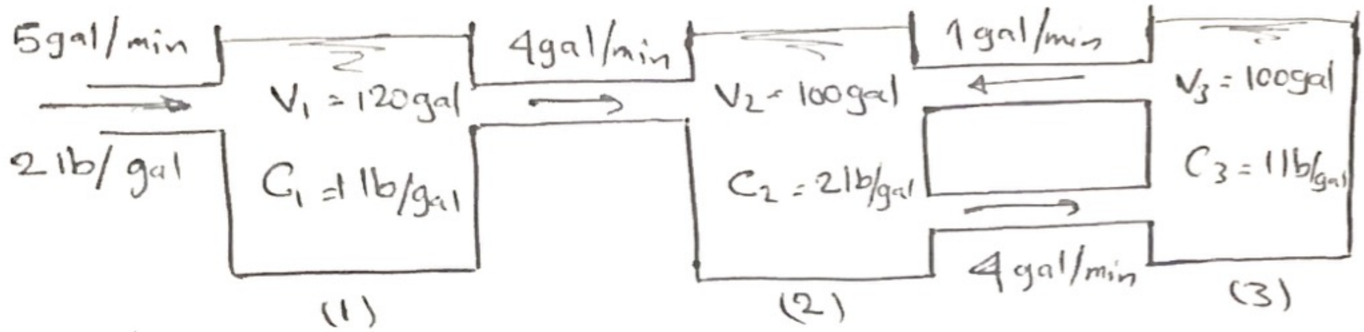
Ex: For the Mass - spring systems shown below write the equation of the amount of salt at any time by using (D-operator) form:



$X(t)$ = amount of salt in tank 1

$y(t)$ = amount of salt in tank 2

$Z(t)$ = amount of salt in tank 3

Tank 1

$$\frac{dx}{dt} = \text{Rate (in)} - \text{Rate (out)}$$

$$= 5 * 2 - 4\left(\frac{x}{120+t}\right)$$

$$\frac{dx}{dt} = 10 - \frac{4x}{120+t}$$

$$\frac{dx}{dt} + \frac{4}{120+t}x = 10$$

$$\left(D + \frac{4}{120+t}\right)x = 10 \quad \text{--- } (1)$$

Tank 2

$$\frac{dy}{dt} = (4) * \frac{x}{120+t} + (1) * \frac{z}{100+t} - (4) * \frac{y}{100+t}$$

$$-\frac{dy}{dt} - \frac{4y}{100+t} + \frac{4x}{120+t} + \frac{z}{100+t} = 0$$

$$\frac{4x}{120+t} - \frac{dy}{dx} - \frac{4y}{100+t} + \frac{z}{100+t} = 0$$

$$\left(\frac{4}{120+t}\right)x - \left(D + \frac{4}{100+t}\right)y + \left(\frac{1}{100+t}\right)z = 0 \quad \text{---} (2)$$

Tank (3)

$$\frac{dz}{dx} = (4) * \frac{y}{100+t} - (1) * \frac{z}{100+t}$$

$$\left(\frac{4}{100+t}\right)y - \left(D + \frac{1}{100+t}\right)z = 0 \quad \text{---} (3)$$

$$\left(D + \frac{4}{120+t}\right)x = 10 \quad \text{---} (1)$$

$$\left(\frac{4}{120+t}\right)x - \left(D + \frac{4}{100+t}\right)y + \left(\frac{1}{100+t}\right)z = 0 \quad \text{---} (2)$$

$$\left(\frac{4}{100+t}\right)x - \left(D + \frac{1}{100+t}\right)z = 0 \quad \text{--} (3)$$

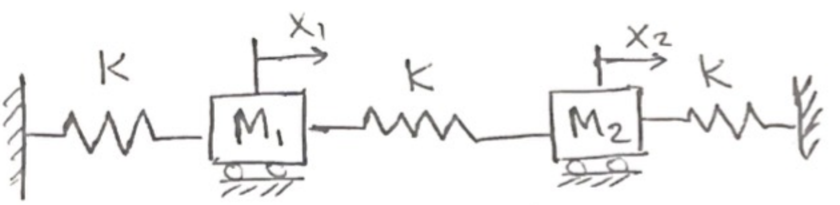
2 Mass - spring system

Ex: The system with two degree of freedom, shown in figure below begins to move under the following initial conditions:

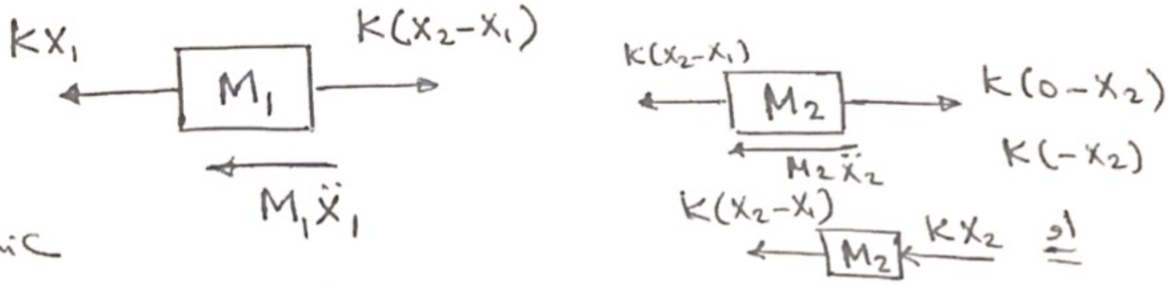
$x_1(0) = 1\ cm$, $x_2(0) = 1\ cm$ } initial displacement

$\dot{x}_1(0) = \sqrt{3K}\ \frac{cm}{sec}$, $\dot{x}_2(0) = -\sqrt{3K}\ \frac{cm}{sec}$ } Initial Velocity

Neglecting friction, derive then solve the differential equations govering the free vibration of the system knowing that $M_1 = M_2 = 1$



Sol.



