E = v^2/2g + y$.
[c] $y_m = (y_1 + y_2)/2$.
[d] $S_f = (qn/1.49 y_m^{1.67})^2$.
[e] $dx = (E_1 - E_2)/(S_f - S_o)$.
[f] $x_2 = x_1 + dx$.

Solution: From Eq. (4.9),

$$y_c = \left(\frac{q^2}{g}\right)^{1/3} = \left(\frac{100}{32.2}\right)^{1/3} = 1.46 \text{ ft}.$$

From Manning's equation (4.23) using $q = vy$,

$$y_n = \left(\frac{qn}{1.49S^{1/2}}\right)^{3/5} = \left(\frac{10(0.025)}{1.49(0.005)^{1/2}}\right)^{3/5} = 1.68.$$

The depth of flow over the brink in Fig. 4.25 is

$$y = 2.5 + y_c = 3.96 \text{ ft}.$$

The solution is carried out by assuming depths and computing Δx. Table 4.15 shows the computations.

Example Problem 4.22 Channel transition 1

A trapezoidal channel with 2 : 1 side slopes and a 4-ft bottom width is flowing at a depth of 1 ft. The channel is concrete and on a slope of 0.1%. If the channel bottom is raised smoothly by 0.1 ft over a short distance, what will be the depth of flow at the exit of the transition?

Solution

$$n = 0.015 \qquad \text{for concrete}$$

$$v_1 = \frac{1.5}{n} R^{2/3} S^{1/2}$$

$$A = bd + zd^2 = 4(1) + 2(1)^2 = 6 \text{ ft}^2$$

$$P = b + 2d\sqrt{z^2 + 1} = 4 + 2(1)\sqrt{2^2 + 1}$$

$$= 8.47 \text{ ft}$$

$$R = A/P = 6/8.47 = 0.71 \text{ ft}$$

$$v_1 = \frac{1.5}{0.015}(0.71)^{2/3}(0.001)^{1/2} = 2.52 \text{ fps}$$

$$\mathbf{F} = \frac{v}{\sqrt{g d_h}}$$

$$d_h = \frac{A}{t} = \frac{6}{b + 2zd} = \frac{6}{4 + 2(2)(1)}$$

$$= 0.75 \text{ ft}$$

$$\mathbf{F} = \frac{2.52}{\sqrt{32.3(0.75)}} = 0.51 \qquad \text{subcritical}$$

$$\frac{v_1^2}{2g} + y_1 = \frac{v_2^2}{2g} + y_2 + \Delta z$$

$$\frac{(2.52)^2}{64.4} + 1.0 = \frac{v_2^2}{64.4} + y_2 + 0.1$$

$$0.9986 = \frac{v_2^2}{64.4} + y_2$$

$$v_2 = \frac{Q}{A} = \frac{v_1 A_1}{A_2} = \frac{2.52(6)}{4y_2 + 2y_2^2}$$

$$0.9986 = \frac{3.54}{\left(4y_2 + 2y_2^2\right)^2} + y_2.$$

Solve by trial

y_2	Right-hand side	
0.90	1.03	
0.75	0.958	
0.84	0.996	OK

$$y_2 = 0.84 \quad \text{ft}$$

Check the Froude number:

$$v_2 = \frac{Q}{A_2} = \frac{2.52(6)}{4(0.84) + 2(0.84)^2} = \frac{15.1}{4.77} = 3.16 \text{ ft}$$

$$d_{\mathrm{h}} = \frac{A}{t} = \frac{4.77}{4 + 2(2)(0.84)} = \frac{4.77}{7.36} = 0.65 \text{ ft}$$

$$\mathbf{F} = \frac{v}{\sqrt{g d_{\mathrm{h}}}} = 0.69 \quad \text{still subcritical.}$$

Solution OK.

Example Problem 4.23 Channel transition 2

A rectangular channel 10 ft wide is carrying 75 cfs. The channel smoothly narrows to 8 ft in width. The flow depth in the 10-ft section is 2.5 ft. What is the depth in the 8 ft section assuming no energy losses?

