



Faculty of Engineering Civil Engineering Department 2nd Class

Strength of Materials I (Mechanics of Materials) (SI Units)

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2017-2018

Mechanics of Material I

Text Books

- Mechanics of Materials, 10th edition (SI version), by: R. C. Hibbeler, 2017
- Mechanics of Materials, 2nd edition (SI version), by: E. Popov, 1990

References

- Strength of Materials, fifth edition, (SI units), Schaum's outlines, by: W. Nash and M. Potter, 1977
- Mechanics of Materials, eighth edition, (SI units), by: James M. Gere, and Barry J. Goodno, 2013.
- Strength of Materials Lectures, Civil Engineering Department, Faculty of Eng., UOM, by: Dr Ali Al-Ghalib, 2014.

Syllabus of Mechanics of Material I – Course No. (50601202):

Ch1. Stress, Axial Loads

The concept of the stress, General stresses in a space element , Types of stresses, Riveted (Bolted) Joints, Allowable stresses; Factor of Safety,

Ch2. Strain, Hooke's Law, Axial Load Problems

True Stress- strain diagram, Ductile and Brittle Materials, Deflection of Axially Loaded Rods (applied within elastic range only), Statically indeterminate members (axially loaded only), Problems Involving Temperature Changes (Thermal changes), Poisson's Ratio, Generalized Hooke's Law, Shear Strain, Hooke's Law in Shear,

Ch3. Axial Force, Shear and Bending moment

Types of supports, Types of loadings, Classification of beams according to their supporting system, Classification of beams according to their analysis procedure, Calculations of beams' reactions, Internal forces and moments in beams, Shear and moment diagrams by equations, Shear and moment diagrams by summation approach,

Ch4. Torsion

Torsion formula of circular sections, Polar moment of inertia, Hollow circular sections, Statically Indeterminate Problems, Torsion of Noncircular Members, Thin-walled members with open cross sections
Thin-walled members with open cross sections

Ch5. Pure Bending of Beams

Curvature of a beam, Bending formula, Elastic section modulus, Beams of Two Materials, Stress Distribution in Composite Sections (Method of Transformed section).

COURSE OBJECTIVES (Learning Outcomes):

- Analyze and design structural members subjected to tension, compression, torsion, bending and combined stresses using the fundamental concepts of stress, strain and elastic behavior of materials.
- Utilize appropriate materials in design considering engineering properties, sustainability, cost and weight.
- Perform engineering work in accordance with ethical and economic constraints related to the design of structures and machine parts.

INTRODUCTION

Strength of materials is a branch of applied mechanics that deals with the behavior of solid bodies subjected to various types of loading. Other names for this field of study are **mechanics of materials** and **solid mechanics**. The solid bodies considered in this book include bars with axial loads, shafts in torsion, beams in bending, and columns in compression. The principal objective of mechanics of materials is to determine the stresses, strains, and displacements in structures and their components due to the loads acting on them. If we can find these quantities for all values of the loads up to the loads that cause failure, we will have a complete picture of the mechanical behavior of these structures

SYMBOLS AND SI UNITS

GREEK ALPHABET

A	α	Alpha	N	ν	Nu
B	β	Beta	Ξ	ξ	Xi
Γ	γ	Gamma	O	o	Omicron
Δ	δ	Delta	Π	π	Pi
E	ϵ	Epsilon	P	ρ	Rho
Z	ζ	Zeta	Σ	σ	Sigma
H	η	Eta	T	τ	Tau
Θ	θ	Theta	Y	υ	Upsilon
I	ι	Iota	Φ	ϕ	Phi
K	κ	Kappa	X	χ	Chi
Λ	λ	Lambda	Ψ	ψ	Psi
M	μ	Mu	Ω	ω	Omega

Prefixes	Value	Standard form	Symbol
Tera	1 000 000 000 000	10^{12}	T
Giga	1 000 000 000	10^9	G
Mega	1 000 000	10^6	M
Kilo	1 000	10^3	k
deci	0.1	10^{-1}	d
centi	0.01	10^{-2}	c
milli	0.001	10^{-3}	m
micro	0.000 001	10^{-6}	μ
nano	0.000 000 001	10^{-9}	n
pico	0.000 000 000 001	10^{-12}	p

Quantity	U.S. Customary	SI Equivalent
Force	lb. kip	4.448N 4.448kN
Length	in ft	25.4mm 0.3048m
Area	in ² ft ²	645.2mm ² 0.0929m ²
Stress	lb/in ² (psi)	6.895kN/m ² (kPa)

CHAPTER ONE – STRESS, AXIAL LOADS

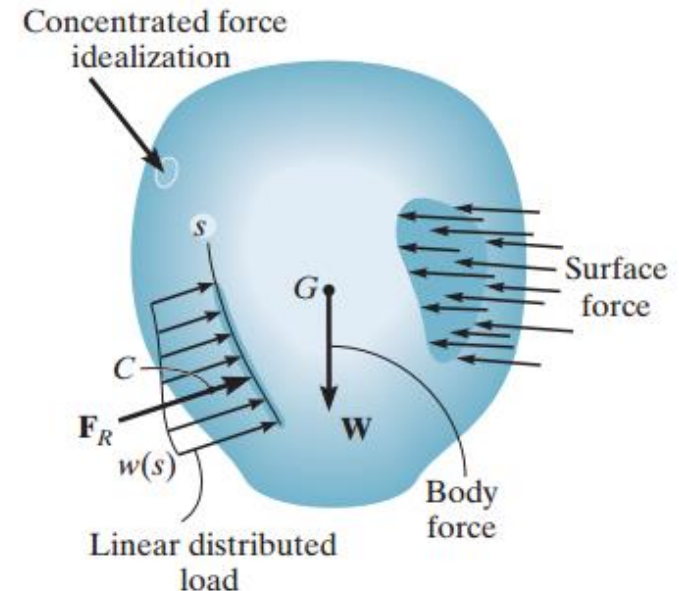
Static Review:

Since statics has an important role in both the development and application of mechanics of materials, it is very important to have a good grasp of its fundamentals. For this reason we will review some of the main principles of statics that will be used throughout the text.

External Loads: A body is subjected to only two types of external loads; namely, surface forces or body forces.

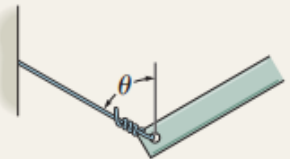
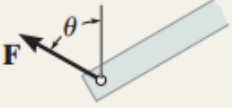
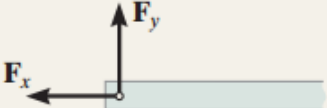
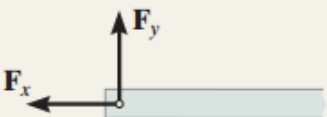
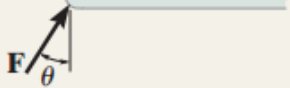
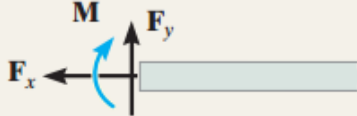
Surface Forces: Surface forces are caused by the direct contact of one body with the surface of another. In all cases these forces are distributed over the area of contact between the bodies. If this area is small in comparison with the total surface area of the body, then the surface force can be idealized as a single concentrated force, which is applied to a point on the body. If the surface loading is applied along a narrow strip of area, the loading can be idealized as a linear distributed load, $w(s)$. Here the loading is measured as having an intensity of force/length along the strip and is represented graphically by a series of arrows along the lines. The resultant force of $w(s)$ is equivalent to the area under the distributed loading curve, and this resultant acts through the centroid C or geometric center of this area.

Body Forces: A body force is developed when one body exerts a force on another body without direct physical contact between the bodies. Examples include the effects caused by the earth's gravitation or its electromagnetic field. In the case of gravitation, this force is called the weight of the body and acts through the body's center of gravity.



Support Reactions:

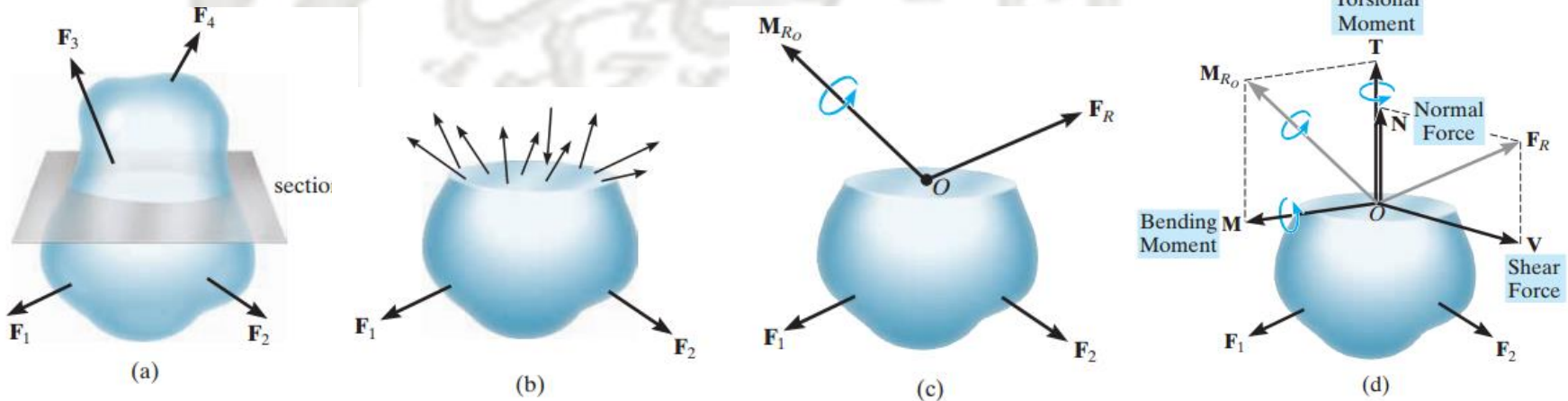
The general rule is: **if the support prevents translation in a given direction, then a force must be developed on the member in that direction. Likewise, if rotation is prevented, a couple moment must be exerted on the member.** For example, the roller support only prevents translation perpendicular or normal to the surface. Hence, the roller exerts a normal force F on the member at its point of contact. Since the member can freely rotate about the roller, a couple moment cannot be developed on the member.

Type of connection	Reaction	Type of connection	Reaction
 Cable	 One unknown: F	 External pin	 Two unknowns: F_x, F_y
 Roller	 One unknown: F	 Internal pin	 Two unknowns: F_x, F_y
 Smooth support	 One unknown: F	 Fixed support	 Three unknowns: F_x, F_y, M

Equations of Equilibrium: Equilibrium of a body requires both a **balance of forces**, to prevent the body from translating or having accelerated motion along a straight or curved path, and a **balance of moments**, to prevent the body from rotating.

$$\begin{aligned}\Sigma F_x &= 0 & \Sigma F_y &= 0 & \Sigma F_z &= 0 \\ \Sigma M_x &= 0 & \Sigma M_y &= 0 & \Sigma M_z &= 0\end{aligned}$$

Internal Resultant Loadings and free body diagram



Normal force, N . This force acts perpendicular to the area. It is developed whenever the external loads tend to push or pull on the two segments of the body.

Shear force, V . The shear force lies in the plane of the area and it is developed when the external loads tend to cause the two segments of the body to slide over one another.

Torsional moment or torque, T . This effect is developed when the external loads tend to twist one segment of the body with respect to the other about an axis perpendicular to the area.

Bending moment, M . The bending moment is caused by the external loads that tend to bend the body about an axis lying within the plane of the area.

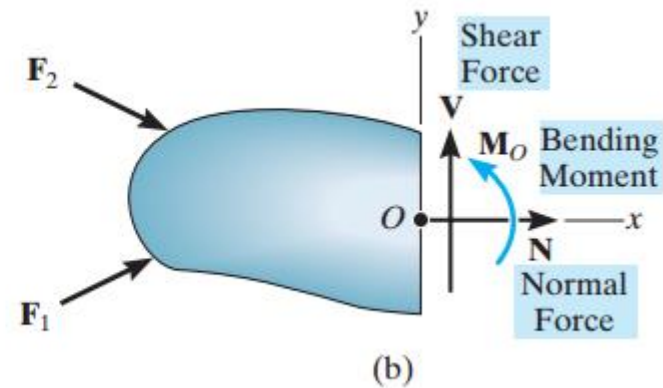
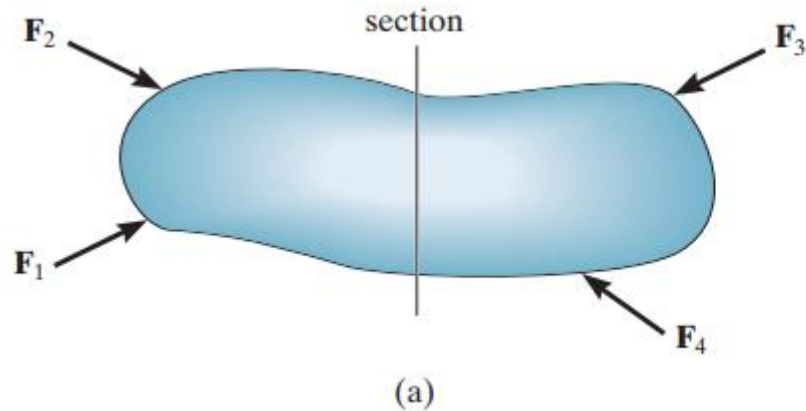
Coplanar Loadings:

If the body is subjected to a coplanar system of forces, then only normal force, shear force, and bending moment components will exist at the section.

$$\Sigma F_x = 0$$

$$\Sigma F_y = 0$$

$$\Sigma M_O = 0$$



EXAMPLE 1-1

Determine the resultant internal loadings acting on the cross section at C of the cantilevered beam shown in the Figure.

$$\frac{w}{6} = \frac{270}{9} \rightarrow w = 180 \text{ N/m}$$

$$F = \frac{1}{2} \times 180 \times 6 = 540 \text{ N}$$

$$(+)\rightarrow \sum F_x = 0$$

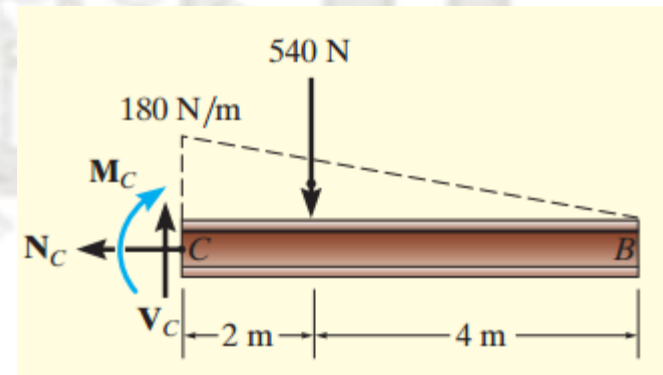
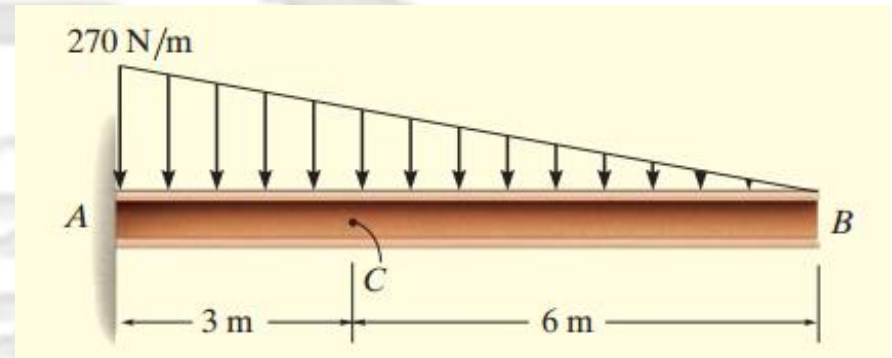
$$-N_c = 0$$

$$(+)\uparrow \sum F_y = 0$$

$$V_c - 540 = 0 \rightarrow V_c = 540 \text{ N}$$

$$\zeta(+)\sum M_c = 0 \rightarrow -M_c - 540 \times 2 = 0$$

$$M_c = -1080 \text{ N.m}$$

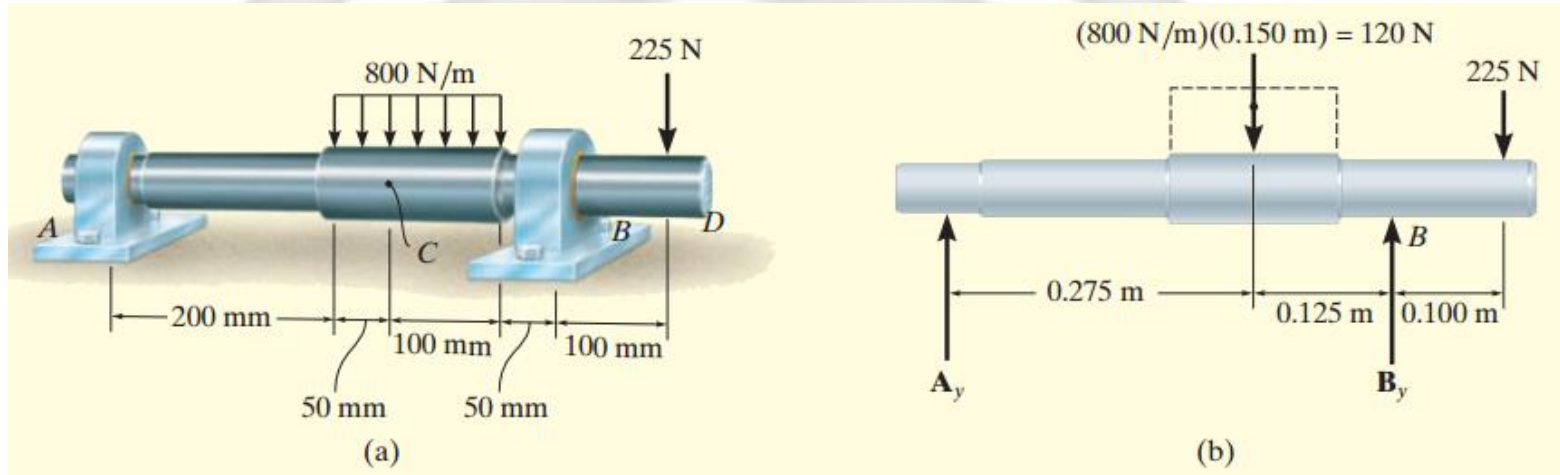


The negative sign indicates that acts in the opposite direction to that shown on the free-body diagram.

H.W. Try solving this problem using segment AC, by first obtaining the support reactions at A.

EXAMPLE 1-2

Determine the resultant internal loadings acting on the cross section at C of the machine shaft shown in the Figure. The shaft is supported by support bearings at A and B, which only give vertical forces on the shaft.



$$\zeta (+) \sum M_B = 0 \rightarrow -A_y \times 0.4 + 120 \times 0.125 - 225 \times 0.1 = 0$$

$$A_y = -18.75 \text{ N}$$

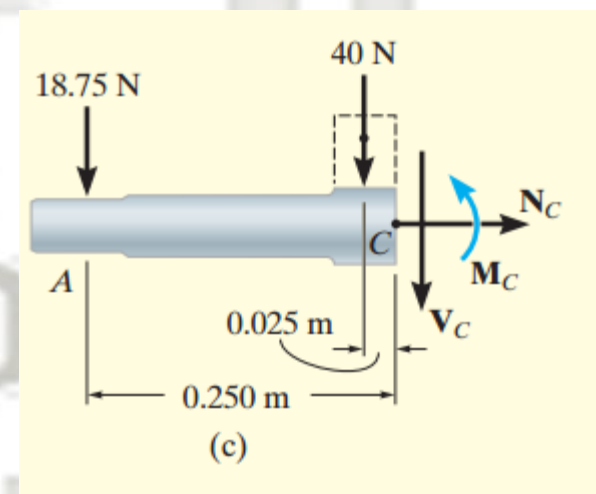
$$(+)\rightarrow \sum F_x = 0 \rightarrow N_c = 0$$

$$(+)\uparrow \sum F_y = 0 \rightarrow -18.75 - 40 - V_c = 0$$

$$V_c = -58.8 \text{ N}$$

$$\zeta (+) \sum M_c = 0 \rightarrow M_c + 40 \times 0.025 + 18.75 \times 0.25 = 0$$

$$M_c = -5.69 \text{ N.m}$$



H.W. Calculate the reaction at B and try to obtain the same results using segment CBD of the shaft.

