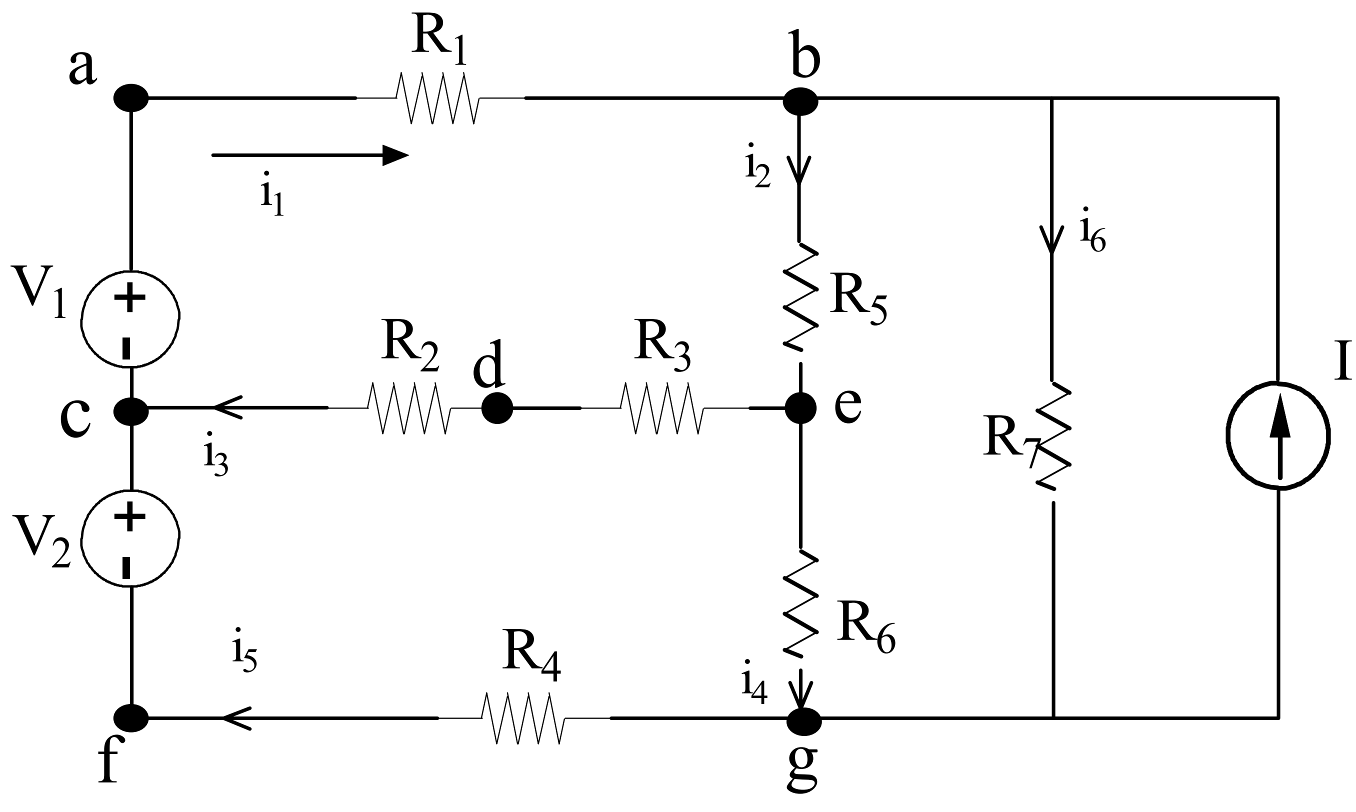
Chapter (3)

Nodal and loop analysis

Consider the following circuit:



Definitions:

Node :

A point where two or more circuit elements join

Ex. a,b,c,d,e,f,g

Essential Node:

A node where three or more circuit element join

Ex. b,c,e,g

Path:

A trace of adjoining elements with no elements included more than once

1. V_1-R_1-R_5-R_6
2. R_5-R_6-R_4-V_2 ,etc

Branch:

A path that connects any two nodes.

Ex. R_1 , V_1 , R_1-R_5 , etc

Essential Branch:

A path that connects two essential nodes without passing through an essential node.

Ex. V_1-R_1 , R_5 , R_2-R_3 , V_2-R_4 , …

Loop :

A path whose last node is the same as its starting node

Ex. (1) V_1-R_1-R_5-R_3-R_2

(2) V_1-R_1-R_5-R_6-R_4-V_2

Mesh :

A loop that doesn't enclose any other loops.

Ex. V_1-R_1-R_5-R_3-R_2

Ex. V_2-R_2-R_3-R_6-R_4 , …

- In chapter (2) we studied circuits containing a single loop or a single node-pair
- Such circuits can be solved easily by one algebraic equation.
- Here , we will study circuit containing multiple node and multiple loops
- Hence we will introduce (2) analysis techniques :

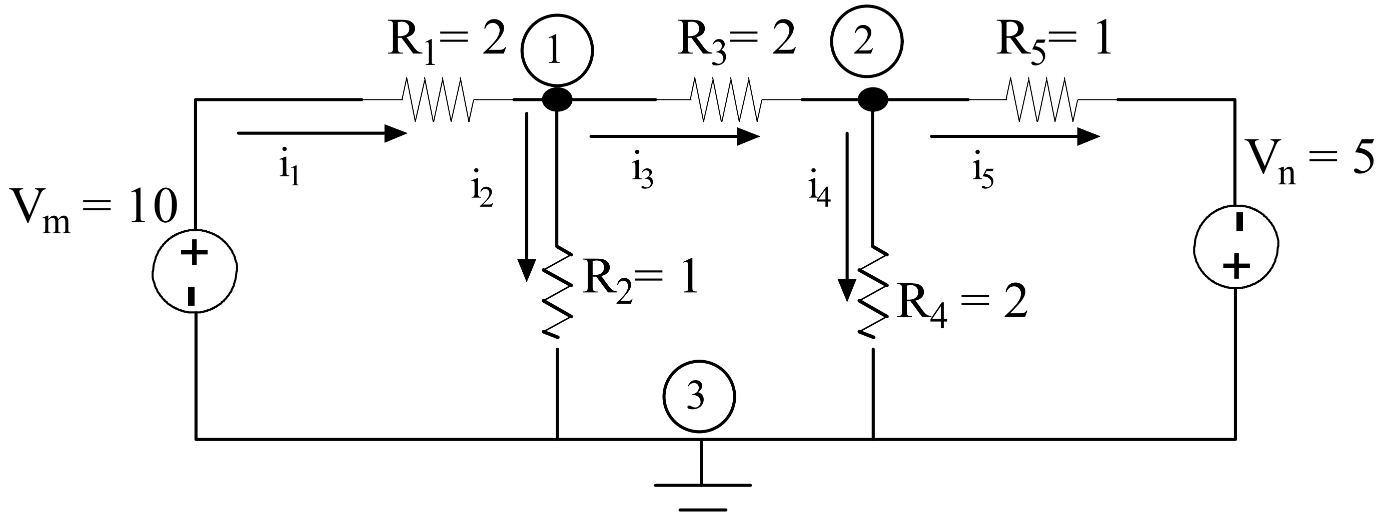
1. **Nodal analysis**
2. **Loop analysis**

(1) Nodal Analysis :

Nodal analysis : is a technique in which KCL is used to determine the nodes' voltages at all essential nodes with respect to the reference node.