Solution

$$\frac{v_1^2}{2g} + y_1 = \frac{v_2^2}{2g} + y_2$$

$$v_1 = \frac{Q}{A} = \frac{75}{10 \times 2.5} = 3 \text{ fps}$$

$$v_2 = \frac{75}{8y_2}$$

$$\frac{3^2}{64.4} + 2.5 = \frac{(75/8y_2)^2}{64.4} + y_2 = \frac{1.36}{y_2^2} + y_2.$$

The solution may be found by trial to be $y_2 = 2.40$ ft. Thus the depth in the 8-ft section is 2.40 ft.

Example

The flow in a 15-ft wide rectangular channel that has a constant bottom slope is 1400 cfs. A computation using Manning's equation indicates that the normal depth is 6.0 ft. At a certain section the depth of flow in the channel is 2.8 ft. Does the depth increase, decrease, or remain the same as one proceeds downstream from this section?

Example 4.A – Graf

A trapezoidal channel with bottom width of 7.0 m and side slopes of m = 1.5 conveys Q = 28 m^3/s with a bed slope of 0.0010 (n = 0.025). The channel is terminated by a sudden drop of the channel bed. Determine what type of water-surface profile is to be expected.

Computation of Water Surface Profiles

- Integration of dh/dx equation from earlier…

$$\frac{dh}{dx} = \frac{S_f - \left(\frac{(Q/A)^2}{C^2 R_h} \right)}{1 - \frac{BQ^2}{gA^3}} = S_f \frac{1 - \left(\frac{(Q/A)^2}{C^2 R_h S_f} \right)}{1 - \frac{BQ^2}{gA^3}}$$

  - Method of successive approximations
  - Method of direct integration
  - Method of graphical integration

Computation of Water Surface Profiles

- Method of successive approximation:

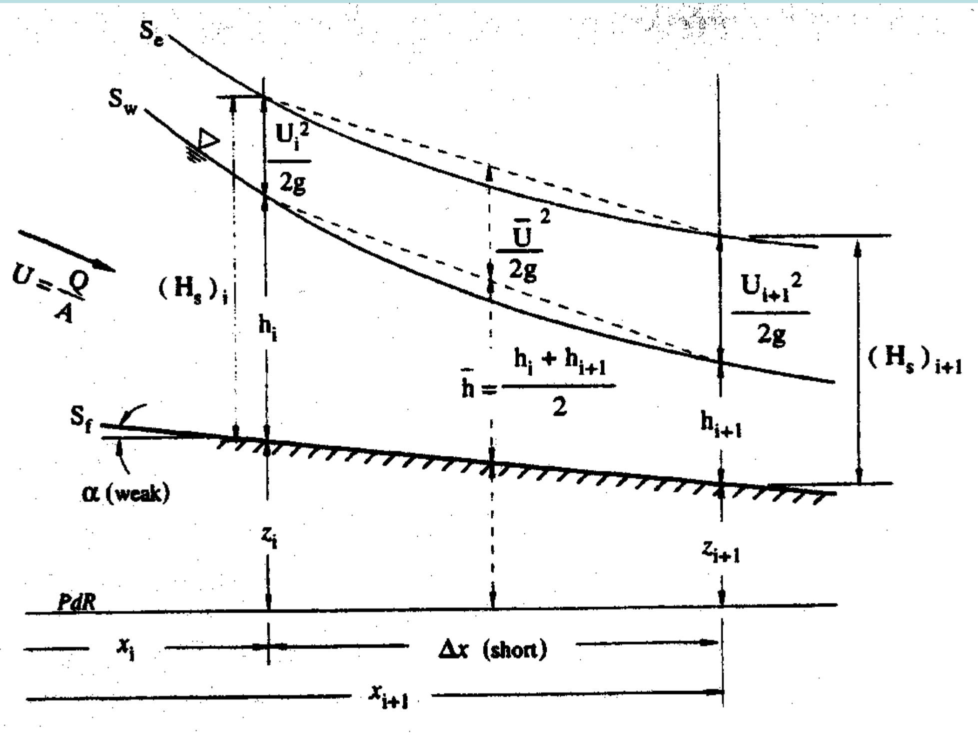


Fig. 4.3 Scheme of a non-uniform flow between two cross sections.