EXAMPLE 1-3

The 500-kg engine is suspended from the crane boom in the Figure. Determine the resultant internal loadings acting on the cross section of the boom at point E.

$$\zeta (+) \sum M_A = 0 \rightarrow F_{CD} \times \frac{3}{5} \times 2 - (500 \times 9.81) \times 3 = 0$$

$$F_{CD} = 12262.5 \text{ N}$$

$$(+)\rightarrow \sum F_x = 0 \rightarrow A_x - (12262.5 \times \frac{4}{5}) = 0$$

$$A_x = 9810 \text{ N}$$

$$(+)\uparrow \sum F_y = 0 \rightarrow -A_y + (12262.5 \times \frac{3}{5}) - (500 \times 9.81) = 0$$

$$A_y = 2452.5 \text{ N}$$

$$(+)\rightarrow \sum F_x = 0 \rightarrow N_E + 9810 = 0$$

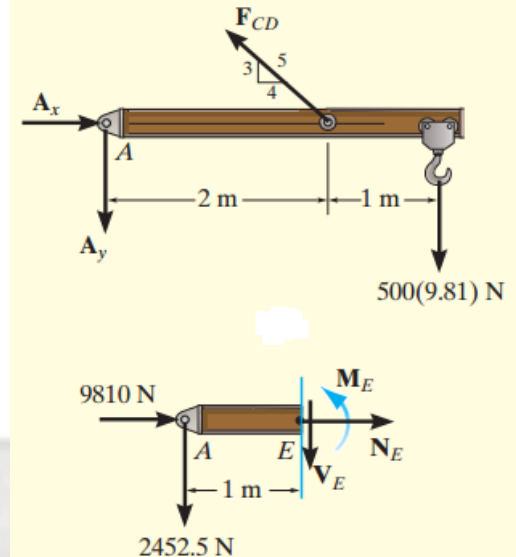
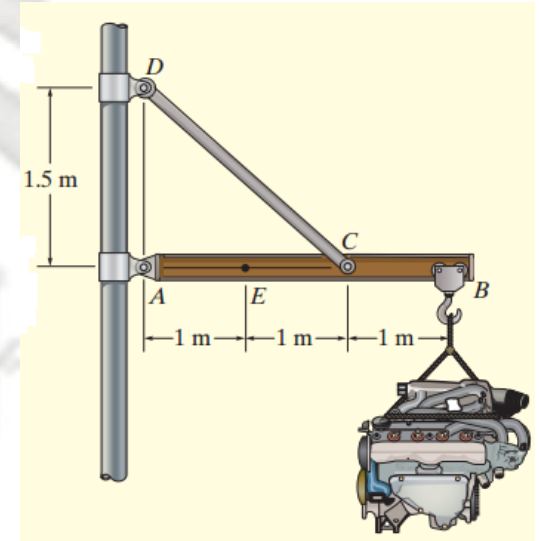
$$N_E = -9810 \text{ N} = -9.81 \text{ kN}$$

$$+\uparrow \sum F_y = 0 \rightarrow -V_E - 2452.5 \text{ N} = 0$$

$$V_E = -2452.5 \text{ N} = -2.45 \text{ kN}$$

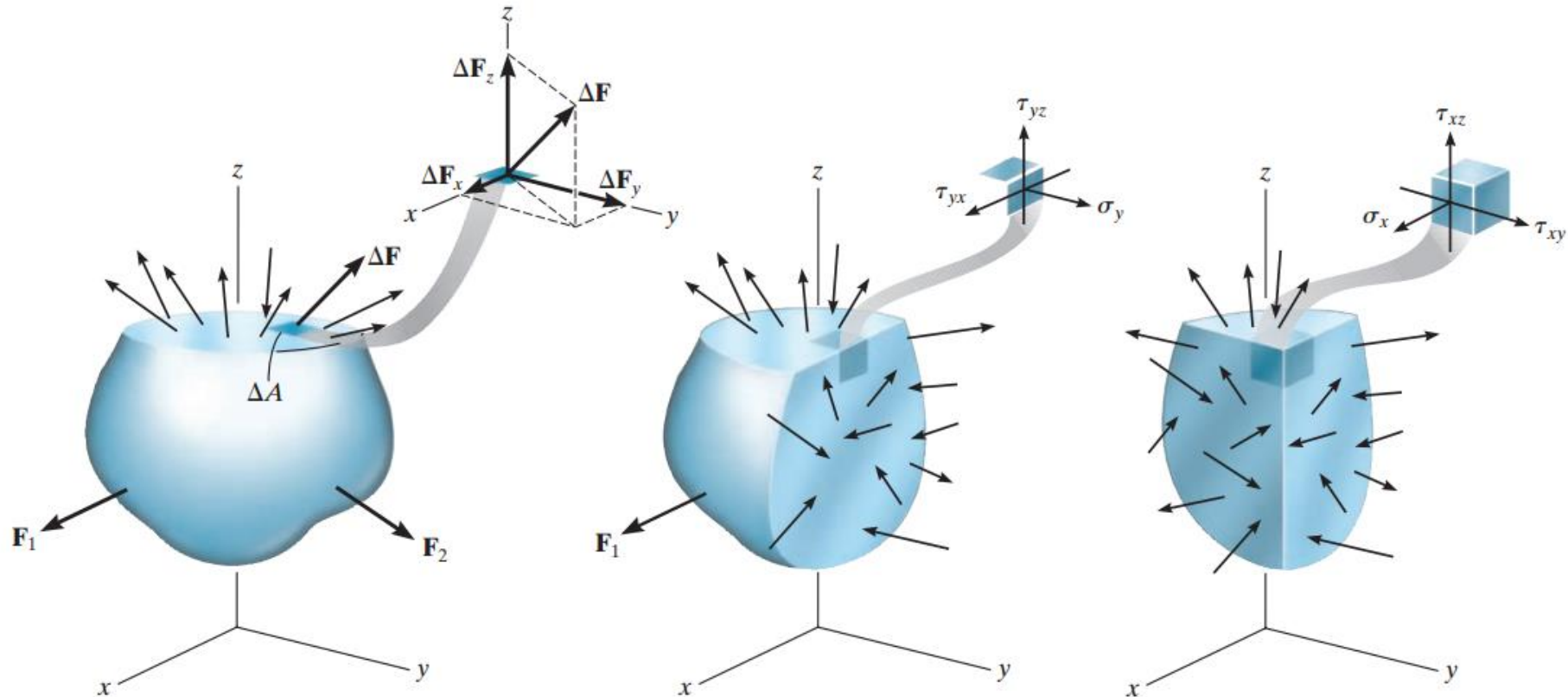
$$\zeta (+) \sum M_E = 0 \rightarrow M_E + 2452.5 \times 1 = 0$$

$$M_E = -2452.5 \text{ N.m} = -2.45 \text{ kN.m}$$



The concept of the stress

We begin by considering the sectioned area to be subdivided into small areas. A typical finite very small force $\Delta \mathbf{F}$ acting on $\Delta \mathbf{A}$ is shown in Figure. This force, like all the others, will have a unique direction and we will replace it by its three components, namely, ΔF_x , ΔF_y , and ΔF_z and which are taken tangent, tangent, and normal to the area, respectively. As $\Delta \mathbf{A}$ approaches zero, so do $\Delta \mathbf{F}$ and its components; however, the quotient of the force and area will, in general, approach a finite limit. This quotient is called **stress**, and as noted, it describes the intensity of the internal force acting on a specific plane (area) passing through a point.



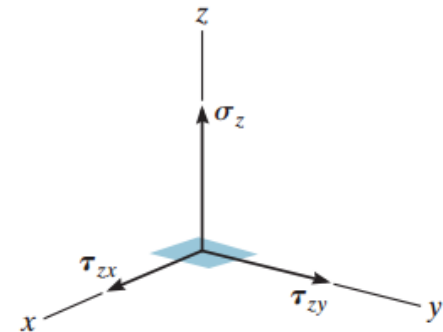
Normal Stress: The intensity of the force acting normal to is defined as the normal stress, σ (sigma). Since is normal to the area then

$$\sigma_z = \lim_{\Delta A \rightarrow 0} \frac{\Delta F_z}{\Delta A}$$

Shear Stress: The intensity of force acting tangent to is called the shear stress, τ (tau). Here we have shear stress components.

$$\tau_{zx} = \lim_{\Delta A \rightarrow 0} \frac{\Delta F_x}{\Delta A}$$

$$\tau_{zy} = \lim_{\Delta A \rightarrow 0} \frac{\Delta F_y}{\Delta A}$$



Note that in this subscript notation z specifies the orientation of the area ΔA , and x and y indicate the axes along which each shear stress acts.

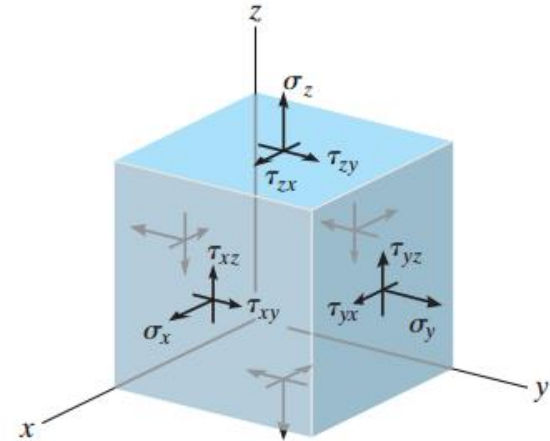
General State of Stress: If the body is further sectioned by planes parallel to the x - z plane and the y - z plane, we can then “cut out” a cubic volume element of material that represents the **state of stress acting around a chosen point in the body**. This state of stress is then characterized by three components acting on each face of the element.

SI Stress Units:

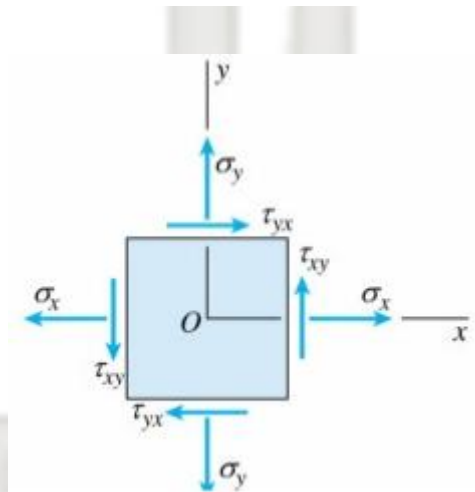
$$\text{Pascal} = \frac{N}{m^2}$$

$$\text{Kilo Pascal} = 1000 \times \frac{N}{m^2} = \frac{kN}{m^2}$$

$$\text{Mega Pascal} = 1000000 \times \frac{N}{m^2} = \frac{N}{mm^2}$$



In plane, there are 4 components of stresses; 2 normal stresses and 2 shear stresses, as in the plane element below:



Type of Stresses

1- Normal Stress:

a) Tensile stress

Where:

P = axial (passes through the centroid) tensile force

A = cross-sectional area

When the applied force is axial and normal, a uniform (equal) maximum normal stress can be achieved through the section.

b) Compressive stress

c) Bearing stress

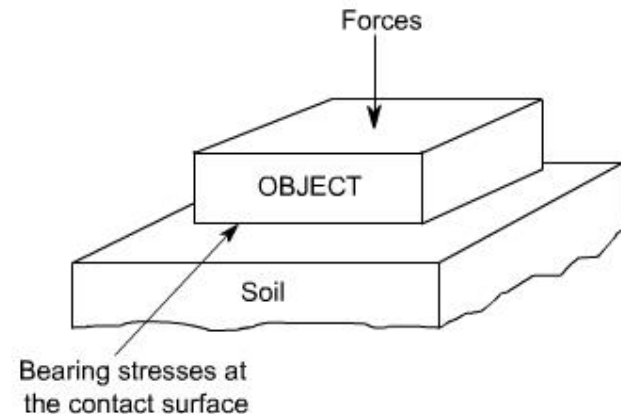
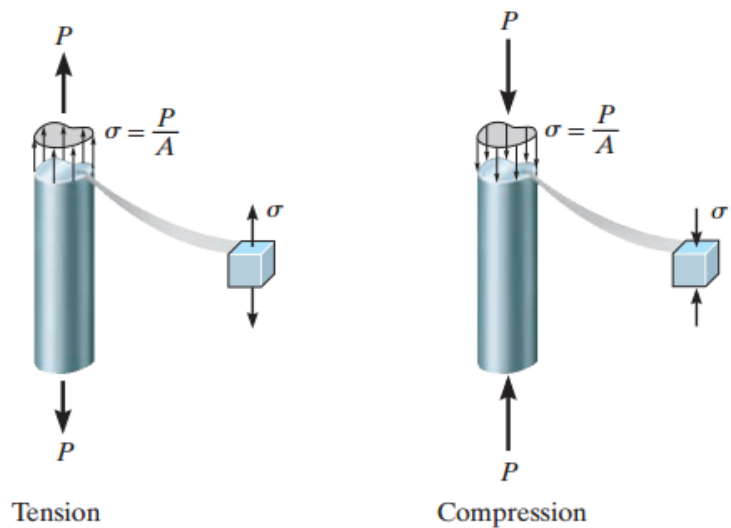
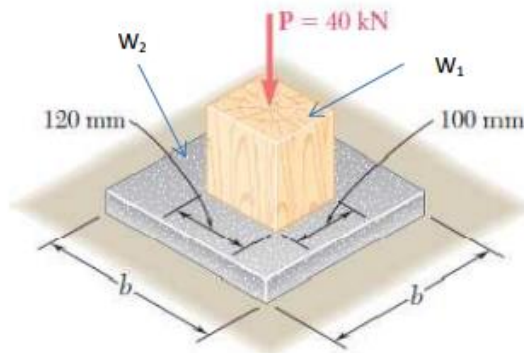
The bearing stress is a normal compressive stress happens **between two surfaces**.

In this example, we have two bearing stresses. First, between the timber block and the steel base, this equals:

$$\sigma_b = \frac{P + W_1}{120 \times 100}$$

Second, between the steel base and the soil, this equals:

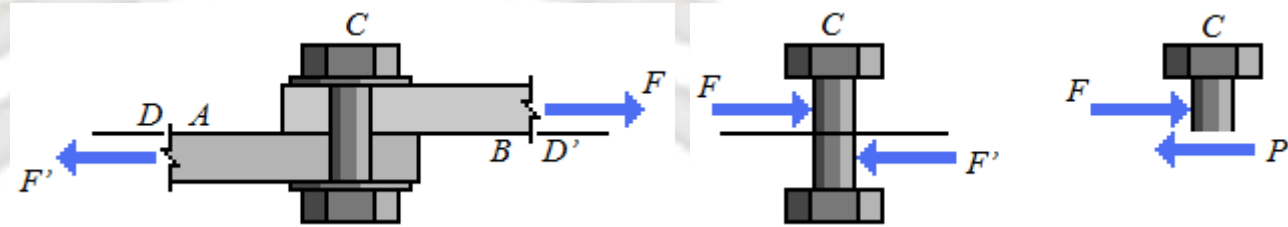
$$\sigma_b = \frac{P + W_1 + W_2}{b \times b}$$



2- Shear Stress

a) Direct Shear stress:

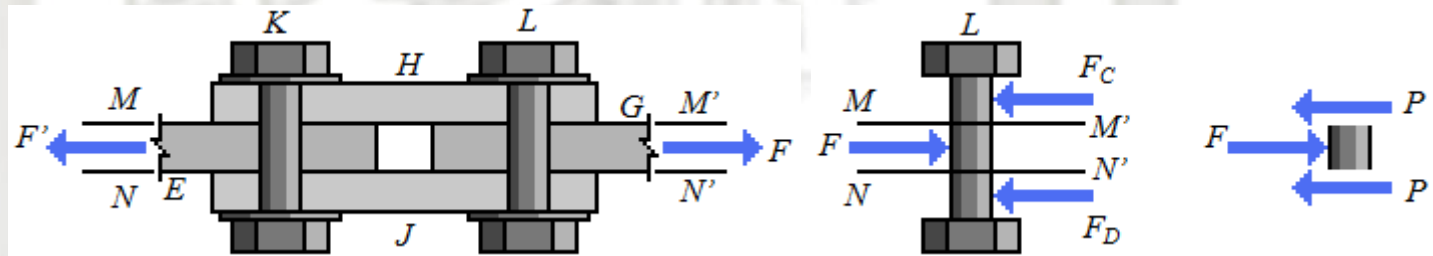
1- Single Shear



If plates A and B are connected by bolt C, shear will take place in bolt C in plane DD'. The bolt is in single shear. Observing that the shear $P = F$, it can be concluded that the average shearing stress is:

$$\tau_{bolt} = \frac{P}{A} = \frac{F}{A} = \frac{F}{\pi r^2}$$

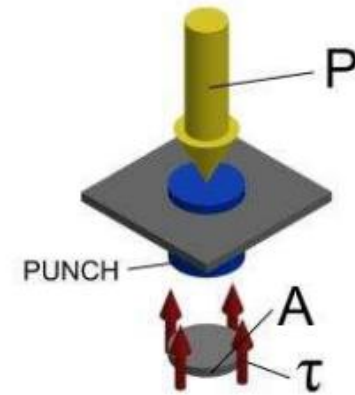
2- Double shear



If splice plates H and J are used to connect plates E and G, shear will take place in bolts K and L in each of the two planes MM' and NN'. The bolts are in double shear. Observing that the shear P in each of the sections is $P = F/2$, it can be concluded that the average shearing stress is:

$$\tau_{bolt} = \frac{P}{A} = \frac{F/2}{A} = \frac{F}{2\pi r^2}$$

b) Bunching Shear Stress



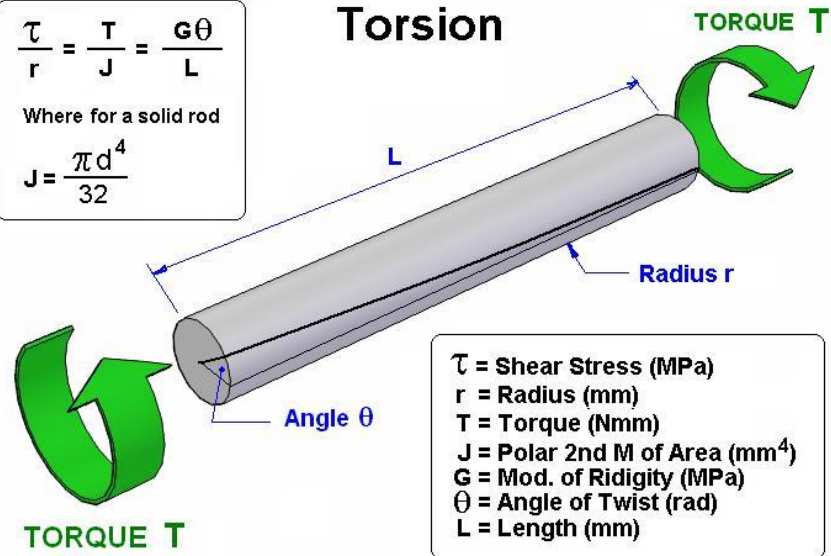
c) Torsional Shear Stress

$$\frac{\tau}{r} = \frac{T}{J} = \frac{G\theta}{L}$$

Where for a solid rod

$$J = \frac{\pi d^4}{32}$$

Torsion

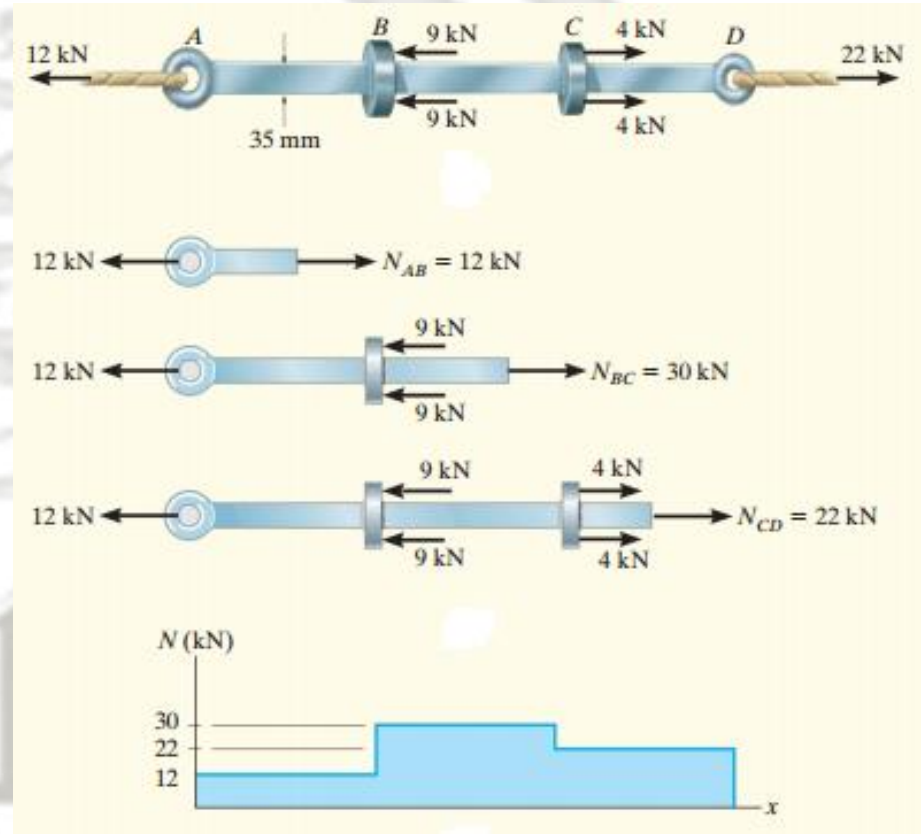
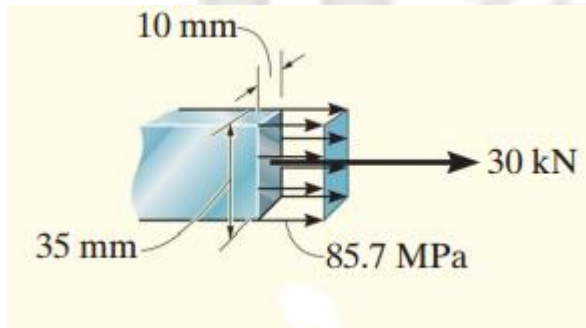


EXAMPLE 1-4

The bar in the Figure has a constant width of 35 mm and a thickness of 10 mm. Determine the maximum average normal stress in the bar when it is subjected to the loading shown.

$$\sigma_{BC} = \frac{N_{BC}}{A}$$

$$\sigma_{BC} = \frac{30 \times 1000}{35 \times 10} = 85.7 \text{ MPa}$$



EXAMPLE 1-5

The 80-kg lamp is supported by two rods AB and BC as shown in the Figure. If AB has a diameter of 10 mm and BC has a diameter of 8 mm, determine the average normal stress in each rod.

$$(+)\rightarrow \sum F_x = 0 \rightarrow \left[F_{BC} \left(\frac{4}{5} \right) - F_{BA} \cos 60^\circ = 0 \right] \times \frac{5}{4} \rightarrow (1)$$

$$(+)\uparrow \sum F_y = 0 \rightarrow \left[F_{BC} \left(\frac{3}{5} \right) - F_{BA} \sin 60^\circ - 784.8 = 0 \right] \times \frac{5}{3} \rightarrow (2)$$

$$F_{BC} - \left(\frac{5}{4} \right) F_{BA} \cos 60^\circ = 0 \rightarrow (1)$$

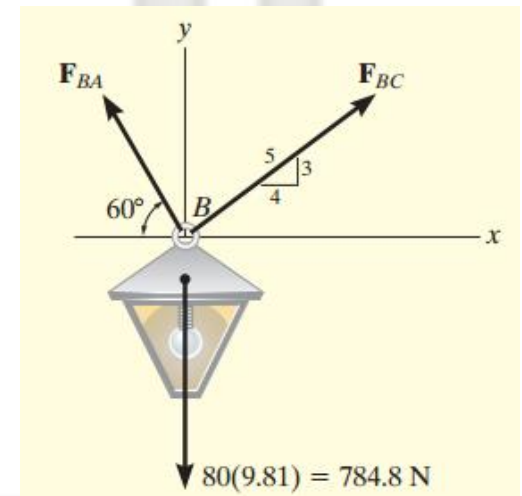
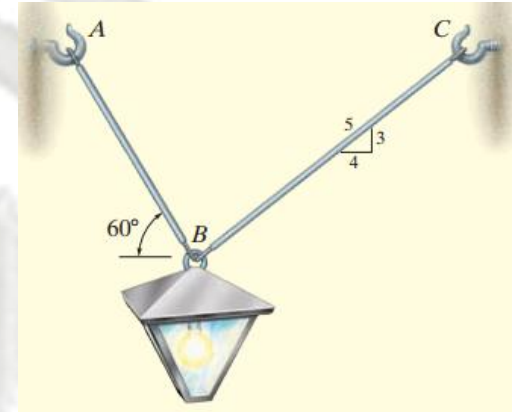
$$F_{BC} - \left(\frac{5}{3} \right) F_{BA} \sin 60^\circ - \left(\frac{5}{3} \right) 784.8 = 0 \rightarrow (2)$$

$$F_{BC} = 395.2 \text{ N}$$

$$F_{BA} = 632.4 \text{ N}$$

$$\sigma_{BC} = \frac{F_{BC}}{A_{BC}} = \frac{395.2}{\pi \times 4^2} = 7.86 \text{ MPa}$$

$$\sigma_{BA} = \frac{F_{BA}}{A_{BA}} = \frac{632.4}{\pi \times 5^2} = 8.05 \text{ MPa}$$



EXAMPLE 1-6

Member AC shown in the Figure. is subjected to a vertical force of 3 kN. Determine the position x of this force so that the average compressive stress at the smooth support C is equal to the average tensile stress in the tie rod AB. The rod has a cross-sectional area of 400 mm² and the contact area at C is 650 mm².

$$(+)\uparrow \sum F_y = 0 \rightarrow F_{AB} + F_C - 3000 = 0 \rightarrow (1)$$

$$\curvearrowleft (+)\sum M_A = 0 \rightarrow -3000 \times x + F_C \times 200 = 0 \rightarrow (2)$$

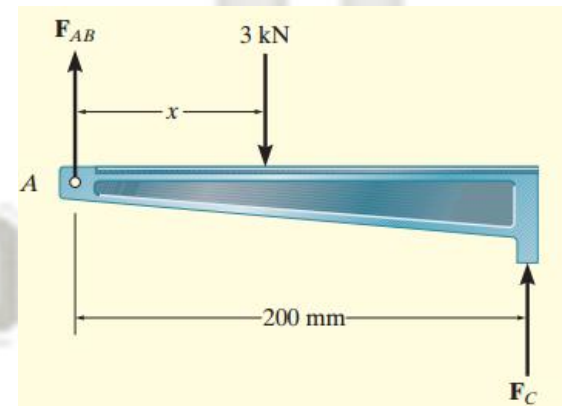
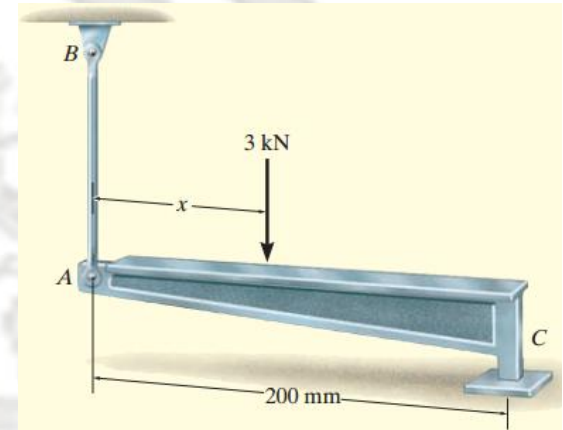
$$\sigma = \frac{F_{AB}}{400} = \frac{F_C}{650}$$

$$F_C = 1.625 F_{AB} \rightarrow (3)$$

$$F_{AB} = 1143 \text{ N}$$

$$F_C = 1857 \text{ N}$$

$$x = 124 \text{ mm}$$



EXAMPLE 1-7

A hollow steel tube with an inside diameter of 100 mm must carry a tensile load of 400 kN. Determine the outside diameter of the tube if the stress is limited to 120 MN/m².

$$P = 400 \text{ kN} = 400000 \text{ N}$$

$$\sigma = 120 \text{ MPa}$$

$$A = \pi \times \frac{D^2}{4} - \pi \times \frac{100^2}{4} = \frac{\pi}{4} (D^2 - 10000)$$

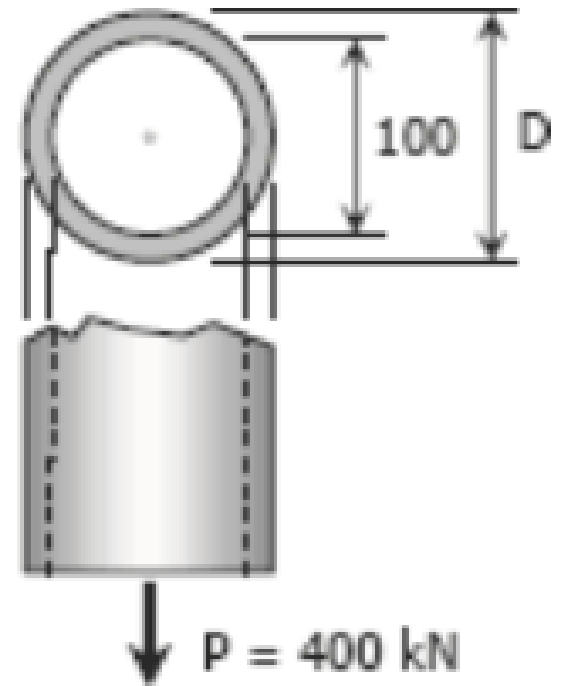
$$P = \sigma \times A$$

$$400000 = 120 \left(\frac{\pi}{4} (D^2 - 10000) \right)$$

$$400000 = 30\pi D^2 - 300000\pi$$

$$D^2 = \frac{400000 + 300000\pi}{30\pi}$$

$$D = 119.35 \text{ mm}$$



EXAMPLE 1-8

A homogeneous 800 kg bar AB is supported at either end by a cable as shown in the Figure. Calculate the smallest area of each cable if the stress is not to exceed 90 MPa in bronze and 120 MPa in steel.

$$W_{bar} = 800 \times 9.81 = 7848 N$$

$$P_{bronze} = P_{steel} = \frac{7848}{2} = 3924 N$$

$$P_{bronze} = \sigma_{bronze} \times A_{bronze}$$

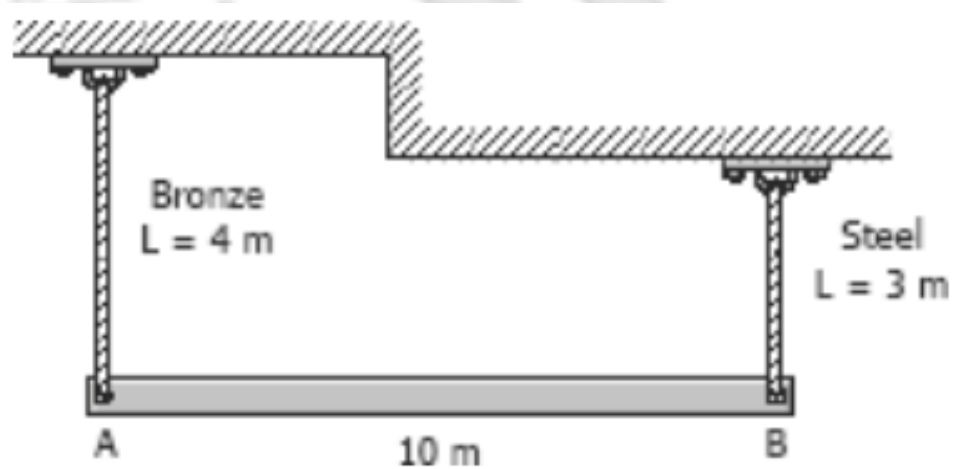
$$3924 = 90 \times A_{bronze}$$

$$A_{bronze} = 43.6 mm^2$$

$$P_{steel} = \sigma_{steel} \times A_{steel}$$

$$3924 = 120 \times A_{steel}$$

$$A_{steel} = 32.7 mm^2$$



EXAMPLE 1-9

An aluminum rod is rigidly attached between a steel rod and a bronze rod as shown in the Figure. Axial loads are applied at the positions indicated. Find the maximum value of P that will not exceed a stress in steel of 140 MPa, in aluminum of 90 MPa, or in bronze of 100 MPa.

$$P_{\text{bronze}} = \sigma_{\text{bronze}} \times A_{\text{bronze}}$$

$$2P_{\text{bronze}} = 100 \times 200$$

$$P_{\text{bronze}} = 10000 \text{ N}$$

$$P_{\text{alum}} = \sigma_{\text{alum}} \times A_{\text{alum}}$$

$$P_{\text{alum}} = 90 \times 400$$

$$P_{\text{alum}} = 36000 \text{ N}$$

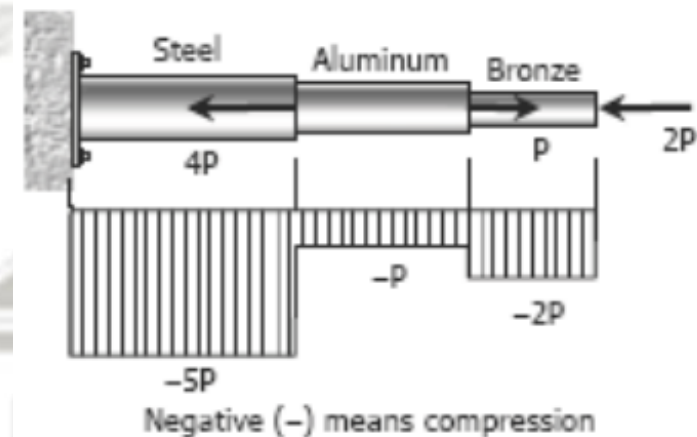
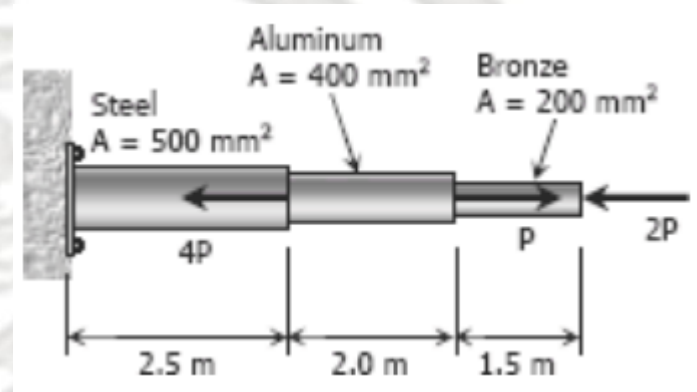
$$P_{\text{steel}} = \sigma_{\text{steel}} \times A_{\text{steel}}$$

$$5P_{\text{steel}} = 140 \times 500$$

$$P_{\text{steel}} = 14000 \text{ N}$$

$$\therefore P_{\text{smallest}} = 10000 \text{ N} = 10 \text{ kN}$$

The safe force is $P=10 \text{ kN}$



EXAMPLE 1-10

Find the stresses in members BC, BD, and CF for the truss shown in the Figure. Indicate the tension or compression. The cross sectional area of each member is 1600 mm^2 .

For member BD: $\sum M_c = 0 \rightarrow 3 \times \frac{4}{5} BD = 3 \times 60$

$$BD = 75 \text{ kN (tension)}$$

$$BD = \sigma_{BD} \times A$$

$$75 \times 1000 = \sigma_{BD} \times 1600$$

$$\sigma_{BD} = 46.875 \text{ MPa (tension)}$$

For member CF: $\sum M_D = 0 \rightarrow 4 \times \frac{1}{\sqrt{2}} CF = 4 \times 90 + 7 \times 60$

$$CF = 195\sqrt{2} = 275.77 \text{ kN (compression)}$$

$$CF = \sigma_{CF} \times A$$

$$275.77 \times 1000 = \sigma_{CF} \times 1600$$

$$\sigma_{CF} = 172.357 \text{ MPa (compression)}$$

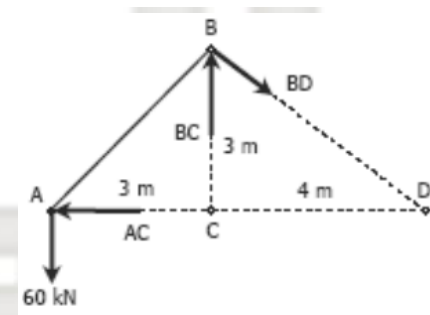
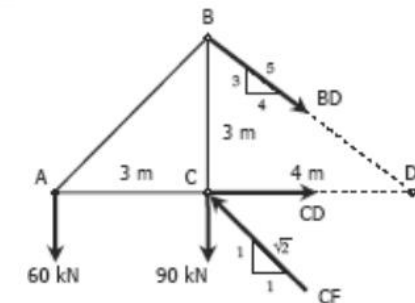
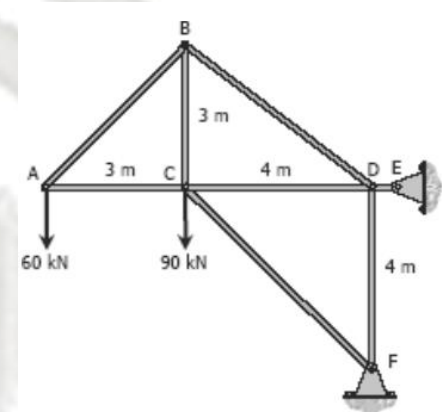
For member BC: $\sum M_D = 0 \rightarrow 4BC = 7 \times 60$

$$BC = 105 \text{ kN (compression)}$$

$$BC = \sigma_{BC} \times A$$

$$105 \times 1000 = \sigma_{BC} \times 1600$$

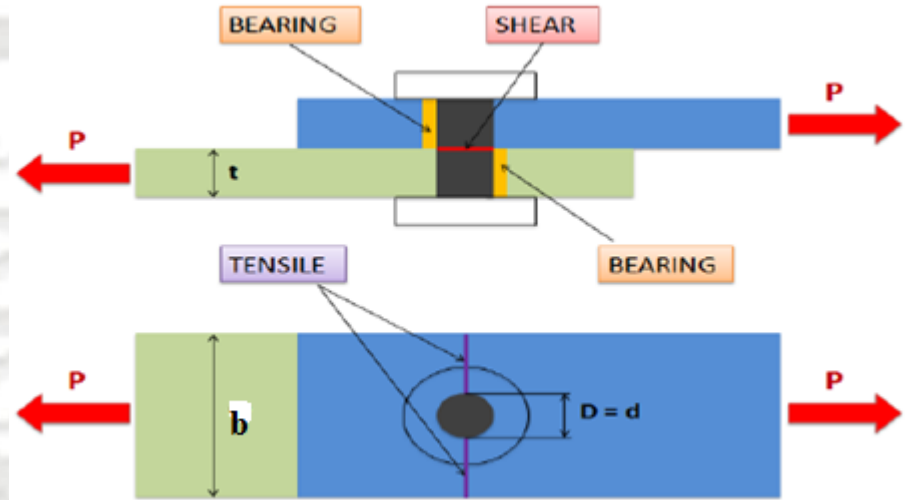
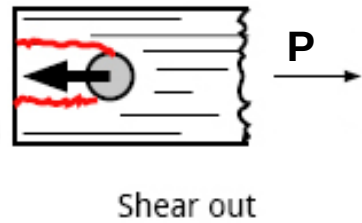
$$\sigma_{BC} = 65.625 \text{ MPa (compression)}$$



Riveted (Bolted) Joints

There are four types of stresses occur at riveted joints. Therefore, the failure is possible in four locations as follows:

- 1- Shearing stress failure in rivets
- 2- Tension stress failure in plate.
- 3- Bearing stress failure between plate and rivet.
- 4- Shearing stress failure in plate.