•Here , node voltage is defined as the voltage of a given node with respect to a reference node

Example:



Essential nodes : 1,2,3

Consider (3) to be the reference node (ground)

At (1) apply KCL:

$$i_1 - i_2 - i_3 = 0$$

$$\frac{V_m - V_1}{R_1} - \frac{V_1}{R_2} - \frac{V_1 - V_2}{R_3} = 0$$

$$\frac{10 - V_1}{2} - \frac{V_1}{1} - \frac{V_1 - V_2}{2} = 0$$

$$5 - \frac{V_1}{2} - V_1 - \frac{V_1}{2} + \frac{V_2}{2} = 0$$

$$5 - 2V_1 + \frac{V_2}{2} = 0$$

$$4V_1 - V_2 = 10 \ldots\ldots \quad (1)$$

At (2), Apply KCL:

$$i_3 - i_4 - i_5 = 0$$

$$\frac{V_1 - V_2}{R_3} - \frac{V_2}{R_4} - \frac{V_2 - V_n}{R_5} = 0$$

$$\frac{(V_1 - V_2)}{2} - \frac{V_2}{2} - \frac{(V_2 + 5)}{1} = 0$$

$$\frac{V_1}{2} - 2V_2 - 5 = 0$$

$$V_1 - 4V_2 = 10 \;\ldots\ldots \quad (2)$$

$$\begin{bmatrix} 4 & -1 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

$$AV = B \quad \Rightarrow \quad V = A^{-1}B \quad \Rightarrow \quad V_1 = 2$$

$$V_2 = -2$$

Note :

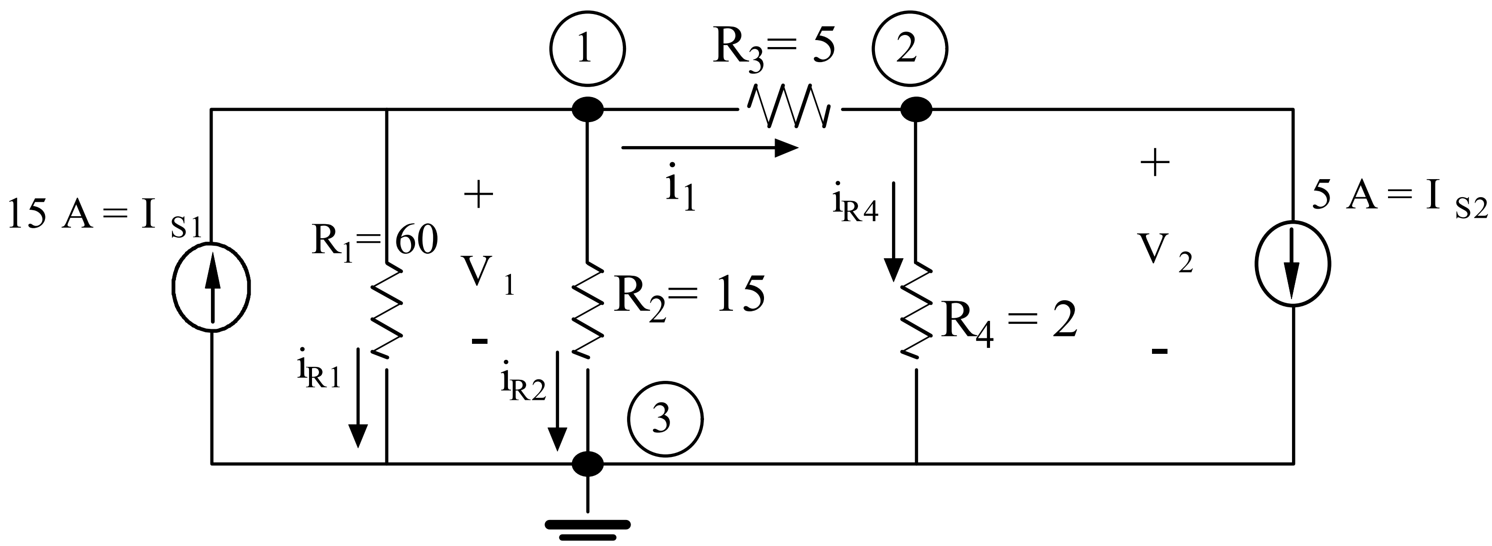
Number of equations = N-1

Where :

N is the number of essential nodes

Example :

Circuit with only independent current source



Find V_1,V_2 and i

# of essential node = N=3
Select (3) as ground (reference node)
# of KCL equations = N-1 = 2

At node (1) apply KCL

$$I_{s1} - i_{R1} - i_{R2} - i_1 = 0$$

$$15 - \frac{V_1}{R_1} - \frac{V_1}{R_2} - \frac{V_1 - V_2}{R_3} = 0$$

$$15 - \frac{V_1}{60} - \frac{V_1}{15} - \frac{V_1}{5} + \frac{V_2}{5} = 0$$

$$\left(\frac{1}{5}+\frac{1}{15}+\frac{1}{60}\right)V_1-\frac{1}{5}V_2=15$$

$$\frac{12+4+1}{60}V_1-\frac{1}{5}V_2=15$$

$$\frac{17}{60}V_1-\frac{1}{5}V_2=15 \qquad \ldots\ldots(1)$$

At node (2)

$$i_1-i_{R4}-I_{s2}=0$$

$$\frac{V_1-V_2}{R_3}-\frac{V_2}{R_4}-5=0$$

$$\frac{V_1-V_2}{5}-\frac{V_2}{2}-5=0$$

$$2V_1-2V_2-5V_2-50=0$$

$$2V_1-7V_2=50 \qquad \ldots\ldots(2)$$

$$\begin{bmatrix} \frac{17}{60} & \frac{-1}{5} \\ 2 & -7 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 15 \\ 50 \end{bmatrix}$$

$$V_1 = 60\,V$$

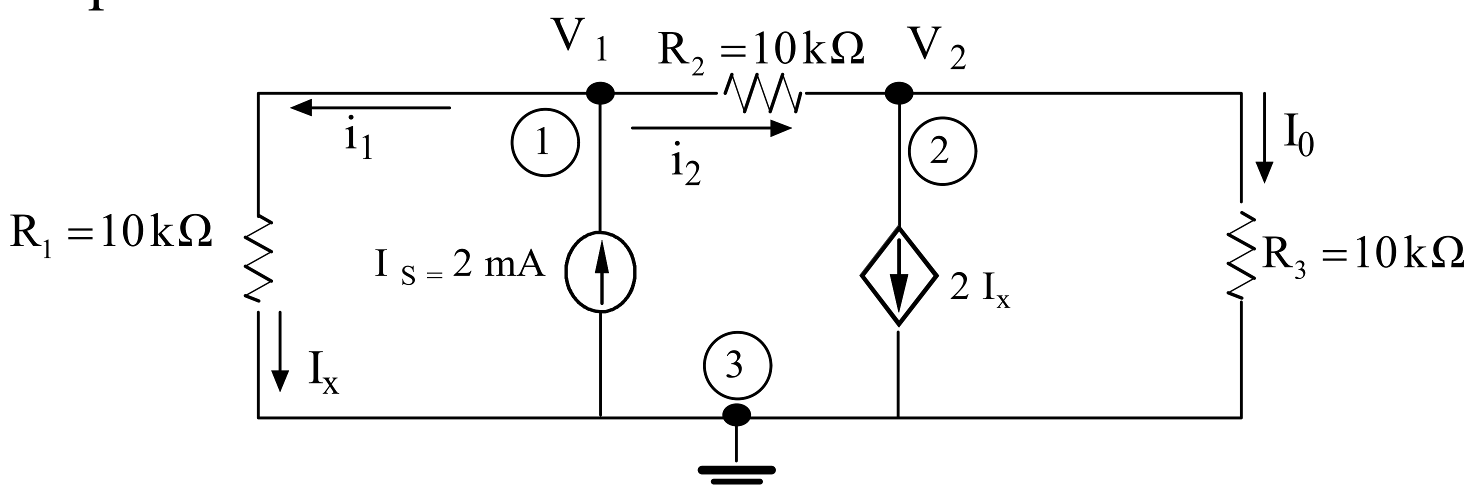
$$V_2 = 10\,V$$

$$i_1 = \frac{V_1 - V_2}{R} = \frac{60 - 10}{5} = 10\,A$$

<u>**Example :**</u>

Circuit with dependent current source

Find I_0



# of essential nodes = N=3
Choose (3) to be reference node (ground)
We have N-1 = 2 KCL equation
at node(1) and(2)

At node (1) , apply KCL

$$I_s - i_1 - i_2 = 0$$

$$2\,mA - \frac{V_1}{R_1} - \frac{V_1 - V_2}{R_2} = 0$$

$$2\,mA = \frac{V_1}{10k} + \frac{V_1 - V_2}{10k} = \frac{V_1}{5k} - \frac{V_2}{10k}$$

