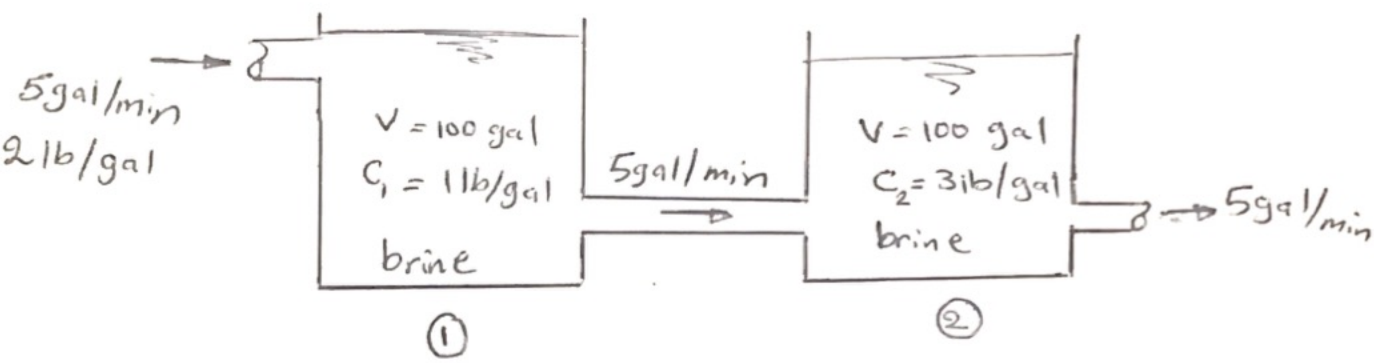
Application of simultaneous linear D.E

1 Tanks - system

Ex:⇒ For the Tanks - system shown in Figure below.
Find the amount of salt at any time in each tank ?



Sol.

For tank ①

x = weight of salt in tank ①

y = weight of salt in tank ②

$$\frac{dx}{dt} = \text{Rate(In)} - \text{Rate(out)}$$

$$= \underset{\frac{gal}{min}\ \frac{lb}{gal}}{5*2} - \underset{\frac{gal}{min}\ \frac{lb}{gal}}{5*\frac{x}{100}} = 10 - 0.05x$$

$$\frac{dx}{dt} + 0.05x = 10$$

$$I.f = e^{\int p(t)\,dt} = e^{\int 0.05\,dt} = e^{0.05t}$$

$$e^{0.05t}x = \int e^{0.05t}(10)\,dt$$

$$x = \left[200e^{0.05t} + C_1\right]e^{-0.05t}$$

$$x = 200 + C_1/e^{0.05t}$$

B.C

at $t=0 \Rightarrow$ تركيز $C_1 = 1\ lb/gal$

$X = 1 * 100 = 100\ lb$

$100 = 200 + C_1 e^{\cdot}$

$C_1 = -100$

$X(t) = 200 - 100 e^{-0.05t}$

For tank ②

$\frac{dy}{dt} = Rate(In) - Rate(out)$

$= 5 * \frac{X}{100} - 5 * \frac{y}{100}$

$= \frac{X}{20} - \frac{y}{20}$

$\frac{-X}{20} + \frac{dy}{dt} + \frac{y}{20} = 0$

$\frac{-X}{20} + (D + \frac{1}{20})y = 0$

$(D + \frac{1}{20})y = + \frac{(200 - 100 e^{-0.05t})}{20}$

$(D + \frac{1}{20})y = 10 - 5e^{-0.05t}$ linear

$I.f = e^{\int \frac{1}{20} dt} = e^{0.05t}$

$e^{0.05t} \cdot y = \int e^{0.05t} (10 - 5e^{-0.05t}) dt$

$e^{0.05t} \cdot y = 200 e^{0.05t} - 5t + C_2$

$y(t) = 200 - 5t e^{-0.05t} + C_2 e^{-0.05t}$

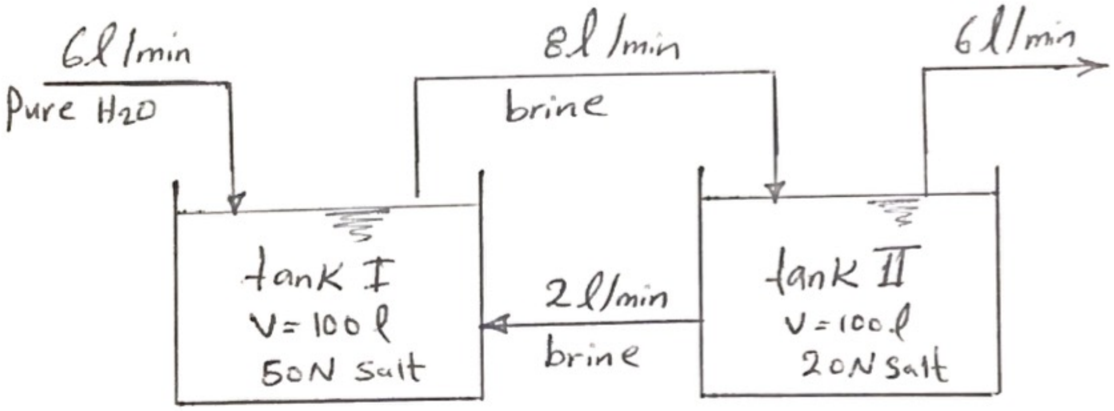
at $t=0$; تركيز $C_2 = 3\ \text{lb/g}$ $\Rightarrow y = 3 * 100 = 300\ \text{lb}$