Assumption:

Shearing stress in rivets is equal and uniform. This assumption is approximately true because the shearing stresses are actually distributed in a non-uniform way across the area of the cut.



1- Shearing stress failure in rivets:

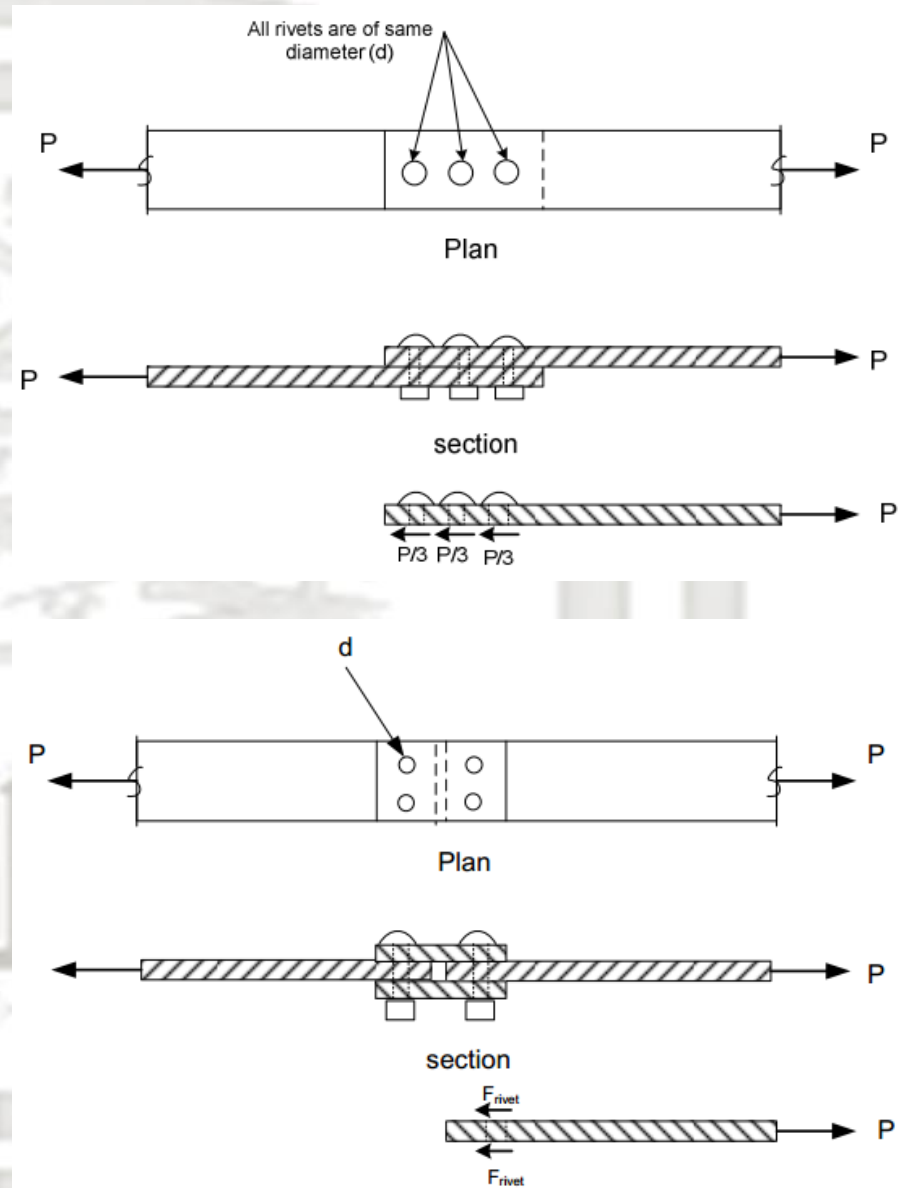
Case 1: All rivets in the same diameter

$$F_{\text{rivet}} = \frac{P}{3}$$

$$\tau_{\text{rivet}} = \frac{P/3}{\pi d^2 / 4}$$

$$F_{\text{rivet}} = \frac{P}{4}$$

$$\tau_{\text{rivet}} = \frac{P/4}{\pi d^2 / 4}$$



Case 2: The rivets with different diameters:

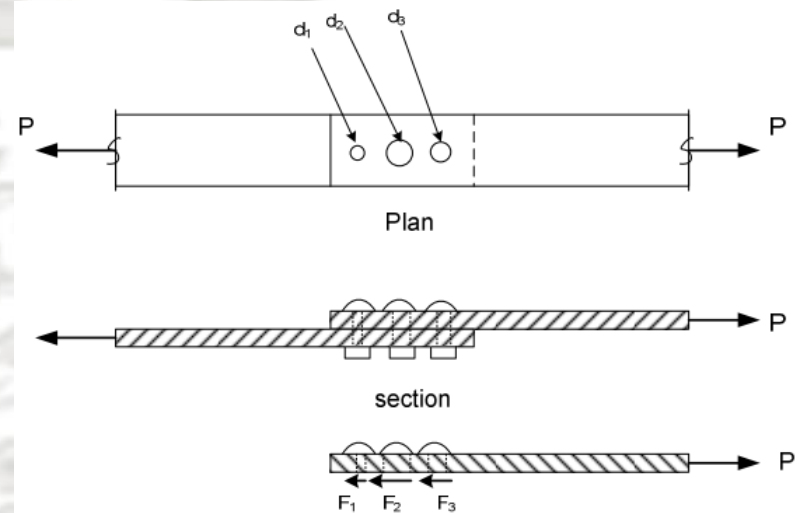
According to the assumption, the shear stress must be equal in the three rivets. Therefore, the shear forces must be different.

$$\tau_{rivet} = \frac{P}{\frac{\pi}{4}(d_1^2 + d_2^2 + d_3^2)}$$

$$F_1 = \tau_{rivet} \times \frac{\pi}{4} d_1^2$$

$$F_2 = \tau_{rivet} \times \frac{\pi}{4} d_2^2$$

$$F_3 = \tau_{rivet} \times \frac{\pi}{4} d_3^2$$



2- Tension stress failure in plate.

$$\sigma_t)_{1-1} = \frac{P}{b \times t}$$

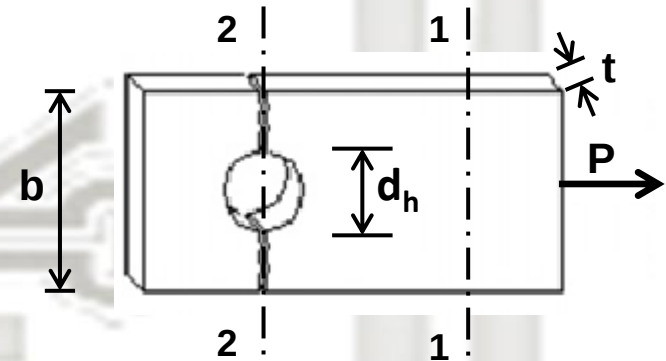
$$\sigma_t)_{2-2} = \frac{P}{(b - d_h) \times t}$$

Where:

t = thickness of plate

d_h = diameter of hole

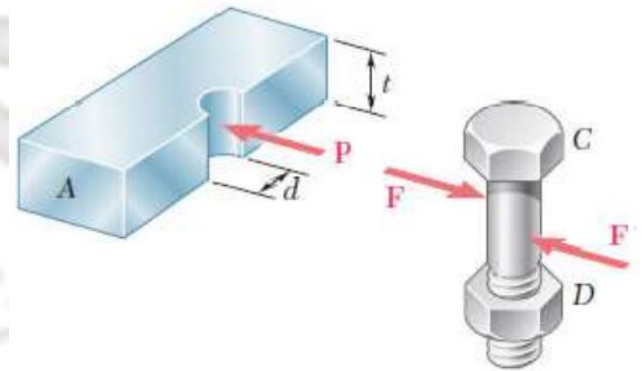
$d_h = d_{rivet} + 3\text{mm}$



$$\sigma_{2-2} > \sigma_{1-1}$$

3- Bearing stress between the plate and rivet

$$\sigma_b = \frac{P}{d \times t}$$



4- Shearing stress failure in plate

$$\sigma_b = \frac{P}{L \times t}$$

Where:

L = Length of the tearing line



Shear out

EXAMPLE 1-11

For the lap joint shown in the Figure, calculate the safe axial tensile force (P); if:

$$\sigma_{\text{tensile}})_{\text{allow.}} = 136 \text{ MPa}$$

$$\tau_{\text{rivet}})_{\text{allow.}} = 102 \text{ MPa}$$

$$\sigma_{\text{bearing}})_{\text{allow.}} = 330 \text{ MPa}$$

Assume the diameter of hole = 25mm.

Sol.

Shear force in the rivet (F_{rivet}) = $P/4$

$$\tau_{\text{rivet}})_{\text{allow.}} = \frac{F_{\text{rivet}}}{A_{\text{rivet}}} = \frac{P/4}{\pi d^2/4} = 102$$

$$P_{\text{safe}} = \pi \times 22^2 \times 102 \times 10^{-3} = 155 \text{ kN}$$

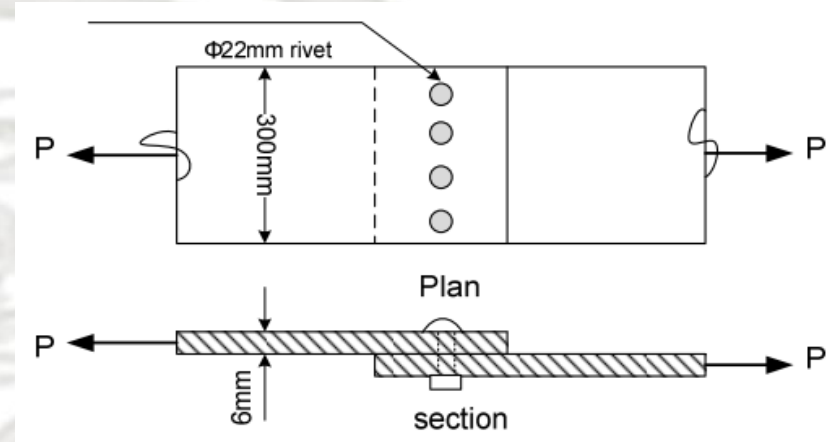
$$\sigma_{\text{tensile}})_{\text{allow}} = \frac{P}{(b - 4h_{\text{hole}}) \times t} = 136$$

$$P_{\text{safe}} = (300 - 4 \times 25) \times 6 \times 136 \times 10^{-3} = 163.2 \text{ kN}$$

$$\sigma_{\text{bearing}})_{\text{allow}} = \frac{P/4}{d \times t} = 330$$

$$P_{\text{safe}} = 4 \times 22 \times 6 \times 330 \times 10^{-3} = 174.2 \text{ kN}$$

The safe force which does not cause failure neither in shear nor in tensile nor in bearing is $P_{\text{safe}} = 155 \text{ kN}$.



EXAMPLE 1-12

Determine the average shear stress in the 20-mm-diameter pin at A and the 30-mm-diameter pin at B that support the beam in Figure.

$$\sum M_A = 0 \rightarrow F_B \times \frac{4}{5} \times 6 - 30 \times 2 = 0 \rightarrow F_B = 12.5 \text{ kN}$$

$$\sum F_x = 0 \rightarrow 12.5 \times \frac{3}{5} - A_x = 0 \rightarrow A_x = 7.5 \text{ kN}$$

$$\sum F_y = 0 \rightarrow A_y + 12.5 \times \frac{4}{5} - 30 = 0 \rightarrow A_y = 20 \text{ kN}$$

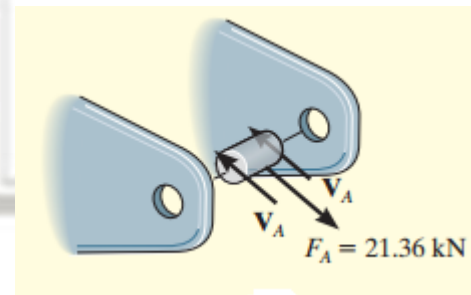
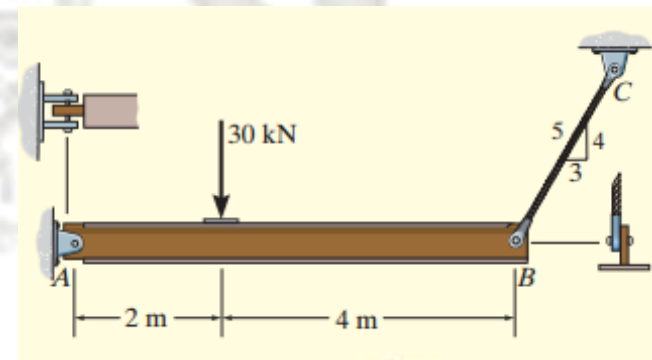
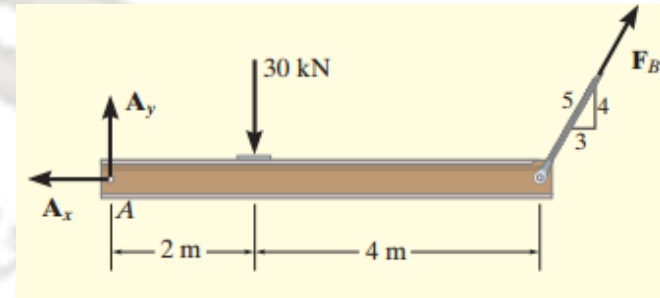
$$F_A = \sqrt{A_x^2 + A_y^2} = \sqrt{7.5^2 + 20^2} = 21.36 \text{ kN}$$

$$V_A = \frac{F_A}{2} = \frac{21.36}{2} = 10.68 \text{ kN}$$

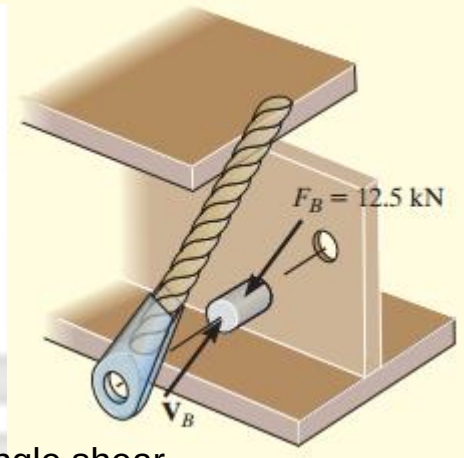
$$V_B = F_B = 12.5 \text{ kN}$$

$$\tau_A = \frac{V_A}{A_A} = \frac{10.68 \times 1000}{\frac{\pi}{4} \times 20^2} = 34 \text{ MPa}$$

$$\tau_B = \frac{V_B}{A_B} = \frac{12.5 \times 1000}{\frac{\pi}{4} \times 30^2} = 17.7 \text{ MPa}$$



Double shear



Single shear

EXAMPLE 1-13

If the wood joint in the Figure has a width of 150 mm, determine the average shear stress developed along shear planes a-a and b-b. For each plane, represent the state of stress on an element of the material.

$$\sum F_x = 0 \rightarrow 6 - F - F = 0$$

$$F = 3 \text{ kN}$$

$$\sum F_x = 0 \rightarrow V_a - 3 = 0$$

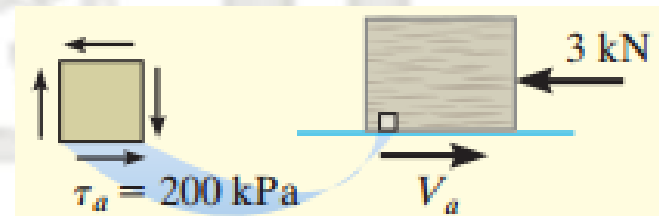
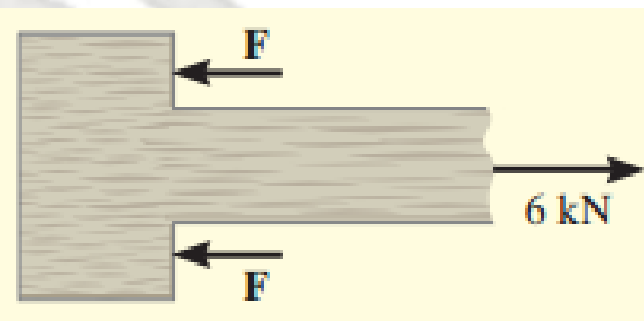
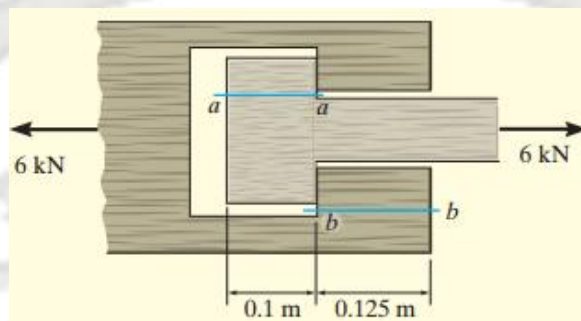
$$V_a = 3 \text{ kN}$$

$$\sum F_x = 0 \rightarrow 3 - V_b = 0$$

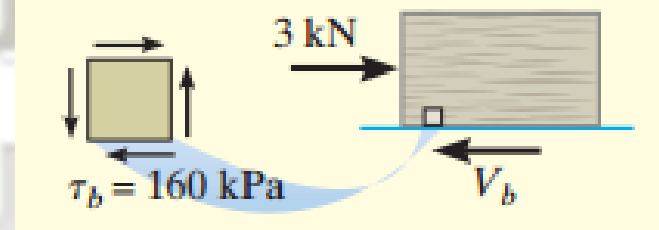
$$V_b = 3 \text{ kN}$$

$$\tau_A = \frac{V_a}{A_a} = \frac{3 \times 1000}{0.1 \times 0.15} = 200 \text{ kPa}$$

$$\tau_B = \frac{V_b}{A_b} = \frac{3 \times 1000}{0.125 \times 0.15} = 160 \text{ kPa}$$



(c)



EXAMPLE 1-14

Find the smallest diameter bolt that can be used in the connector shown in the Figure, if $P = 400 \text{ kN}$. The shearing strength of the bolt is 300 MPa . If the thickness of plate = 30 mm , find the bearing stress.