$$0.2\ V_1 - 0.1\ V_2 = 2\ A \quad \ldots\ldots(1)$$

At node (2) apply KCL,

$$i_2 - 2I_x - I_0 = 0$$

$$i_2 - 2I_x - I_0 = 0$$

$$\frac{V_1 - V_2}{R_2} - 2I_x - I_0 = 0$$

where $\quad I_x = i_1 = \dfrac{V_1}{R_1}$

$$I_0 = \frac{V_2}{R_3}$$

$$\Rightarrow \quad \frac{V_1 - V_2}{10k} - 2\frac{V_1}{10k} - \frac{V_2}{10k} = 0$$

$$1\ V_1 + 2\ V_2 = 0 \quad \ldots\ldots(2)$$

$$\begin{bmatrix} 0.2 & -0.1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} V1 \\ V2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$V_1 = 8\,V$$

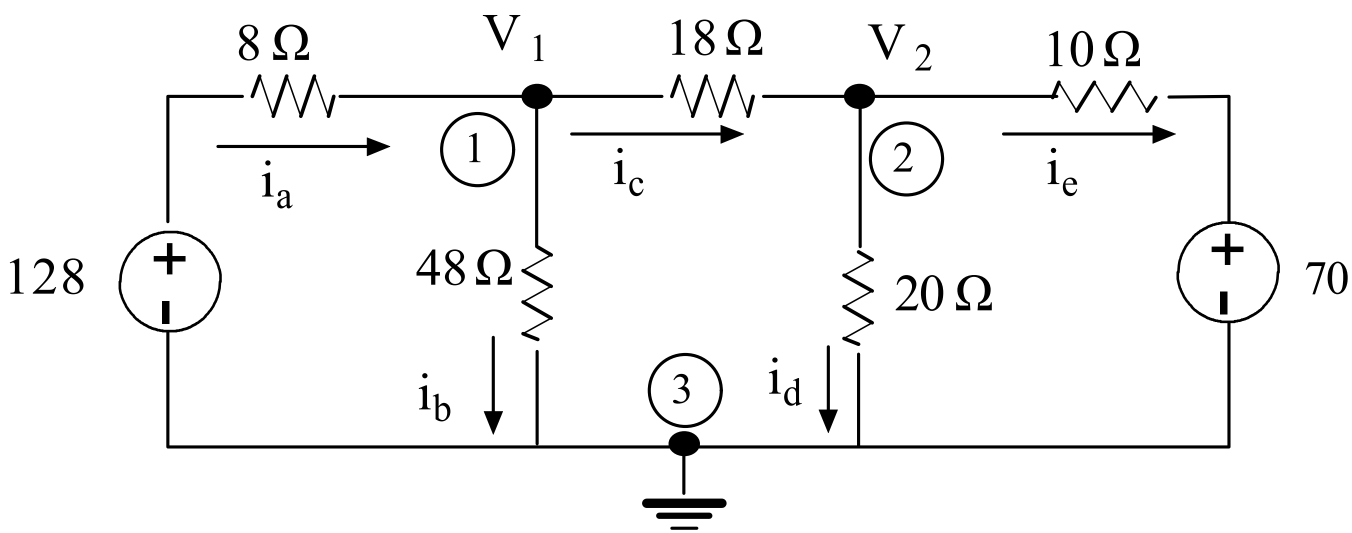
$$V_2 = -4\,V$$

$$\Rightarrow \quad I_0 = \frac{V_2}{R_3} = \frac{-4}{10\,k} = -0.4\,mA$$

Example:

(circuit with independent voltage source)

Find i_a, i_b, i_c, i_d, i_e



N=3 ⟹ N-1 = 2

Choose (3) to be ground

KCL at (1)

$$\frac{128-V_1}{8}-\frac{V_1}{48}-\frac{V_1-V_2}{18}=0$$

$$16-\frac{V_1}{8}-\frac{V_1}{48}-\frac{V_1}{18}+\frac{V_2}{18}=0$$

$$0.201388\,V_1-\frac{1}{18}V=16 \quad \ldots\ldots(1)$$

KCL at (2)

$$\frac{V_1-V_2}{18}-\frac{V_2}{20}-\frac{V_2-70}{10}=0$$

$$\frac{1}{18}V_1-0.20555\,V_2=-7 \quad \ldots\ldots(2)$$

$$\begin{bmatrix} 0.201338 & \dfrac{-1}{18} \\ \dfrac{1}{18} & -0.2055 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 16 \\ -7 \end{bmatrix}$$

$$V_1 = 96\,V$$

$$V_2 = 60\,V$$

$$\therefore \quad i_a = \frac{128 - V_1}{8} = \frac{128 - 96}{8} = 4\,A$$

$$i_b = \frac{V_1}{48} = \frac{96}{48} = 2\,A$$

$$i_c = \frac{V_1 - V_2}{18} = \frac{96 - 60}{18} = 2\,A$$

$$i_d = \frac{V_2}{20} = \frac{60}{20} = 4\,A$$

$$i_e = \frac{V_2 - 70}{10} = -1\,A$$

$$\therefore \quad i_a = \frac{128 - V_1}{8} = \frac{128 - 96}{8} = 4\,A$$

$$i_b = \frac{V_1}{48} = \frac{96}{48} = 2\,A$$

$$i_c = \frac{V_1 - V_2}{18} = \frac{96 - 60}{18} = 2\,A$$

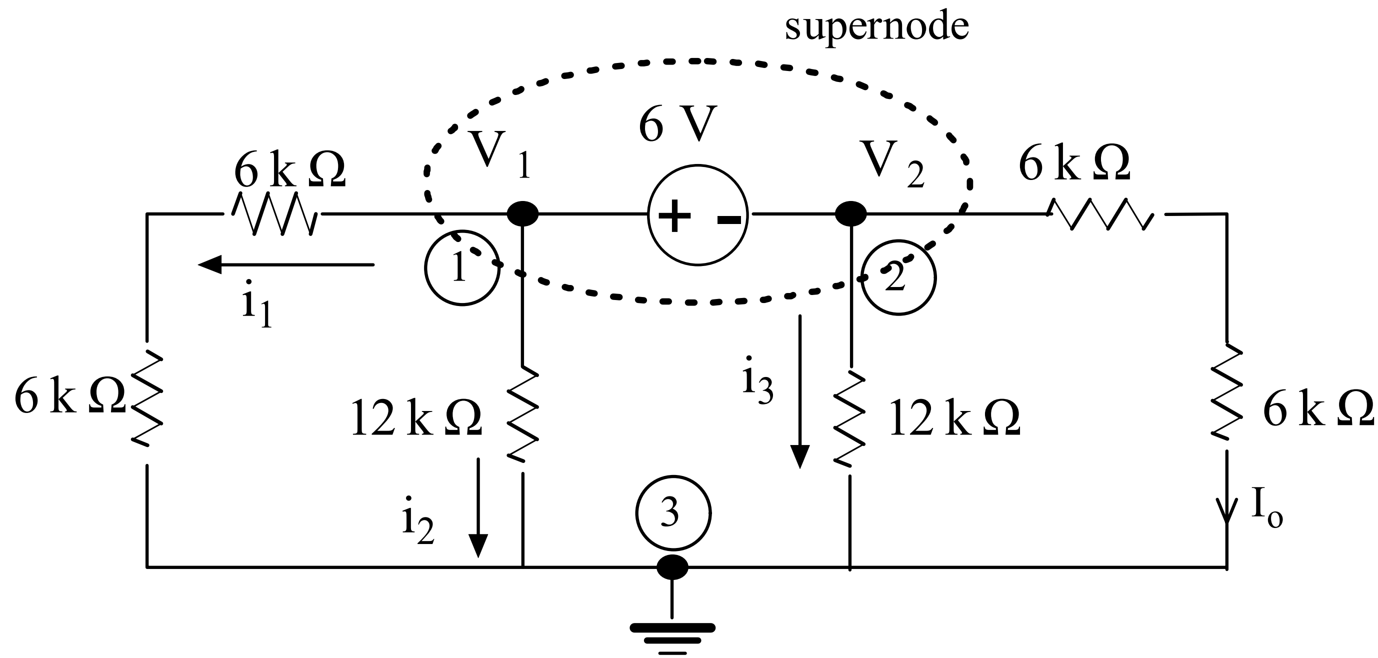
$$i_d = \frac{V_2}{20} = \frac{60}{20} = 4\,A$$

