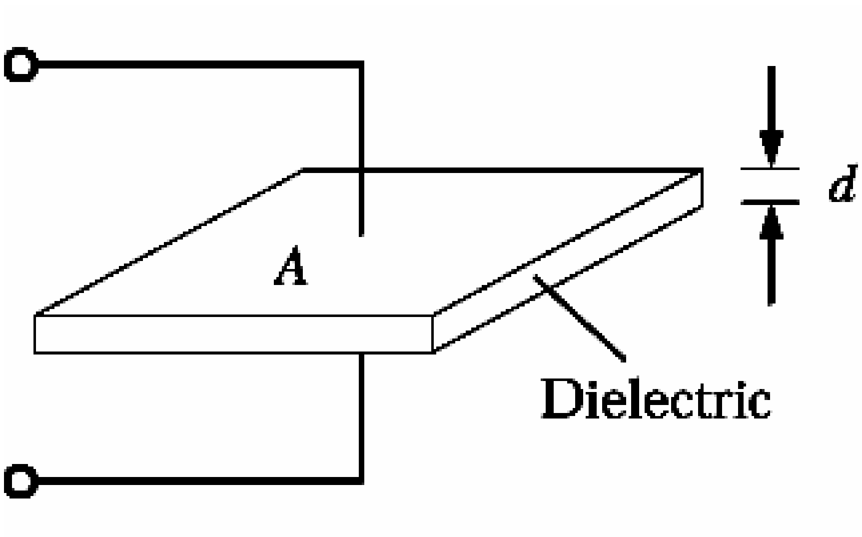
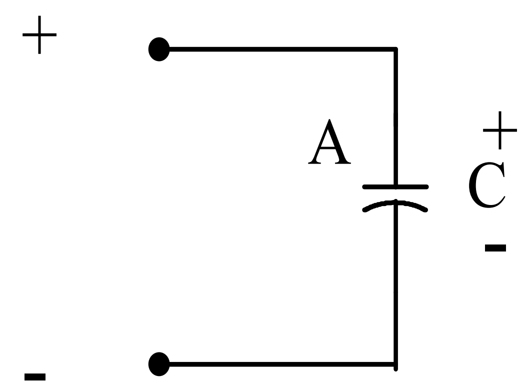
Chapter (5)
Capacitors and Inductors

Capacitors :
A circuit element that is composed of two conducting plates or surfaces separated by a dielectric (non conducting) materials





circuit symbol

Let A : surface area of each plate

d : distance between the two plates

Capacitance $\uparrow$ As Area $\uparrow$

Capacitance $\downarrow$ As distance $\uparrow$

$$\therefore \quad C \propto \frac{A}{d}$$

It is found that $C = \dfrac{\varepsilon_0 A}{d}$

Where :

$\varepsilon_0 \equiv$ Permittivity of free space

$$\varepsilon_0 \equiv 8.85 * 10^{-12} \ {}^{F}\!/\!{}_{m}$$

- If a voltage source (v) is connected to the capacitor , +ve charge will be transferred to one plate while –ve charge will be transferred to the other plate.

Let the charge stored at the capacitor $\equiv q$

If $v \uparrow$, $q \uparrow$

$\therefore \quad v \propto q$

It has been found that

$$q = c\,v$$

C is the capacitance

$$\Rightarrow \quad c = \frac{q}{v}$$

Current in capacitor :

We know that
$$i(t) = \frac{d\,q(t)}{d\,t}$$

$$\therefore \quad i_c(t) = \frac{d}{d\,t}\left(c\,v_c(t)\right)$$

$$\Rightarrow \boxed{i_c(t) = C\,\frac{dv_c(t)}{d\,t}}$$

Voltage of capacitors

$$\because \quad i_c(t) = C\,\frac{dv_c(t)}{dt}$$

$$\therefore \quad i_c(t)\,dt = C\,dv_c(t)$$

$$\Rightarrow d\,v_c(t) = \frac{1}{C}\,i_c(t)dt$$

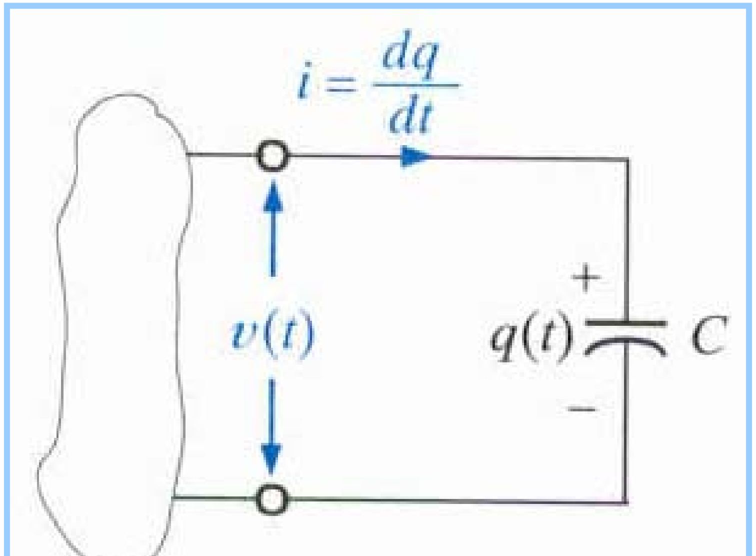
$$\boxed{v_c(t) = v_c(t_0) + \frac{1}{C}\int_{\tau=t_0}^{\tau=t} i_c(\tau)\,d\tau}$$

Where t_0 : initial time

Note:

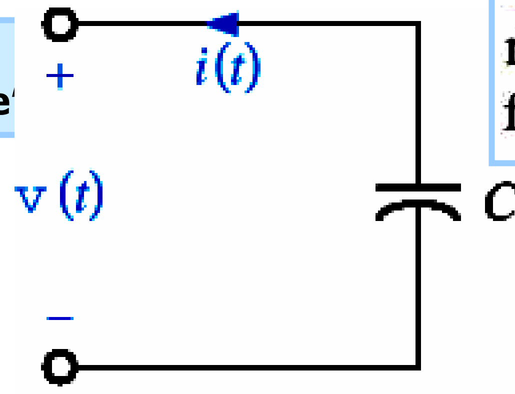
Capacitors only store and release ELECTROSTATIC energy. They do not “create

The capacitor is a passive element and follows the passive sign convention



Linear capacitor circuit representation

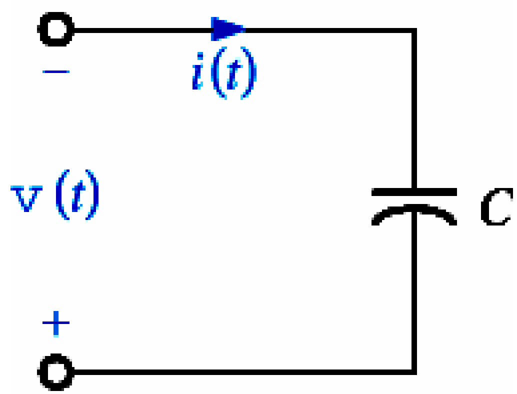
$$i(t) = C\frac{dv}{dt}(t)$$



(a)

Write the i-v relationship for the following capacitors.

$$i(t) = -C\frac{dv(t)}{dt}$$



(b)

$$i(t) = -C\frac{dv(t)}{dt}$$

Power of the capacitors :

$$\boxed{P_c(t) = v_c(t)\, i_c(t)}$$

$$P_c(t) = v_c(t)\, C\, \frac{dv_c(t)}{d\,t} = C\, v_c(t)\, \frac{dv_c(t)}{dt}$$

or

$$P_c(t) = i_c(t)\left[v_c(t_0) + \frac{1}{C} \int_{\tau=t_0}^{\tau=t} i_c(\tau)\, d\tau \right]$$

Energy of capacitor

$$w_c(t) = \int_{\tau=-\infty}^{\tau=t} P_c(\tau)\,d\tau$$

$$= \int_{\tau=-\infty}^{\tau=t} \left(C\, v_c(\tau) \frac{dv_c(\tau)}{d\,\tau} \right) d\tau$$

$$= \int_{v_c(-\infty)}^{v_c(t)} C\, v_c(\tau)\, dv_c(\tau)$$

$$= \frac{1}{2} C\, v_c^{\,2}\,(\tau)\Big|_{v_c(-\infty)}^{v_c(t)}$$

$$= \frac{1}{2} C\, v_c^{\,2}\,(t) - \frac{1}{2} C\, v_c^{\,2}(-\infty)$$

Assuming

$$v_c\left(-\infty\right) = 0$$

$$\therefore \quad \boxed{w_c(t) = \frac{1}{2} C\, v_c^{\,2}(t)}$$

$$v_c(t) = \frac{q(t)}{C}$$

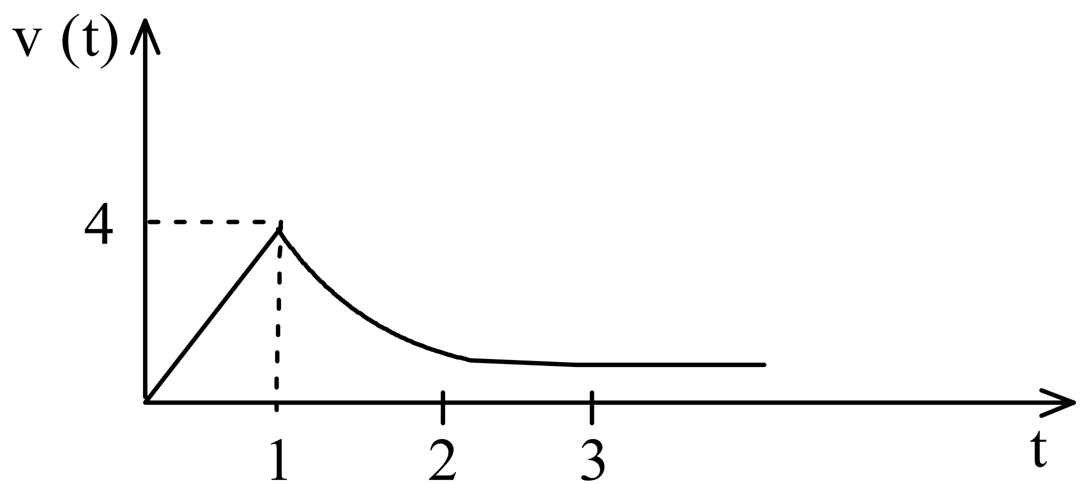
$$\therefore \quad w_c(t) = \frac{1}{2} C \frac{q^2(t)}{C^2}$$

$$\boxed{w_c\left(t\right) = \frac{1}{2C} q^2(t)}$$

Example:

The following voltage is imposed across the terminals of a 0.5 μF capacitor.

$$v_c(t) = \begin{cases} 0 \quad V \quad , & t \le 0 \\ 4t \quad V \quad , & 0 \le t \le 1 \\ 4e^{-(t-1)} \ V, & 1 \le t < \infty \end{cases}$$

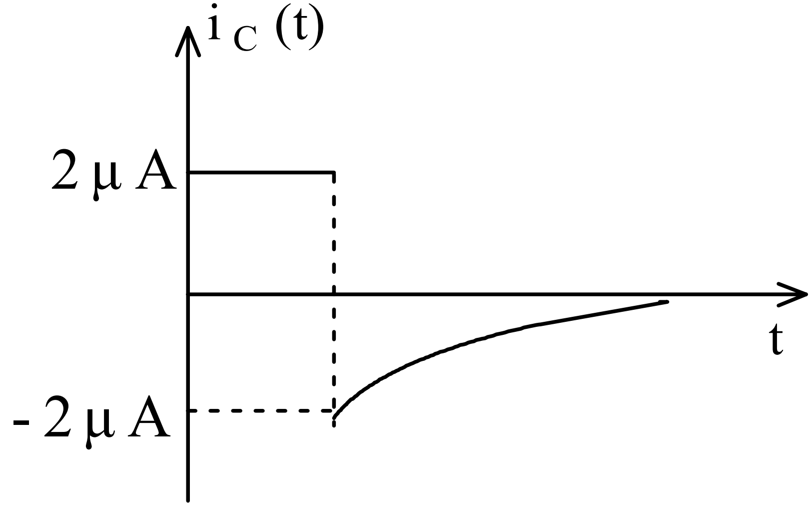


Find the following:

1. $i_c(t)$
2. $P_c(t)$
3. $w_c(t)$

(1) $$i_c(t) = C\frac{dv_c(t)}{dt}$$

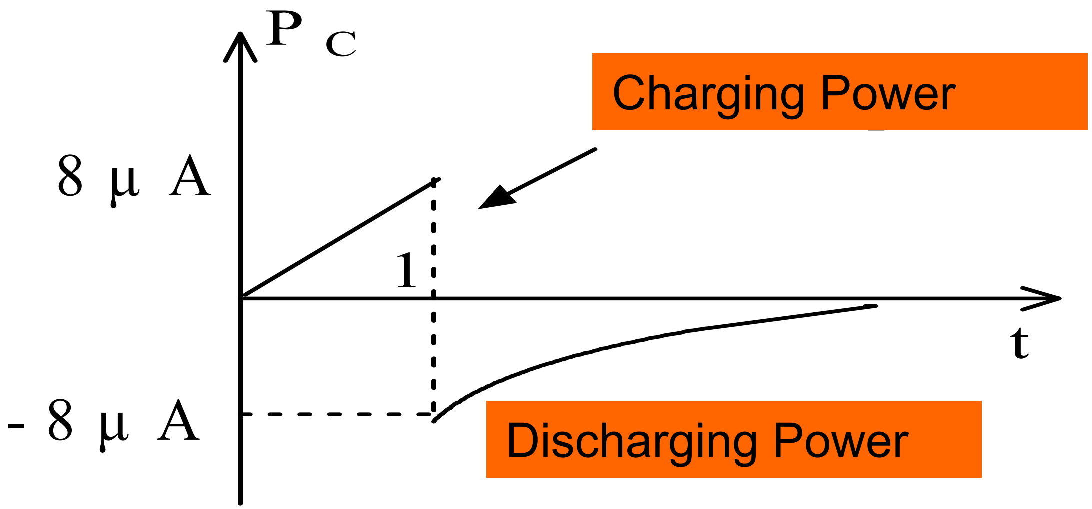
$$\therefore \quad i_c(t) = \begin{cases} 0 & t \langle 0 \\ C\dfrac{d}{dt}(4t) = 4C = 2\,\mu A & 0 \langle t \langle 1 \\ C\dfrac{d}{dt}\left[4e^{-(t-1)}\right] = (4C)(-1)e^{-(t-1)} = -2e^{-(t-1)}\mu A & 1 \langle t \langle \infty \end{cases}$$



(2)

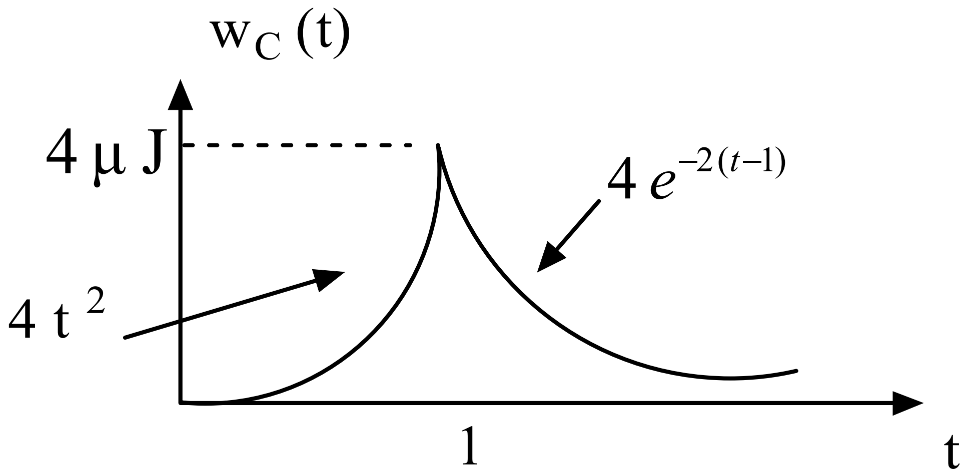
$$P_c(t) = C\, v_c(t)\, \frac{dv_c(t)}{dt}$$

$$= \begin{cases} (0.5\mu F)(0)\dfrac{dv_c(t)}{dt} = 0 \quad W & t \le 0 \\ (0.5\mu F)(4t)(4) = 8t \quad \mu W & 0 \le t < 1 \\ (0.5\mu F)\left(4e^{-(t-1)}\right)(-4)e^{-(t-1)} = -8e^{-2(t-1)} \quad \mu W & 1 < t < \infty \end{cases}$$



(3)
$$w_c(t) = \frac{1}{2} C\, v_c^{\,2}(t)$$

$$w_c(t) = \begin{cases} 0 & t \le 0 \\ \frac{1}{2}(0.5\,\mu\,F)(4\,t)^2 = 4\,t^2 \quad \mu J & 0 \le t \le 1 \\ \frac{1}{2}(0.5\,\mu\,F)\left[4e^{-(t-1)}\right]^2 = 4e^{-2(t-1)} \quad \mu J & 1 \le t < \infty \end{cases}$$



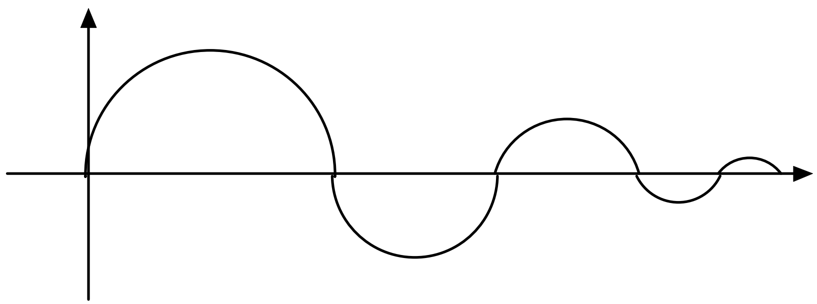
Example :

The voltage at the terminals of a 0.5 μF capacitor is

$$v_c(t) = \begin{cases} 0 & t \le 0 \\ 100\,e^{-20000t}\,\sin(40000\,t) \quad V & t \ge 0 \end{cases}$$

Find:

1. i(0)
2. Power delivered to the capacitors at t = Π/80 m S.
3. Energy stored in the capacitor at t = Π/80 m S



$$i(t) = C\frac{dv_c(t)}{dt}$$

$$= C\frac{d}{dt}\left[100\,e^{-20{,}000\,t}\,\sin(40{,}000\,t)\right]$$

$$= C\left[100\,e^{-20{,}000t}\,\cos(40{,}000\,t)(40{,}000) + \sin(40{,}000\,t)\,(-2*10^{6}\,e^{-20{,}000t})\right]$$

$$i_c(0) = 0.5*10^{-6}\,\left[100\,(1)\,(1)\,(40{,}000) + 0\right]$$

$$i_c(0) = 2\ A$$

<u>(2) Find</u> $P_C\left(\frac{\Pi}{80}m\right)$?