Computation of Water Surface Profiles

- Method of successive approximation:

$$\frac{d}{dx}\left(\frac{(Q/A)^2}{2g}\right)+\frac{dh}{dx}-S_f=-S_e=-\frac{(Q/A)^2}{C^2R_h}$$

$$dh=\left(S_f-\frac{(Q/A)^2}{C^2R_h}\right)dx-\frac{Q^2}{2g}d\left(\frac{1}{A^2}\right)$$

$$h_{i+1}-h_i=\left(S_f-\frac{Q^2}{\overline{C}^2\overline{A}^2\overline{R}_h}\right)(x_{i+1}-x_i)-\frac{Q^2}{2g}\left(\frac{1}{A_{i+1}^{\,2}}-\frac{1}{A_i^{\,2}}\right)$$

Computation of Water Surface Profiles

- Three forms of this equation:

All Channels:

$$(1) \quad h_{i+1} - h_i = \left(S_f - \frac{Q^2}{\overline{C}^2 \overline{A}^2 \overline{R}_h} \right) (x_{i+1} - x_i) - \frac{Q^2}{2g} \left(\frac{1}{A_{i+1}{}^2} - \frac{1}{A_i{}^2} \right)$$

$$(2) \quad (h + z)_{i+1} - (h + z)_i = \left(- \frac{Q^2}{\overline{C}^2 \overline{A}^2 \overline{R}_h} \right) (x_{i+1} - x_i) - \frac{Q^2}{2g} \left(\frac{1}{A_{i+1}{}^2} - \frac{1}{A_i{}^2} \right)$$

Prismatic Channels:

$$(3a) \quad (H_s)_{i+1} - (H_s)_i = \left(S_f - \frac{U^2}{\overline{C}^2 \overline{R}_h} \right) (x_{i+1} - x_i) \leftarrow Chezy$$

$$(3b) \quad (H_s)_{i+1} - (H_s)_i = \left(S_f - \frac{n^2 U^2}{\overline{R}_h{}^{4/3}} \right) (x_{i+1} - x_i) \leftarrow Manning's$$

Computation of Water Surface Profiles

- If you arbitrarily select Δx, solve for the variations in the flow depth, Δh –
  - **Standard Step Method** (Method of Reaches)
- If you arbitrarily select Δh, solve for the variations in the distance, Δx –
  - **Direct Step Method** (Method of Depth Variation)

- Before you use either method:

1. Establish control points (known relationship between flow depth and discharge)
2. Computations proceed upstream for subcritical flow, $Fr<1$ and downstream for supercritical flow, $Fr>1$
3. When you are closer to the critical depth (curvature of water surface more pronounced), you must use smaller steps

Direct Step Method
(Explicit Method)

- Use known flow depth, h_i, at x_i
- Select h_{i+1} (should be very close to h_i)
- Calculate x_{i+1} using finite-difference equations

Standard Step Method (Implicit Method)

- Use known flow depth, h_i, at x_i
- Select x_{i+1}
- Guess value of h_{i+1}
  - Calculate average C, A, R_h, and U with corresponds to the average flow depth
  - Use difference equations given above to calculate h_{i+1}
  - Use calculated h_{i+1} as new guess
  - Continue with successive approximations until the calculated h_{i+1} matches previously calculated value

Method of Direct Integration

- Note that integration is a direct solution method:
  - You can proceed from one section to another whatever the distance between the sections
  - In Methods of Successive Approximations, you must use small distances to avoid computational inaccuracy

Method of Direct Integration

- Chow (1959):

$$\frac{dh}{dx} = S_f \frac{1-\left(\frac{h_n}{h}\right)^N}{1-\left(\frac{h_c}{h}\right)^M}$$

  - N = hydraulic exponent for the conveyance (function of cross-section and type of friction coefficient)
  - M = second hydraulic exponent