$$i_e = \frac{V_2 - 70}{10} = -1\,A$$

Special case:

What if a branch between two essential non-reference node contain a voltage source ?

This case is called “**super node" case**.



Find I_0

- # of essential nodes = N = 3
- "Super node: is the voltage source and the two connecting nodes
- # of equations = N-1-1 = 3-1-1=1

Reference node Super node

But we need (2) equations to find the two unknowns V_1 and V_2

⇒ There is an equation that describe the super node.

Apply KCL at the super node :

$$i_1 + i_2 + i_3 + I_0 = 0$$

$$\frac{V_1}{12\,k} + \frac{V_1}{12\,k} + \frac{V_2}{12k} + \frac{V_2}{12k} = 0$$

$$\frac{V_1}{6k} + \frac{V_2}{6k} = 0 \quad \ldots\ldots(1)$$

The super node is described by:

$$V_2 + 6 = V_1 \quad \ldots\ldots(2)$$

$$\begin{bmatrix} \frac{1}{6k} & \frac{1}{6k} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} V1 \\ V2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

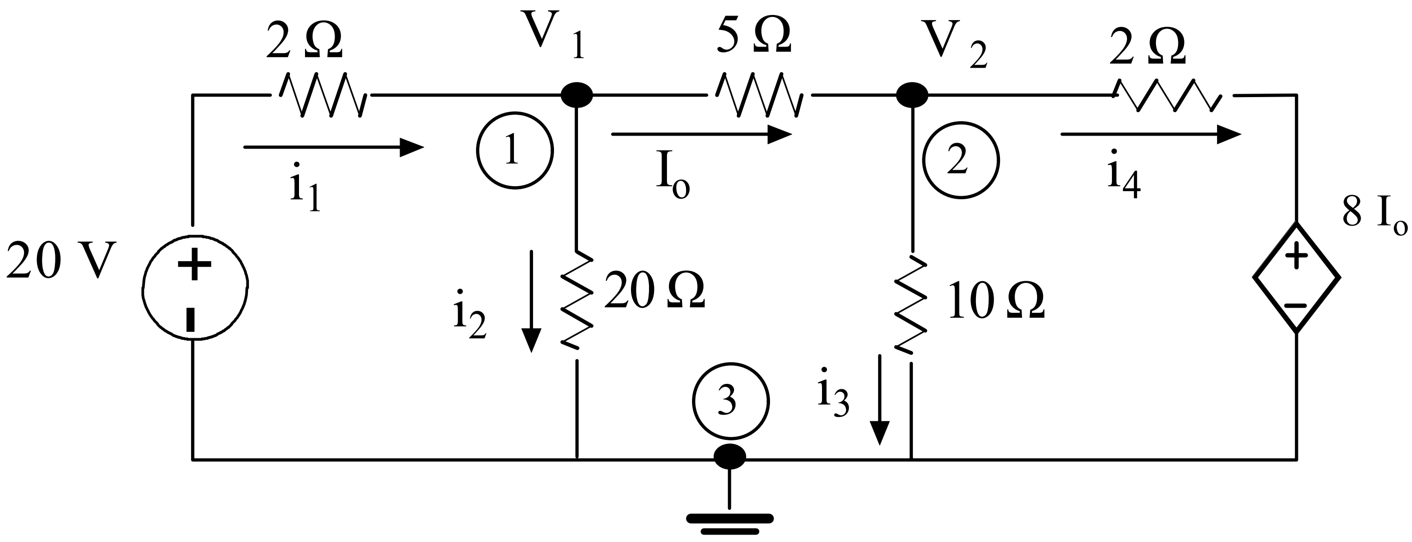
$$V_1 = 3\,V$$

$$V_2 = -3\,V$$

$$\Rightarrow \quad I_0 = \frac{V2}{12k} = \frac{-3}{12k} = -0.25\,mA$$

Example:

Circuits with dependent voltage sources



Find I_0?

$N = 3$ ⇨ $N-1 = 2$ equations

KCL at node (1):

$$i_1 = i_2 + I_0$$

$$\frac{20 - V_1}{2} = \frac{V_1}{20} + \frac{V_1 - V_2}{5}$$

$$10 - \frac{V_1}{2} = \frac{V_1}{20} + \frac{V_1}{5} - \frac{V_2}{5}$$

$$\left(\frac{1}{2} + \frac{1}{20} + \frac{1}{5}\right)V_1 - \frac{1}{5}V_2 = 10$$

$$\frac{3}{4}V_1 - \frac{1}{5}V_2 = 10 \quad \ldots\ldots(1)$$

$$I_0 = i_3 + i_4$$

$$\frac{V_1 - V_2}{5} = \frac{V_2}{10} + \frac{V_2 - 8I_0}{2}$$

$$\frac{V_1}{5} - \frac{V_2}{5} = \frac{V_2}{10} + \frac{V_2}{2} - 4\left(\frac{V_1 - V_2}{5}\right)$$

$$5\left(\frac{V_1 - V_2}{5}\right) = \frac{V_2}{10} + \frac{V_2}{2}$$

$$V_1 - V_2 = \frac{V_2}{10} + \frac{V_2}{2}$$

$$V_1 - \left(1 + \frac{1}{10} + \frac{1}{2}\right)V_2 = 0$$

$$V_1 - \frac{8}{5}V_2 = 0 \quad \ldots\ldots(2)$$

$$\begin{bmatrix} \frac{3}{4} & \frac{-1}{5} \\ 1 & \frac{-8}{5} \end{bmatrix} \begin{bmatrix} V1 \\ V2 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

$$V_1 = 16\,V$$

$$V_2 = 10\,V$$

$$I_0 = \frac{V_1 - V_2}{5} = \frac{6}{5} = 1.2\,A$$

Loop Analysis (Mesh)

Mesh analysis : It is a technique in which KVL is used to determine the current in all meshes

