Engineering analysis
and numerical.

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Contents:-

chapter one

Determinants and matrices.

chapter 1

Intoduction of determinants and matrices.

consider the following set of equations:-

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1$$
$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$
$$a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n = b_n$$

ترميز

In matrix notation :-

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \Rightarrow AX = b$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{12} \\ a_{21} & a_{22} & \cdots & a_{21} \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

*If $b \neq 0$ they are called non-homogeneous, if $b = 0$ the equation is called homogeneous.

If $b \neq 0$ and $|A| \neq 0$ then we have a unique solution $X = A^{-1} b$

Determinants:-

If A is a square matrix then det. A or $|A|$ is a number calculated from A and found as follows.

$A \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $|A| = ad - bc$

If $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 3 \\ 3 & 0 & 5 \end{bmatrix} \Rightarrow |A| = 2\begin{vmatrix} 5 & 3 \\ 0 & 5 \end{vmatrix} - 3\begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix} + 4\begin{vmatrix} 1 & 5 \\ 3 & 0 \end{vmatrix}$

$\Rightarrow |A| = 2(25) - 3(5-9) + 4(-15) = 50 + 12 - 60 = 2$

H.W If $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix}$, find det.(A)?

ans: 240

Ex

Evaluate the determinant $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 2 & 0 \\ 2 & 0 & 1 & 3 \\ 1 & 2 & 1 & 0 \end{bmatrix}$

Solution

$$\begin{array}{c} \begin{array}{cccc} + & - & + & - \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 2 & 0 \\ 2 & 0 & 1 & 3 \\ 1 & 2 & 1 & 0 \end{bmatrix} \end{array} = 0 \begin{bmatrix} 0 & 2 & 0 \\ 0 & 1 & 3 \\ 2 & 1 & 0 \end{bmatrix} - 1 \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 1 & 1 & 0 \end{bmatrix}$$

$$+2 \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= 0 - \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 1 & 1 & 0 \end{bmatrix} \end{array} + 2 \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix} \end{array} - 3 \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} \end{array}$$

$$= -\left[(-3) - 2(-3)\right] + 2\left[-6\right] - 3\left[(-2) + 2(4)\right]$$

$$= -3 - 12 - 18 = \underline{\underline{-33}}$$

Solution of Linear equations :-

(A) Direct methode :-

1. Cramer rule.
2. matrix inversion method.

في المرحلة الاولى

3. Gauss-elimination method. [Row-operater]

حذف

4. Gauss-Jordan elimination method.
5. Triangularization method <u>or</u> factorization method <u>or</u> decomposition method.

(B) Indirect methode :-

1. Jacobi method of iteration.
2. Gauss-seidel method of iteration.

Ex2 Solve by Gauss-elimination method:-

$$3X + 4Y + 5Z = 18 \;,\quad 2X - Y + 8Z = 13$$
$$5X - 2Y + 7Z = 20 .$$

Solution

$$\because \boxed{AX = b}$$

$$\begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$$

* شروط الطريقة :-

① $A_{3\times3}$

② $|A| \neq 0$

$$\because |A| = 3(-7+16) - 4(14-40) + 5(-4+5) = 27 + 104 + 5 = 136 \neq 0$$

$$\left[\begin{array}{ccc|c} 3 & 4 & 5 & 18 \\ 2 & -1 & 8 & 13 \\ 5 & -2 & 7 & 20 \end{array}\right] \xrightarrow{R_2 = 3R_2 - 2R_1} \left[\begin{array}{ccc|c} 3 & 4 & 5 & 18 \\ 0 & -11 & 14 & 3 \\ 5 & -2 & 7 & 20 \end{array}\right] \xrightarrow{R_3 = 3R_3 - 5R_1}$$

$$\left[\begin{array}{ccc|c} 3 & 4 & 5 & 18 \\ 0 & -11 & 14 & 3 \\ 0 & -26 & -4 & -30 \end{array}\right] \xrightarrow{R_3 = \frac{R_3}{2}} \left[\begin{array}{ccc|c} 3 & 4 & 5 & 18 \\ 0 & -11 & 14 & 3 \\ 0 & -13 & -2 & -15 \end{array}\right] \xrightarrow{R_3 = R_3 - \left(\frac{13}{11}\right) R_2}$$

$$\left[\begin{array}{ccc|c} 3 & 4 & 5 & 18 \\ 0 & -11 & 14 & 3 \\ 0 & 0 & -\frac{204}{11} & -\frac{204}{11} \end{array}\right] \xrightarrow{R_3 = \left(\frac{-11}{204}\right) R_3} \left[\begin{array}{ccc|c} 3 & 4 & 5 & 18 \\ 0 & -11 & 14 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Now:-

$$3X + 4Y + 5Z = 18$$
$$-11Y + 14Z = 3$$
$$Z = 1$$

$$\therefore \boxed{Z = 1} \;;\; \boxed{Y = 1} \;;\; 3X = 9 \rightarrow \boxed{X = 3}$$

Ⓐ Direct method :-

(*) الطريقة الاولى Cramer والثانية matrix inversion تم شرحها بالتفصيل في المرحلة الاولى .

③ Gauss- elimination method (Row operater method):-

① For the matrix $[a_{ij}/b_i]$ $i = 1, 2, --- n$, $j = 1, 2, --- m$

شروط الطريقة
① المصفوفة مربعة
② $|A| \neq 0$

② Multiply the first row R_1 by $\frac{-a_{i1}}{a_{11}}$ and add the ith row for $i = 1, 2, 3 --- n$

③ Repeat the step (2) for second row (R_2) to $(n-1)$ th row.

④ we will get upper triangular matrix, $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$

⑤ Solve for X_n from the n.th equation and solve for X_{n-1}, ---, X_1

Ex/ Use Gauss elimination method to solve the following linear equation :-

$$2X_1 + 4X_2 - 8X_3 = 6$$
$$-X_1 - 3X_2 + 6X_3 = 4$$
$$5X_1 + 7X_2 - 2X_3 = 24$$

شروط الطريقة :
① المصفوفة مربعة $A_{3\times3}$
② $|A| \neq 0$

Solution

$$\begin{array}{ccc} \oplus & \ominus & \oplus \end{array}$$
$$\begin{bmatrix} 2 & 4 & -8 \\ -1 & -3 & 6 \\ 5 & 7 & -2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 24 \end{bmatrix}$$

(*)
$\because |A| = 2(6-42) - 4(2-30) - 8(-7+15) =$
$= -72 + 112 - 64 = -24 \neq 0$

$$\left[\begin{array}{ccc|c} 2 & 4 & -8 & 6 \\ -1 & -3 & 6 & 4 \\ 5 & 7 & -2 & 24 \end{array}\right] \quad R_2 = R_2 + R_1\left(\frac{1}{2}\right)$$

$$\left[\begin{array}{ccc|c} 2 & 4 & -8 & 6 \\ 0 & -1 & 2 & 7 \\ 5 & 7 & -2 & 24 \end{array}\right] \quad R_3 = R_3 + R_1\left(\frac{-5}{2}\right)$$

$$\left[\begin{array}{ccc|c} 2 & 4 & -8 & 6 \\ 0 & -1 & 2 & 7 \\ 0 & -3 & 18 & 9 \end{array}\right] \Rightarrow R_3 = R_3 + (R_2)(-3)$$

$$\left[\begin{array}{ccc|c} 2 & 4 & -8 & 6 \\ 0 & -1 & 2 & 7 \\ 0 & 0 & 12 & -12 \end{array}\right]$$