Method of Direct Integration

- Chow (1959):

$$N(h) = \frac{2h}{3A}\left(5B - 2R_h \frac{dP}{dh}\right) \rightarrow 2.0 < N < 5.3$$

$$M(h) = \frac{h}{A}\left(3B - \frac{A}{B}\frac{dB}{dh}\right) \rightarrow 3 < M < 4.8$$

$$\frac{h}{h_n} = \eta \rightarrow dh = h_n d\eta$$

$$dx = \frac{1}{S_f}\left[1 - \left(\frac{1}{1-\eta^N}\right) + \left(\frac{h_c}{h}\right)^M \left(\frac{\eta^{N-M}}{1-\eta^N}\right)\right] h_n d\eta$$

Method of Direct Integration

- Back to N and M:
  - Assume trapezoidal channel…

$$P = b + 2h\sqrt{1+m^2} \rightarrow \frac{dP}{dh} = 2\sqrt{1+m^2}$$

$$B = b + 2mh \rightarrow \frac{dB}{dh} = 2m$$

  - See equations 4.26a on page 196 of Graf

Method of Direct Integration

- Chow (1959):

$$\int dx = \int \frac{1}{S_f}\left[1-\left(\frac{1}{1-\eta^N}\right)+\left(\frac{h_c}{h}\right)^M\left(\frac{\eta^{N-M}}{1-\eta^N}\right)\right]h_n d\eta$$

$$x_i - x_{i+1} = \frac{h_n}{S_f}\left\{\left(\eta_i - \eta_{i+1}\right)-\int_0^{\eta}\left(\frac{1}{1-\eta^N}\right)d\eta+\left(\frac{h_c}{h}\right)^M\int_0^{\eta}\left(\frac{\eta^{N-M}}{1-\eta^N}\right)d\eta\right\}$$

$$First\ Integral \rightarrow \int\left(\frac{d\eta}{1-\eta^N}\right) = -\int\left(\frac{d\eta}{\eta^N-1}\right) = \Phi(\eta, N) \rightarrow Table\ 4.1$$

$$Second\ Integral \rightarrow \int_0^{\eta}\left(\frac{\eta^{N-M}d\eta}{1-\eta^N}\right) = \frac{J}{N}\int_0^{\xi}\left(\frac{d\xi}{1-\xi^J}\right) = \frac{J}{N}\Phi(\xi, J) \rightarrow Table\ 4.1$$

$$\xi = \eta^{N/J} \qquad J = \frac{N}{N-M+1}$$

Method of Direct Integration

- Chow (1959):

$$x_i' = \frac{h_n}{S_f}\left\{(\eta_i) - [\Phi(\eta_i, N_i)] + \left(\frac{h_c}{h_n}\right)^{M_i} \frac{J_i}{N_i}[\Phi(\xi_i, J_i)]\right\}$$

$$x_{i+1}' = \frac{h_n}{S_f}\left\{(\eta_{i+1}) - [\Phi(\eta_{i+1}, N_{i+1})] + \left(\frac{h_c}{h_n}\right)^{M_{i+1}} \frac{J_{i+1}}{N_{i+1}}[\Phi(\xi_{i+1}, J_{i+1})]\right\}$$

$$\Delta x = x_i' - x_{i+1}'$$

BAE 6333 – Fluvial Hydraulics
Evaluation of Chow (1959) Integral for Direct-Integration Method
Non-Uniform, Gradually Varied Flow

Integral given by Graf (1998) for Chow (1959) integral is given by…

$$-\int\left(\frac{d\eta}{\eta^N-1}\right)=\Phi(\eta,N)\rightarrow Table\ 4.1 \qquad (1)$$

Maple solves this integral to be:

$$\Phi(\eta,N)=-\int\left(\frac{d\eta}{\eta^N-1}\right)=\frac{\eta}{N}LerchiPhi\left(\eta^N,1,\frac{1}{N}\right) \qquad (2)$$

LerchPhi(z,a,v) is given by the following as long as $|z|$**<1** (Erdelyi, 1953), where the limit of the ratio q/K_s is unity:

$$LerchPhi(z,a,v)=\sum_{n=0}^{\infty}\frac{z^n}{(v+n)^a} \qquad (3)$$

This gives you a solution for the left-side of Table 4.1 in Graf (1998).