$$300 = 200 - (0) + C_2 e^{0}$$

$$C_2 = 100$$

$$y(t) = 200 - 5te^{-t/20} + 100e^{-t/20}$$

EX8 Two tanks are connected as shown in Fig. below. The first tank contains initially (100 l) of brine containing (50N) salt while the second tank contain (100 l) brine in which (20N) salt is dissolved. starting from the time $t=0$, the pumping process shown in figure was applied. If the brine in each tank is uniform by stirring. Find amount of salt in each tank as a function of time (t)



نلاحظ ان الخزان (I) يدخل له 8 l/min و يخرج منه 8 l/min

∴ لا يحدث تغيير بالحجم

وكذلك (II) يدخل له (8) ويخرج منه (8)

X = amount of salt in tank I at any time

Y = " " " " tank II " " "

For tank (I)

$$\frac{dx}{dt} = \text{Rate (IN)} - \text{Rate (out)}$$

$$2\frac{l}{min} * \frac{y}{100} - 8\frac{l}{min} * \frac{x}{100}$$

$$= \frac{2y}{100} - \frac{8x}{100}$$

$$\frac{dx}{dt} + \frac{8x}{100} - \frac{2y}{100}$$

$$(D + 0.08)x - 0.02y = 0 \quad --- (1)$$

For tank (II)

$$\frac{dy}{dt} = \text{Rate (IN)} - \text{Rate (out)}$$

$$= 8 * \frac{x}{100} - 2 * \frac{y}{100} - 6 * \frac{y}{100}$$

$$\frac{dy}{dt} = 0.08x - 0.08y$$

$$\frac{dy}{dt} + 0.08y - 0.08x = 0$$

$$-0.08x + (D + 0.08)y = 0 \quad --- (2)$$

$$\begin{vmatrix} D+0.08 & -0.02 \\ -0.08 & D+0.08 \end{vmatrix} x = 0$$

$$\begin{vmatrix} D+0.08 & -0.02 \\ -0.08 & D+0.08 \end{vmatrix} y = 0$$

$$(D^2 + 0.16D + 0.0048)\,x = 0$$

$$(D^2 + 0.16D + 0.0048)\,y = 0$$

$$(D + 0.12)(D + 0.04) = 0 \Rightarrow D_1 = -0.12,\ D_2 = -0.04$$

$$\left.\begin{aligned} x(t) &= C_1 e^{-0.12t} + C_2 e^{-0.04t} \\ y(t) &= k_1 e^{-0.12t} + k_2 e^{-0.04t} \end{aligned}\right] \text{sub into eq (1)}$$

$$(D + 0.08)(C_1 e^{-0.12t} + C_2 e^{-0.04t}) - 0.02(k_1 e^{-0.12t} + k_2 e^{-0.04t}) = 0$$

$$-0.12 C_1 e^{-0.12t} - 0.04 C_2 e^{-0.04t} + 0.08 C_1 e^{-0.12t} + 0.08 C_2 e^{-0.04t} - 0.02 k_1 e^{-0.12t} - 0.02 k_2 e^{-0.04t} = 0$$

$$-0.04 C_1 - 0.02 k_1 = 0 \Rightarrow k_1 = -2C_1$$

$$0.04 C_2 - 0.02 k_2 = 0 \Rightarrow k_2 = 2C_2$$

$$\therefore\ x(t) = C_1 e^{-0.12t} + C_2 e^{-0.04t}$$

$$y(t) = -2C_1 e^{-0.12t} + 2C_2 e^{-0.04t}$$

Initial Conditions

$x(0) = 50,\ y(0) = 20$

$$\left.\begin{aligned} 50 &= C_1 + C_2 \\ 20 &= -2C_1 + 2C_2 \end{aligned}\right]\ \begin{aligned} C_2 &= 30 \\ C_1 &= 20 \end{aligned}$$

$$x(t) = 20 e^{-0.12t} + 30 e^{-0.04t}$$

$$y(t) = -40 e^{-0.12t} + 60 e^{-0.04t}$$