$$P_c(t) = C\,v_c(t)\frac{dv_c(t)}{dt} \qquad ,t \ge 0$$

$$= (0.5\,\mu F)\left(100\,e^{-20{,}000t}\,\sin(40{,}000\,t)\right)$$

$$*\left[100\,e^{-20{,}000t}\,\cos(40{,}000\,t)(40{,}000) + \sin(40{,}000\,t)(-2*10^{6}\,e^{-20{,}000t})\right]$$

$$P_C\left(\frac{\Pi}{80}\,m\right) = -20.79\ W \qquad \text{(discharging)}$$

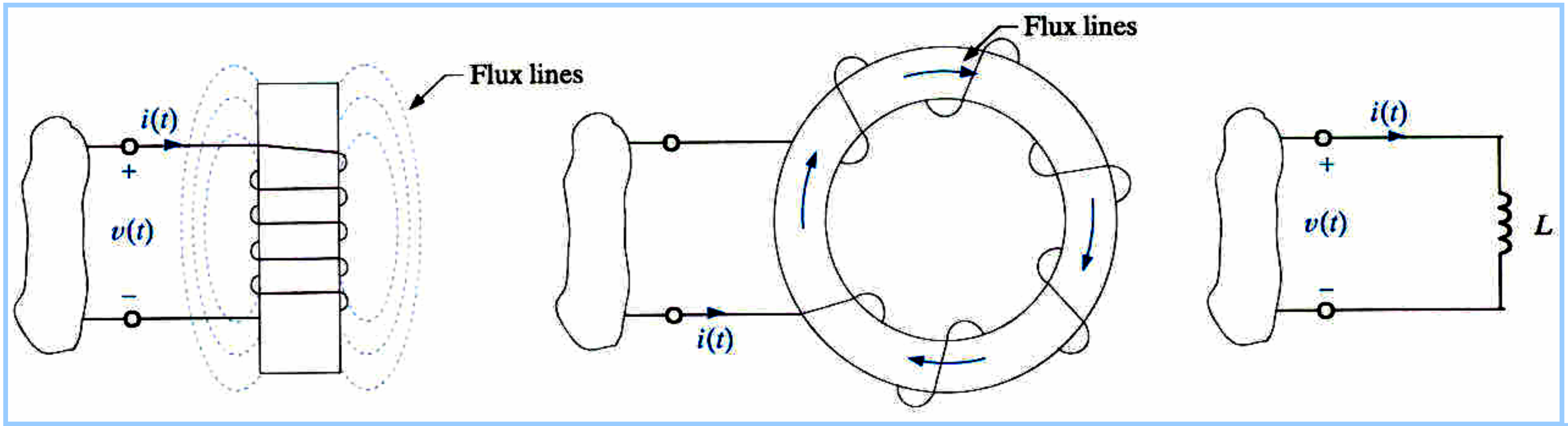
(3) Find $W_C\left(\frac{\Pi}{80}\,m\right)$?

$$W_C(t) = \frac{1}{2}\,C\,v_C^{\,2}(t) = \frac{1}{2}\,C\left[100e^{-20{,}000t}\ \sin(40{,}000\,t)\right]^2$$

$$W_C\left(\frac{\Pi}{80}\,m\right) = 519.2\ \mu J$$

Inductors :

Inductors are circuit elements that consist of a conducting wire in the shape of a coil



Circuit representation for an inductor

• If a current is flowing in the inductor, it produce a magnetic field ,Φ.

$$\Phi(t) = \mathrm{Li(t)}$$

Where L is the inductance and measured in Henry [H]

- The direction of (Φ) depends on the right-hand rule.
- As the current increases or decreases, the magnetic field spreads or collapse
- The change in magnetic field induces a voltage across the inductor.

$$\mathrm{V_L(t)} = \frac{\mathrm{d}\Phi\,\mathrm{(t)}}{\mathrm{dt}}$$

$$\boxed{\therefore \mathrm{V_L(t)} = \mathrm{L}\frac{\mathrm{di_L(t)}}{\mathrm{dt}}}$$

Current in inductors :

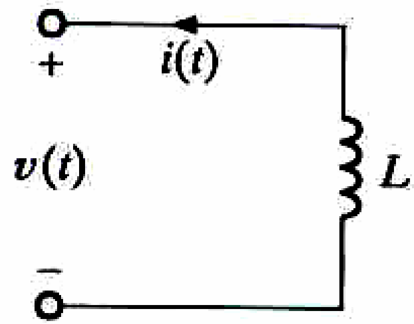
$$v_L(t) = L\frac{di_L(t)}{dt}$$

$$di_L(t) = \frac{1}{L}v_L(t)\,dt$$

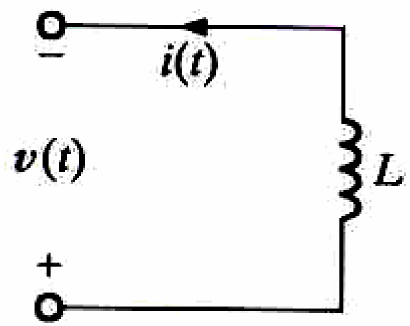
Integrate both sides as before

$$\boxed{i_L(t) = i_L(t_0) + \frac{1}{L}\int_{\tau=t_0}^{\tau=t} v(\tau)\,d\tau}$$

Write the i-v relationship for the following inductors.



(a)



(b)

$$v(t) = -L\frac{di(t)}{dt}$$

$$v(t) = L\frac{di(t)}{dt}$$

Power in inductor :

$$\boxed{P_L(t) = v_L(t)\, i_L(t)}$$

$$= i_L(t)\left[L\frac{di_L(t)}{dt}\right]$$

$$P_L(t) = L\, i_L(t)\,\frac{di_L(t)}{dt}$$

$$P_L(t) = v_L(t)\left[i_L(t_0) + \frac{1}{L}\int_{\tau=t_0}^{\tau=t} v(\tau)\, d\tau\right]$$

Energy in inductors:

$$w_L(t) = \int_{\tau=-\infty}^{\tau=t} P_L(\tau)\, d\tau$$

$$= \int_{\tau=-\infty}^{\tau=t} L\, i_L(\tau) \frac{di_L(\tau)}{d\tau}\, d\tau$$

As before

$$\boxed{w_L(t) = \frac{1}{2} L\, i_L^{\,2}(t)}$$

Example :

The current flow through an 100 m H inductor

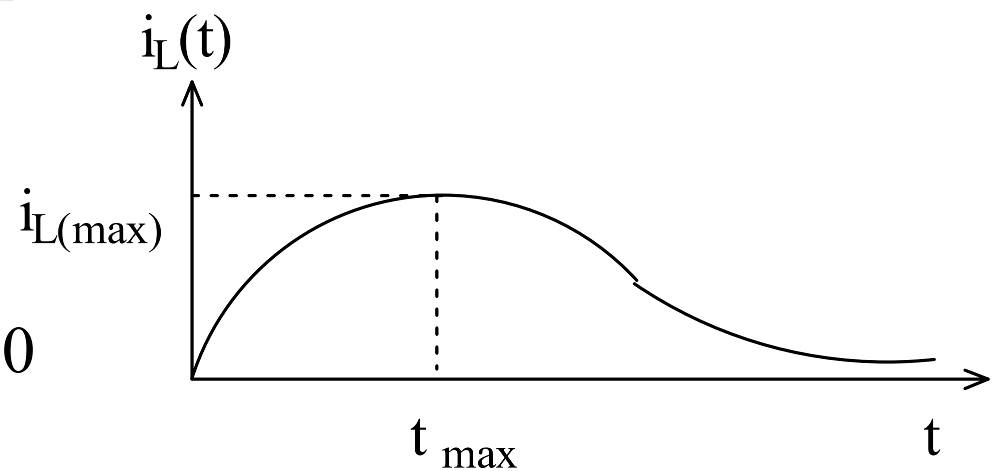
$$i_L(t) = \begin{cases} 0 \quad A & t \le 0 \\ 10\, t\, e^{-5t} \quad A & t \ge 0 \end{cases}$$

Find :

(1) Maximum value of current.

(2) $v_L(t)$, (3) $P_L(t)$, (4) $w_L(t)$

First, find t_{max}



$$\text{let } \frac{di_L(t)}{dt} = 0$$

$$(10\,t)\left(-5e^{-5t}\right) + e^{-5t}\,(10) = 0$$

$$\therefore \qquad e^{-5t}\left[-50\,t + 10\right] = 0$$

$$t_{max} = 0.2 \ \text{ sec}$$

$$i_{Lmax} = i_L(0.2) = 10(0.2)\,e^{-5(0.2)}$$

$$= 2\,e^{-1} = 0.736\ \text{A}$$

(2) $$v_L(t) = L\,\frac{di_L(t)}{dt}$$

$$= (0.1\ \text{H})\frac{d}{dt}\left[10\,t\,e^{-5t}\right]$$

$$= (0.1)\,e^{-5t}\,(-50\,t + 10)$$

$$= e^{-5t}\,(1 - 5\,t)$$

$$v_L(t) = \begin{cases} 0 & ,t < 0 \\ e^{-5t}\,(1 - 5\,t) & ,t > 0 \end{cases}$$

(3) $\quad P_L(t)$?

$$P_L(t) = L\, i_L(t)\, \frac{di_L(t)}{dt}$$

$$P_L(t) = 0 \qquad\qquad , \quad t \le 0$$

$$P_L(t) = (0.1)\left(10\, t\, e^{-5t}\right)\left[e^{-5t}\,(-50\, t + 10)\right] \quad , \quad t \ge 0$$

$$P_L(t) = t\, e^{-10t}\,\left(10 - 50\, t\right)$$

$$P_L(t) = \begin{cases} 0 & , \quad t \le 0 \\ 10\, t\, e^{-10t}\,(1 - 5\, t) & , \quad t \ge 0 \end{cases}$$

(4) $w_L(t)$?

$$w_L(t) = \frac{1}{2} L\, i_L^{\,2}(t)$$

$$= \begin{cases} 0 & , t \le 0 \\ \frac{1}{2}(0.1)\left(10\, t\, e^{-5t}\right)^2 = 5\, t^2\, e^{-10t} & , t \ge 0 \end{cases}$$

Summary of results :

	Capacitor	**Inductor**
v (t)	$v_C(t_0) + \frac{1}{c}\int_{\tau=t_0}^{\tau=t} i_c(\tau)\,d\tau$	$L\,\frac{di_L(t)}{dt}$
i (t)	$c\,\frac{d\,v_C(t)}{dt}$	$i_L(t_0) + \frac{1}{L}\int_{\tau=t_0}^{\tau=t} v_L(\tau)\,d\tau$
P (t)	$c\,v_C(t)\,\frac{dv_C(t)}{dt}$	$L\,i_L(t)\,\frac{di_L(t)}{dt}$
w (t)	$\frac{1}{2}\,c\,v_C^{\,2}(t)$	$\frac{1}{2}\,L\,i_L^{\,2}(t)$

Notes on capacitor :

1. If $v_C(t) = \text{constant}$, $i_C(t) = 0$

$\Rightarrow$ Capacitor will be open circuit

2. $v_C(t)$ cannot change instantaneously (no sudden change)

because $$\int_{\tau=t_0}^{\tau=t} i_C(\tau)\,d\tau = 0$$

3. Capacitors can store energy

$$w_C(t) = \frac{1}{2} c\, v_C^{\,2}(t)$$

Notes on inductors :

1. If $i_L(t) = \text{constant}$, $v_L(t) = 0$

$\Longrightarrow$ Inductor will be short circuit

2. $i_L(t)$ cannot change instantaneously (no sudden change)