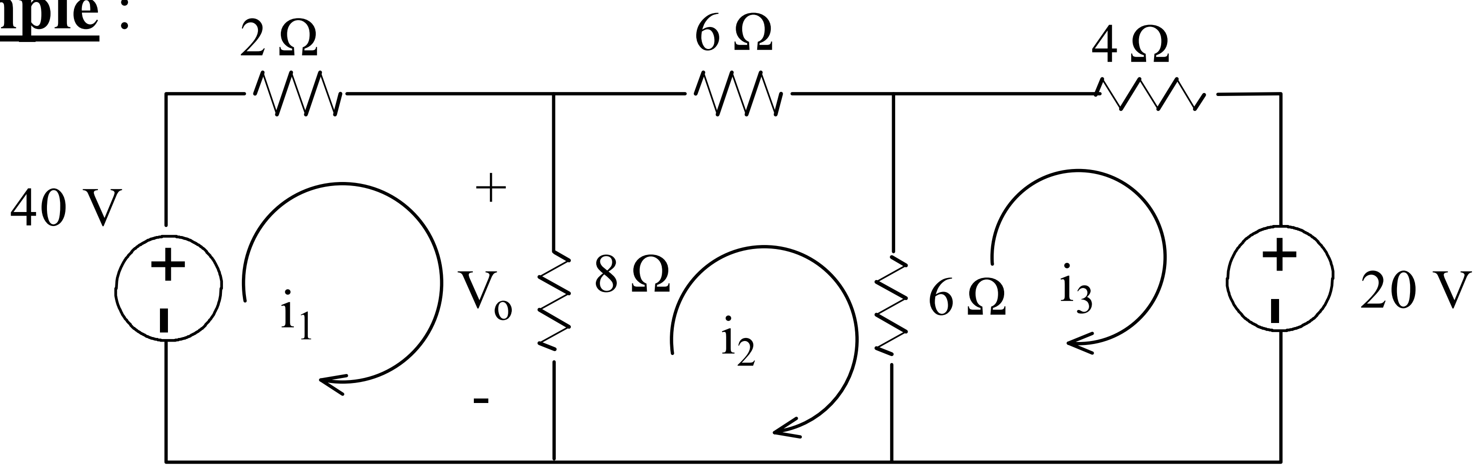
\# of equations needed = # of meshes = b_e-(n_e-1)

Where :

b_e : # of essential branches

n_e: # of essential nodes

Example :



Find V_0?

We have (3) meshes

KVL left loop :

$$-40+2i_1+8(i_1-i_2)=0$$

$$10i_1-8i_2+0i_3=40 \quad \ldots\ldots(1)$$

KVL middle loop:

$$8(i_2 - i_1) + 6i_2 + 6(i_2 - i_3) = 0$$

$$-8i_1 + 20i_2 - 6i_3 = 0$$

KVL right loop :

$$20 + 6(i_3 - i_2) + 4i_3 = 0$$

$$0i_1 - 6i_2 + 10i_3 = -20$$

$$\begin{bmatrix} 10 & -8 & 0 \\ -8 & 20 & -6 \\ 0 & -6 & 10 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 40 \\ 0 \\ -20 \end{bmatrix}$$

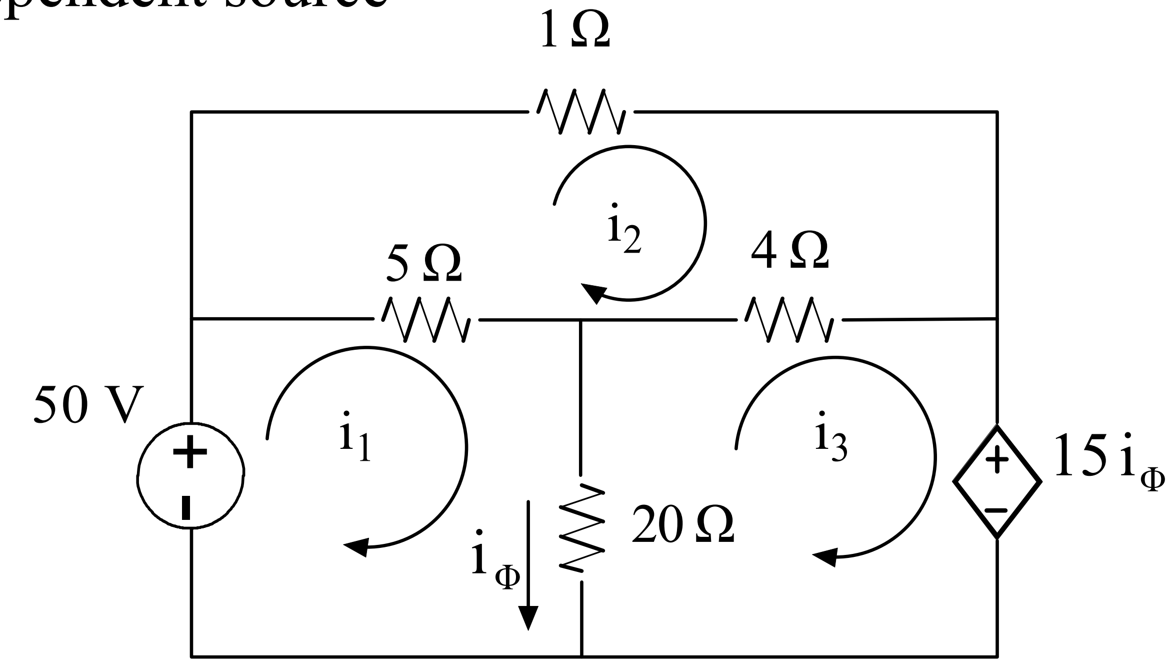
$$i_1 = 5.6\,\text{A} \quad , i_2 = 2\,\text{A} \quad , i_3 = -0.8\,\text{A}$$

$$V_0 = 8(i_1 - i_2) = 8(5.6 - 2) = 8(3.6)$$

$$V_0 = 28.8\,\text{V}$$

Example:

Mesh with dependent source



# of meshes = 3   so   3 equations are needed

KVL around mesh 1:

$$-50+5(i_1-i)+20(i_1-i_3)$$

$$25i_1-5i_2-20i_3=50 \quad \ldots\ldots(1)$$

KVL around mesh 2: $1i_2 + 4(i_2 - i_3) + 5(i_2 - i_1) = 0$

$$-5i_1 + 10i_2 - 4i_3 = 0 \quad \ldots\ldots(2)$$

KVL around mesh 3: $15i_\Phi + 20(i_3 - i_1) + 4(i_3 - i_2) = 0$

where $i_\Phi = i_1 - i3$

$$\therefore \quad 15(i_1 - i_3) + 20(i_3 - i_1) + 4(i_3 - i_2) = 0$$

$$-5(i_1 - i_3) + 4(i_3 - i_2) = 0$$

$$-5i_1 - 4i_2 + 9i_3 = 0$$

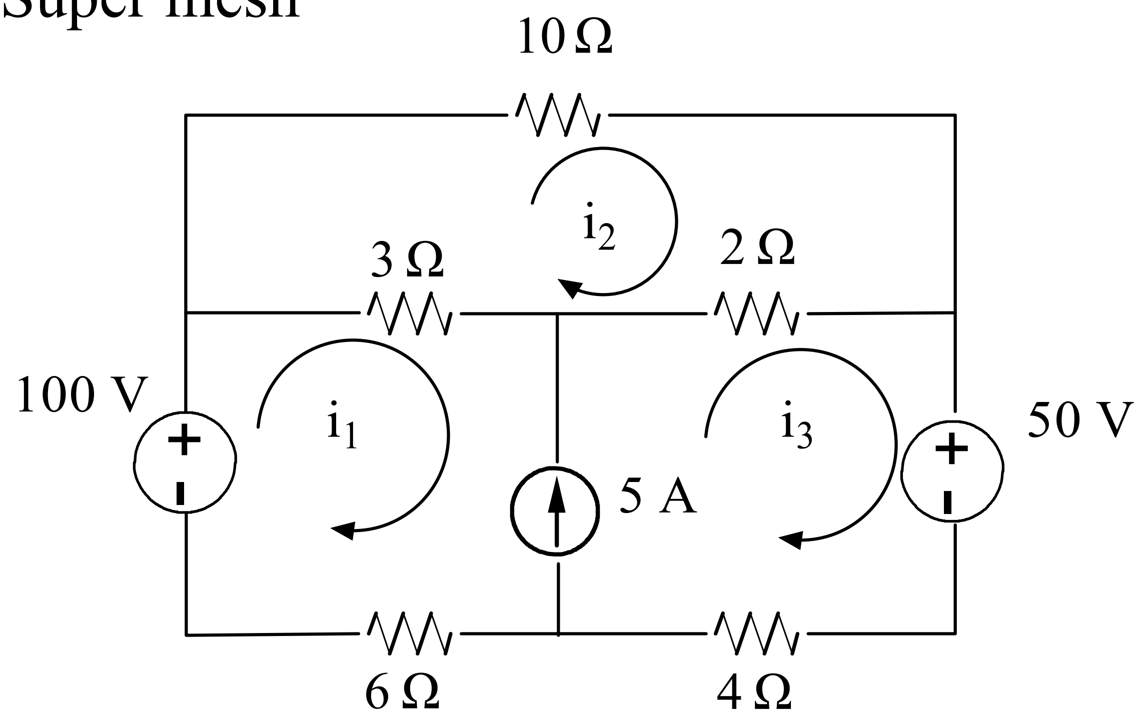
$$\begin{bmatrix} 25 & -5 & -20 \\ -5 & 10 & -4 \\ -5 & -4 & 9 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 50 \\ 0 \\ 0 \end{bmatrix}$$

$$i_1 = 29.6\,\text{A}, \quad i_2 = 26\,\text{A}, \quad i_3 = 28\,\text{A}$$

Special Case :