NOTE: I have actually used this function before for solving for unsaturated flow beneath a stream due to alluvial well pumping (Fox and Gordji, 2007) to solve the following differential equation:

$$s_w-M-H_w=\int_0^{h_{cl}}\frac{\partial h_c}{1-\frac{q}{K_s}\left(\frac{h_c}{h_e}\right)^\eta}$$

The integral in equation (12) can be expressed as a Lerch Phi function for ease of incorporation into MODFLOW (Erdelyi, 1953):

$$s_w-M-H_w=\frac{h_{cl}}{\eta}LerchPhi\left(\frac{q}{K_s}\left(\frac{h_{cl}}{h_e}\right)^\eta,1,\frac{1}{\eta}\right)$$

Fox, G.A. and L. Gordji. 2007. Consideration for unsaturated flow beneath a streambed during alluvial well depletion. *Journal of Hydrologic Engineering – ASCE* 12(2): 139-145.

Table 4.1 Functions for gradually varied flow.

$$\Phi(\eta, N) = -\int_0^{\eta} \frac{d\eta}{\eta^N - 1}$$

The constant of integration is adjusted for $\Phi(0,N) = 0$ and $\Phi(\infty,N) = 0$

η \ N	2.8	3.0	3.2	3.6	4.0	5.0
0.10	0.100	0.100	0.100	0.100	0.100	0.100
0.20	0.201	0.200	0.200	0.200	0.200	0.200
0.30	0.303	0.302	0.302	0.301	0.300	0.300
0.40	0.408	0.407	0.405	0.403	0.402	0.401
0.44	0.452	0.450	0.448	0.445	0.443	0.441
0.48	0.497	0.494	0.492	0.488	0.485	0.482
0.52	0.544	0.540	0.536	0.531	0.528	0.523
0.56	0.593	0.587	0.583	0.576	0.572	0.565
0.58	0.618	0.612	0.607	0.599	0.594	0.587
0.60	0.644	0.637	0.631	0.623	0.617	0.608
0.61	0.657	0.650	0.644	0.635	0.628	0.619
0.62	0.671	0.663	0.657	0.647	0.640	0.630
0.63	0.684	0.676	0.669	0.659	0.652	0.641
0.64	0.698	0.690	0.683	0.672	0.664	0.652
0.65	0.712	0.703	0.696	0.684	0.676	0.663
0.66	0.727	0.717	0.709	0.697	0.688	0.675
0.67	0.742	0.731	0.723	0.710	0.701	0.686
0.68	0.757	0.746	0.737	0.723	0.713	0.698
0.69	0.772	0.761	0.751	0.737	0.726	0.710
0.70	0.787	0.776	0.766	0.750	0.739	0.722
0.71	0.804	0.791	0.781	0.764	0.752	0.734
0.72	0.820	0.807	0.796	0.779	0.766	0.746
0.73	0.837	0.823	0.811	0.793	0.780	0.759
0.74	0.854	0.840	0.827	0.808	0.794	0.771
0.75	0.872	0.857	0.844	0.823	0.808	0.784
0.76	0.890	0.874	0.861	0.839	0.823	0.798
0.77	0.909	0.892	0.878	0.855	0.838	0.811
0.78	0.929	0.911	0.896	0.872	0.854	0.825
0.79	0.949	0.930	0.914	0.889	0.870	0.839
0.80	0.970	0.950	0.934	0.907	0.887	0.854
0.81	0.992	0.971	0.954	0.925	0.904	0.869
0.82	1.015	0.993	0.974	0.945	0.922	0.885
0.83	1.039	1.016	0.996	0.965	0.940	0.901
0.84	1.064	1.040	1.019	0.985	0.960	0.918
0.85	1.091	1.065	1.043	1.007	0.980	0.935
0.86	1.119	1.092	1.068	1.031	1.002	0.954
0.87	1.149	1.120	1.095	1.055	1.025	0.973
0.88	1.181	1.151	1.124	1.081	1.049	0.994
0.89	1.216	1.183	1.155	1.110	1.075	1.015
0.90	1.253	1.218	1.189	1.140	1.103	1.039
0.91	1.294	1.257	1.225	1.173	1.133	1.064
0.92	1.340	1.300	1.266	1.210	1.166	1.092
0.93	1.391	1.348	1.311	1.251	1.204	1.123
0.94	1.449	1.403	1.363	1.297	1.246	1.158
0.95	1.518	1.467	1.423	1.352	1.296	1.199
0.96	1.601	1.545	1.497	1.417	1.355	1.248
0.97	1.707	1.644	1.590	1.501	1.431	1.310
0.98	1.855	1.783	1.720	1.617	1.536	1.395
0.99	2.106	2.017	1.940	1.814	1.714	1.537
0.995	2.355	2.250	2.159	2.008	1.889	1.678