Now we have the eq.s:-

$2X_1 + 4X_2 - 8X_3 = 6$ — ①

$-X_2 + 2X_3 = 7$ — ②

$12X_3 = -12$ — ③

$\therefore \boxed{X_3 = -1}$; $\boxed{X_2 = -9}$; $\boxed{X_1 = 17}$

Inversion of a matrix using Gauss-elimination method:-

$A \cdot A^{-1} = I$, I: Identity matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

If X is the inverse of A, then $AX = I$. Now, we have to find the elements of X. $[A \mid I]$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{---} \quad (1)$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{---} \quad (2)$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{---} \quad (3)$$

From eqs. (1), (2) and (3) we can solve by using Gauss-elimination method. The solution set of each system (1), (2) and (3) will be the corresponding column of the inverse matrix X.

Ex Find by Gaussian elimination method, the inverse of

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

بنفس الشروط

① $A_{3\times 3}$ = مصفوفة مربعة

② $|A| = 1 \neq 0$

Solution

متغير ثابت

$$(A, I) = \left[\begin{array}{ccc|ccc} 3 & -1 & 1 & 1 & 0 & 0 \\ -15 & 6 & -5 & 0 & 1 & 0 \\ 5 & -2 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 = R_3 - \left(\frac{5}{3}\right) R_1]{R_2 = R_2 + 5R_1}$$

* our aim to reduse the matrix A to an upper triangular matrix.

$$= \left[\begin{array}{ccc|ccc} 3 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 5 & 1 & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -\frac{5}{3} & 0 & 1 \end{array}\right] \xrightarrow{R_3 = R_3 + \left(\frac{1}{3}\right) R_2} \left[\begin{array}{ccc|ccc} 3 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 5 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 1 \end{array}\right]$$

$$\begin{bmatrix} 3 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} X_{11} \\ X_{21} \\ X_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 3 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} X_{12} \\ X_{22} \\ X_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \frac{1}{3} \end{bmatrix}; \quad \begin{bmatrix} 3 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} X_{13} \\ X_{23} \\ X_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$3X_{11} - X_{21} + X_{31} = 1 \qquad 3X_{12} - X_{22} + X_{32} = 0 \qquad 3X_{13} - X_{23} + X_{33} = 0$$

$$X_{21} = 5 \qquad X_{22} = 1 \qquad -X_{23} = 0$$

$$\frac{1}{3}X_{31} = 0 \qquad \frac{1}{3}X_{32} = \frac{1}{3} \qquad \frac{1}{3}X_{33} = 1$$

$$\boxed{\therefore X_{31} = 0,\ X_{21} = 5;\ X_{11} = 2} \qquad \boxed{X_{32} = 1,\ X_{22} = 1;\ X_{12} = 0} \qquad \boxed{X_{33} = 3,\ X_{23} = 0;\ X_{13} = -1}$$

Hence $A^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ $\quad$ $\boxed{\text{by check } A \cdot A^{-1} = I}$

<u>H.W</u> By Gaussian elimination, find the A^{-1}

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 3 & -1 & -4 \end{bmatrix}$$

ans: $\begin{bmatrix} 8/3 & -1 & 2/3 \\ -4/3 & 1 & -1/3 \\ 7/3 & -1 & 1/3 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 3 & -1 & -4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 = 3R_2 - R_3} \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 7 & 4 & 0 & 3 & -1 \\ 3 & -1 & -4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 = R_1 + R_3} \left[\begin{array}{ccc|ccc} 3 & 0 & -3 & 1 & 0 & 1 \\ 0 & 7 & 4 & 0 & 3 & -1 \\ 3 & -1 & -4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 = R_3 - R_1} \left[\begin{array}{ccc|ccc} 3 & 0 & -3 & 1 & 0 & 1 \\ 0 & 7 & 4 & 0 & 3 & -1 \\ 0 & -1 & -1 & -1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 = 7R_3 + R_2} \left[\begin{array}{ccc|ccc} 3 & 0 & -3 & 1 & 0 & 1 \\ 0 & 7 & 4 & 0 & 3 & -1 \\ 0 & 0 & -3 & -7 & 3 & -1 \end{array}\right] \Rightarrow \begin{bmatrix} 3 & 0 & -3 \\ 0 & 7 & 4 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -7 \end{bmatrix}$$

$\therefore 3x_{11} - 3x_{31} = 1 \qquad 7x_{21} + 4x_{31} = 0, \; x_{31} = \frac{7}{3}$

$\therefore 3x_{11} - 3\left(\frac{7}{3}\right) = 1 \Rightarrow \qquad 7x_{21} + 4\left(\frac{7}{3}\right) = 0$

$\therefore x_{11} = 8/3 \qquad \therefore x_{21} = \frac{-28}{21} = \frac{-4}{3}$

$$\begin{bmatrix} 3 & 0 & -3 \\ 0 & 7 & 4 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$

$\therefore 3x_{12} - 3x_{32} = 0 \Rightarrow x_{12} = -1$

$7x_{22} + 4x_{32} = 3 \Rightarrow x_{22} = 1$

$x_{32} = -1$

$$\begin{bmatrix} 3 & 0 & -3 \\ 0 & 7 & 4 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$3x_{13} - 3x_{23} = 1 \Rightarrow x_{13} = 2/3$

$7x_{23} + 4x_{33} = -1 \Rightarrow 7x_{23} = -1 - \frac{4}{3}$

$x_{23} = -1/3$

$-3x_{33} = -1$

$\therefore x_{33} = \frac{1}{3}$

Hence, $A^{-1} = \begin{bmatrix} 8/3 & -1 & 2/3 \\ -4/3 & 1 & -1/3 \\ 7/3 & -1 & 1/3 \end{bmatrix}$

(4) Gauss - Jordan elimination method :-

For the matrix $[A/b]$, and by some elimination steps change the matrix in (1) to another matrix which is $[I/b]$ ، I = Identity matrix