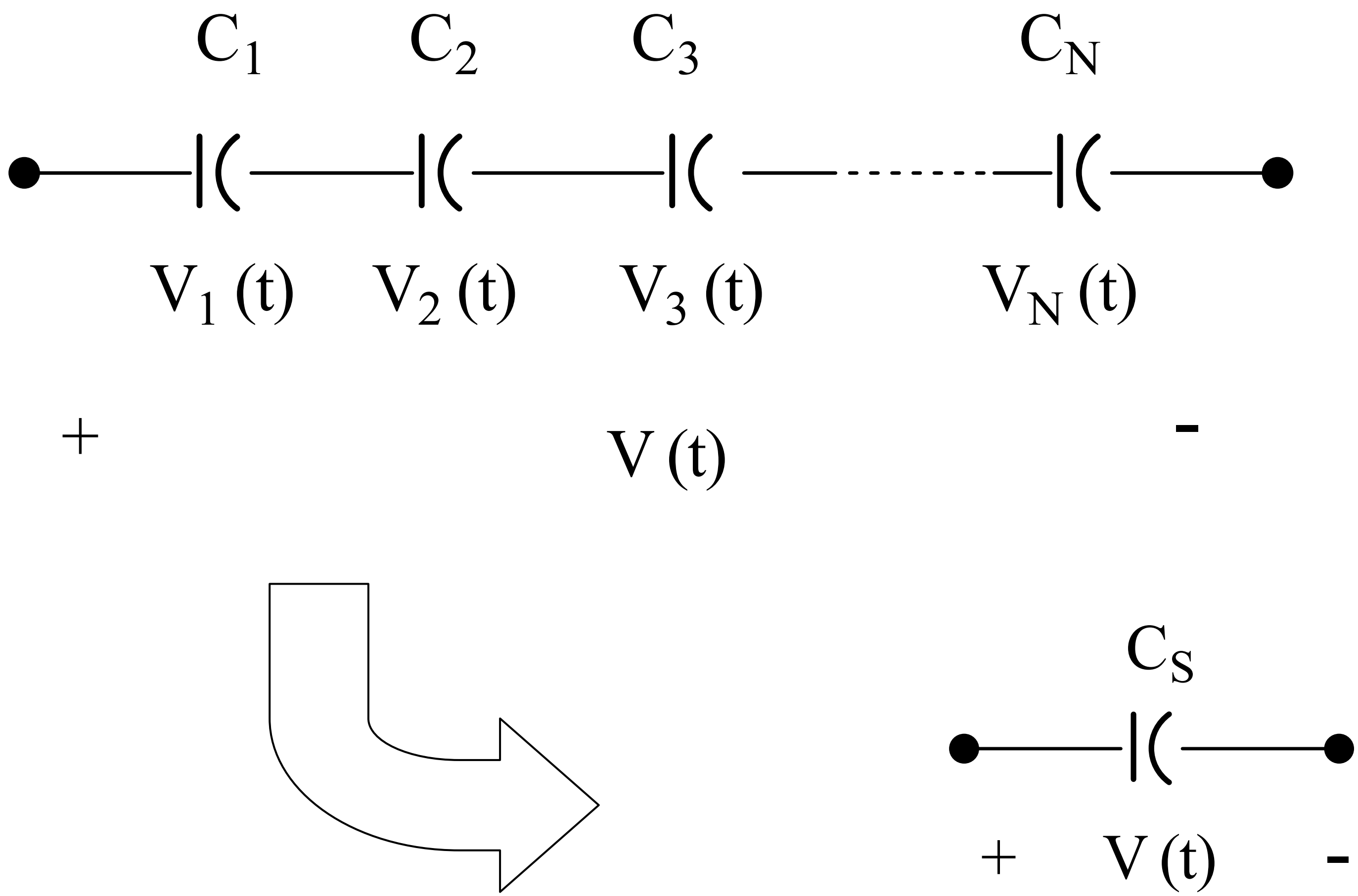
because $$\int_{\tau=t_0}^{\tau=t} v_L(\tau)\,d\tau = 0$$

3. Inductors can store energy

$$w_L(t) = \frac{1}{2} L\, i_L^{\,2}(t)$$

Capacitors and Inductors combinations :

1. Series capacitors :



$$v(t) = v_1(t) + v_2(t) + v_3(t) + \cdots + v_N(t)$$

For each capacitor ,

$$v_k(t) = v_k(t_0) + \frac{1}{c_k} \int_{\tau=t0}^{\tau=t} i(\tau)\, d\tau$$

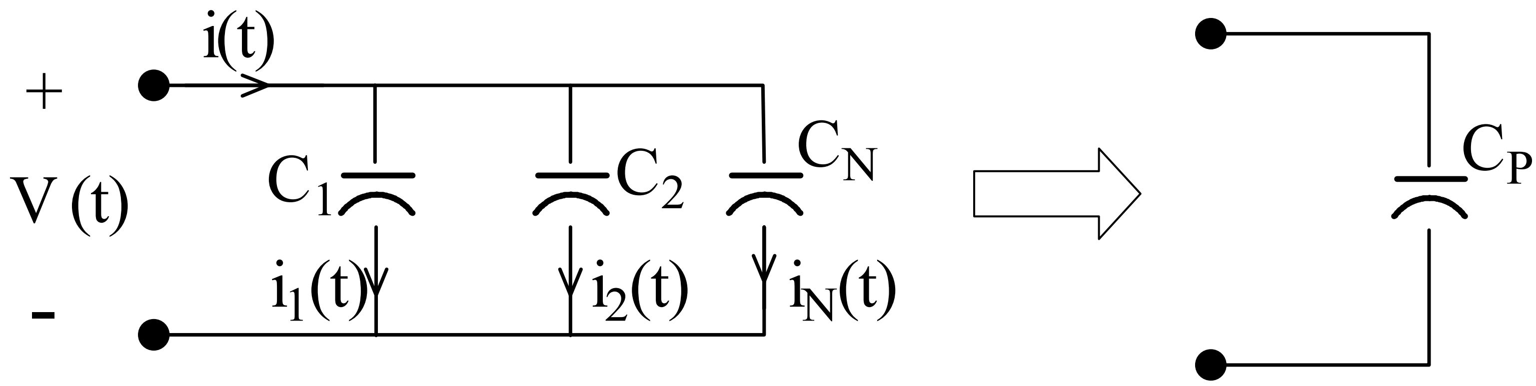
$$v(t) = v_1 + v_2 + \cdots + v_N$$

$$v(t) = \left[\sum_{k=1}^{N} v_k(t_0) \right] + \left(\sum_{k=1}^{N} \frac{1}{c_k} \right) \int_{\tau=t_0}^{\tau=t} i(\tau)\, d\tau$$

The equivalent capacitance C_S is

$$\boxed{\frac{1}{C_S} = \sum_{k=1}^{N} \frac{1}{C_k} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}}$$

Parallel capacitors :



$$i(t) = i_1(t) + i_2(t) + \cdots + i_N(t)$$

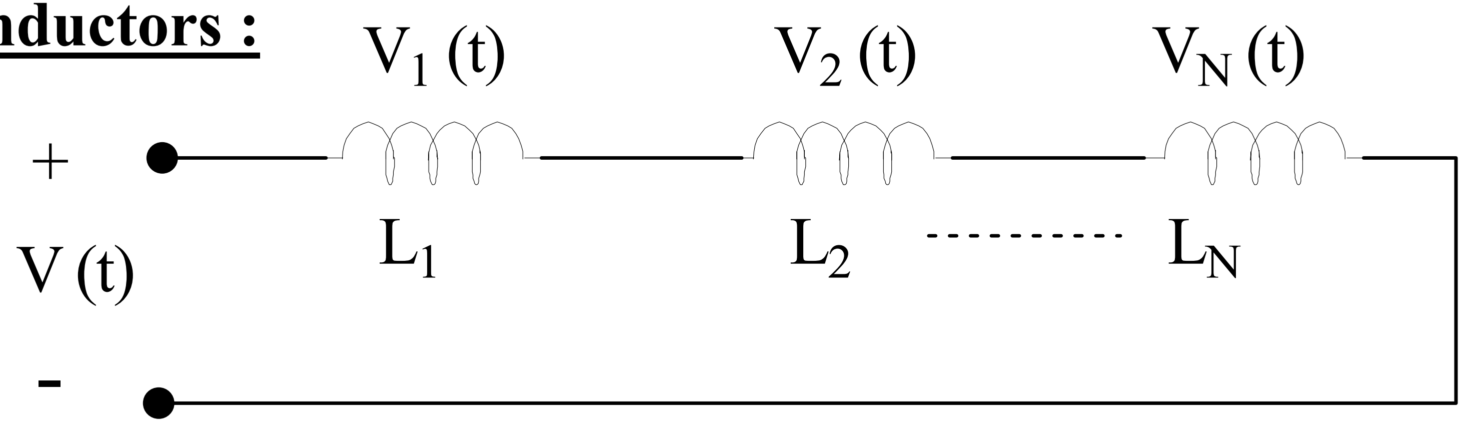
$$= C_1 \frac{dv(t)}{dt} + C_2 \frac{dv(t)}{dt} + \cdots + C_N \frac{dv(t)}{dt}$$

$$i(t) = \frac{dv(t)}{dt}\left[\sum_{k=1}^{N} C_k\right] = C_P \frac{dv(t)}{dt}$$

The equivalent capacitance ,C_P

$$\boxed{C_P = C_1 + C_2 + C_3 + \cdots + C_N}$$

Series Inductors :