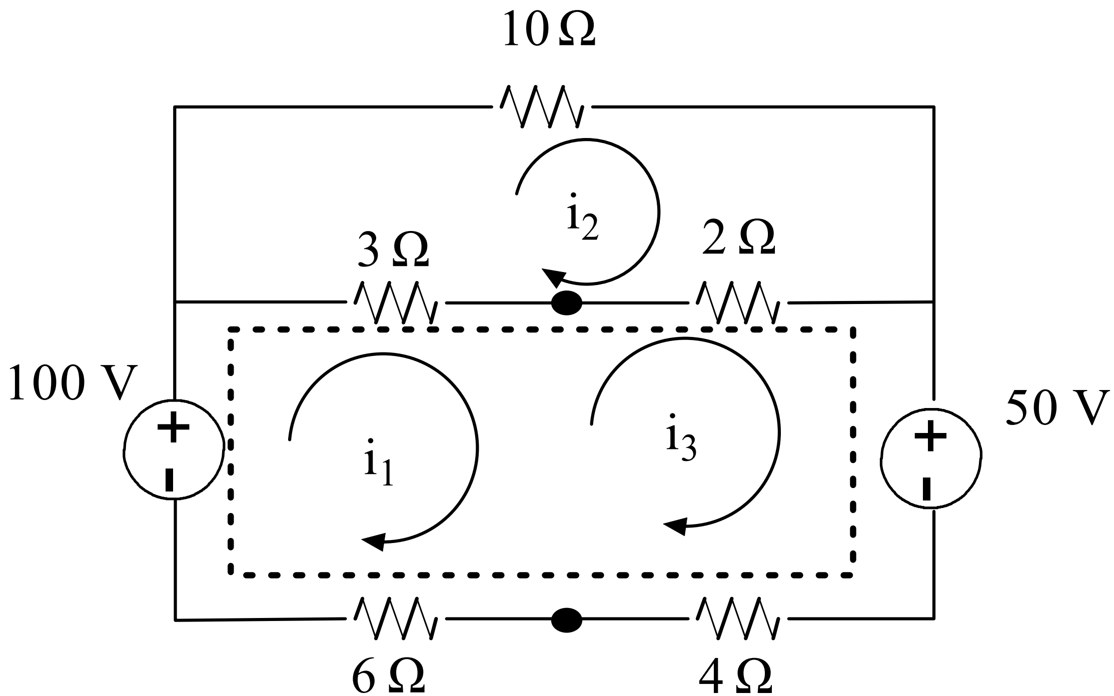
What happens if a current source is located between two meshes

⇨ "Super mesh"



You don't know the voltage across the current source !!

⇨ Remove the whole branch that includes the current source



Apply KVL around the super mesh

$$-100+3(i_1-i_2)+2(i_3-i_2)+50+4i_3+6i_1=0$$

$$9i_1-5i_2+6i_3=50 \quad \ldots\ldots(1)$$

KVL around the upper loop:

$$10i_2 + 2(i_2 - i_3) + 3(i_2 - i_1) = 0$$

$$-3i_1 + 15i_2 - 2i_3 = 0 \qquad \ldots\ldots(2)$$

We also know

$$i_3 - i_1 = 5\ \text{A}$$

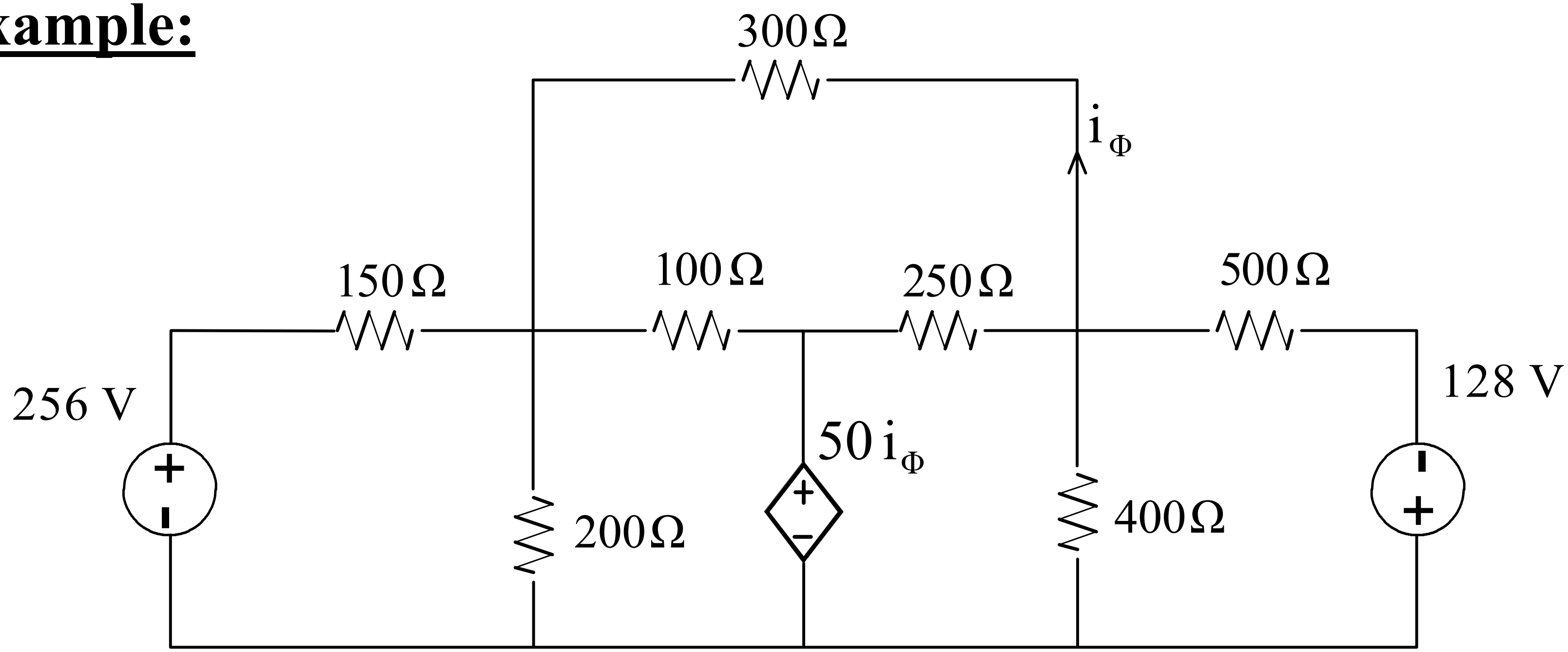
$$-i_1 + 0i_2 + i_3 = 5 \qquad \ldots\ldots(3)$$