η \ N	2.8	3.0	3.2	3.6	4.0	5.0
1.005	1.818	1.649	1.506	1.279	1.107	0.817
1.01	1.572	1.419	1.291	1.089	0.936	0.681
1.02	1.327	1.191	1.078	0.900	0.766	0.546
1.03	1.186	1.060	0.955	0.790	0.668	0.469
1.04	1.086	0.967	0.868	0.714	0.600	0.415
1.05	1.010	0.896	0.802	0.656	0.548	0.374
1.06	0.948	0.838	0.748	0.608	0.506	0.342
1.07	0.896	0.790	0.703	0.569	0.471	0.315
1.08	0.851	0.749	0.665	0.535	0.441	0.292
1.09	0.812	0.713	0.631	0.506	0.415	0.272
1.10	0.777	0.681	0.601	0.480	0.392	0.254
1.11	0.746	0.652	0.575	0.457	0.372	0.239
1.12	0.718	0.626	0.551	0.436	0.354	0.225
1.13	0.692	0.602	0.529	0.417	0.337	0.212
1.14	0.669	0.581	0.509	0.400	0.322	0.201
1.15	0.647	0.561	0.490	0.384	0.308	0.191
1.16	0.627	0.542	0.473	0.369	0.295	0.181
1.17	0.608	0.525	0.458	0.356	0.283	0.173
1.18	0.591	0.509	0.443	0.343	0.272	0.165
1.19	0.574	0.494	0.429	0.331	0.262	0.157
1.20	0.559	0.480	0.416	0.320	0.252	0.150
1.22	0.531	0.454	0.392	0.299	0.235	0.138
1.24	0.505	0.431	0.371	0.281	0.219	0.127
1.26	0.482	0.410	0.351	0.265	0.205	0.117
1.28	0.461	0.391	0.334	0.250	0.193	0.108
1.30	0.442	0.373	0.318	0.237	0.181	0.100
1.32	0.424	0.357	0.304	0.225	0.171	0.093
1.34	0.408	0.342	0.290	0.214	0.162	0.087
1.36	0.393	0.329	0.278	0.204	0.153	0.081
1.38	0.378	0.316	0.266	0.194	0.145	0.076
1.40	0.365	0.304	0.256	0.185	0.138	0.071
1.42	0.353	0.293	0.246	0.177	0.131	0.067
1.44	0.341	0.282	0.236	0.169	0.125	0.063
1.46	0.330	0.273	0.227	0.162	0.119	0.059
1.48	0.320	0.263	0.219	0.156	0.113	0.056
1.50	0.310	0.255	0.211	0.149	0.108	0.053
1.60	0.269	0.218	0.179	0.123	0.087	0.040
1.70	0.236	0.189	0.153	0.103	0.072	0.031
1.80	0.209	0.166	0.133	0.088	0.060	0.024
1.90	0.188	0.147	0.117	0.076	0.050	0.020
2.00	0.169	0.132	0.104	0.066	0.043	0.016
2.20	0.141	0.107	0.083	0.051	0.032	0.011
2.40	0.119	0.089	0.068	0.040	0.024	0.008
2.60	0.102	0.076	0.057	0.033	0.019	0.005
2.80	0.089	0.065	0.048	0.027	0.015	0.004
3.00	0.078	0.056	0.041	0.022	0.012	0.003
3.50	0.059	0.041	0.029	0.015	0.008	0.002
4.00	0.046	0.031	0.022	0.010	0.005	0.001
5.00	0.031	0.020	0.013	0.006	0.003	0.000
10.00	0.009	0.005	0.003	0.001	0.000	0.000