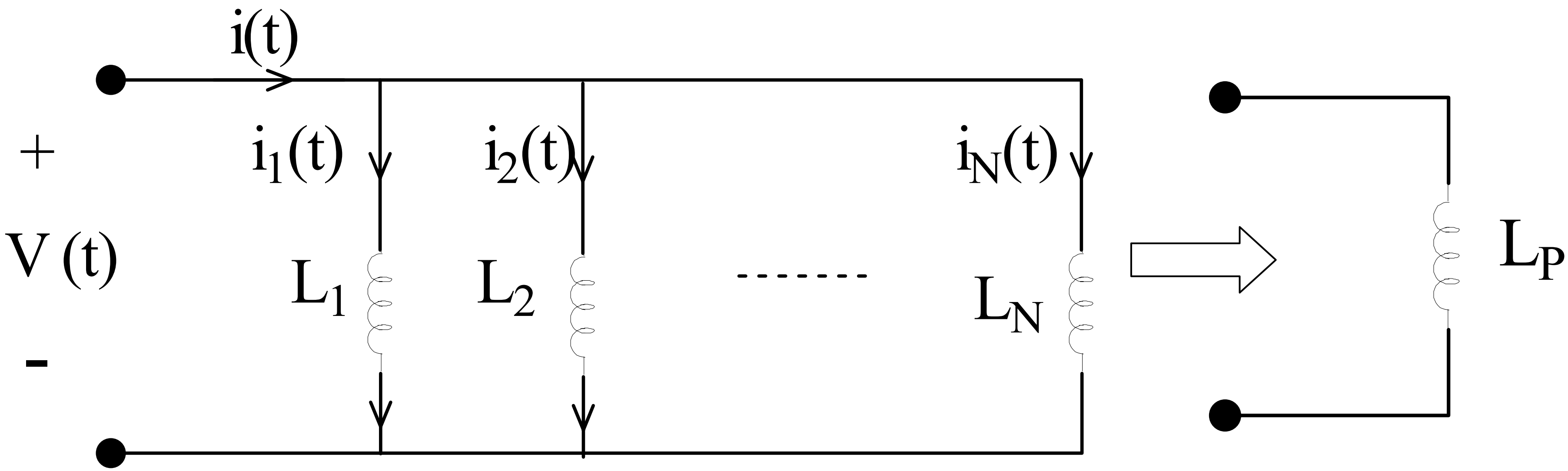
$$v(t) = v_1(t) + v_2(t) + \cdots + v_N(t)$$

$$= L_1 \frac{di(t)}{dt} + L_2 \frac{di(t)}{dt} + \cdots + L_N \frac{di(t)}{dt}$$

$$v(t) = \left(\sum_{k=1}^{N} L_k \right) \frac{di(t)}{dt} = L_S \frac{di(t)}{dt}$$

$$\boxed{L_S = L_1 + L_2 + L_3 + \cdots + L_N}$$

Parallel Inductors :



$$i(t) = i_1(t) + i_2(t) + \cdots + i_N(t)$$

where $$i_k(t) = i_k(t_0) + \frac{1}{L_k} \int_{\tau=t_0}^{\tau=t} v(\tau)\, d\tau$$

$$i(t) = \left[\sum_{k=1}^{N} i_k(t_0) \right] + \left[\sum_{k=1}^{N} \frac{1}{L_k} \right] \int_{\tau=t_0}^{\tau=t} v(\tau)\, d\tau$$

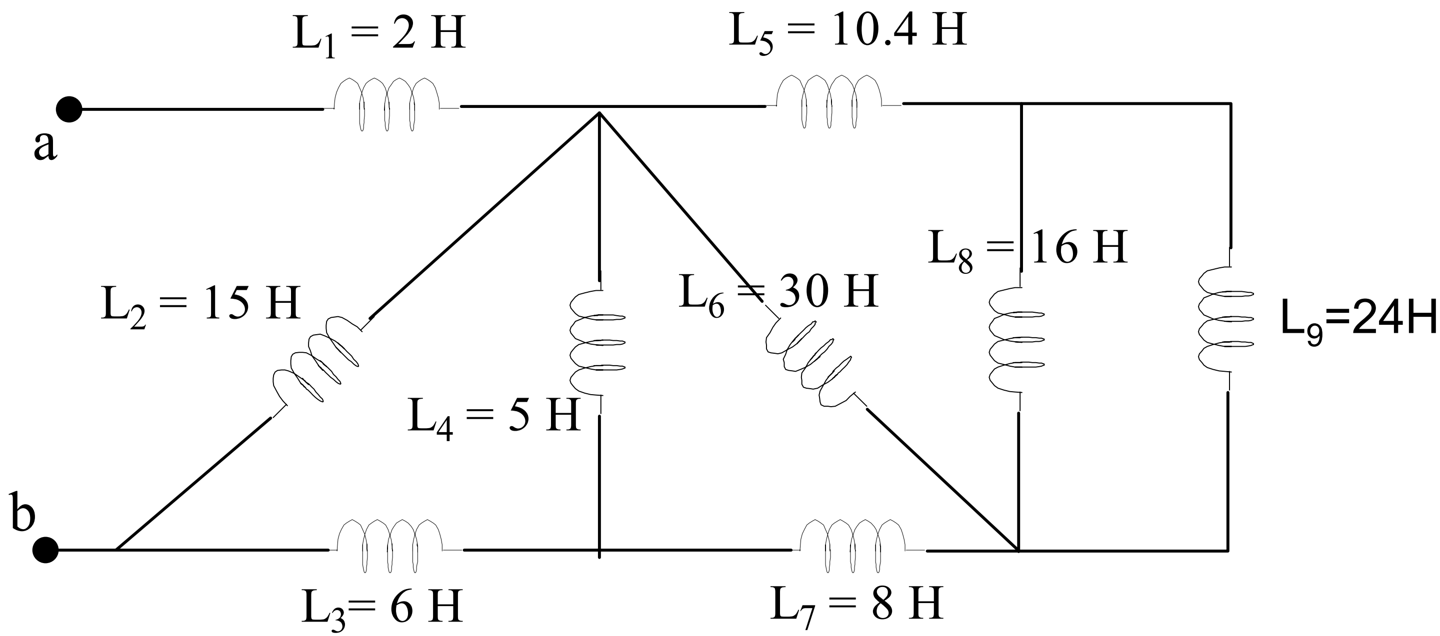
$$i(t) = i(t_0) + \frac{1}{L_P} \int_{\tau=t_0}^{\tau=t} v(\tau)\, d\tau$$

The equivalent inductance , L_P

$$\boxed{\frac{1}{L_P} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N}}$$

Example:

Find the equivalent inductance with respect to the terminals a ,b?



- L_{89}=L_8 in parallel with L_9

$$\therefore \quad L_{89} = \frac{L8\ L9}{L8 + L9} = 9.6\ \text{H}$$

- L_{589}=L_5 in series with L_{89}

$$L_{589} = L_5 + L_{89} = 10.4 + 9.6 = 20\ \text{H}$$

- L_{6589}=L_6 in parallel with L_{589}

$$L_{6589} = \frac{L_6\ L_{589}}{L_6 + L_{589}} = 12\ \text{H}$$

- L_{76589}=L_7 in series with L_{6589}

$$L_{76589} = L_7 + L_{6589} = 8 + 12 = 20\ \text{H}$$

- L_{476589}=L_4 in parallel with L_{76589}

$$L_{476589} = \frac{L_4\ L_{76589}}{L_4 + L_{76589}} = 4\ \text{H}$$

- $L_{3476589}=L_3$ in series with L_{476589}

$$L_{3476589} = L_3 + L_{346589} = 6 + 4 = 10 \text{ H}$$

- $L_{23476589}=L_2$ in parallel with $L_{3476589}$

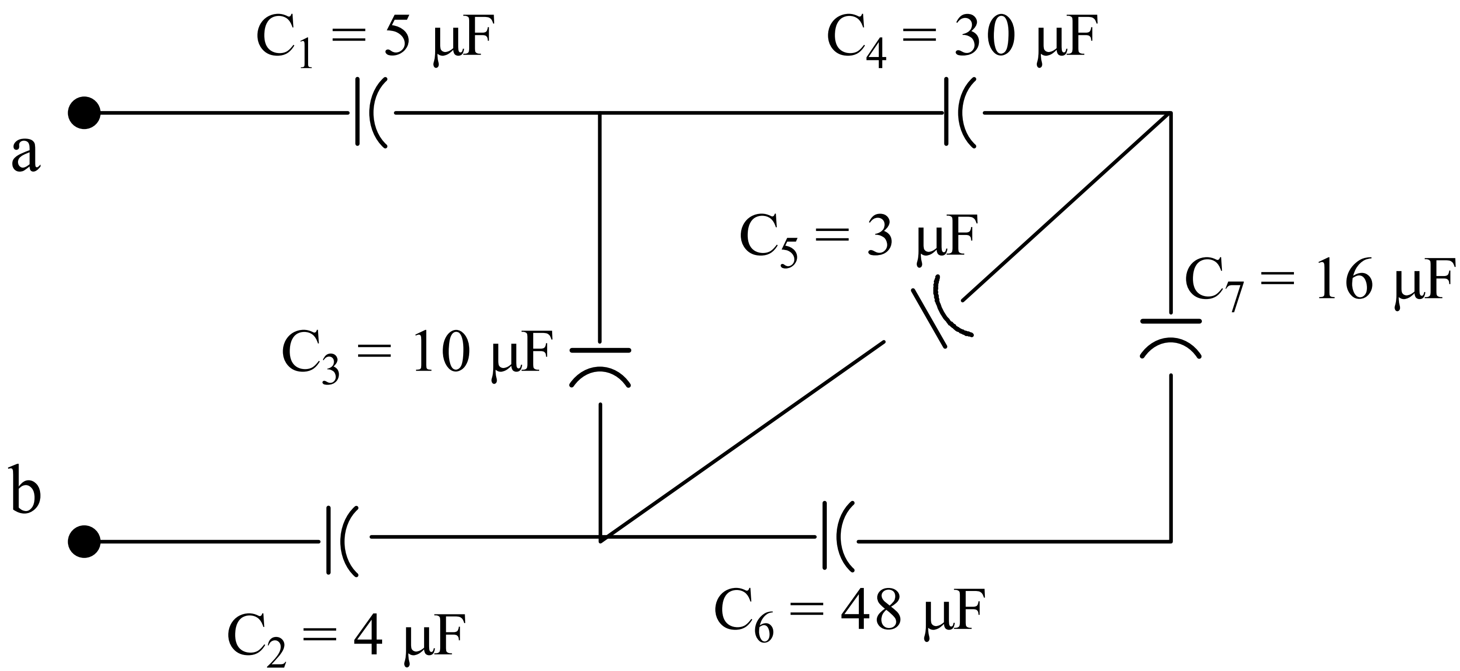
$$L_{23476589} = \frac{L_2 \; L_{3476589}}{L_2 + L_{3476589}} = 6 \text{ H}$$

- $L_{ab}=L_1$ in series with $L_{23476589}$

$$L_{ab} = L_1 + L_{2346589} = 2 + 6 = 8 \text{ H}$$

Example :

Find the equivalent capacitance at the terminals a and b ?