$$i_1 = 1.75\ \text{A}$$

$$i_2 = 1.25\ \text{A}$$

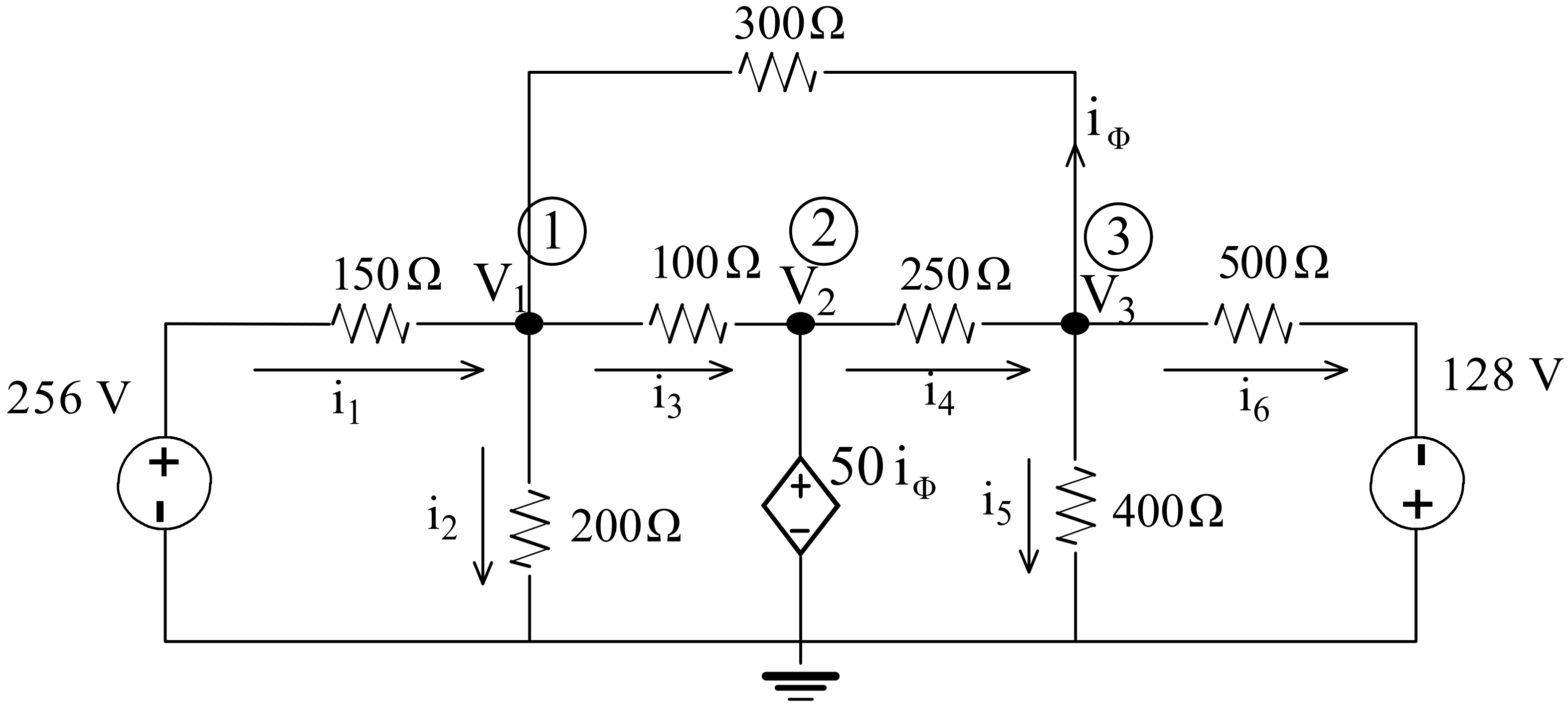
$$i_3 = 6.75\ \text{A}$$

Example:



Use nodal analysis and loop analysis to find power in the 300 (Ω) resistor ?

Nodal Analysis :



<u>KCL at node (1):</u> $i_1 + i_\Phi - i_2 - i_3 = 0$

$$\frac{256 - V_1}{150} + \frac{V_3 - V_1}{300} - \frac{V_1}{200} - \frac{V_1 - V_2}{100} = 0$$

$$V_1\left(\frac{1}{150} + \frac{1}{300} + \frac{1}{200} + \frac{1}{100}\right) - \left(\frac{1}{100}\right)V_2 - \left(\frac{1}{300}\right)V_3 = \frac{256}{150}$$

$$0.0250\ V_1 - 0.01\ V_2 - 0.00333\ V_3 = 1.7067 \qquad \ldots\ldots(1)$$

<u>KCL at node (3):</u>

$$i_4 - i_5 - i_6 - i_\Phi = 0$$

$$\frac{V_2 - V_3}{150} - \frac{V_3}{400} - \frac{V_3 - (-128)}{500} - \frac{V_3 - V1}{300} = 0$$

$$\frac{V_1}{300} + \frac{V_2}{250} + V_3\left(\frac{-1}{250} - \frac{1}{400} - \frac{1}{500} - \frac{1}{300}\right) = \frac{128}{500}$$

$$0.0033\,V_1 + 0.004\,V_2 - 0.0118\,V_3 = 0.256 \qquad \ldots\ldots(2)$$

You can notice that

$$V_2 = 50\, i_\Phi$$

$$V_2 = 50\left(\frac{V_3 - V_1}{300}\right) = \frac{1}{6}V_3 - \frac{1}{6}V_1$$

$$0.166\ V_1 + 1\ V_2 - 0.1667\ V_3 = 0 \qquad \ldots\ldots(3)$$

$$\begin{bmatrix} 0.0383 & -0.01 & -0.0033 \\ 0.0033 & 0.004 & -0.0118 \\ 0.1667 & 1 & -0.1667 \end{bmatrix} \begin{bmatrix} V1 \\ V2 \\ V3 \end{bmatrix} = \begin{bmatrix} 1.7067 \\ 0.256 \\ 0 \end{bmatrix}$$