نفس شروط الطريقة السابقة

Ex Use the Gauss - Jordan method to solve the following system :-

$$2X_1 + 4X_2 - 8X_3 = 6 \quad , \quad -X_1 - 3X_2 + 6X_3 = 4$$

$$5X_1 + 7X_2 - 2X_3 = 24$$

*شروط ① المصفوفة مربعة ② $|A| \neq 0$

Solution $\boxed{|A| = -24 \neq 0}$

$$\left[\begin{array}{ccc|c} 2 & 4 & -8 & 6 \\ -1 & -3 & 6 & 4 \\ 5 & 7 & -2 & 24 \end{array}\right] \xrightarrow{R_1 = \frac{R_1}{2}} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 3 \\ -1 & -3 & 6 & 4 \\ 5 & 7 & -2 & 24 \end{array}\right] \xrightarrow[R_3 = R_3 - 5R_1]{R_2 = R_2 + R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 3 \\ 0 & -1 & 2 & 7 \\ 0 & -3 & 18 & 9 \end{array}\right] \xrightarrow[R_3 = \frac{R_3}{3}, \; R_2 = -R_2]{R_1 = R_1 + 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 17 \\ 0 & 1 & -2 & -7 \\ 0 & -1 & 6 & 3 \end{array}\right] \xrightarrow{R_3 = R_3 + R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 17 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & 4 & 4 \end{array}\right] \xrightarrow{R_3 = \frac{R_3}{4}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 17 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{R_2 = R_2 + 2R_3}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 17 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & -1 \end{array}\right] \qquad \boxed{\begin{array}{l} X_1 = 17 \\ X_2 = -9 \\ X_3 = -1 \end{array}}$$

⑤ Triangularization method or factorization method (decomposition method :-

حلول إلى العوامل

طريقة التحليل

$$\because \; [A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

نفس الشروط السابقة :

① المصفوفة مربعة

② $|A| \neq 0$

let $L = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix}$ is Lower triangular matrix ,

or $\begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix}$.

& $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$ is upper triangular matrix.

or $\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$

"اعتماداً على القيم الموجودة"

$\because AX = b$, let $A = LU$

$\Rightarrow LUX = b$, let $UX = y \Rightarrow \boxed{Ly = b}$

Ex 1 By the method of triangularization, solve the following system.

$5x - 2y + z = 4$, $7x + y - 5z = 8$; $3x + 7y + 4z = 10$.

Solution :-

$$\begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}$$

$\boxed{UX = y}$

x
y
z

$\because AX = b \quad ; \quad \text{let } LU = A$

that is $\begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ L_{21}U_{11} & L_{21}U_{12}+U_{22} & L_{21}U_{13}+U_{23} \\ L_{31}U_{11} & L_{31}U_{12}+L_{32}U_{22} & L_{31}U_{13}+L_{32}U_{23}+U_{33} \end{bmatrix} = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$$

Equating coefficients,

$\boxed{U_{11} = 5}$

$\boxed{u_{12} = -2}$

$\boxed{u_{13} = 1}$

$L_{21}U_{11} = 7$

$\Rightarrow \boxed{L_{21} = \frac{7}{5}}$

$L_{21}U_{12} + U_{22} = 1$

$\frac{7}{5} * -2 + U_{22} = 1$

$\boxed{u_{22} = 1 + \frac{14}{5} = \frac{19}{5}}$

$L_{21}U_{13} + U_{23} = -5$

$\frac{7}{5}(1) + U_{23} = -5$

$\boxed{U_{23} = -5 - \frac{7}{5} = \frac{-32}{5}}$

$L_{31}U_{11} = 3$

$\boxed{\therefore L_{31} = \frac{3}{5}}$

$L_{31}U_{12} + L_{32}U_{22}$

$\frac{3}{5}(-2) + L_{32}\left(\frac{19}{5}\right) = 7$

$\therefore \boxed{L_{32} = \frac{7 + \frac{6}{5}}{\frac{19}{5}} = \frac{41}{19}}$

$L_{31}U_{13} + L_{32}U_{23} + U_{33} = 4$

$\frac{3}{5}(1) + \left(\frac{41}{19}\right)\left(\frac{-32}{5}\right) + U_{33} = 4$

$U_{33} = 4 - \frac{3}{5} + \frac{1312}{95}$

$\therefore \boxed{U_{33} = \frac{327}{19}}$

$\because AX = b$ & $LU = A \Rightarrow LUX = b$

let $\boxed{UX = y} \Rightarrow \boxed{Ly = b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{7}{5} & 1 & 0 \\ \frac{3}{5} & \frac{41}{19} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}$$

$\boxed{y_1 = 4}$, $\frac{7}{5}y_1 + y_2 = 8$; $\frac{3}{5}y_1 + \frac{41}{19}y_2 + y_3 = 10$

$\therefore y_2 = 8 - \frac{7}{5}(4) \Rightarrow \boxed{y_2 = 8 - \frac{28}{5} = \frac{12}{5}}$

$y_3 = 10 - \frac{3}{5}(4) - \frac{41}{19}\left(\frac{12}{5}\right) \Rightarrow \boxed{y_3 = \frac{46}{19}}$

$\boxed{\because UX = y} \Rightarrow \begin{bmatrix} 5 & -2 & 1 \\ 0 & \frac{19}{5} & \frac{-32}{5} \\ 0 & 0 & \frac{327}{19} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ \frac{12}{5} \\ \frac{46}{19} \end{bmatrix}$

$\Rightarrow 5x - 2y + z = 4$, $\frac{19}{5}y - \frac{32}{5}z = \frac{12}{5}$; $\frac{327}{19}z = \frac{46}{19}$

$\Rightarrow \boxed{z = \frac{46}{327}}$; $\frac{19}{5}y = \frac{12}{5} + \frac{32}{5}\left(\frac{46}{327}\right) \Rightarrow \boxed{y = \frac{284}{327}}$

$\Rightarrow 5x = 4 + 2\left(\frac{284}{327}\right) - \frac{46}{327} \Rightarrow \boxed{x = \frac{366}{327}}$

Ex 2 solve by Triangularization method, the following system :-

$x+5y+z=14$, $2x+y+3z=13$; $3x+y+4z=17$

Solution $\because AX=b$

$$\begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 17 \end{bmatrix}$$; let $LU=A$

$$\begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \underset{1}{u_{11}} & \underset{5}{u_{12}} & \underset{1}{u_{13}} \\ L_{21}u_{11}=2 & L_{21}u_{12}+u_{22}=1 & L_{21}u_{13}+u_{23}=3 \\ \underset{=3}{L_{31}u_{11}} & \underset{=1}{L_{31}u_{12}+L_{32}u_{22}} & \underset{=4}{L_{31}u_{13}+L_{32}u_{23}+u_{33}} \end{bmatrix} = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix}$$

$\boxed{u_{11}=1}$; $\boxed{u_{12}=5}$ $\boxed{u_{13}=1}$

$\therefore \boxed{L_{21}=2}$ $\boxed{u_{22}=-9}$ $\boxed{u_{23}=1}$

$\boxed{L_{31}=3}$

$15+L_{32}(-9)=1$

$\boxed{L_{32}=\frac{14}{9}}$

$3+\frac{14}{9}(1)+u_{33}=4$

$\boxed{u_{33}=1-\frac{14}{9}=\frac{-5}{9}}$

$\because \underline{LUX = b}$ & $\therefore y = UX \Rightarrow Ly = b$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{14}{9} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 17 \end{bmatrix}$$

$\therefore \boxed{y_1 = 14}$ $\qquad 2y_1 + y_2 = 13$ & $\quad 3y_1 + \frac{14}{9} y_2 + y_3 = 17$