Dynamic

$\xrightarrow{+} \Sigma F = M * \text{acceleration}$

$$-KX_1 + K(X_2 - X_1) = M_1 \ddot{X}_1$$

$$M_1 \ddot{X}_1 + 2KX_1 - KX_2 = 0 \qquad \boxed{M_1 = 1}$$

$$\ddot{X}_1 + 2KX_1 - KX_2 = 0$$

$$(D^2 + 2K) X_1 - K X_2 = 0 \quad --- (1)$$

$$-K(X_2 - X_1) - KX_2 = M_2 \ddot{X}_2$$

$$M_2 \ddot{X}_2 + 2KX_2 - KX_1 = 0 \qquad \boxed{M_2 = 1}$$

$$-KX_1 + \ddot{X}_2 + 2KX_2 = 0$$

$$-KX_1 + (D^2 + 2K)X_2 = 0 \quad \text{----} ②$$

$$(D^2 + 2K)X_1 - KX_2 = 0 \quad \text{--} ①$$

$$-KX_1 + (D^2 + 2K)X_2 = 0 \quad \text{---} ②$$

$$\begin{vmatrix} D^2+2K & -K \\ -K & D^2+2K \end{vmatrix} X_1 = 0$$

$$\begin{vmatrix} D^2+2K & -K \\ -K & D^2+2K \end{vmatrix} X_2 = 0$$

$$(D^2+2K)^2 - K^2 = D^4 + 4KD^2 + 4K^2 - K^2$$

$$D^4 + 4KD^2 + 3K^2$$

$$(D^4 + 4KD^2 + 3K^2)X_1 = 0 \quad , \quad (D^4 + 4KD^2 + 3K^2)X_2 = 0$$

$$D^2 = -3K \qquad D_{1,2} = \pm\sqrt{3K}\, i$$

$$D^2 = -K \qquad D_{3,4} = \mp\sqrt{K}\, i$$

$$X_1(t) = C_1 \cos\sqrt{3K}t + C_2 \sin\sqrt{3K}t + C_3 \cos\sqrt{K}t + C_4 \sin\sqrt{K}t$$

$$X_2(t) = K_1 \cos\sqrt{3K}t + K_2 \sin\sqrt{3K}t + K_3 \cos\sqrt{K}t + K_4 \sin\sqrt{K}t$$

sub in eq. ①

$$(D^2+2K)(C_1\cos\sqrt{3K}t + C_2\sin\sqrt{3K}t + C_3\cos\sqrt{K}t + C_4\sin\sqrt{K}t)$$

$$-K(K_1\cos\sqrt{3K}t + K_2\sin\sqrt{3K}t + K_3\cos\sqrt{K}t + K_4\sin\sqrt{K}t) = 0$$

$$K_1 = -C_1 \;,\; K_2 = -C_2 \;,\; K_3 = C_3 \;,\; K_4 = C_4$$

$$X_1(t) = C_1\cos\sqrt{3K}t + C_2\sin\sqrt{3K}t + C_3\cos\sqrt{K}t + C_4\sin\sqrt{K}t$$

$$X_2(t) = -C_1\cos\sqrt{3K}t - C_2\sin\sqrt{3K}t + C_3\cos\sqrt{K}t + C_4\sin\sqrt{K}t$$

Initial condition

$$X_1(0) = 1 = C_1\cos(0) + \cancel{C_2\sin(0)} + C_3\cos(0) + \cancel{C_4\sin(0)} = C_1 + C_3$$

$$X_2(0) = 1 = -C_1\cos(0) - C_2\sin(0) + C_3\cos(0) + C_4\sin(0) = -C_1 + C_3$$

$$\left.\begin{array}{l} C_1 + C_3 = 1 \\ -C_1 + C_3 = 1 \end{array}\right\} \begin{array}{l} C_3 = 1 \\ C_1 = 0 \end{array}$$

$$X_1'(0) = \sqrt{3K} = -\cancel{\sqrt{3K}C_1\sin\sqrt{3K}t} + \sqrt{3K}C_2\cos\sqrt{3K}t - \cancel{\sqrt{K}C_3\sin\sqrt{K}t} + \sqrt{K}C_4\cos\sqrt{K}t$$

$$X_2'(0) = -\sqrt{3K} = \cancel{\sqrt{3K}C_1\sin\sqrt{3K}t} - \sqrt{3K}C_2\cos\sqrt{3K}t - \cancel{\sqrt{K}C_3\sin\sqrt{K}t} + \sqrt{K}C_4\cos\sqrt{K}t$$

$$\left.\begin{array}{l} X_1'(0) = \sqrt{3K} = \sqrt{3K}C_2 + \sqrt{K}C_4 \\ X_2'(0) = -\sqrt{3K} = -\sqrt{3K}C_2 + \sqrt{K}C_4 \end{array}\right\} \begin{array}{l} C_2 = 1 \\ C_4 = 0 \end{array}$$

$$X_1(t) = \sin\sqrt{3K}t + \cos\sqrt{K}t$$

$$X_2(t) = -\sin\sqrt{3K}t + \cos\sqrt{K}t$$