- $C_{67} = C_6$ in series with C_7

$$C_{67} = \frac{C_6 \, C_7}{C_6 + C_7} = 12 \, \mu \text{ F}$$

- $C_{567} = C_5$ in parallel with C_{67}

$$C_{567} = C_5 + C_{67} = (3 + 12)\,\mu\text{ F} = 15\,\mu\text{ F}$$

- $C_{4567} = C_4$ in series with C_{567}

$$C_{4567} = \frac{C_4\ C_{567}}{C_4 + C_{567}} = 10\,\mu\text{ F}$$

- $C_{34567} = C_3$ in parallel with C_{4567}

$$C_{34567} = C_3 + C_{4567} = (10 + 10)\mu\text{ F} = 20\,\mu\text{ F}$$

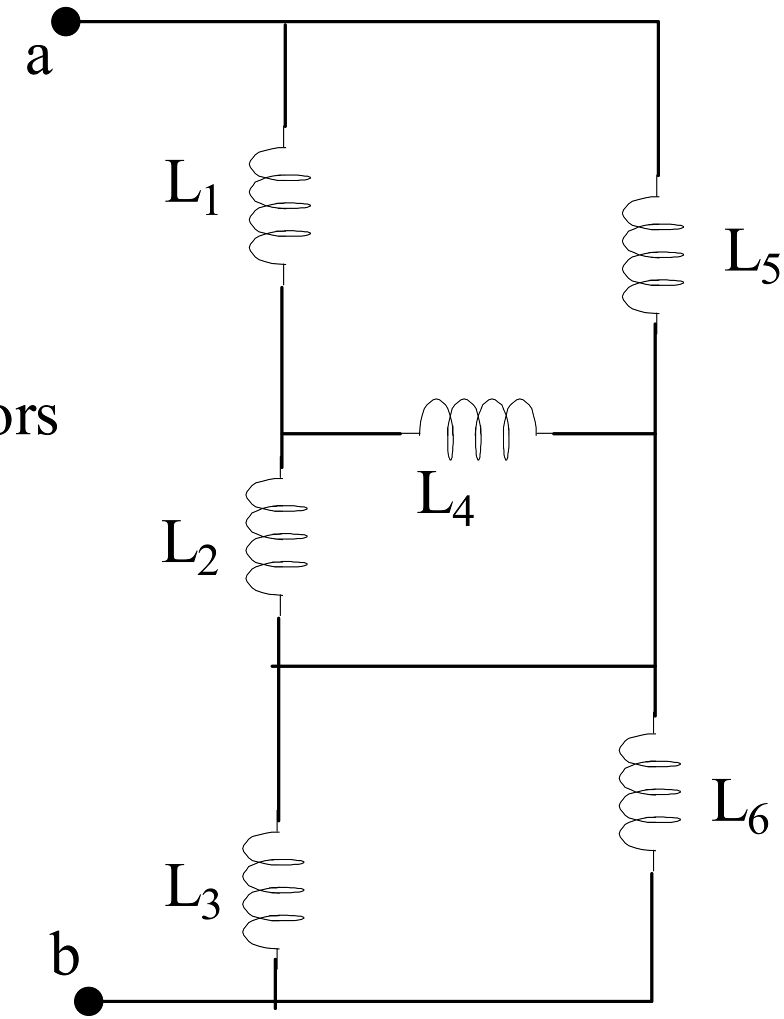
- $C_{ab} = C_1$ in series with C_{34567} in series with C_2

$$\frac{1}{C_{ab}} = \frac{1}{C_1} + \frac{1}{C_{34567}} + \frac{1}{C_2} = \frac{1}{5\mu} + \frac{1}{20\mu} + \frac{1}{4\mu}$$

$$\Rightarrow \qquad C_{ab} = 2\,\mu\text{ F}$$

Example :

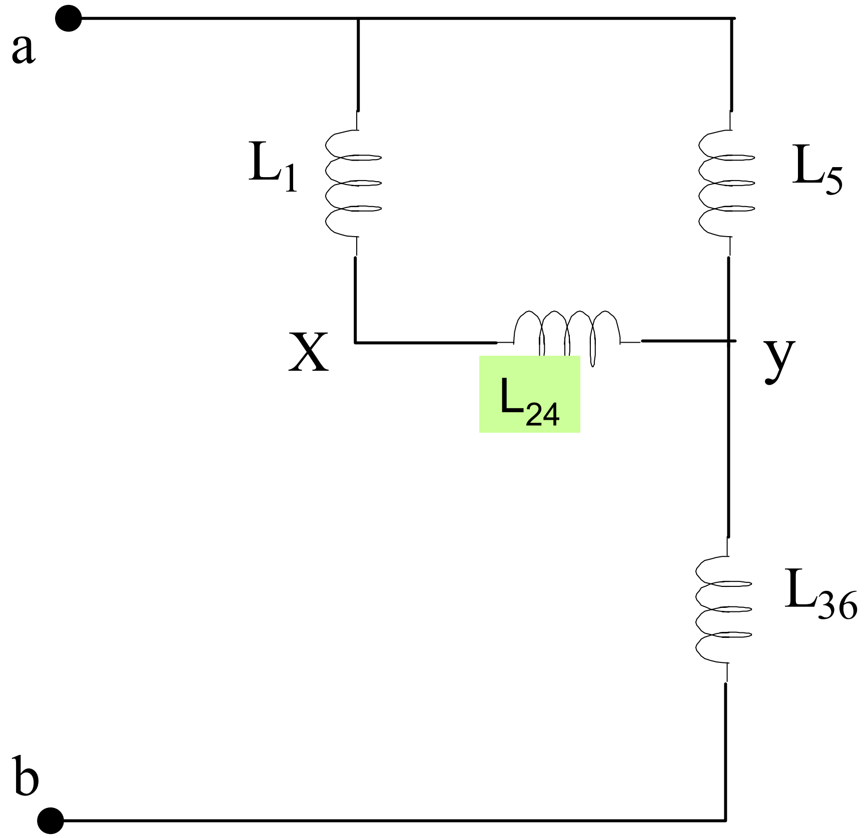
Find the equivalent inductance at a,b

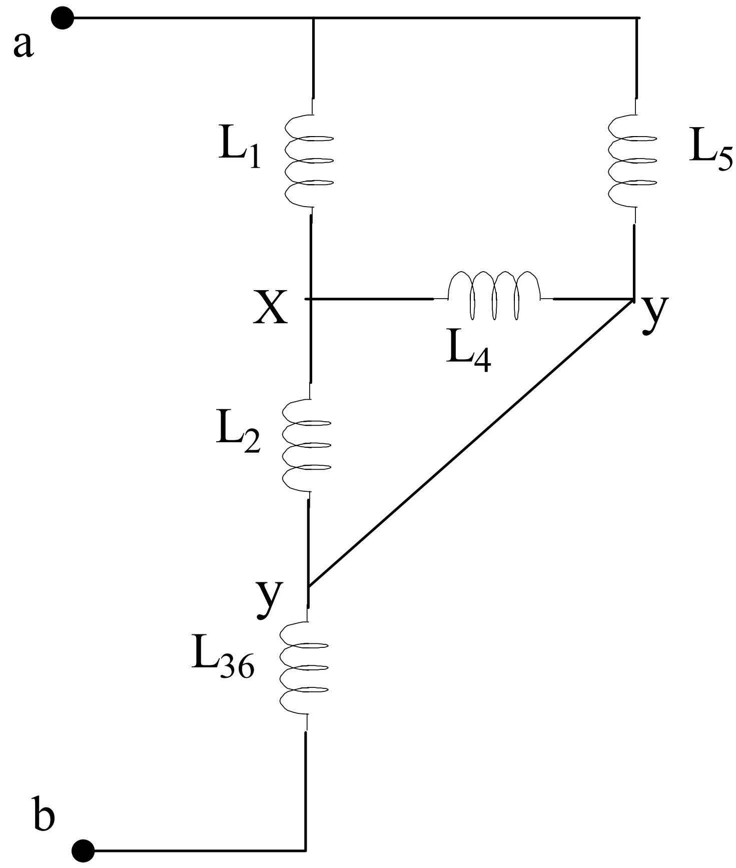


in all inductors are 4 mH

- $L_{36} = L_3$ in parallel with L_6

$$L_{36} = \frac{L_3 \, L_6}{L_3 + L_6} = 2 \text{ m H}$$





- $L_{24} = L_2$ in parallel with L_4

$$L_{24} = \frac{L_2 \; L_4}{L_2 + L_4} = 2 \text{ m H}$$

- $L_{124} = L_1$ in series with L_{24}

$$L_{124} = L_1 + L_{24} = (4 + 2)\ \text{m H} = 6\ \text{m H}$$

- $L_{1245} = L_{124}$ in parallel with L_5

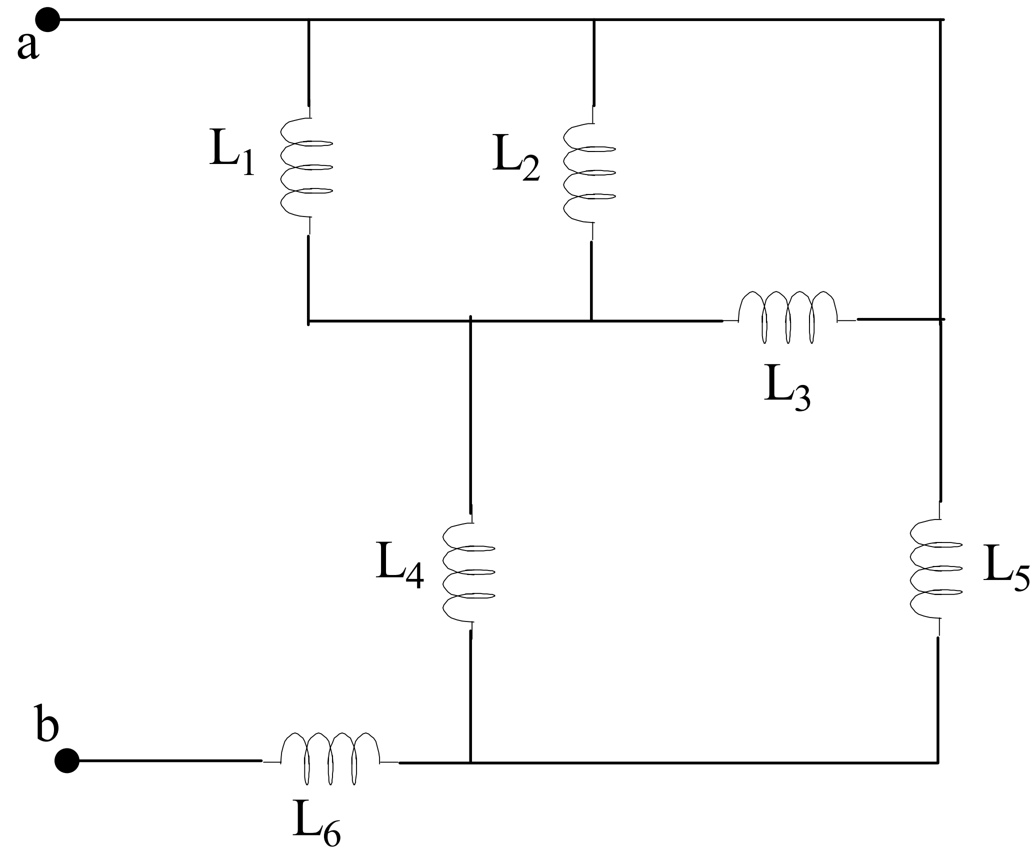
$$L_{1245} = \frac{L_{124}\ L_5}{L_{124} + L_5} = 2.4\ \text{m H}$$