$\boxed{\therefore y_2 = 13 - 28 = -15}$ $\qquad 3(14) + \left(\frac{14}{9}\right)(-15) + y_3 = 17$

$\boxed{\therefore y_3 = \frac{-5}{3}}$

$\therefore UX = y$

$$\begin{bmatrix} 1 & 5 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & -\frac{5}{9} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ -15 \\ -\frac{5}{3} \end{bmatrix}$$

$\therefore x + 5y + z = 14 \quad ; \quad -9y + z = -15$

$-\frac{5}{9} z = \frac{-5}{3} \Rightarrow \boxed{z = 3} \quad ; \quad -9y = -15 - 3 = -18$

$\boxed{y = 2} \Rightarrow x = 14 - 10 - 3 \Rightarrow \boxed{x = 1}$

(B) Indirect methods (iterative methods) for solving linear system:-

The system :- $a_{11}X_1 + a_{12}X_2 + a_{13}X_3 = b_1$

$$a_{21}X_1 + a_{22}X_2 + a_{23}X_3 = b_2$$

$$a_{31}X_1 + a_{32}X_2 + a_{33}X_3 = b_3$$

will be solvable by this method if :- الفقر

شرط

شرط الطريقة الغير مباشرة

$$\left\{\begin{array}{l} |a_{11}| > |a_{12}| + |a_{13}| \\ |a_{22}| > |a_{21}| + |a_{23}| \\ |a_{33}| > |a_{31}| + |a_{32}| \end{array}\right.$$

أكبر فقط بدون مساواة

ملاحظة :- يجب ان يكون العنصر القطري لـ A اكبر من مجموع قيم بقية الصف بعد أخذ القيمة المطلقة لها .

In other words, the solution will exist (iteration will converge) if the absolute values of the leading diagonal elements of the coefficient matrix A of the system AX=b are greater than the sum of absolute values of the other coefficients of the row.

ملاحظة :- اذا لم تتحقق أحد هذه الشروط يتم التبديل بين صف وصف لأنه في حالة عدم تحقق أحد الشروط فان الحالة هذه تكون متباعدة وليست متقاربة .

Note:- Sometimes the iterative methods does not work. let us (دعنا) experiment and see that a rearrangement of the original linear system, can result in a system of iteration equations that will produce a divergent sequence of points because the strictly (بدقة) diagonally dominant not satisfied.

Ex

$$\left.\begin{array}{l} X + 5Z = 1 \\ x + 3y + z = 2 \\ -4X + 2y + Z = 3 \end{array}\right\} \Rightarrow \begin{array}{ll} |1| > |5| & \times \\ |3| > |1| + |1| & \checkmark \\ |1| > |-4| + |2| & \times \end{array}$$

* يجب اعادة ترتيب المعادلات لتحقيق الشرط [بالاعتماد على أكبر القيم القطر].

$$\begin{array}{lll} -4X + 2y + Z = 3 & \Rightarrow |-4| > |2| + |1| & \checkmark \\ X + 3y + Z = 2 & \Rightarrow |3| > |1| + |1| & \checkmark \\ X + 0y + 5Z = 1 & \Rightarrow |5| > |1| & \checkmark \end{array}$$

∴ تحقق الشرط

---

accuracey :- we must satisfied the accuracy condition.

K : is iteration $= 1, 2, 3 ----$

ϵ : error نسبة الخطأ تعطى في السؤال وهي كمية موجبة مثل: $0.001, 0.0001 ---$

$$\max. |X^{k+1} - X^{k}| \leq \epsilon$$

$$\max. \{ |X^{k+1} - X^{k}|, |y^{k+1} - y^{k}|, |z^{k+1} - z^{k}| \} \leq \epsilon$$

لازم الكل اقل من ϵ واذا واحد اكبر ∴ نعيد التكرار.

① Jacobi method of iteration or Gauss-Jacobi method:

$$X_1^{k+1} = \frac{b_1 - a_{12} X_2^{k} - a_{13} X_3^{k}}{a_{11}}$$

[illegible]

$$X_2^{k+1} = \frac{b_2 - a_{21} X_1^{k} - a_{23} X_3^{k}}{a_{22}}$$

$$X_3^{k+1} = \frac{b_3 - a_{31} X_1^{k} - a_{32} X_2^{k}}{a_{33}}$$

Where ; X_1^0 , X_2^0 and X_3^0 is constant. وهن معلومات السؤال واذا لم تعطى تكون صفر.

EX solve the following system by Gauss-Jacobi :-

$10X - 5y - 2z = 3$, $4X - 10y + 3z = -3$; $X + 6y + 10z = -3$.

Correct to 3 decimal places? 0.0001 $\boxed{\text{معناه } \epsilon = 0.0001}$

Solution $\boxed{\because AX = b}$

$$\begin{bmatrix} 10 & -5 & -2 \\ 4 & -10 & 3 \\ 1 & 6 & 10 \end{bmatrix} \begin{bmatrix} X \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix}$$

لازم: $|10| > |-5| + |-2|$ ✓

$|-10| > |4| + |3|$ ✓

$|10| > |1| + |6|$ ✓

$$\therefore \boxed{X^{k+1} = \frac{3 + 5y^{k} + 2z^{k}}{10}} \quad , \quad \boxed{y^{k+1} = \frac{-3 - 4X^{k} - 3z^{k}}{-10}}$$

$$\boxed{z^{k+1} = \frac{-3 - X^{k} - 6y^{k}}{10}}$$

Let the initial values $(X^0, Y^0, Z^0) = (0,0,0)$ اذا لم تعطى بالسؤال

$\therefore$ at $k=0$
First iteration

$$X^1 = \frac{3+5(0)+2(0)}{10} = 0.3$$

$$y^1 = \frac{-3-4(0)-3(0)}{-10} = 0.3$$

$$Z^1 = \frac{-3-0-6(0)}{10} = -0.3$$

---

at $k=1$
Second iteration

$$X^2 = \frac{3+5(0.3)+2(-0.3)}{10} = 0.39$$

$$y^2 = \frac{-3-4(0.3)-3(-0.3)}{-10} = 0.33$$

$$Z^2 = \frac{-3-0.3-6(0.3)}{10} = -0.51$$

$$|X^2 - X^1| = |0.39 - 0.3| = 0.09 > 0.0001 \; \alpha$$

اذا شرط واحد لم يتحقق لنسبة الخطأ $\therefore$ نعيد التكرار

iteration	X	y	Z
1	0.3	0.3	-0.3
2	0.39	0.33	-0.51
3	0.363	0.303	-0.537
4	0.3441	0.2841	-0.5181
5	0.33843	0.2822	-0.50487
6	0.340126	0.283911	-0.503163
7	0.3413229	0.2851015	-0.504359
8	0.34167891	0.2852214	-0.505193
9	0.341572	0.28511136	-0.5053007