The bolt is subjected to double shear

$$V = \frac{P}{2}$$

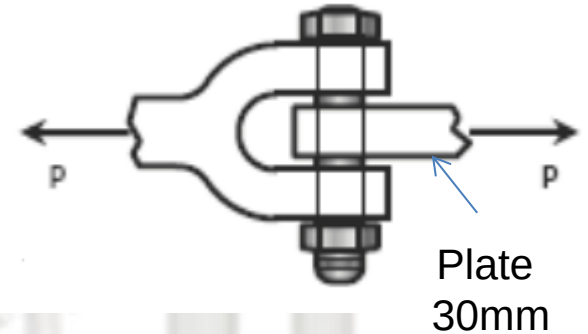
$$\tau = \frac{V}{A} \rightarrow A = \frac{V}{\tau}$$

$$\frac{\pi}{4} d^2 = \frac{400000}{2 \times 300}$$

$$d^2 = 848.826$$

$$d = 29.13 \text{ mm}$$

$$\sigma_b = \frac{P}{dt} = \frac{400000}{29.13 \times 30} = 457.72 \text{ MPa}$$



EXAMPLE 1-15

Compute the shearing stress in the pin at B for the member supported as shown in the Figure. The pin diameter is 20 mm.

$$\sum M_c = 0 \rightarrow 0.25B_y = 0.25 \times 40 \sin 35^\circ + 0.2 \times 40 \cos 35^\circ$$

$$B_y = 49.156 \text{ kN}$$

$$\sum F_x = 0 \rightarrow B_x = 40 \cos 35^\circ = 32.766 \text{ kN}$$

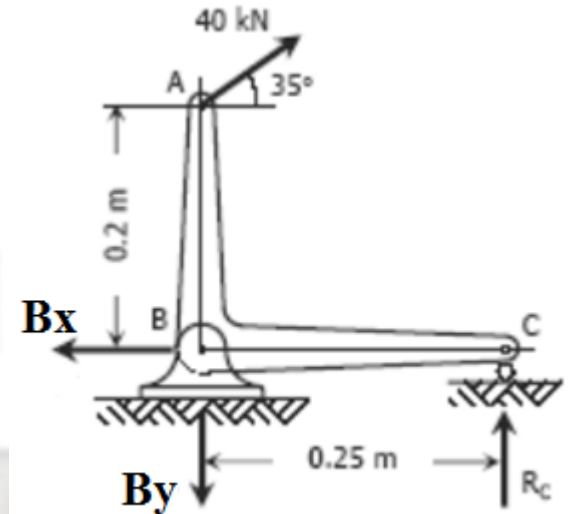
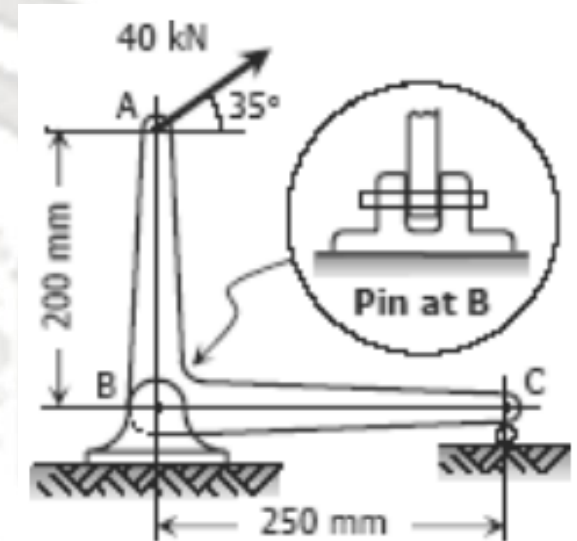
$$R_B = \sqrt{B_x^2 + B_y^2} = \sqrt{32.766^2 + 49.156^2} = 59.076 \text{ kN}$$

$$V_B = \frac{R_B}{2} = \frac{59.076}{2} = 29.538 \text{ kN}$$

$$\tau_B = \frac{29.538}{A}$$

$$\tau_B = \frac{29.538 \times 1000}{\frac{\pi}{4} \times 20^2}$$

$$\tau_B = 94.02 \text{ MPa}$$



EXAMPLE 1-16

In the Figure shown, assume that a 20-mm-diameter rivet joins the plates that are each 110 mm wide. The allowable stresses are 120 MPa for bearing in the plate material and 60 MPa for shearing of rivet. Determine (a) the minimum thickness of each plate; and (b) the largest average tensile stress in the plates. (Assume $d_h = 25\text{mm}$).

a) From shearing of rivet

$$P = \tau \times A_{\text{rivet}} = 60 \times \frac{\pi}{4} \times 20^2 = 18849.5\text{N}$$

From bearing of plate

$$P = \sigma_b \times A_b$$

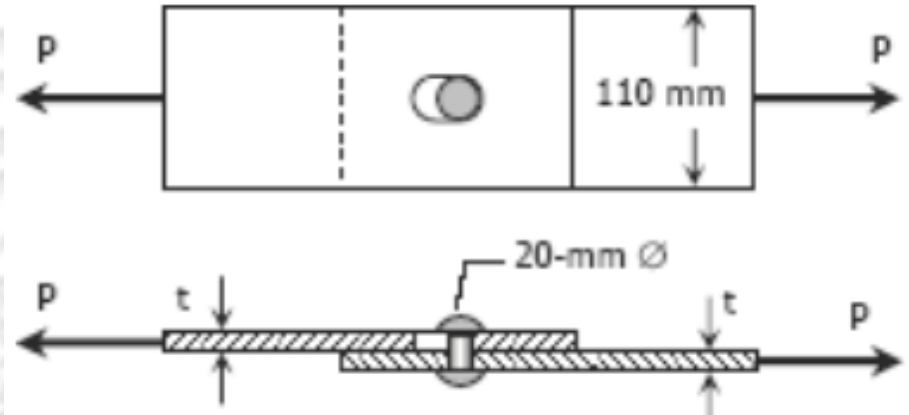
$$18849.5 = 120 \times 20 \times t \rightarrow t = 7.85\text{mm}$$

b) Largest average tensile stress in the plate

$$\sigma = \frac{P}{A}$$

$$\sigma = \frac{18849.5}{7.85(110 - 25)}$$

$$\sigma = 28.25\text{MPa}$$



Allowable stresses; Factor of Safety

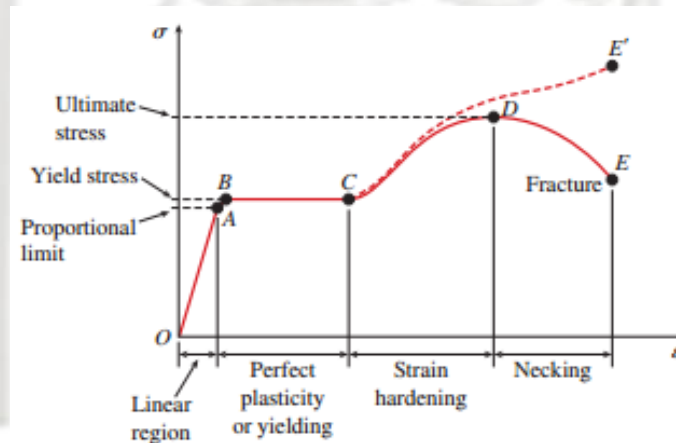
To properly design a structural member or mechanical element it is necessary to restrict the stress in the material to a level that will be safe. To ensure this safety, it is therefore necessary to choose an allowable stress that restricts the applied load to one that is less than the load the member can fully support. One method of specifying the allowable load for a member is to use a number called the **factor of safety**. The factor of safety (F.S.) is a ratio of the **ultimate load** to the **allowable load**. Here the ultimate load is found from experimental testing of the material. The factor of safety must be greater than 1 in order to avoid the potential for failure. the F.S. used in the design of aircraft or space vehicle components may be close to 1 in order to reduce the weight of the vehicle. Or, in the case of a nuclear power plant, the factor of safety for some of its components may be as high as 3 due to uncertainties in loading or material behavior. In design, the area of the member or element is the unknown, while the force is known. However, information on the parameter, stress, must be provided. In fact, information on the stress of material can be gathered from tests.

$$F.S = \frac{F_{failure}}{F_{allowable}}$$

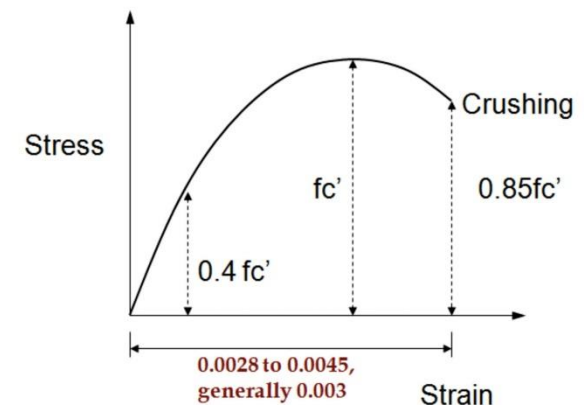
Or

$$F.S = \frac{F_{ultimate}}{F_{allowable}}$$

$$F.S = \frac{Stress_{ultimate}}{Stress_{allowable}}$$



Stress Strain Curve of Concrete



Practically, the stress reaches its maximum value and the corresponding stress (at point D for an example of structural steel in tension) is called **ultimate stress**. However, the stress value used in design is set significantly lower than the ultimate stress and known as **allowable stress** by use of **factors of safety**.

EXAMPLE 1-17

The rigid bar AB shown in the Figure is supported by a steel rod AC having a diameter of 20 mm and an aluminum block having a cross sectional area of 1800 mm². The 18-mm-diameter pins at A and C are subjected to single shear. If the failure stress for the steel and aluminum is 680 MPa and 70 MPa respectively, and the failure shear stress for each pin is 900 MPa, determine the largest load P that can be applied to the bar. Apply a factor of safety of F.S=2.

$$(\sigma_{st})_{allow} = \frac{(\sigma_{st})_{failure}}{F.S} = \frac{680}{2} = 340 \text{ MPa}$$

$$(\sigma_{al})_{allow} = \frac{(\sigma_{al})_{failure}}{F.S} = \frac{70}{2} = 35 \text{ MPa}$$

$$\tau_{allow} = \frac{\tau_{failure}}{F.S} = \frac{900}{2} = 450 \text{ MPa}$$

$$\sum M_B = 0 \rightarrow P \times 1.25 = F_{AC} \times 2$$

$$\sum M_A = 0 \rightarrow P \times 0.75 = F_B \times 2$$

Rod AC: $F_{AC} = (\sigma_{st})_{allow} \times A_{AC} = 340 \times 10^{-3} \times \pi \times 10^2 = 106.8 \text{ kN}$

$$P = \frac{106.8 \times 2}{1.25} = 171 \text{ kN}$$

Block B: $F_B = (\sigma_{al})_{allow} \times A_B = 35 \times 10^{-3} \times 1800 = 63 \text{ kN}$

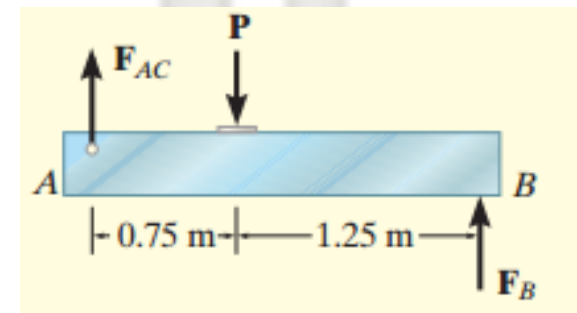
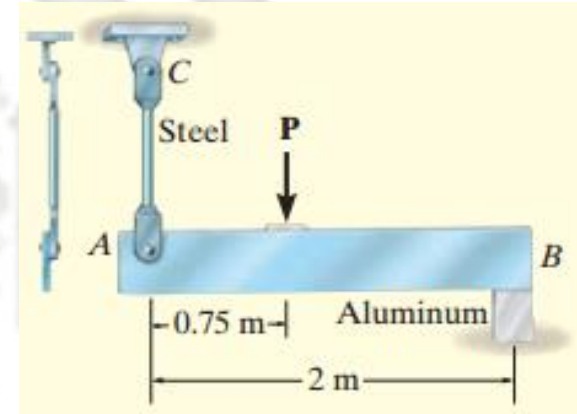
$$P = \frac{63 \times 2}{0.75} = 168 \text{ kN}$$

Pin A or C, due to single shear:

$$F_{AC} = V = \tau_{allow} \times A = 450 \times 10^{-3} \times \pi \times 9^2 = 114.5 \text{ kN}$$

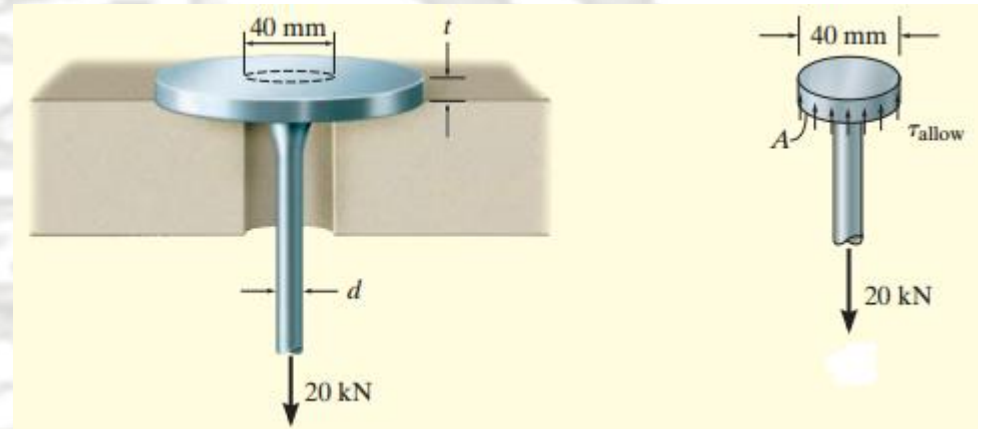
$$P = \frac{114.5 \times 2}{1.25} = 183 \text{ kN}$$

By comparison, as P reaches its smallest value (168 kN), the allowable normal stress will first be developed in the aluminum block. Hence, **P = 168 kN**.



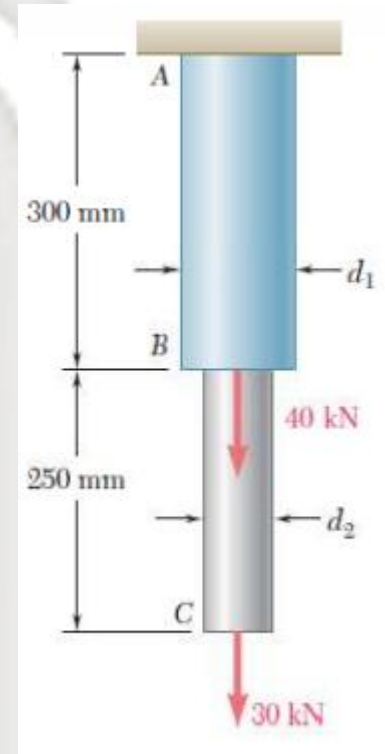
H.W.

The suspender rod is supported at its end by a fixed-connected circular disk as shown in Figure. If the rod passes through a 40-mm-diameter hole, determine the minimum required diameter of the rod and the minimum thickness of the disk needed to support the 20-kN load. The allowable normal stress for the rod is 60 MPa and the allowable shear stress for the disk is 35 MPa.



Class Work

The two solid cylindrical steel rods AB and BC are welded together at B and loaded as shown in the figure. Knowing that the normal stress must not exceed 175MPa in rod AB and 150MPa in rod BC determine the smallest allowable values of d_1 and d_2 .



CHAPER TWO – STRAIN, HOOK’S LAW, AXIAL LOAD PROBLEMS

Introduction:

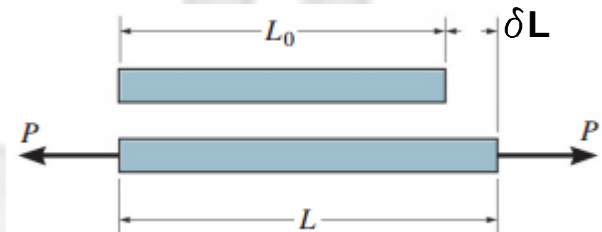
In engineering the deformation of a body is specified using the concepts of normal and shear strain. Whenever a force is applied to a body, it will tend to change the body's shape. These changes are referred to as deformation, and they may be highly visible or practically unnoticeable. Deformation of a body can also occur when the temperature of the body is changed. The figure shows the before and after positions of three different line segments on this rubber membrane which is subjected to tension. The vertical line is lengthened, the horizontal line is shortened, and the inclined line changes its length and rotates.



Normal Strain (Axial Strain):

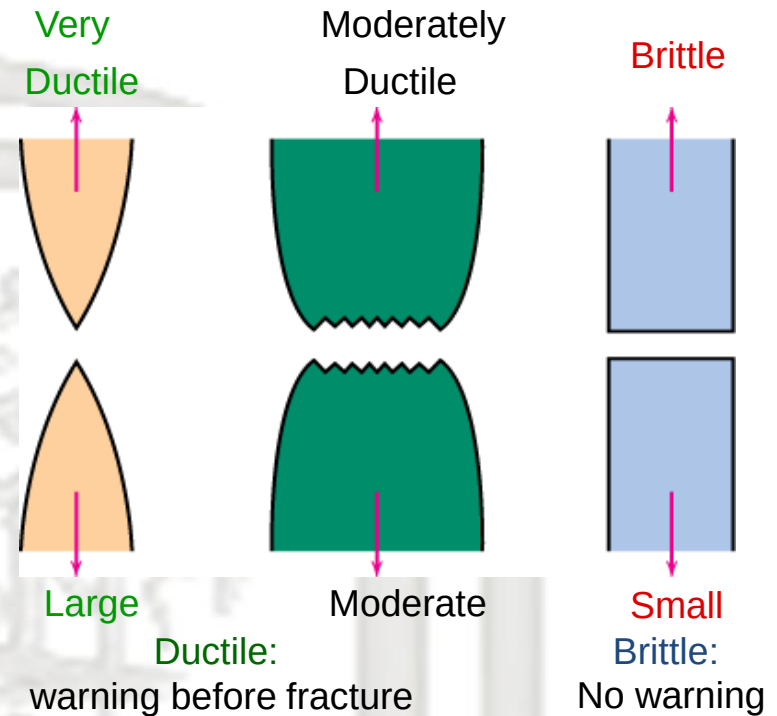
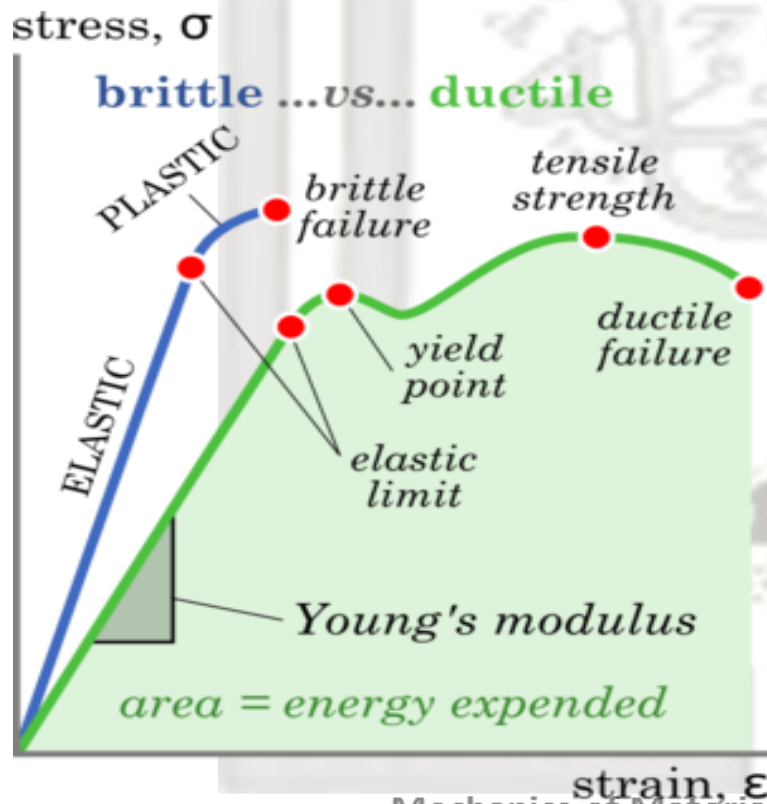
If an axial load P is applied to the bar in the Figure, it will change the bar's length L_0 to a length L . We will define the average normal strain ε (epsilon) of the bar as the change in its length δ (delta) = $L - L_0$ divided by its original length, that is:

$$\varepsilon = \frac{L - L_0}{L_0} = \frac{\delta L}{L_0} \text{ (dimensionless quantity)}$$



Ductile and Brittle Materials

Engineering materials are commonly classified as either ductile or brittle materials. A ductile material lies one having a relatively large tensile strain up to the point of rupture (for example, structural **steel** or **aluminum**) whereas a brittle material has a relatively small strain up to this same point. **Cast iron** and **concrete** are examples of brittle materials.