- $L_{124536} = L_{1245}$ in series with L_{36}

$$L_{124536} = L_{1245} + L_{36} = (2.4 + 2)\ \text{m H}$$
$$L_{124536} = 4.4\ \text{m H}$$

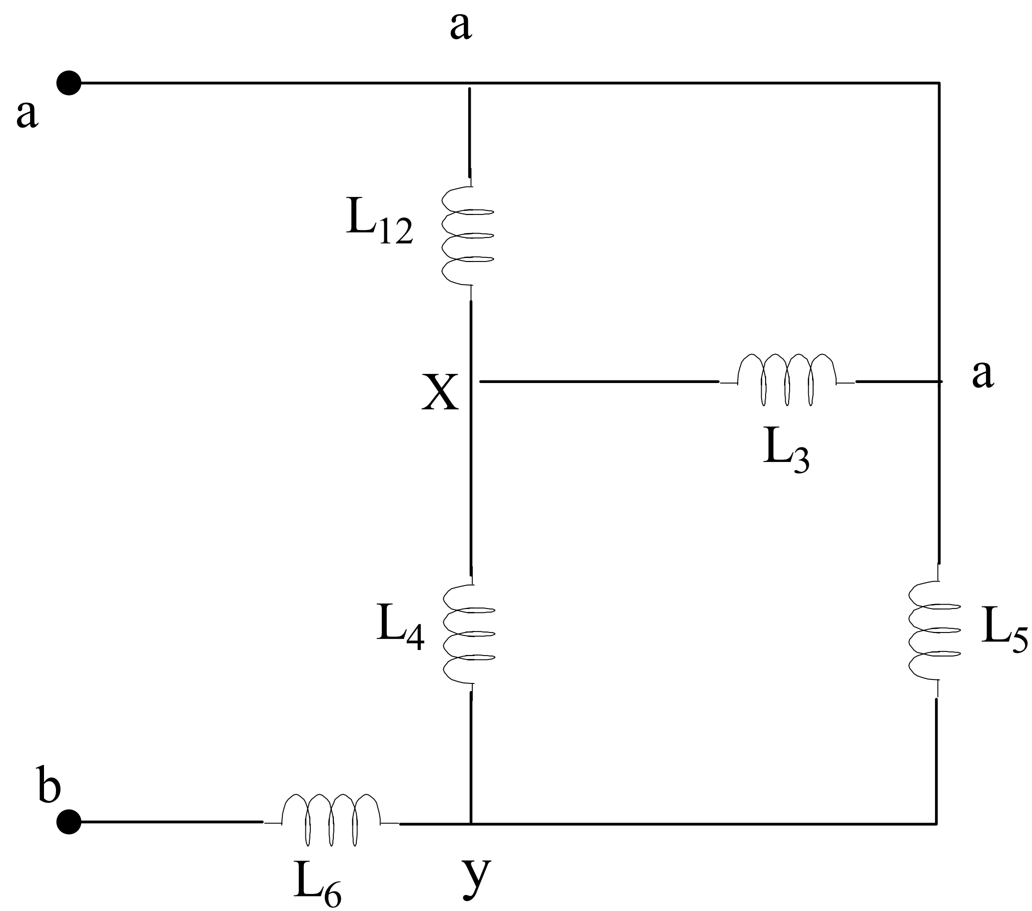
Example :

Find the equivalent inductance at a , b if all L are 6 m H



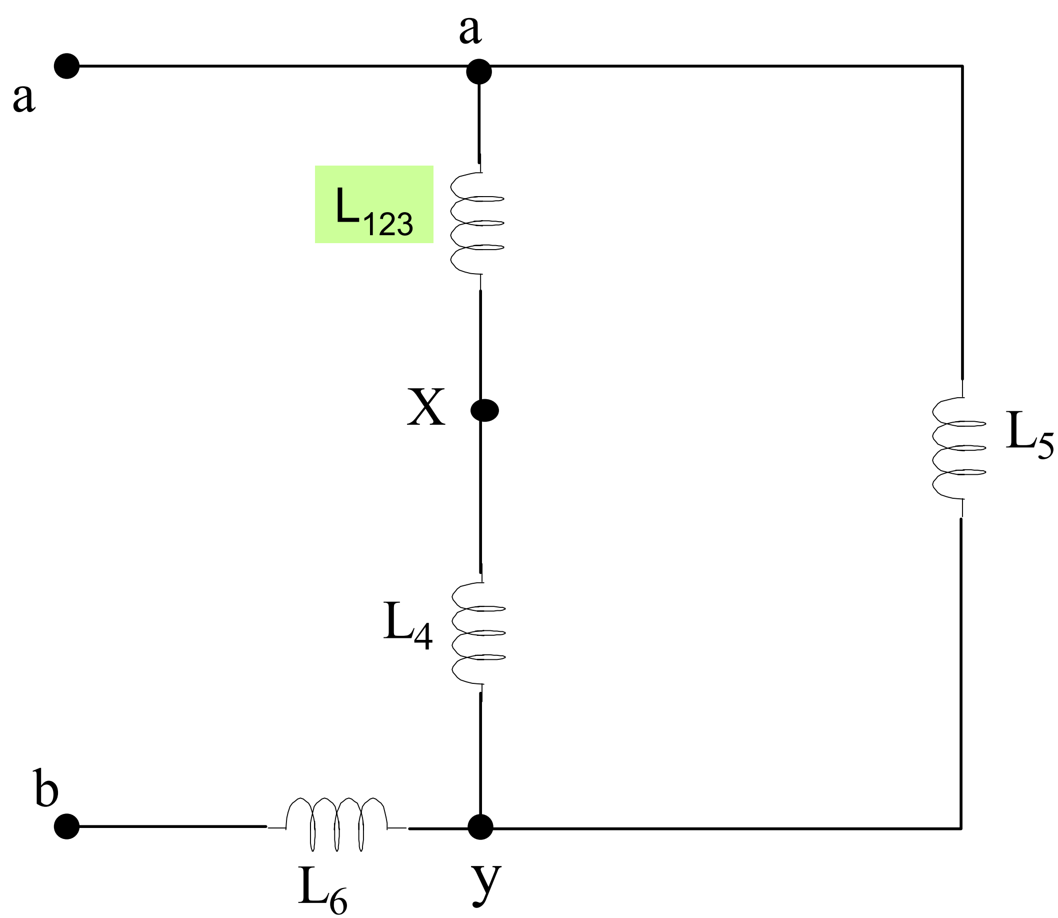
$L_{12} = L_1$ in parallel with L_2

$$L_{12} = \frac{L_1 \, L_2}{L_1 + L_2} = 3 \text{ m H}$$



$L_{123} = L_{12}$ in parallel with L_3

$$L_{123} = \frac{L_{12}\ L_3}{L_{12} + L_3} = \frac{(3)(6)}{(3)+(6)}\,\text{m H} = 2\ \text{m H}$$



$L_{1234} = L_{123}$ in series with L_4

$$L_{1234} = L_{123} + L_4 = 2 + 6 = 8 \text{ m H}$$

$L_{12345} = L_{1234}$ in parallel with L_5

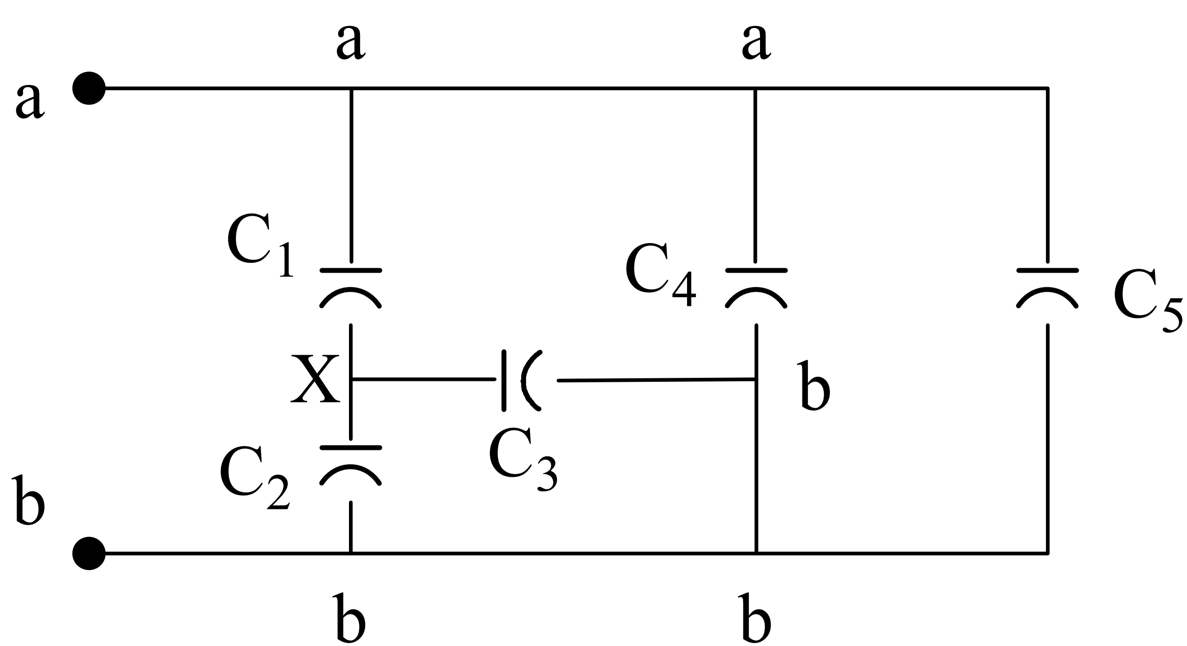
$$L_{12345} = \frac{L_{1234}\ L_5}{L_{1234} + L5} = \frac{(8)(6)}{(8)+(6)} \text{ m H} = 3.42 \text{ m H}$$