$] \in$

* عندما تثبت ثلاث مراتب حيث بالطرح بين القيمة الاخيرة وسابقتها يكون الناتج اقل من $\in = 0.0001$

$$\{ |X^9 - X^8| \text{ و } |y^9 - y^8| \text{ و } |Z^9 - Z^8| \} \leq \in$$

$$\{ 0.0001 \text{ و } 0.0001 \text{ و } 0.0001 \}$$

الكل ثلاث مراتب عشرية

Hence the values correct to 3 decimal places are:- $\boxed{x = 0.342}$, $\boxed{y = 0.285}$; $\boxed{z = -0.505}$

⊛ <u>Note</u>:- After getting the values of the unknowns, substitute these values in the given equations, and check the correctness of the results. ⊛

المفروض هذه القيم تحقق المعادلة الأصلية

② Gauss-Seidel method of iteration :-

ولنفس الشرط السابق :

* عدد خطوات هذه الطريقة اقل وهي اكثر دقة من الطريقة السابقة

$$|a_{11}| > |a_{12}| + |a_{13}|$$

$$|a_{22}| > |a_{21}| + |a_{23}|$$

$$|a_{33}| > |a_{31}| + |a_{32}|$$

$$X_1^{k+1} = \frac{b_1 - a_{12} X_2^{k} - a_{13} X_3^{k}}{a_{11}}$$

$$X_2^{k+1} = \frac{b_2 - a_{21} X_1^{k+1} - a_{23} X_3^{k}}{a_{22}}$$

$$X_3^{k+1} = \frac{b_3 - a_{31} X_1^{k+1} - a_{32} X_2^{k+1}}{a_{33}}$$

Ex * أعادة حل نفس المثال السابق ص21 بطريقة Gauss-Seidel

$\because A X = b$

$$\begin{bmatrix} 10 & -5 & -2 \\ 4 & -10 & 3 \\ 1 & 6 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix}$$

* الشرط

$|10| > 7$ ✓

$|-10| > 7$ ✓

$|10| > 7$ ✓

$$x^{k+1} = \frac{3 + 5y^{k} + 2z^{k}}{10} , \quad y^{k+1} = \frac{-3 - 4x^{k+1} - 3z^{k}}{-10}$$

$$z^{k+1} = \frac{-3 - x^{k+1} - 6y^{k+1}}{10}$$

Let the initial values $y^0 = 0$, $z^0 = 0$ اذا لم تعطى بالسؤال

$\therefore$ at $k=0$

First iteration

$$x^1 = \frac{3 + 5(0) + 2(0)}{10} = 0.3$$

$$y^1 = \frac{-3 - 4(0.3) - 3(0)}{-10} = 0.42$$

$$z^1 = \frac{-3 - (0.3) - 6(0.42)}{10} = -0.582$$

---

at $k=1$

second iteration

$$x^2 = \frac{3 + 5(0.42) + 2(-0.582)}{10} = 0.3936$$

$$y^2 = \frac{-3 - 4(0.3936) - 3(-0.582)}{-10} = 0.28284$$

$$z^2 = \frac{-3 - (0.3936) - 6(0.28284)}{10} = -0.509064$$

$|x^2 - x^1| = |0.3936 - 0.3| = 0.0936 > 0.0001\ \alpha$ $\therefore$ نعيد التكرار وهكذا.

iteration	x	y	z
1	0.3	0.42	−0.582
2	0.3936	0.28284	−0.509064
3	0.3396072	0.2831236	−0.503834
4	0.340794	0.285167	−0.5051799
5	0.3415547	0.2850679	−0.5051962
6	0.3414947	0.285039	−0.5051728
7			

$|0.3414947 - 0.3415547| = 0.000$ ✓

$|0.285039 - 0.2850671| = 0.000$ ✓

$|-0.5051728 + 0.5051961| = 0.000$ ✓

Eigen values and Eigen vectors :-

Definition:- Let $A = [a_{ij}]$ be an $n \times n$ matrix.
A non-zero vector, X is said to be a characteristic vector of A if there exists a scalar λ such that $AX = \lambda X$.
This is referred to as the eigen value, values of the scalar λ for which non-trivial solutions exist are called eigen values and corresponding solutions $\boxed{X \neq 0}$ are called the eigen vectors.

The characteristic equation:- The set of simultaneous equations, $AX = \lambda X$, where $A_{n \times n}$, $X_{n \times 1}$ column vector can be written form,

$$(A - \lambda I)X = 0 \quad ; \text{where } I: \text{identity matrix}$$

The non-trivial solution exist for the homo. equation if $\phi(\lambda) = |A - \lambda I| = 0$ where $\phi(\lambda) = \lambda^n + C_{n-1}\lambda^{n-1} \cdots + C_1\lambda + C_0 = 0$ is called the characteristic equation of A.

Ex// $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ Find characteristic equation.

$$\because \phi(\lambda) = |A - \lambda I| = 0 \Rightarrow \phi(\lambda) = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{array}{c} \begin{matrix} \oplus & \ominus & \oplus \end{matrix} \\ \begin{bmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{bmatrix} \end{array}$$

$$= (1-\lambda)[(2-\lambda)(-1-\lambda) - 1] - [(1+\lambda)] - 2[-1] = 0 \Rightarrow (1-\lambda)(-3-\lambda+\lambda^2) + 1 - \lambda = 0$$

$$= \boxed{\lambda^3 - 2\lambda^2 - \lambda + 2 = 0} \text{ char. eq.}$$

Ex Find the eigen values and the corresponding vectors of

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

Solution If λ is an eigen value of A and X is the corresponding eigen vector then :-

$$(A-\lambda I)X = 0 \Rightarrow \begin{bmatrix} 1-\lambda & 2 & -1 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{---} (1)$$

The characteristic eq. of A is $\boxed{|A-\lambda I| = 0}$

$$\therefore |A-\lambda I| = \begin{vmatrix} \overset{\oplus}{1-\lambda} & \overset{\ominus}{2} & \overset{\oplus}{-1} \\ 0 & 2-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)(-2-\lambda) + 0 + 0 = 0$$

$$\therefore (1-\lambda)(2-\lambda)(-2-\lambda) = 0$$

$$\therefore \boxed{\lambda = 1}, \ \boxed{\lambda = 2} \text{ and } \boxed{\lambda = -2}$$

$\therefore$ The eigen values of A are $1, 2, -2$

Eigen vector of A :-

(1) If $\boxed{\lambda = 1}$, put $\lambda = 1$ in eq. (1)

$$\Rightarrow \begin{bmatrix} 0 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} 2x_2 - x_3 &= 0 \\ x_2 + 2x_3 &= 0 \\ -3x_3 &= 0 \end{aligned}$$

$$\Rightarrow x_3 = 0 \ \& \ x_2 = 0$$

Note that we can not find X_1 from these equations, because X_1 is not present in any of these equations.

Hence $\boxed{X_1=\alpha}$, $X_2=0$; $X_3=0$

Then $\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$; where $\alpha \neq 0$

Hence $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the eigen vector of A at $\lambda=1$.