Example 4.A – Graf

A trapezoidal channel with bottom width of 7.0 m and side slopes of m = 1.5 conveys Q = 28 m^3/s with a bed slope of 0.0010 (n = 0.025). The channel is terminated by a sudden drop of the channel bed. Calculate and plot the profile upstream from the drop using:

(i) method of direct integration (Chow)

(ii) direct step method

(iii) standard step method

Method of Direct Integration - Chow (1959)

b	7 m
m	1.5
n	0.025
S_f	0.001
Q	28 m^3/s
h_n	1.866 m
h_c	1.085 m

	h	h/b	M	N	J	η	ζ	$\Phi(\eta,N)$	$\Phi(\zeta,J)$	x'	x
For Control Point:	1.085	0.155	3.249	3.484	2.820	0.581	0.512	0.60	0.53	98.35	0.00
Select new h	1.200	0.171	3.274	3.506	2.846	0.643	0.581	0.68	0.62	93.22	5.13
Select new h	1.800	0.257	3.400	3.620	2.967	0.965	0.957	1.46	1.54	-551.55	649.90
Select new h	1.864	0.266	3.412	3.632	2.978	0.999	0.999	2.45	2.93	-2002.84	**2101.19**

What happens if I go to 1.866 m (i.e., h = h $_c$)?

	h	h/b	M	N	J	η	ζ	$\Phi(\eta,N)$	$\Phi(\zeta,J)$	x'	x
Select new h	1.866	0.267	3.413	3.632	2.978	1.000	1.000	Infinity	Infinity	N/A	N/A

Direct Step Method

b	7 m
m	1.5
n	0.025
S_f	0.001
Q	28 m^3/s
h_n	1.866 m
h_c	1.085 m

	h	B	P	A	Rh	U	$U^2/2g$	Hs	ΔHs	Se	Avg. Se	Avg. Se-Sf	Δx	x
For Control Point:	1.085	10.255	10.912	9.361	0.858	2.991	0.456	1.541	----	0.007	----	----		
Select new h	1.100	10.300	10.966	9.515	0.868	2.943	0.441	1.541	0.000	0.007	0.007	0.006	0.060	0.060
Select new h	1.150	10.450	11.146	10.034	0.900	2.791	0.397	1.547	0.006	0.006	0.006	0.005	1.093	1.153
Select new h	1.200	10.600	11.327	10.560	0.932	2.652	0.358	1.558	0.011	0.005	0.005	0.004	2.713	3.866
Select new h	1.300	10.900	11.687	11.635	0.996	2.407	0.295	1.595	0.037	0.004	0.004	0.003	11.396	15.262
Select new h	1.400	11.200	12.048	12.740	1.057	2.198	0.246	1.646	0.051	0.003	0.003	0.002	22.962	38.224
Select new h	1.500	11.500	12.408	13.875	1.118	2.018	0.208	1.708	0.061	0.002	0.002	0.001	40.978	79.202
Select new h	1.600	11.800	12.769	15.040	1.178	1.862	0.177	1.777	0.069	0.002	0.002	0.001	71.432	150.634
Select new h	1.700	12.100	13.129	16.235	1.237	1.725	0.152	1.852	0.075	0.001	0.002	0.001	131.246	281.880
Select new h	1.800	12.400	13.490	17.460	1.294	1.604	0.131	1.931	0.079	0.001	0.001	0.000	294.203	576.083
Select new h	1.866	12.598	13.728	18.285	1.332	1.531	0.120	1.986	0.054	0.001	0.001	0.000	779.917	**1356.000**

What happens if I don't take small steps?

	h	B	P	A	Rh	U	$U^2/2g$	Hs	ΔHs	Se	Avg. Se	Avg. Se-Sf	Δx	x
For Control Point:	1.085	10.255	10.912	9.361	0.858	2.991	0.456	1.541	----	0.007	----	----		
Select new h	1.200	10.600	11.327	10.560	0.932	2.652	0.358	1.558	0.017	0.005	0.006	0.005	3.575	3.575
Select new h	1.866	12.598	13.728	18.285	1.332	1.531	0.120	1.986	0.427	0.001	0.003	0.002	223.389	**226.964**