$L_{ab} = L_6$ in series with L_{12345}

$$L_{ab} = L_6 + L_{12345}$$
$$= 6 + 3.429 \text{ m H} = 9.429 \text{ m H}$$

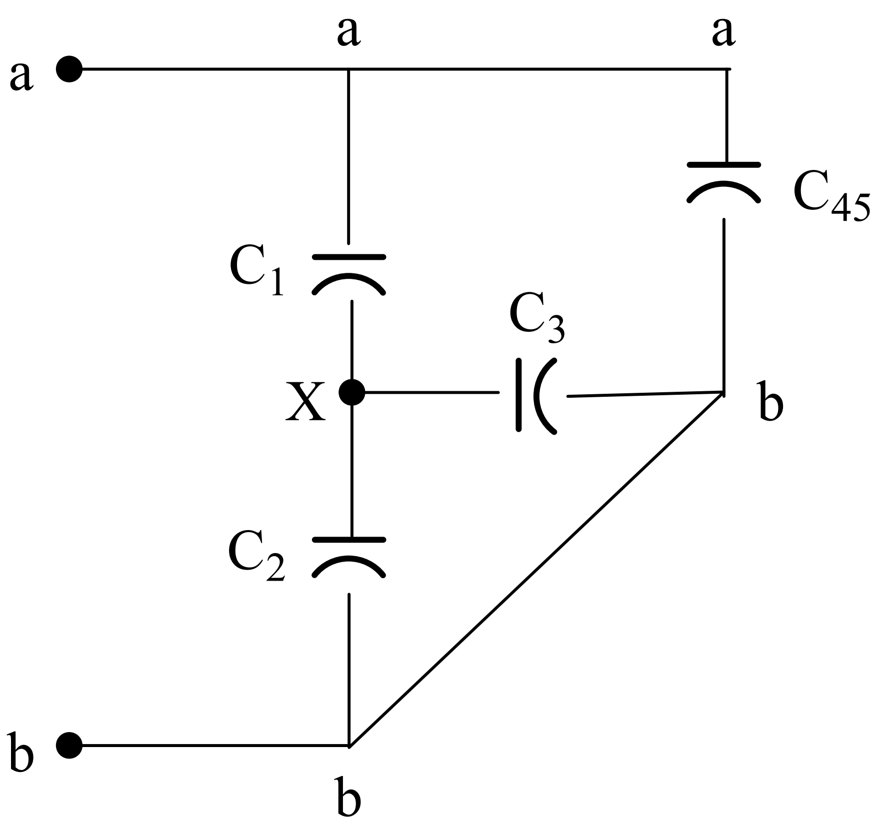
Example:

Find the equivalent capacitance w.r.t. a, b if all C's are 4 μ F

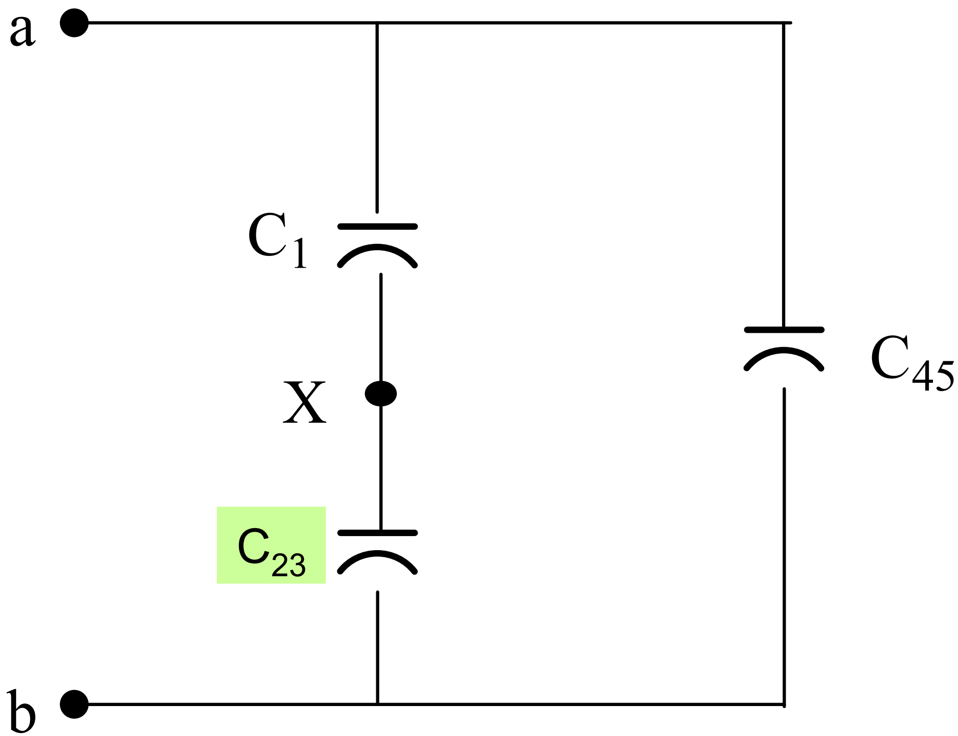


$C_{45} = C_4$ in parallel with C_5

$= C_4 + C_5 = 8\ \mu\text{F}$



$$C_{23} = C_2 \text{ in parallel with } C_3$$
$$= C_2 + C_3 = 8\ \mu\ F$$



$$C_{123} = C_1 \text{ in series with } C_{23}$$

$$C_{123} = \frac{C_1\,C_{23}}{C_1 + C_{23}} = \frac{(4)(8)}{4+8}\,\mu\text{ F} = \frac{32}{12}\,\mu\text{ F} = \frac{8}{3}\,\mu\text{ F}$$

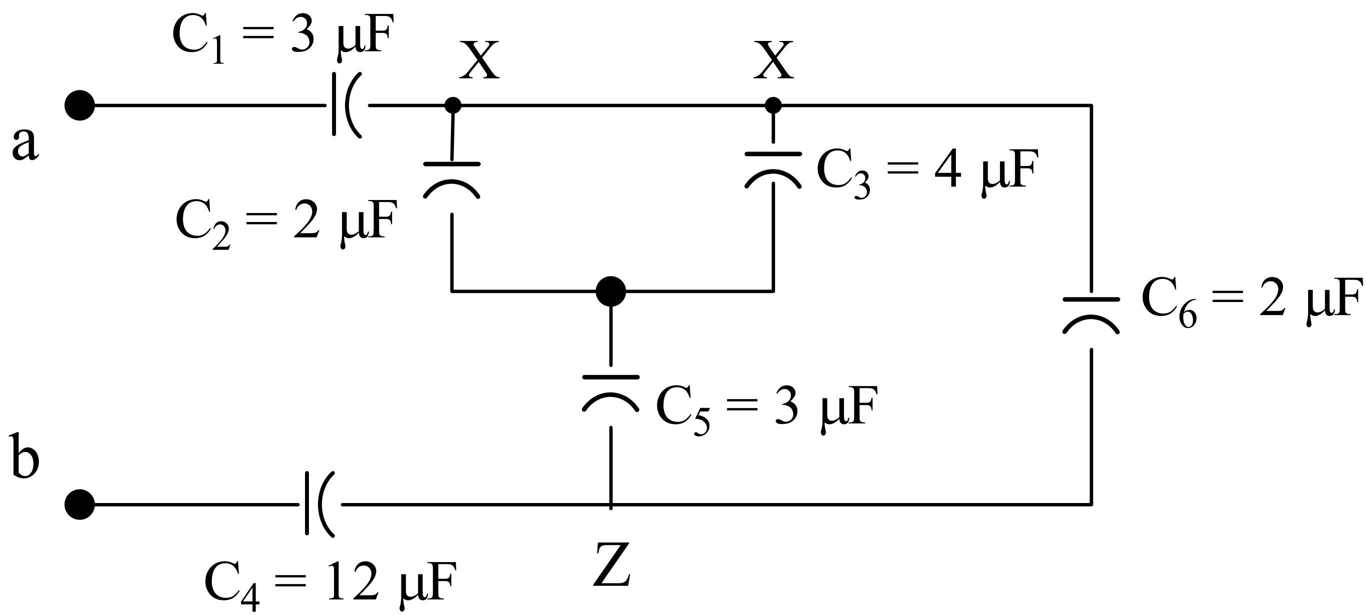
$C_{ab} = C_{123}$ in parallel with C_{45}

$$C_{ab} = C_{123} + C_{45} = \frac{8}{3}\,\mu\,\mathrm{F} + 8\mu\,\mathrm{F} = \frac{8 + 24}{3}\,\mu\,\mathrm{F}$$

$$= \frac{32}{3}\,\mu\,\mathrm{F}$$

Example :

Find the equivalent capacitance w.r.t a, b ?



$C_{23} = C_2$ in parallel with C_3

$= C_2 + C_3 = 6\ \mu\ F$

$C_{235} = C_{23}$ in series with C_5

$$C_{235} = \frac{C_{23}\ C_5}{C_{23} + C_5} = \frac{(6)(3)}{6+3}\mu\ F = 2\ \mu\ F$$

$C_{2356} = C_{235}$ in parallel with C_6

$$= C_{235} + C_6 = (2+2)\ \mu\ F = 4\ \mu\ F$$

$C_{ab} = C_1$ in series with C_{2356} in series with C_4

$$\frac{1}{C_{ab}} = \frac{1}{C_1} + \frac{1}{C_{2356}} + \frac{1}{C_4} = \frac{1}{3\mu} + \frac{1}{4\mu} + \frac{1}{12\mu}$$

$$C_{ab} = 1.5\ \mu\ F$$