---

② If $\boxed{\lambda=2}$, put $\lambda=2$ in eq. ① :

$$\Rightarrow \begin{bmatrix} -1 & 2 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} -X_1+2X_2-X_3 &= 0 \\ 2X_3 &= 0 \\ -4X_3 &= 0 \end{aligned}$$

$\Rightarrow X_3=0$ & $-X_1+2X_2=0$ or $X_1=2X_2$; let $\boxed{X_2=\beta}$

then $X_1=2\beta$. Hence $\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 2\beta \\ \beta \\ 0 \end{bmatrix} = \beta \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ is eigen vector of A at $\lambda=2$, where $\beta \neq 0$

---

③ If $\boxed{\lambda=-2}$, put $\lambda=-2$ in eq. ① :

$$\Rightarrow \begin{bmatrix} 3 & 2 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} 3X_1+2X_2-X_3 &= 0 \\ 4X_2+2X_3 &= 0 \\ 0 &= 0 \end{aligned}$$

$\Rightarrow X_2=-\frac{1}{2}X_3$ & $3X_1=2X_3$ $\therefore X_1=\frac{2X_3}{3}$; let $\boxed{X_3=\gamma}$

then $X_2=\frac{-\gamma}{2}$; $X_1=\frac{2\gamma}{3}$.

Hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2\gamma/3 \\ -\gamma/2 \\ \gamma \end{bmatrix} = \gamma \begin{bmatrix} 2/3 \\ -1/2 \\ 1 \end{bmatrix}$ is eigen vector of A at $\lambda = -2$, where, $\gamma \neq 0$

Notes

① The sum of the eigen values = sum of the diagonal elements of A.

② The product of the $\boxed{\text{eigen values}}$ = $|A|$

Ex Find the eigen valus and eigen Vector of the matrix

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} ?$$

solution The char. eq. of A is $|A-\lambda I| = 0 \Rightarrow$

$$\begin{array}{ccc} \oplus & \ominus & \oplus \end{array}$$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (6-\lambda)\left[(3-\lambda)^2 - 1\right] + 2\left[-2(3-\lambda) + 2\right] + 2\left[2 - 2(3-\lambda)\right] = 0$$

$$\Rightarrow (6-\lambda)\left[9 - 6\lambda + \lambda^2 - 1\right] + 2\left[-6 + 2\lambda + 2\right] + 2\left[2 - 6 + 2\lambda\right] = 0$$

$$\Rightarrow (6-\lambda)\left[\lambda^2 - 6\lambda + 8\right] + 2\left[2\lambda - 4\right] + 2\left[2\lambda - 4\right] = 0$$

$$\Rightarrow \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

1 2 3

* ثلاث جذور موجبة

وبفرض $-\lambda$ لإيجاد الجذور السالبة

$$-\lambda^3 - 12\lambda^2 - 36\lambda - 32 = 0$$

* ولا جذر سالب

$$\begin{array}{r|l} & \lambda^2 - 10\lambda + 16 \\ \hline \lambda - 2 & \lambda^3 - 12\lambda^2 + 36\lambda - 32 \\ & \mp\lambda^3 \pm 2\lambda^2 \\ \hline & -10\lambda^2 + 36\lambda - 32 \\ & \pm 10\lambda^2 \mp 20\lambda \\ \hline & 16\lambda - 32 \\ & 16\lambda - 32 \\ \hline & 0 \quad 0 \end{array}$$

لازم

$\therefore$ let $\lambda=2 \Rightarrow (\lambda-2)(\lambda^2-10\lambda+16)=0$

$(\lambda-2)(\lambda-8)(\lambda-2)=0 \Rightarrow \boxed{\lambda=2}, \boxed{\lambda=8}$ and $\boxed{\lambda=2}$

$\therefore$ The eigen values of A are $2, 2, 8$.

Eigenvector of A :-

$$\therefore (A-\lambda I)X=0 \Rightarrow \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

① If $\lambda=2 \Rightarrow \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ by Row operater method.

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \xrightarrow[R_3=2R_3-R_1]{R_2=2R_2+R_1} \begin{bmatrix} 4 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 4X_1-2X_2+2X_3=0 \Rightarrow 2X_1-X_2+X_3=0$

let $X_2=\alpha_1$; $X_3=\alpha_2 \Rightarrow X_1=\frac{\alpha_1}{2}-\frac{\alpha_2}{2}$

$$\therefore \text{Hence } \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} \frac{\alpha_1}{2}-\frac{\alpha_2}{2} \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \frac{\alpha_1}{2} \\ \alpha_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{\alpha_2}{2} \\ 0 \\ \alpha_2 \end{bmatrix} \Rightarrow \begin{bmatrix} X_1 \\ X_2 \\ X_2 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$$

is the eigen vector of A at $\lambda=2$, where $\alpha_1 \neq 0$, $\alpha_2 \neq 0$

Here we are getting $\underbrace{\alpha_1\begin{bmatrix}1/2\\1\\0\end{bmatrix}}$ and $\underbrace{\alpha_2\begin{bmatrix}-1/2\\0\\1\end{bmatrix}}$ as eigen vectors of A at $\lambda = 2$.

② If $\lambda = 8$ $\Rightarrow \begin{bmatrix}-2 & -2 & 2\\ -2 & -5 & -1\\ 2 & -1 & -5\end{bmatrix}\begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$$\xrightarrow[R_3 = R_3 + R_1]{R_2 = R_2 - R_1} \begin{bmatrix}-2 & -2 & 2\\ 0 & -3 & -3\\ 0 & -3 & -3\end{bmatrix} \xrightarrow{R_3 = R_3 - R_2} \begin{bmatrix}-2 & -2 & 2\\ 0 & -3 & -3\\ 0 & 0 & 0\end{bmatrix}$$

$$= \begin{bmatrix}-2 & -2 & 2\\ 0 & -3 & -3\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \Longrightarrow \begin{aligned}-2X_1 - 2X_2 + 2X_3 &= 0\\ -3X_2 - 3X_3 &= 0\end{aligned}$$

$= X_2 = -X_3$ let $X_3 = \beta$

& $-X_1 - X_2 + X_3 = 0 \Rightarrow X_1 = X_3 - X_2 = \beta + \beta = 2\beta$

$\therefore$ Hence $\begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix} = \begin{bmatrix}2\beta\\-\beta\\\beta\end{bmatrix} = \beta\begin{bmatrix}2\\-1\\1\end{bmatrix}$ is the eigen vector of A at $\lambda = 8$. where $\beta \neq 0$

Ex Find eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 3 & -3 & 2 \\ -1 & 5 & -2 \\ -1 & 3 & 0 \end{bmatrix} ?$$

solution

$$\because |A-\lambda I| = 0 \Rightarrow \begin{array}{c} \begin{matrix} + & \quad - & \quad + \end{matrix} \\ \begin{vmatrix} 3-\lambda & -3 & 2 \\ -1 & 5-\lambda & -2 \\ -1 & 3 & -\lambda \end{vmatrix} \end{array} = 0$$

$$\Rightarrow (3-\lambda)\left[(5-\lambda)(-\lambda)+6\right] + 3\left[\lambda-2\right] + 2\left[\overbrace{-3+(5-\lambda)}^{2-\lambda}\right] = 0$$

$$\therefore (3-\lambda)\left[\lambda^2-5\lambda+6\right] + (3\lambda-6) + 4-2\lambda = 0$$

$$\Rightarrow (3-\lambda)(\lambda-3)(\lambda-2) + (\lambda-2) = 0$$

$$(\lambda-2)\left[-(\lambda-3)(\lambda-3)+1\right] = 0 \Rightarrow (\lambda-2)\left[-\lambda^2+6\lambda-8\right] = 0$$

$$\Rightarrow (\lambda-2)(\lambda^2-6\lambda+8) = 0 \Rightarrow (\lambda-2)(\lambda-2)(\lambda-4) = 0$$

$\Rightarrow \lambda = 2, 2, 4$. $\therefore$ The eigen values are $2, 2, 4$.