Ductile material	Brittle material
Is the ability of a Material to withstand Plastic deformation Under tensile stress Without fracture.	brittle material fractures at or near its proportional limit.
Fracture occur far Away from P.L	Fracture occur at or near P.L

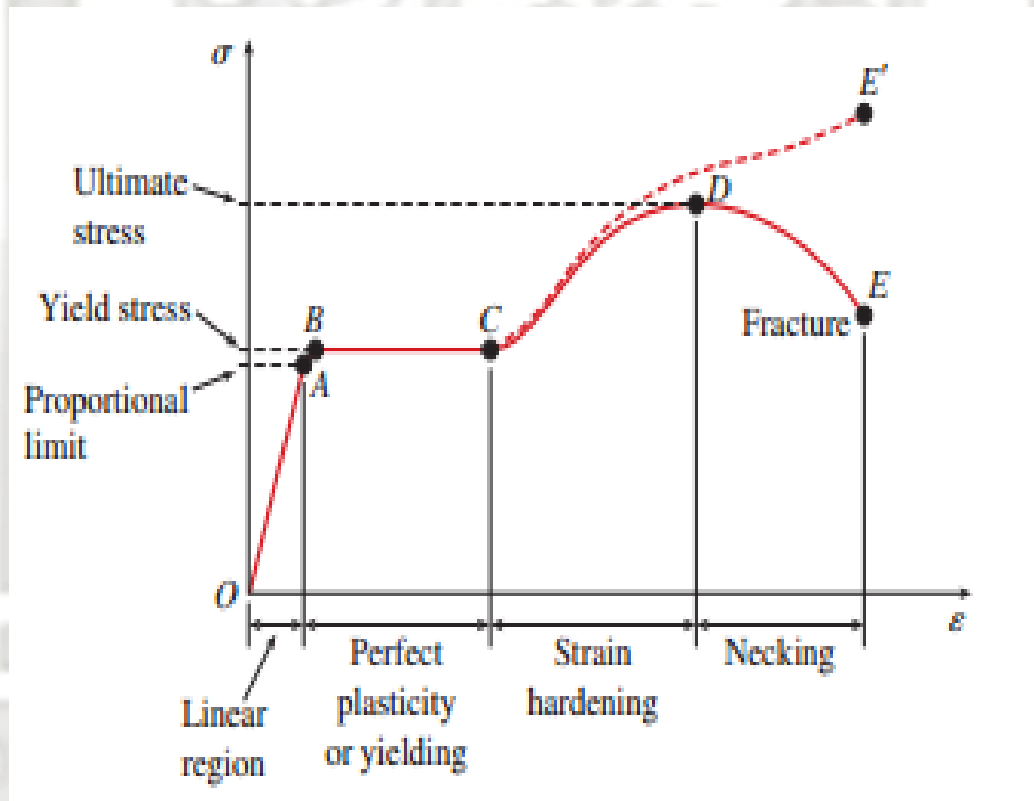
Hooke's Law

From the origin O to the point called proportional limit, the stress-strain curve is a straight line. This linear relation between elongation and the axial force causing was first noticed by Sir Robert Hooke in 1678 and is called Hooke's law that within the proportional limit, the stress is directly proportional to strain.

$$\sigma = E\varepsilon$$

Where: E denotes the slope of the straight-line portion OA.

The quantity E is equal to the slope of the stress-strain diagram from O to A it is called the modulus of elasticity of the material in tension, or, as it is often called, Young's modulus of elasticity.



EXAMPLE 2-1

Determine the average normal strains in the two wires in the Figure shown if the ring at A moves to A'.

The original length of each wire is

$$L_{AB} = L_{AC} = \sqrt{3^2 + 4^2} = 5m$$

The final lengths are:

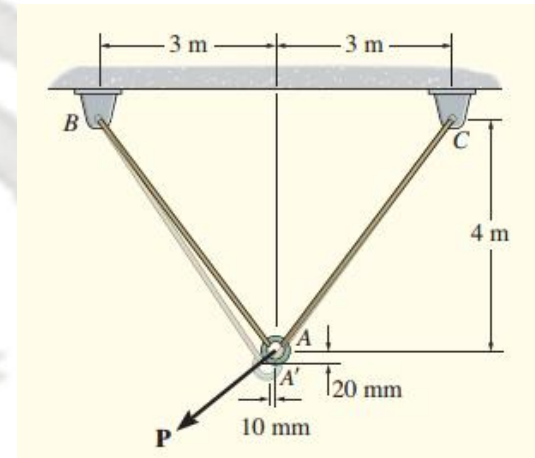
$$L_{A'B} = \sqrt{(3 - 0.01)^2 + (4 + 0.02)^2} = 5.01004m$$

$$L_{A'C} = \sqrt{(3 + 0.01)^2 + (4 + 0.02)^2} = 5.022m$$

The normal strains:

$$\epsilon_{AB} = \frac{L_{A'B} - L_{AB}}{L_{AB}} = \frac{5.01004 - 5}{5} = 2.01 \times 10^{-3} m/m$$

$$\epsilon_{AC} = \frac{L_{A'C} - L_{AC}}{L_{AC}} = \frac{5.022 - 5}{5} = 4.4 \times 10^{-3} m/m$$



EXAMPLE 2-2

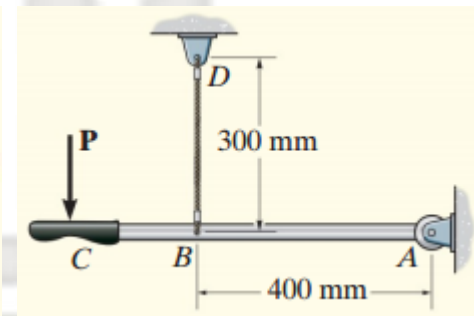
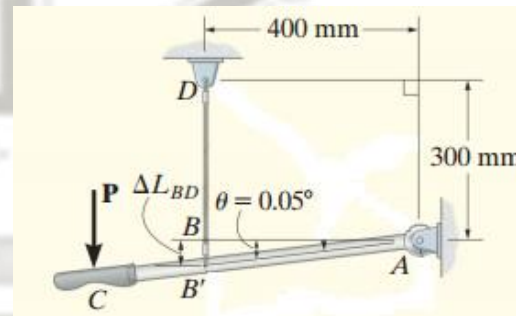
When force **P** is applied to the rigid lever arm ABC shown in the Figure, the arm rotates counterclockwise about pin A through an angle of 0.05° . Determine the normal strain in wire BD.

$$\tan(0.05) = \frac{BB'}{400}$$

$$BB' = 400 \times \tan(0.05) = 0.349mm$$

$$L_{DB'} = 300 + 0.349 = 300.349mm$$

$$\epsilon_{DB'} = \frac{L_{DB'} - L_{DB}}{L_{DB}} = \frac{300.349 - 300}{300} = 0.00116m/m$$



Deflection of Axially Loaded Rods

Using Hooke's law and the definitions of stress and strain, we will now develop an equation that can be used to determine the elastic displacement of a member subjected to axial loads. The free-body diagram of the axially loaded element is shown in Figure. The resultant internal axial force will be a function of x since the external distributed loading will cause it to vary along the length of the bar.

Provided the stress does not exceed the proportional limit, we can apply Hooke's law

$$\sigma = \frac{P_x}{A_x}$$

$$\varepsilon = \frac{d\delta}{dx}$$

$$\frac{P_x}{A_x} = E_x \left(\frac{d\delta}{dx} \right) \rightarrow d\delta = \frac{P_x dx}{A_x E_x}$$

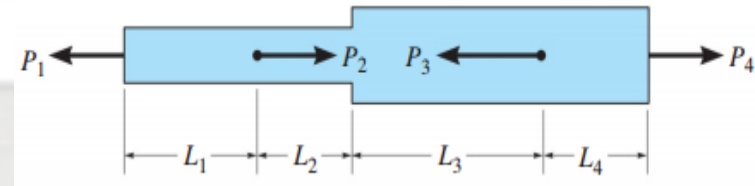
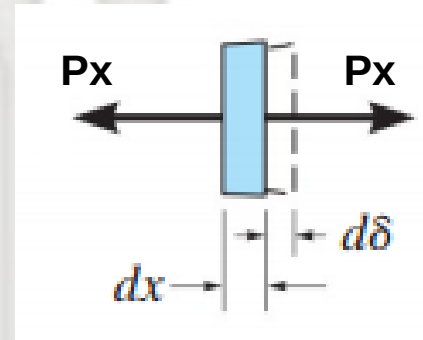
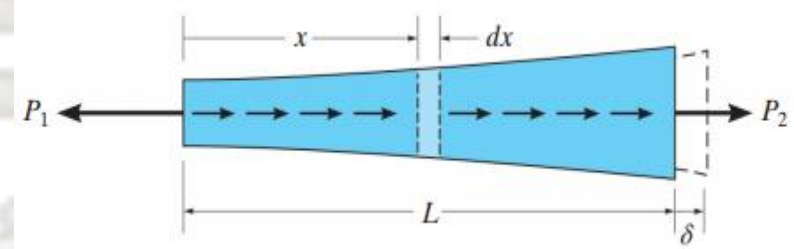
$$\delta = \int_0^L \frac{P_x dx}{A_x E_x}$$

For constant load P_x , cross-sectional area A_x and Modulus of elasticity E_x , the above equation becomes:

$$\delta = \frac{PL}{AE}$$

If the structural member has some segments with different load, area, and E in each segment, the total deflection equals the sum of the deflection of each segment.

$$\delta = \sum \frac{PL}{AE}$$



EXAMPLE 2-3

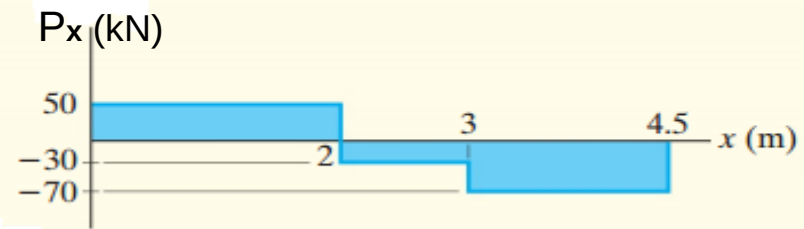
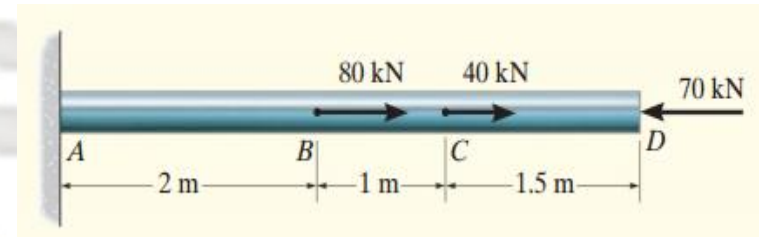
The uniform steel bar shown in the Figure has a diameter of 50 mm and is subjected to the loading shown. Determine the displacement at D. Assume $E=200$ GPa.

$$\delta_D = \sum \frac{PL}{AE}$$

$$\delta_D = \frac{-70 \times 1000 \times 1500}{\pi \times 25^2 \times 200000} + \frac{-30 \times 1000 \times 1000}{\pi \times 25^2 \times 200000} + \frac{50 \times 1000 \times 2000}{\pi \times 25^2 \times 200000}$$

$$\delta_D = -0.0891 \text{ mm}$$

This negative result indicates that point D moves to the left.



EXAMPLE 2-4

A member is made of a material that has a specific weight of $\gamma = 6$ kN/m³ and modulus of elasticity of $E=9$ GPa. Determine how far of the cone's end is displaced due to gravity when it is suspended in the vertical position.

$$\frac{x}{y} = \frac{0.3}{3} \rightarrow x = 0.1y \rightarrow Vol. = \frac{1}{3} \pi x^2 y = \frac{\pi \times (0.1y)^2 \times y}{3} = 0.01047 y^3$$

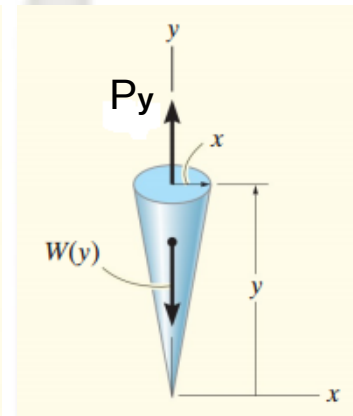
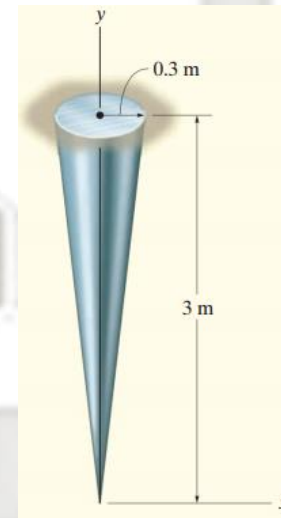
$$W = \gamma \times V$$

$$\sum Fy = 0 \rightarrow P_y = 6 \times 1000 \times 0.01047 y^3 = 62.83 y^3$$

$$A_y = \pi x^2 = 0.03142 y^2$$

$$\delta = \int_0^L \frac{P dy}{AE} = \int_0^3 \frac{62.83 y^3 dy}{0.03142 y^2 \times 9 \times 10^9} = 222.2 \times 10^{-9} \int_0^3 y dy = \frac{222.2 \times 10^{-6}}{2} [y^2]_0^3$$

$$\delta = 1 \times 10^{-6} \text{ m}$$



very small amount Mechanics of Materials – 2nd Class Dr. Ashraf Alfeehan

EXAMPLE 2-5

A bronze bar is fastened between a steel bar and an aluminum bar as shown in the Figure. Axial loads are applied at the positions indicated. Find the largest value of P that will not exceed an overall deformation of 3.0 mm, or the following stresses: 140 MPa in the steel, 120 MPa in the bronze, and 80 MPa in the aluminum. Use $E_{st} = 200$ GPa, $E_{al} = 70$ GPa, and $E_{br} = 83$ GPa.

$$P_{st} = \sigma_{st} \times A_{st} = 140 \times 480 = 67200 N$$

$$2P_{bro} = \sigma_{bro} \times A_{bro} = 120 \times 650 \rightarrow P_{bro} = 39000 N$$

$$2P_{al} = \sigma_{al} \times A_{al} = 80 \times 320 \rightarrow P_{al} = 12800 N$$

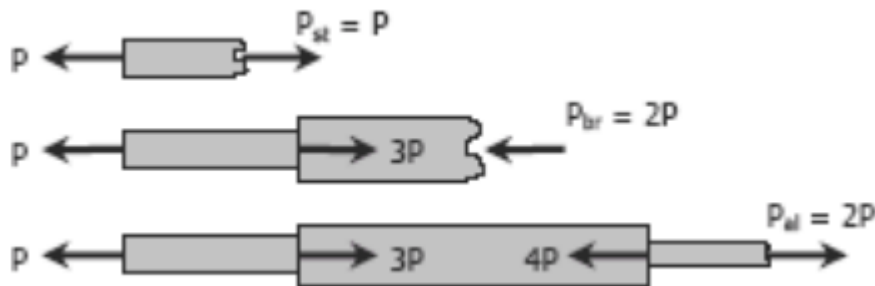
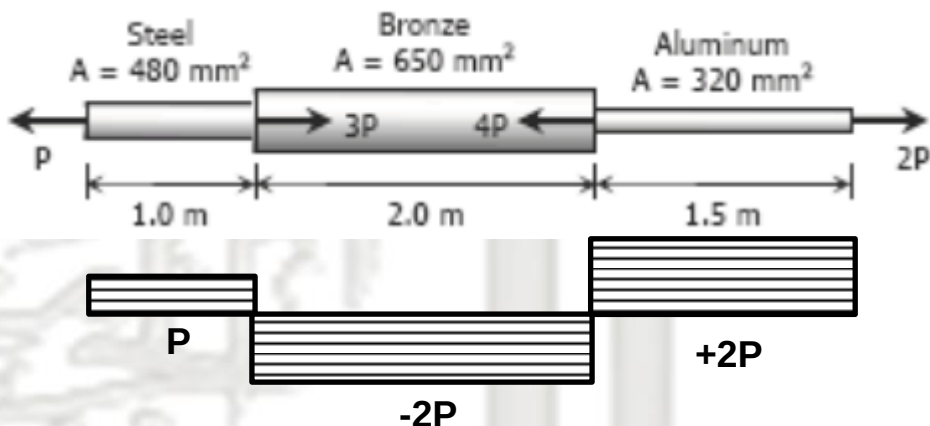
$$\delta = \delta_{st} - \delta_{bro} + \delta_{al}$$

$$3 = \frac{P \times 1000}{480 \times 200000} - \frac{2P \times 2000}{650 \times 70000} + \frac{2P \times 1500}{480 \times 83000}$$

$$3 = P \left(\frac{1}{96000} - \frac{1}{11375} + \frac{3}{26560} \right)$$

$$P = 84611 N$$

Use the smallest $P = 12800$ N



EXAMPLE 2-6

Rigid beam AB rests on the two short posts shown in Figure.. AC is made of steel and has a diameter of 20 mm, and BD is made of aluminum and has a diameter of 40 mm. Determine the displacement of point F on AB if a vertical load of 90 kN is applied over this point. Take $E_{st} = 200 \text{ GPa}$, $E_{al} = 70 \text{ GPa}$.