$$V_1 = 62.5 \text{ V}$$

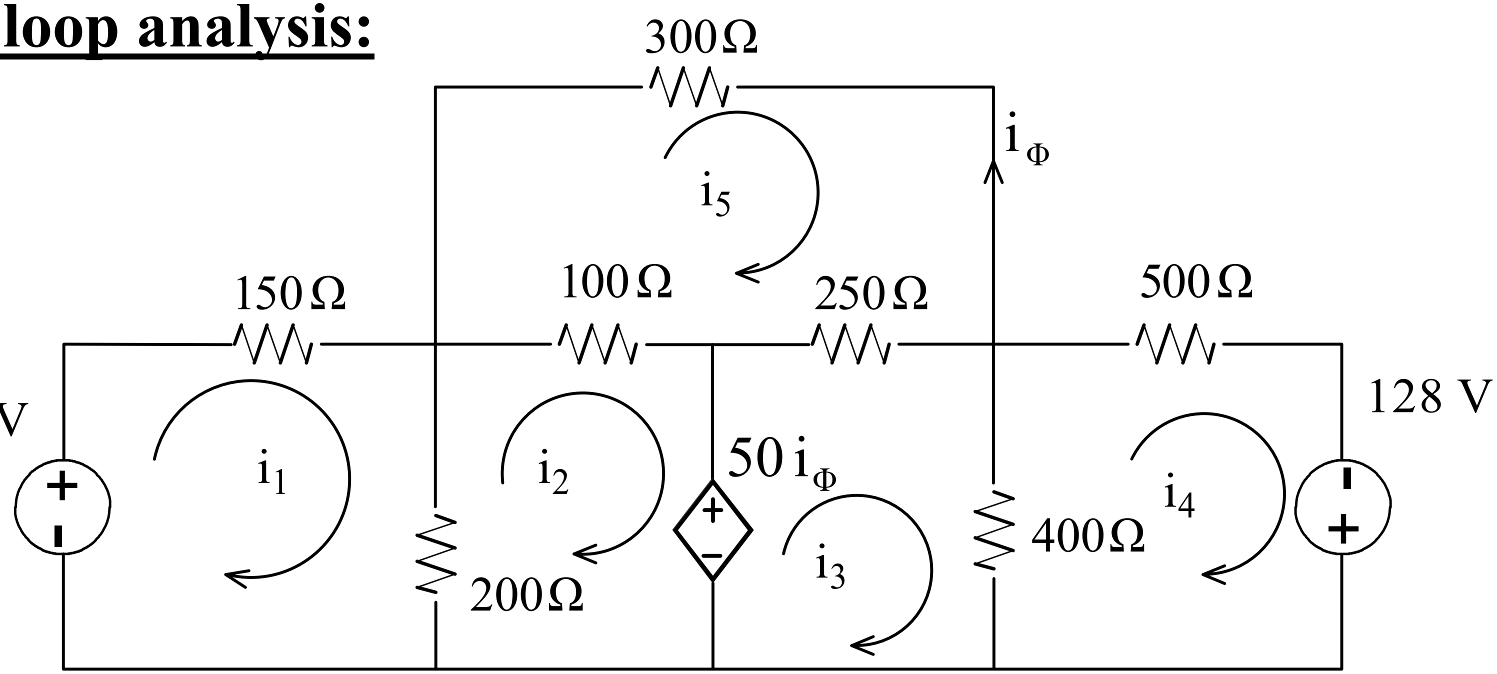
$$V_2 = -11.75 \text{ V}$$

$$V_3 = -8 \text{ V}$$

$$i_\Phi = \frac{V_2}{50} \quad \Rightarrow \quad i_\Phi = -0.235 \text{ A}$$

$$P_{300\Omega} = i_\Phi^{\ 2} \ R = (-0.235)^2 (300) = 16.5675 \text{ W}$$

Using loop analysis:



5 meshes ⇨ 5 equations

KVL around loop (1):

$$150\, i_1 + 200\,(i_1 - i_2) = 256$$

$$350\, i_1 - 200\, i_2 = 256 \qquad \ldots\ldots(1)$$

KVL around loop (2):

$$100\,(i_2 - i_5) + 50\,i_\Phi + 200\,(i_2 - i_1) = 0$$

$$i_\Phi = -\,i_5$$

$$100\,(i_2 - i_5) - 50\,i_5 + 200\,(i_2 - i_1) = 0$$

$$-\,200\,i_1 + 300\,i_2 - 150\,i_5 = 0 \qquad \ldots\ldots(2)$$

KVL around loop (3)

$$500\,i_4 - 128 + 400\,(i_4 - i_3) = 0$$

$$-\,400\,i_3 + 900\,i_4 = 128 \qquad \ldots\ldots(4)$$

KVL around loop (4)

$$250\,(i_3 - i_5) + 400\,(i_3 - i_4) - 50\,i_\Phi = 0$$

$$650\,i_3 - 400\,i_4 - 200\,i_5 = 0 \qquad \ldots\ldots(3)$$

since $\qquad i_\Phi = -\,i_5$

KVL around loop (5)

$$300\,i_5 + 250\,(i_5 - i_3) + 100\,(i_5 - i_2) = 0$$

$$-100\,i_2 - 250\,i_3 + 650\,i_5 = 0 \qquad \ldots\ldots(5)$$

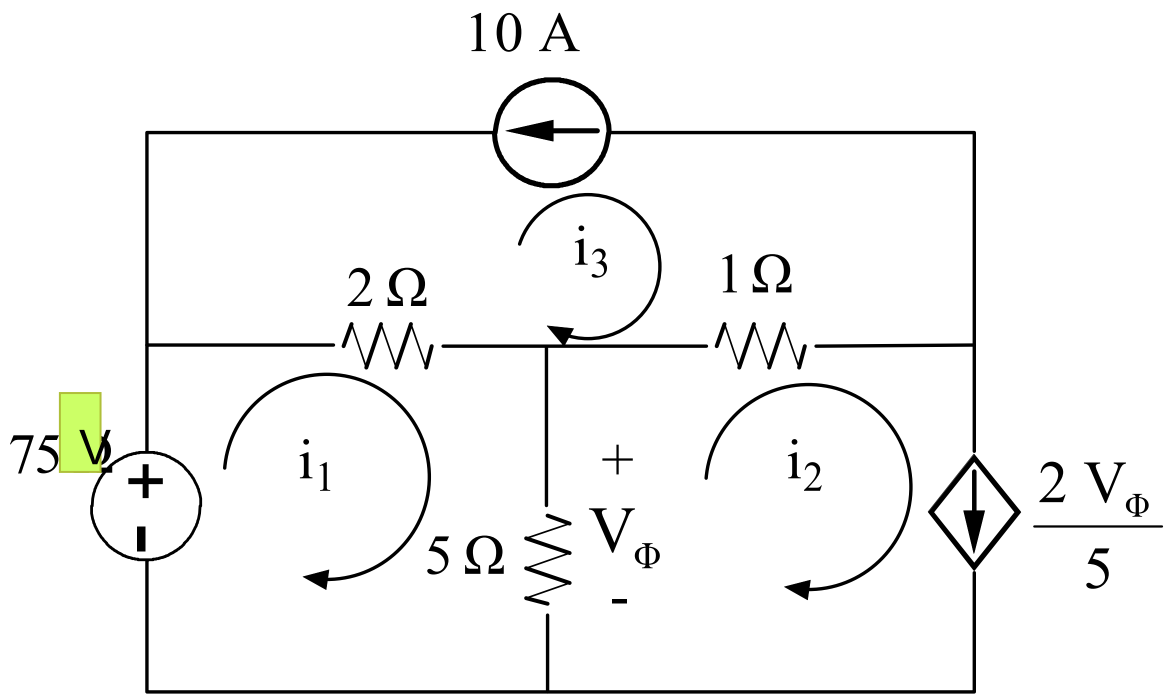
$$\begin{bmatrix} 350 & -200 & 0 & 0 & 0 \\ -200 & 300 & 0 & 0 & -150 \\ 0 & 0 & 650 & -400 & -200 \\ 0 & 0 & -400 & 900 & 0 \\ 0 & -100 & -250 & 0 & 650 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = \begin{bmatrix} 256 \\ 0 \\ 0 \\ 128 \\ 0 \end{bmatrix}$$

$$i_1 = 1.29 \text{ A}$$
$$i_2 = 0.9775 \text{ A}$$
$$i_3 = 0.22 \text{ A}$$
$$i_4 = 0.24 \text{ A}$$
$$i_5 = 0.235 \text{ A}$$

$$P_{300\Omega} = (i_5)^2 \ R = (0.235)^2 \ (300)$$
$$= 16.5675 \text{ W}$$

Example:

Use the loop analysis to find V_Φ



(3) Meshes ⇒ (3) equations

$i_3 = -10A$

KVL around loop (1):

$$2\,(i_1 - i_3) + 5\,(i_1 - i_2) - 75 = 0$$

$$7\,i_1 - 5\,i_2 + 20 - 75 = 0$$

$$7\,i_1 - 5\,i_2 = 55 \qquad \ldots\ldots (1)$$

Equation of dependent source:

$$i_2 = \frac{2\,V_\Phi}{5}$$

$$V_\Phi = 5\,(i_1 - i_2)$$

$$i_2 = \left(\frac{2}{5}\right)(5)\left(i_1 - i_2\right) = 2\,i_1 - 2\,i_2$$

$$2\,i_1 - 3\,i_2 = 0 \qquad \ldots\ldots (2)$$

$$\begin{bmatrix} 7 & -5 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 55 \\ 0 \end{bmatrix}$$

$$i_1 = 15 \text{ A}$$

$$i_2 = 10 \text{ A}$$

$$V_\Phi = 5\,(i_1 - i_2)$$

$$= 5\,(15 - 10) = 25 \text{ V}$$

$$V_\Phi = 25 \text{ V}$$