The eigen vactor :-

① If $\lambda = 2 \Rightarrow \begin{bmatrix} 1 & -3 & 2 \\ -1 & 3 & -2 \\ -1 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\therefore \begin{bmatrix} 1 & -3 & 2 \\ -1 & 3 & -2 \\ -1 & 3 & -2 \end{bmatrix} \xrightarrow[R_3 = R_3 - R_2]{R_2 = R_2 - R_3} \begin{bmatrix} 1 & -3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \therefore X_1 - 3X_2 + 2X_3 = 0$$

let $X_2 = \alpha_1$, $X_3 = \alpha_2 \Rightarrow X_1 = 3X_2 - 2X_3$

$\therefore X_1 = 3\alpha_1 - 2\alpha_2$

Hence $\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 3\alpha_1 - 2\alpha_2 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \alpha_1 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

Here we are getting $\alpha_1 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\alpha_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ as eigen vectors of A at $\lambda = 2$, where α_1 and $\alpha_2 \neq 0$

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(2) If $\lambda = 4 \Rightarrow \begin{bmatrix} -1 & -3 & 2 \\ -1 & 1 & -2 \\ -1 & 3 & -4 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\xrightarrow[R_3 = R_3 - R_2]{R_2 = R_2 - R_3} \begin{bmatrix} -1 & -3 & 2 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$

$-X_1 - 3X_2 + 2X_3 = 0$

$-2X_2 + 2X_3 = 0$

$2X_2 - 2X_3 = 0$

let $X_3 = \alpha_3$

$\Rightarrow X_2 = X_3 = \alpha_3$; $\therefore -X_1 - 3X_3 + 2X_3 = 0 \Rightarrow X_1 = -\alpha_3$

Hence $\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} -\alpha_3 \\ \alpha_3 \\ \alpha_3 \end{bmatrix} = \alpha_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

The Cayley – Hamilton theorem :- كيلي هملتن

Definition :- an expression of the form :-

$$f(x) = A_0 + A_1 x + A_2 x^2 + \cdots A_m x^m , \quad A_m \neq 0 ,$$

where $A_0, A_1, A_2, \ldots, A_m$ are matrices each of order $n \times n$, is called a matric polynomial of degree m.

Every square matrix satisfies its own characteristic equation, $|A - \lambda I| = 0$.

Let $|A - \lambda I| = (-1)^n [\lambda^n + a_1 \lambda^{n-1} + \cdots + a_n]$

Determination of A^{-1} by using Cayley-Hamilton theorem:

A satisfies it characteristic equation i.e.

$$(-1)^n [A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots a_n I] = 0$$

$$\Rightarrow A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots a_n I = 0] * A^{-1}$$

$$\Rightarrow A^{-1} [A^n + a_1 A^{n-1} + a_2 A^{n-2} \cdots a_n I] = 0$$

If A is non-singular, then we have :-

$$a_n A^{-1} = -A^{n-1} - a_1 A^{n-2} - \cdots - a_{n-1} I$$

$$\Rightarrow A = \left[\frac{-1}{a_n}\right] [A^{n-1} - A^{n-2} + \cdots + a_{n-1} I]$$

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* تنفع في إيجاد $A^{-1}, A^{-2}, A^{-3}, \ldots$

$$\boxed{A A^{-1} = A^{-1} A = I}$$

$$\boxed{I A = A}$$

$$\boxed{I A^{-1} = A^{-1}}$$

$$\left\{ \begin{array}{l} \lambda \longrightarrow A \\ a \longrightarrow aI \\ \lambda^2 \longrightarrow A^2 \end{array} \right.$$

a : constant

Ex: If $A = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$ verify Cayley-Hamilton theorem?

Find A^{-1}?

solution

$\because |A - \lambda I| = 0 \Rightarrow \begin{array}{c} \begin{array}{ccc} \oplus & \ominus & \oplus \end{array} \\ \begin{bmatrix} 2-\lambda & 1 & 2 \\ 5 & 3-\lambda & 3 \\ -1 & 0 & -2-\lambda \end{bmatrix} \end{array} = 0$

$$\Rightarrow (2-\lambda)\left[(3-\lambda)(-2-\lambda)\right] - \left[5(-2-\lambda)+3\right] + 2\left[3-\lambda\right] = 0$$

$$\Rightarrow (2-\lambda)\left[-6-\lambda+\lambda^2\right] - \left[-10-5\lambda+3\right] + 6-2\lambda = 0$$

$$\Rightarrow -12 - 2\lambda + 2\lambda^2 + 6\lambda + \lambda^2 - \lambda^3 + 7 + 5\lambda + 6 - 2\lambda = 0$$

$$\Rightarrow -\lambda^3 + 3\lambda^2 + 7\lambda + 1 = 0 \Rightarrow \boxed{\lambda^3 - 3\lambda^2 - 7\lambda - 1 = 0}$$

- To verify Cayley-Hamilton theorem, we have to show that:-

نعوض مكان λ ب A
والثابت نضرب $I*$

$$\Rightarrow A^3 - 3A^2 - 7A - I = 0.$$

Now $A^2 = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 4+5-2 & 2+3 & 4+3-4 \\ 10+15-3 & 5+9 & 10+9-6 \\ -2+2 & -1 & -2+4 \end{bmatrix}$

$\Rightarrow A^2 = \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix}$; $\because A^3 = A^2 \cdot A = \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$

$$\therefore A^3 = \begin{bmatrix} 14+25-3 & 7+15 & 14+15-6 \\ 44+70-13 & 22+42 & 44+42-26 \\ -5-2 & -3 & -3-4 \end{bmatrix} = \begin{bmatrix} 36 & 22 & 23 \\ 101 & 64 & 60 \\ -7 & -3 & -7 \end{bmatrix}$$

Now, $A^3 - 3A^2 - 7A - I = \begin{bmatrix} 36 & 22 & 23 \\ 101 & 64 & 60 \\ -7 & -3 & -7 \end{bmatrix} - 3\begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix}$

$$-7\begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 36-21-14-1 & 22-15-7 & 23-9-14 \\ 101-66-35 & 64-42-21-1 & 60-39-21 \\ -7+7 & -3+3 & -7-6+14-1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore$ This verifies Cayley-Hamilton theorem.