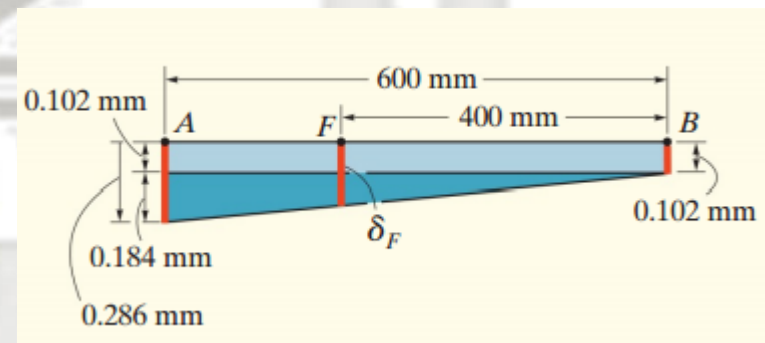
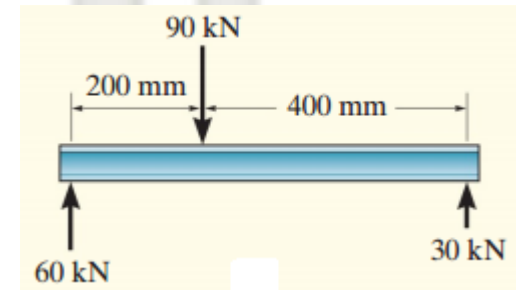
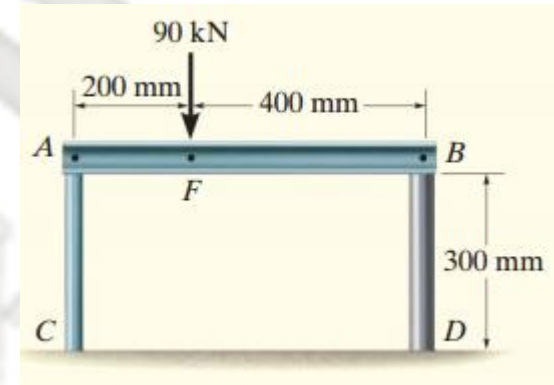
$$\delta_A = \frac{P_{AC} L_{AC}}{A_{AC} E_{st}}$$

$$\delta_A = \frac{-60 \times 1000 \times 300}{\pi \times 10^2 \times 200000} = -0.286 \text{ mm} = 0.286 \text{ mm} \downarrow$$

$$\delta_B = \frac{P_{BD} L_{BD}}{A_{BD} E_{al}}$$

$$\delta_B = \frac{-30 \times 1000 \times 300}{\pi \times 20^2 \times 70000} = -0.102 \text{ mm} = 0.102 \text{ mm} \downarrow$$

$$\delta_F = 0.102 + 0.184 \times \frac{400}{600} = 0.225 \text{ mm} \downarrow$$



EXAMPLE 2-7

A uniform concrete slab of total weight W is to be attached, as shown in the Figure, to two rods whose lower ends are on the same level. Determine the ratio of the areas of the rods so that the slab will remain level. Take $E_{st} = 200 \text{ GPa}$, $E_{al} = 70 \text{ GPa}$.

$$\sum M_{al} = 0 \rightarrow 6P_{st} = 2W$$

$$P_{st} = \frac{W}{3}$$

$$\sum M_{st} = 0 \rightarrow 6P_{al} = 4W$$

$$P_{al} = \frac{2W}{3}$$

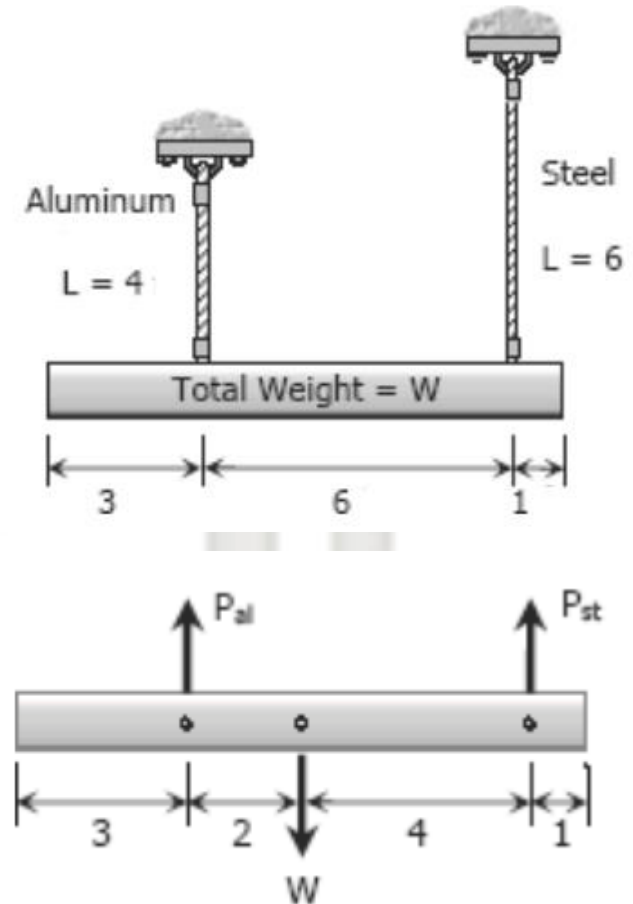
$$\delta_{st} = \delta_{al}$$

$$\left(\frac{PL}{AE}\right)_{st} = \left(\frac{PL}{AE}\right)_{al}$$

$$\frac{\frac{W}{3} \times 6}{A_{st} \times 200} = \frac{\frac{2W}{3} \times 4}{A_{al} \times 70}$$

$$\frac{2W}{A_{st} \times 200} = \frac{8W}{A_{al} \times 70 \times 3}$$

$$\frac{A_{al}}{A_{st}} = \frac{200}{70} \times \frac{8}{6} = 3.809$$



EXAMPLE 2-8

The horizontal rigid beam ABCD is supported by vertical bars BE and CF and is loaded by vertical forces $P_1=400$ kN and $P_2=360$ kN acting at points A and D, respectively. Bars BE and CF are made of steel ($E=200$ GPa) and have cross-sectional areas $A_{BE}=11,100$ mm² and $A_{CF}=9,280$ mm². The distances between various points on the bars are shown in the figure. Determine the vertical displacements of points A and D.

$$\sum M_B = 0 \rightarrow 400 \times 1.5 + F_{CF} \times 1.5 = 360 \times 3.6$$

$$F_{CF} = 464 \text{ kN}$$

$$\sum M_C = 0 \rightarrow 400 \times 3 = F_{BE} \times 1.5 + 360 \times 2.1$$

$$F_{BE} = 296 \text{ kN}$$

Shortening of bar BE:

$$\delta_{BE} = \frac{F_{BE} L_{BE}}{A_{BE} E} = \frac{296 \times 3}{11100 \times 200} = 0.4 \text{ mm}$$

Shortening of bar CF:

$$\delta_{CF} = \frac{F_{CF} L_{CF}}{A_{CF} E} = \frac{464 \times 2.4}{9280 \times 200} = 0.6 \text{ mm}$$

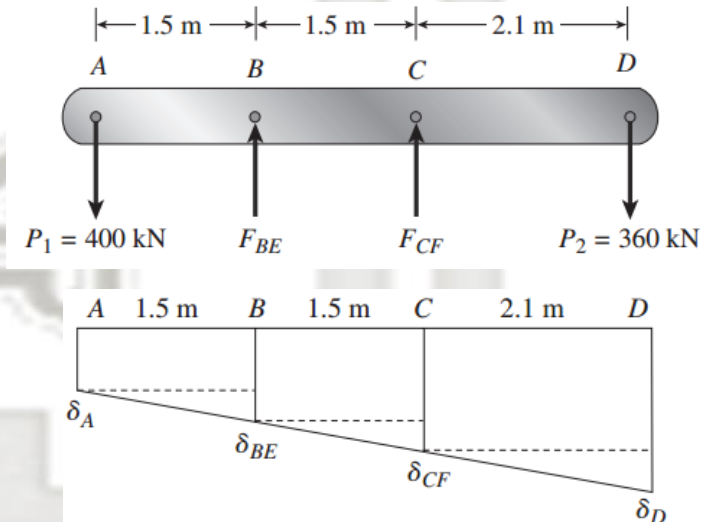
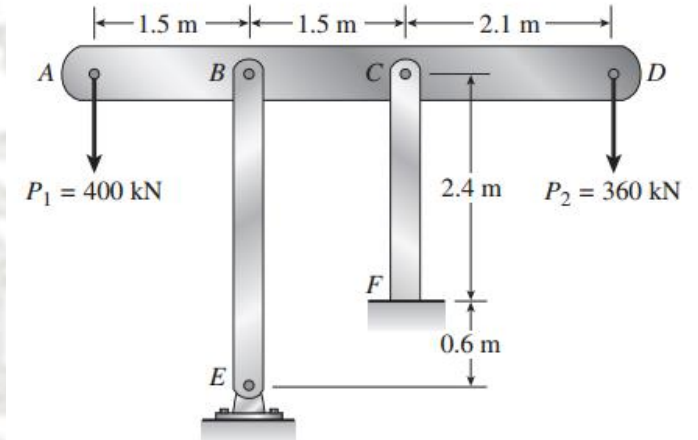
From displacement diagram:

$$\delta_{BE} - \delta_A = \delta_{CF} - \delta_{BE}$$

$$\delta_A = 2\delta_{BE} - \delta_{CF} = 2 \times 0.4 - 0.6 = 0.2 \text{ mm}$$

$$\delta_D - \delta_{CF} = \frac{2.1}{1.5} (\delta_{CF} - \delta_{BE})$$

$$\delta_D = \frac{12}{5} \delta_{CF} - \frac{7}{5} \delta_{BE} = \frac{12}{5} \times 0.6 - \frac{7}{5} \times 0.4 = 0.88 \text{ mm}$$



EXAMPLE 2-9

For the composite column shown:

1. Determine the maximum load P if the total deflection $\Delta=0.25\text{mm}$.
2. Draw the vertical deflection diagram.

$$\delta_{total} = \left(\frac{PL}{AE}\right)_{st} + \left(\frac{PL}{AE}\right)_{al}$$

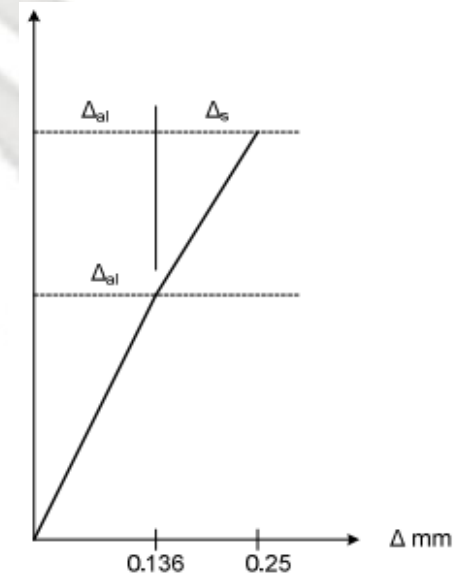
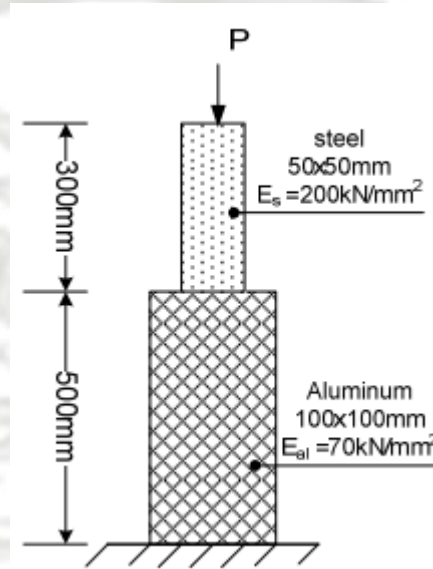
$$0.25 = \frac{P \times 300}{200 \times 50 \times 50} + \frac{P \times 500}{70 \times 100 \times 100}$$

$$0.25 = 0.0006P + 0.00071P$$

$$P = 190.8\text{kN}$$

$$\delta_{al} = \frac{190.8 \times 500}{70 \times 100 \times 100} = 0.136\text{mm}$$

$$\delta_{st} = \frac{190.8 \times 300}{200 \times 50 \times 50} = 0.114\text{mm}$$



Statically indeterminate members (axially loaded only)

When the reactive forces or the internal resisting forces over a cross section exceed the number of independent equations of equilibrium, the structure is called **statically indeterminate**. These cases require the use of additional relations that depend on the elastic deformations in the members.

Consider the bar shown in Figure, which is fixed supported at both of its ends. From its free-body diagram, there are two unknown support reactions. Equilibrium requires:

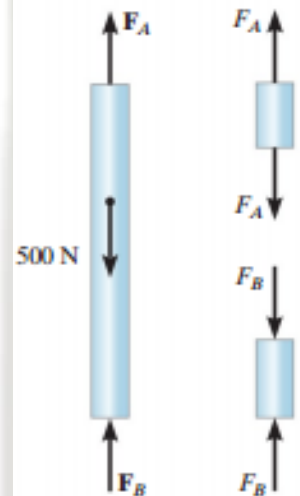
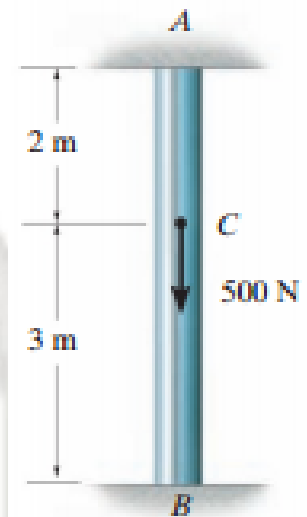
$$\sum F_y = 0 \rightarrow F_B + F_A - 500 = 0$$

This type of problem is called **statically indeterminate**, since the equilibrium equation is not sufficient to determine both reactions on the bar. An additional equation needed for solution called **compatibility equation** that specifies the conditions for displacement. In this case, the displacement of end A of the bar with respect to end B to equal zero, since the end supports are fixed. Realizing that the internal force in segment AC is (+F_A), and in segment CB it is (-F_B), then the compatibility equation can be written as

$$\frac{F_A \times 2}{AE} - \frac{F_B \times 3}{AE} = 0 \rightarrow F_A = 1.5F_B$$

$$F_A = 300\text{ N}$$

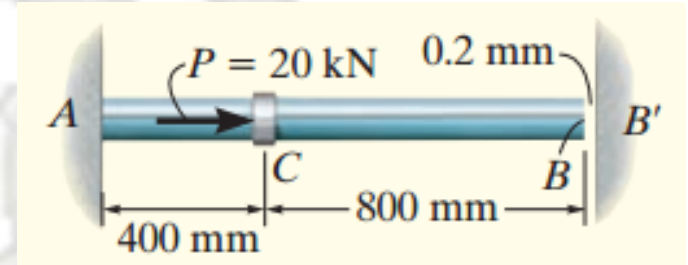
$$F_B = 200\text{ N}$$



EXAMPLE 2-10

The steel rod shown in Figure, has a diameter of 10 mm. It is fixed to the wall at A, and before it is loaded, there is a gap of 0.2 mm between the wall at B and the rod. Determine the reactions on the rod if it is subjected to an axial force of $P = 20$ kN. (Neglect the size of the collar at C). Take $E_{st} = 200$ GPa.

We will assume that force P is large enough to cause the rod's end B to contact the wall at B'. When this occurs, the problem becomes statically indeterminate since there are two unknowns and only one equation of equilibrium.



$$\sum F_x = 0$$

$$F_A + F_B = 20000 \rightarrow (1)$$

$$\frac{F_A \times L_{AC}}{AE} - \frac{F_B \times L_{CB}}{AE} = 0.0002$$

$$\frac{F_A \times 0.4}{AE} - \frac{F_B \times 0.8}{AE} = 0.0002$$

$$0.4F_A - 0.8F_B = 0.0002AE$$

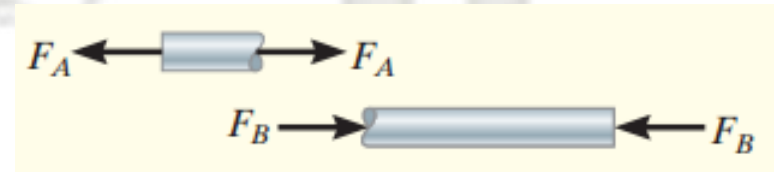
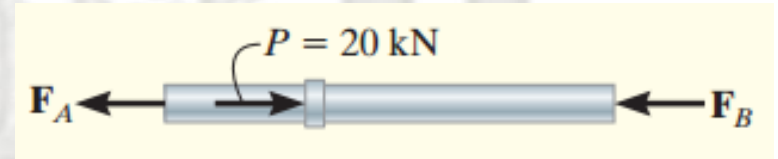
$$0.4F_A - 0.8F_B = 0.0002 \times \pi \times 0.005^2 \times 200 \times 10^9$$

$$0.4F_A - 0.8F_B = 3141.6$$

$$F_A = 2F_B + 7854 \rightarrow (2)$$

$$2F_B + 7854 + F_B = 20000 \rightarrow F_B = 4048.67 \text{ N} = 4.048 \text{ kN}$$

$$F_A = 2 \times 4048.67 + 7854 = 15951.33 \text{ N} = 15.951 \text{ kN}$$



NOTE:

Since the answer for F_B is positive, indeed end B contacts the wall at B as originally assumed. If F_B were a negative quantity, the problem would be statically determinate, so that $F_B = 0$ and $F_A = 20$ kN.

EXAMPLE 2-11

The three steel bars shown in the Figure, are pin connected to a rigid member. If the applied load on the member is 15 kN, determine the force developed in each bar. Bars AB and EF each have a cross-sectional area of 50 mm², and bar CD has a cross-sectional area of 30 mm².

$$\sum F_y = 0 \rightarrow F_A + F_C + F_E = 15 \rightarrow (1)$$

$$\sum M_C = 0 \rightarrow F_A \times 0.4 = 15 \times 0.2 + F_E \times 0.4$$

$$0.4F_A = 3 + 0.4F_E \rightarrow (2)$$

$$\frac{\delta_A - \delta_E}{0.8} = \frac{\delta_C - \delta_E}{0.4}$$

$$\delta_C = \frac{1}{2}\delta_A + \frac{1}{2}\delta_E$$

$$\frac{F_C L}{30E_{st}} = \frac{F_A L}{2 \times 50 \times E_{st}} + \frac{F_E L}{2 \times 50 \times E_{st}}$$

$$F_C = 0.3F_A + 0.3F_E \rightarrow (3)$$

Solving Eqs. 1–3 simultaneously yields:

$$F_A = 15 - F_C - F_E$$

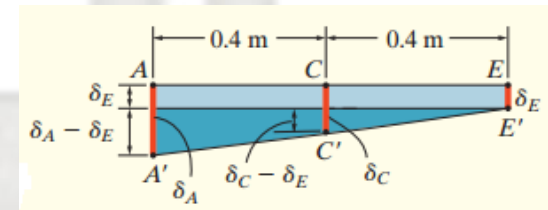
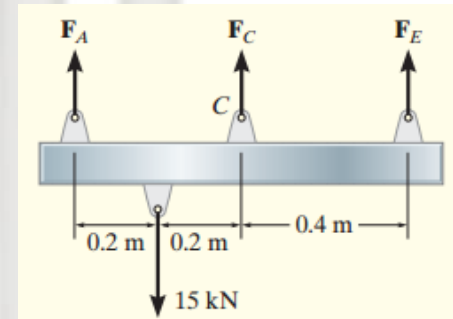
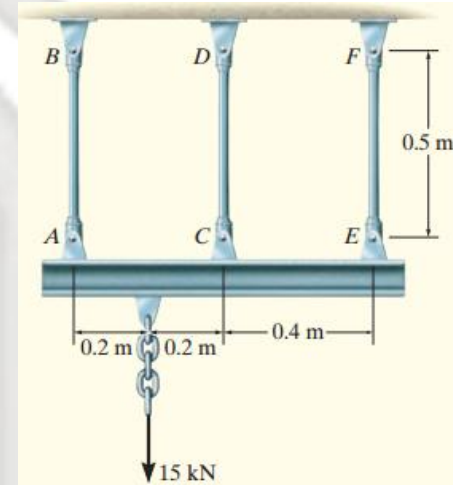
$$F_C = 0.3(15 - F_C - F_E) + 0.3F_E$$

$$F_C = 4.5 - 0.3F_C \rightarrow F_C = 3.46 \text{ kN}$$

$$F_A = 15 - 3.46 - F_E = 11.538 - F_E$$

$$0.4(11.538 - F_E) = 3 + 0.4F_E \rightarrow F_E = 2.02 \text{ kN}$$

$$F_A = 15 - 3.46 - 2.02 = 9.52 \text{ kN}$$



EXAMPLE 2-12

A reinforced concrete column 200 mm in diameter is designed to carry an axial compressive load of 300 kN. Determine the required area of the reinforcing steel if the allowable stresses are 6 MPa and 120 MPa for the concrete and steel, respectively. Use $E_{co} = 14 \text{ GPa}$ and $E_{st} = 200 \text{ GPa}$.