To find A^{-1} : $\because [A^3 - 3A^2 - 7A - I = 0] * A^{-1}$ (* نضرب A^{-1})

$$\therefore A^{-1}[A^3 - 3A^2 - 7A - I = 0] \Rightarrow A^2 - 3A - 7I - A^{-1} = 0$$

$$\boxed{A^{-1} = A^2 - 3A - 7I} \Rightarrow A^{-1} = \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix} - 3\begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 7-6-7 & 5-3 & 3-6 \\ 22-15 & 14-9-7 & 13-9 \\ 3 & -1 & 2+6-7 \end{bmatrix} = \begin{bmatrix} -6 & 2 & -3 \\ 7 & -2 & 4 \\ 3 & -1 & 1 \end{bmatrix}$$

Check $AA^{-1} = I$

Ex Using Cayley-Hamilton theorem to find A^{-1} and A^4 if :-

$$A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$$

Solution

$$\because |A-\lambda I| = 0 \Rightarrow \begin{array}{c} \begin{array}{ccc} \oplus & \ominus & \oplus \end{array} \\ \begin{bmatrix} 7-\lambda & 2 & -2 \\ -6 & -1-\lambda & 2 \\ 6 & 2 & -1-\lambda \end{bmatrix} \end{array} = 0$$

$$\Rightarrow (7-\lambda)\left[(-1-\lambda)^2-4\right] - 2\left[-6(-1-\lambda)-12\right] - 2\left[-12-6(-1-\lambda)\right] = 0$$

$$\Rightarrow (7-\lambda)\left[1+2\lambda+\lambda^2-4\right] - 2\left[6\lambda-6\right] - 2\left[6\lambda-6\right] = 0$$

$$\Rightarrow (7-\lambda)(\lambda^2+2\lambda-3) - 24(\lambda-1) = 0$$

$$\Rightarrow 7\lambda^2 + \underline{14\lambda} - 21 - \lambda^3 - 2\lambda^2 + \underline{3\lambda} - 24\lambda + 24 = 0$$

$$\Rightarrow -\lambda^3 + 5\lambda^2 - 7\lambda + 3 = 0 \Rightarrow \boxed{\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0}, \text{ Char. eq.}$$

By Cayley-Hamilton theorem, we must have :-

$$A^3 - 5A^2 + 7A - 3I = 0 \;\Big]\; * A^{-1}$$

(*) نعوض مكان كل λ بـ A
نضرب الثابت $* I$
نضرب الكل $* A^{-1}$

$$\therefore A^2 - 5A + 7I - 3A^{-1} = 0 \Rightarrow \boxed{A^{-1} = \frac{1}{3}\left[A^2 - 5A + 7I\right]}$$

$$\therefore A^2 = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 49-12-12 & 14-2-4 & -14+4+2 \\ -42+6+12 & -12+1+4 & 12-2-2 \\ 42-12-6 & 12-2-2 & -12+4+1 \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} \qquad \because A^{-1} = \frac{1}{3}[A^2 - 5A + 7I]$$

$$\therefore A^2 - 5A + 7I = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} - 5\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 25-35+7 & 8-10 & -8+10 \\ -24+30 & -7+5+7 & 8-10 \\ 24-30 & 8-10 & -7+5+7 \end{bmatrix} = \begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$$

To Find A^4? $\quad \because [A^3 - 5A^2 + 7A - 3I = 0] * A$

$$\Rightarrow A^4 - 5A^3 + 7A^2 - 3A = 0 \Rightarrow \boxed{A^4 = 5A^3 - 7A^2 + 3A}$$

$$\because A^3 = A^2 \cdot A = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix}\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix}$$

$$\therefore A^4 = 5\begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix} - 7\begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} + 3\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 395 & 130 & -130 \\ -390 & -125 & 130 \\ 390 & 130 & -125 \end{bmatrix} - \begin{bmatrix} 175 & 56 & -56 \\ -168 & -49 & 56 \\ 168 & 56 & 69 \end{bmatrix} + \begin{bmatrix} 21 & 6 & -6 \\ -18 & -3 & 6 \\ 18 & 6 & -3 \end{bmatrix}$$

$$\therefore A^4 = \begin{bmatrix} 241 & 80 & -80 \\ -240 & -79 & 80 \\ 240 & 80 & -79 \end{bmatrix}$$

<u>Check :-</u> للتحقق فقط

$$A^4 = A^2 \cdot A^2 = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} = \begin{bmatrix} 241 & 80 & -80 \\ -240 & -79 & 80 \\ 240 & 80 & -79 \end{bmatrix}$$

Ex Using Cayley-Hamilton theorem to find A^{-2}

if $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

Solution

$\because |A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 5 \\ 3 & 5 & 6-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)[(4-\lambda)(6-\lambda)-25] - 2[2(6-\lambda)-15] + 3[10-3(4-\lambda)] = 0$

$\Rightarrow (1-\lambda)[24-10\lambda+\lambda^2-25] - 2[12-2\lambda-15] + 3[10-12+3\lambda] = 0$

$\Rightarrow 24-10\lambda+\lambda^2-25-24\lambda+10\lambda^2-\lambda^3+25\lambda+\cancel{6}+4\lambda-\cancel{6}+9\lambda = 0$

$\Rightarrow -\lambda^3+11\lambda^2+4\lambda-1 = 0 \Rightarrow \boxed{\lambda^3-11\lambda^2-4\lambda+1=0}$ char. eq.

By Cayley-Hamilton theorem, we must have :-

$A^3-11A^2-4A+I=0 \;] * A^{-1} \Rightarrow A^2-11A-4I+A^{-1}=0$

$\boxed{\therefore A^{-1} = -A^2+11A+4I} \; * A^{-1} \Rightarrow \boxed{A^{-2} = -A+11I+4A^{-1}}$

$\therefore A^{-2} = -A+11I+4[-A^2+11A+4I] \Rightarrow \boxed{A^{-2} = -4A^2+43A+27I}$

$$\because A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1+4+9 & 2+8+15 & 3+10+18 \\ 2+8+15 & 4+16+25 & 6+20+30 \\ 3+10+18 & 6+20+30 & 9+25+36 \end{bmatrix}$$

$$\therefore A^2 = \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} \quad \& \because A^{-2} = -4A^2 + 43A + 27I$$

$$\therefore A^{-2} = -4\begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 65 & 70 \end{bmatrix} + 43\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 4 \\ 3 & 5 & 6 \end{bmatrix} + 27\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore A^{-2} = \begin{bmatrix} 14 & -14 & 5 \\ -14 & 19 & -9 \\ 5 & -9 & 5 \end{bmatrix}$$

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check :-

للتحقق فقط

$$A^{-2} = (A^{-1})^2$$

$$\because A^{-1} = -A^2 + 11A + 4I = -\begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} + 11\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix} \quad \& \because A^{-2} = (A^{-1})^2 = A^{-1} \cdot A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

$$\therefore A^{-2} = \begin{bmatrix} 14 & -14 & 5 \\ -14 & 19 & -9 \\ 5 & -9 & 5 \end{bmatrix}$$