$$\delta_{con} = \delta_{st}$$

$$\left(\frac{PL}{AE}\right)_{con} = \left(\frac{PL}{AE}\right)_{st}$$

$$\left(\frac{\sigma L}{E}\right)_{con} = \left(\frac{\sigma L}{E}\right)_{st}$$

$$\left(\frac{\sigma L}{14000}\right)_{con} = \left(\frac{\sigma L}{200000}\right)_{st} \rightarrow 100\sigma_{con} = 7\sigma_{st}$$

$$100\sigma_{con} = 7 \times 120$$

$$\sigma_{con} = 8.4 \text{ MPa} > 6 \text{ MPa} \quad \text{Not o.k}$$

$$\sigma_{con} = 6 \text{ MPa} \rightarrow 100 \times 6 = 7\sigma_{st}$$

$$\sigma_{st} = 85.71 \text{ MPa} < 120 \text{ MPa}$$

Use $\sigma_{con} = 6 \text{ MPa}$

and $\sigma_{st} = 85.71 \text{ MPa}$

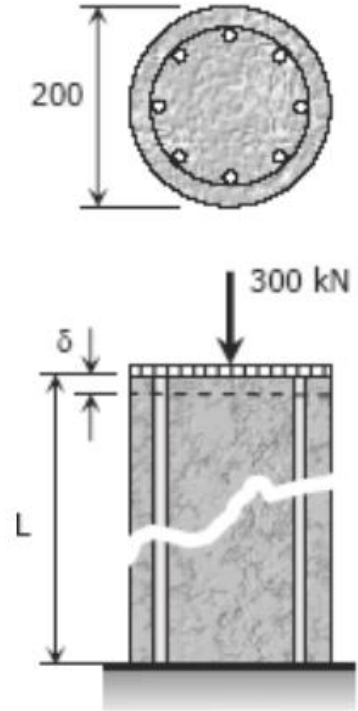
$$\sum F_y = 0 \rightarrow P_{con} + P_{st} = 300$$

$$\sigma_{st} A_{st} + \sigma_{con} A_{con} = 300$$

$$85.71 A_{st} + 6(\pi \times 100^2 - A_{st}) = 300 \times 1000$$

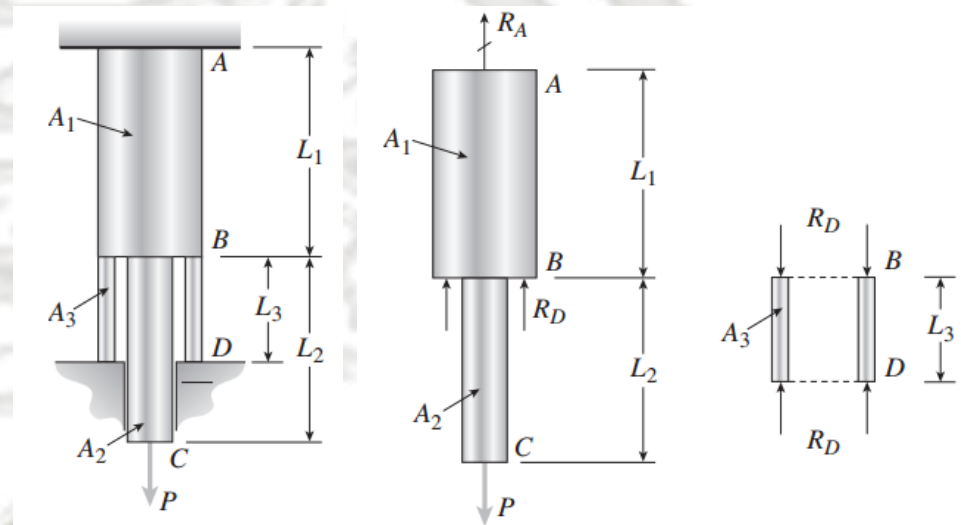
$$79.71 A_{st} + 600000\pi = 300000$$

$$A_{st} = 1398.9 \text{ mm}^2$$



H.W

A circular steel bar ABC ($E = 200 \text{ GPa}$) has cross sectional area A_1 from A to B and cross-sectional area A_2 from B to C. The bar is supported rigidly at end A and is subjected to a load P equal to 40 kN at end C. A circular steel collar BD having cross-sectional area A_3 supports the bar at B. The collar fits snugly at B and D when there is no load. Determine the elongation AC of the bar due to the load P . (Assume $L_1=2L_3=250 \text{ mm}$, $L_2=225 \text{ mm}$, $A_1=2A_3=960 \text{ mm}^2$, and $A_2=300 \text{ mm}^2$.)



Problems Involving Temperature Changes

A change in temperature can cause a body to change its dimensions. Generally, if the temperature increases, the body will expand, whereas if the temperature decreases, it will contract. If the material is homogeneous and isotropic, it has been found from experiment that the displacement of the end of a member having a length (L) can be calculated using the formula:

$$\delta_T = \alpha \Delta T L$$

Where:

α = Linear coefficient of thermal expansion $1/C^\circ$.

ΔT = Algebraic change in temperature of the member ($T_{\text{final}} - T_{\text{initial}}$).

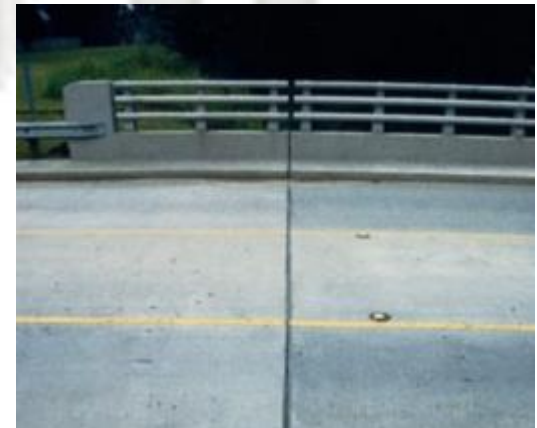
L = Original length of the member

δ_T = Algebraic change in the length of the member

$$\epsilon_{\text{thermal}} = \frac{\delta_T}{L} = \alpha \Delta T$$

$$\sigma_{\text{thermal}} = \alpha \Delta T E$$

The effect of temperature changing is only **important in statically indeterminate members**. This means, temperature change causes stresses and strains only in indeterminate elements, while determinate elements elongate and shrink freely without any stress and strain.



EXAMPLE 2-13

A plastic bar ACB having two different solid circular cross sections is held between rigid supports as shown in the figure. The diameters in the left and right parts are 50 mm and 75 mm, respectively. The corresponding lengths are 225 mm and 300 mm. Also, the modulus of elasticity E is 6.0 GPa, and the coefficient of thermal expansion is $100 \times 10^{-6}/^\circ\text{C}$. The bar is subjected to a uniform temperature increase of 30°C . Calculate the following quantities:

- (a) the compressive force P in the bar.
- (b) the maximum compressive stress σ_c .
- (c) the displacement of point C .

$$\delta_T = \alpha \Delta T L = 100 \times 10^{-6} \times (30 - 0) \times (225 + 300) = 1.575 \text{ mm}$$

$$\delta_P = \frac{PL_1}{A_1 E} + \frac{PL_2}{A_2 E} = P \left(\frac{225}{\pi \times 25^2 \times 6000} + \frac{300}{\pi \times 37.5^2 \times 6000} \right)$$

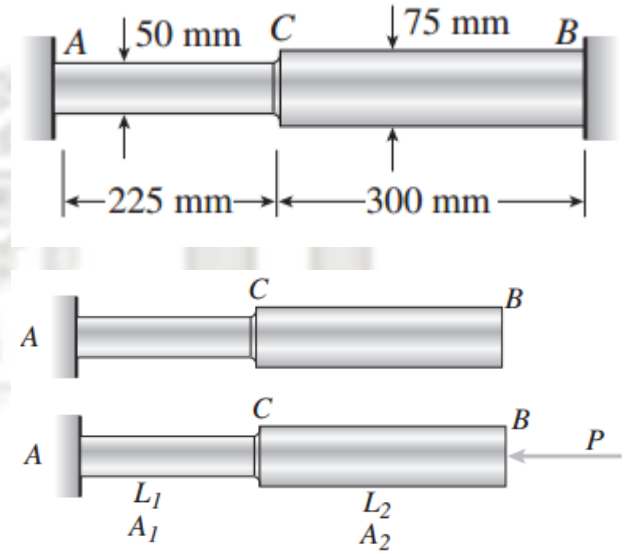
$$\delta_T = \delta_P$$

$$1.575 = P \left(\frac{225}{\pi \times 25^2 \times 6000} + \frac{300}{\pi \times 37.5^2 \times 6000} \right)$$

$$P = 51781 \text{ N}$$

$$\sigma_c = \frac{P}{A_1} = \frac{51781}{1963.5} = 26.4 \text{ MPa}$$

$$\delta_c = -\frac{PL_1}{A_1 E} + \alpha \Delta T L_1 = -0.989 + 0.675 = -0.314 \text{ mm}$$



EXAMPLE 2-14

The rigid beam shown in the Figure is fixed to the top of the three posts made of steel and aluminum. The posts each have a length of 250 mm when no load is applied to the beam, and the temperature is $T_1 = 20^\circ\text{C}$. Determine the force supported by each post if the bar is subjected to a uniform distributed load of 150 kN/m and the temperature is raised to $T_2 = 80^\circ\text{C}$. ($E_{st} = 200 \text{ GPa}$ and $E_{al} = 73.1 \text{ GPa}$.)

$$\sum F_y = 0$$

$$2F_{st} + F_{al} = 90000$$

$$\delta_{st} = -(\delta_{st})_F + (\delta_{st})_T$$

$$\delta_{al} = -(\delta_{al})_F + (\delta_{al})_T$$

$$-(\delta_{st})_F + (\delta_{st})_T = -(\delta_{al})_F + (\delta_{al})_T$$

$$-\left(\frac{FL}{AE}\right)_{st} + (\alpha\Delta TL)_{st} = -\left(\frac{FL}{AE}\right)_{al} + (\alpha\Delta TL)_{al}$$

$$-\frac{F_{st} \times 0.25}{\pi \times 0.02^2 \times 200 \times 10^9} + 12 \times 10^{-6} (80 - 20) \times 0.25 =$$

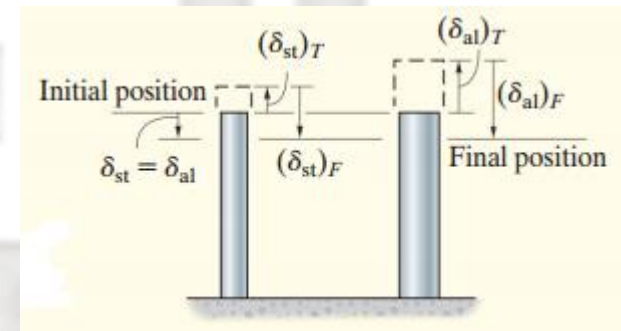
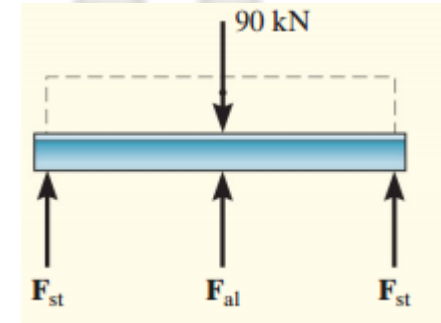
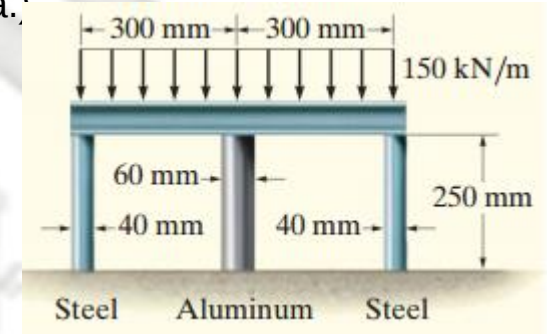
$$-\frac{F_{al} \times 0.25}{\pi \times 0.03^2 \times 73.1 \times 10^9} + 23 \times 10^{-6} (80 - 20) \times 0.25$$

$$F_{st} = 1.216F_{al} - 165900$$

$$F_{st} = -16.4 \text{ kN}$$

$$F_{al} = 123 \text{ kN}$$

The negative value for F_{st} indicates that this force acts opposite to that shown in the Figure. In other words, the steel posts are in tension and the aluminum post is in compression.



EXAMPLE 2-15

The rigid bar ABC in Figure is pinned at B and attached to the two vertical rods. Initially, the bar is horizontal and the vertical rods are stress-free. Determine the stress in the aluminum rod if the temperature of the steel rod is decreased by 40°C. Neglect the weight of bar ABC.

$$(\delta_{st})_T = \alpha \times \Delta T \times L = 11.7 \times 10^{-6} \times 40 \times 900 = 0.4212 \text{ mm}$$

$$\frac{\delta_A}{0.6} = \frac{\delta_{al}}{1.2} \rightarrow \delta_A = 0.5\delta_{al}$$

$$(\delta_{st})_T - \delta_{st} = 0.5\delta_{al}$$

$$0.4212 - \left(\frac{PL}{AE}\right)_{st} = 0.5\left(\frac{PL}{AE}\right)_{al}$$

$$0.4212 - \frac{P_{st} \times 900}{300 \times 200000} = 0.5 \times \frac{P_{al} \times 1200}{1200 \times 70000}$$

$$28080 - P_{st} = 0.4762 P_{al} \rightarrow (1)$$

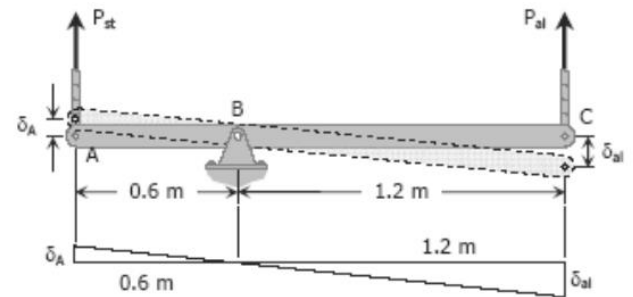
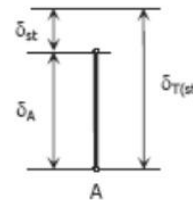
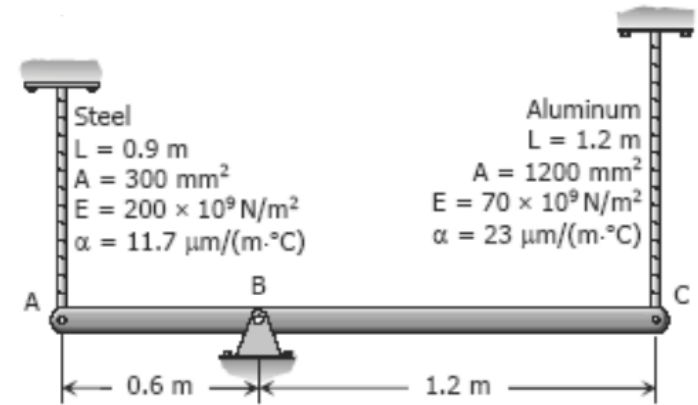
$$\sum M_B = 0$$

$$0.6 P_{st} = 1.2 P_{al} \rightarrow P_{st} = 2 P_{al} \rightarrow (2)$$

$$280800 - 2 P_{al} = 0.4762 P_{st}$$

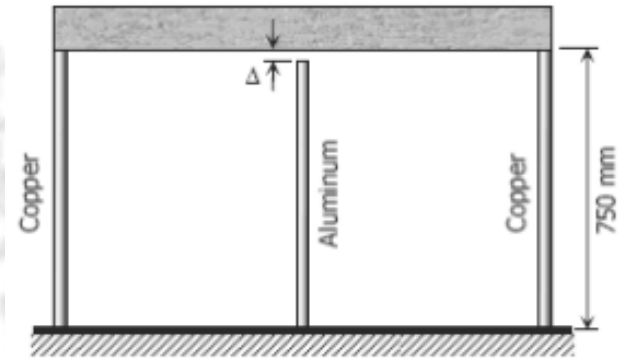
$$P_{al} = 11340 \text{ N}$$

$$\sigma_{al} = \frac{P_{al}}{A_{al}} = \frac{11340}{1200} = 9.45 \text{ MPa}$$



H.W

As shown in Figure, there is a gap between the aluminum bar and the rigid slab that is supported by two copper bars. At 10°C , $\delta = 0.18 \text{ mm}$. Neglecting the mass of the slab, calculate the stress in each rod when the temperature in the assembly is increased to 95°C . For each copper bar, $A = 500 \text{ mm}^2$, $E = 120 \text{ GPa}$, and $\alpha = 16.8 \mu\text{m}/(\text{m}\cdot^{\circ}\text{C})$. For the aluminum bar, $A = 400 \text{ mm}^2$, $E = 70 \text{ GPa}$, and $\alpha = 23.1 \mu\text{m}/(\text{m}\cdot^{\circ}\text{C})$.



EXAMPLE 2-16

A steel rod of diameter 15 mm is held (without any initial stresses) between rigid walls by the arrangement shown in the Figure. Calculate the temperature drop ΔT (degrees Celsius) at which the average shear stress in the 12-mm diameter bolt becomes 45 MPa. (For the steel rod, use $\alpha = 12 \times 10^{-6}/^{\circ}\text{C}$ and $E = 200 \text{ GPa}$.)

$$\delta = \alpha \times \Delta T \times L$$

$$\frac{PL}{AE} = \alpha \times \Delta T \times L$$

$$P = AE \times \alpha \times \Delta T$$

$$V = \frac{P}{2} \quad \text{Double Shear}$$

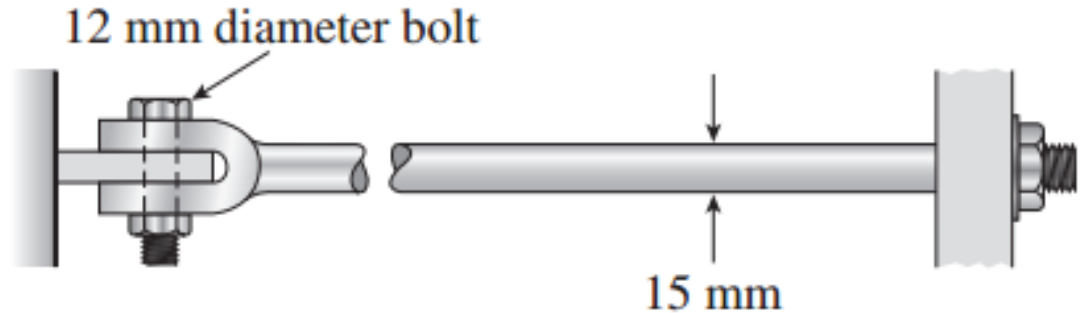
$$V = \frac{AE \times \alpha \times \Delta T}{2}$$

$$\tau = \frac{V}{A_{\text{bolt}}} = \frac{AE \times \alpha \times \Delta T}{2A_{\text{bolt}}}$$

$$\Delta T = \frac{2 \times \tau \times A_{\text{bolt}}}{AE \times \alpha}$$

$$\Delta T = \frac{2 \times 45 \times \frac{\pi}{4} 12^2}{200 \times \frac{\pi}{4} 15^2 \times 12 \times 10^{-6}}$$

$$\Delta T = 24^{\circ}\text{C}$$



Poisson's Ratio

When a deformable body is subjected to a tensile force (or a compressive force), not only does it elongate but it also contracts laterally. In the early 1800s, the French scientist S. Poisson realized that within the elastic range the ratio of the strain in the lateral or radial direction to the strain in the longitudinal or axial direction is a constant. This ratio is referred to as Poisson's ratio.

Poisson's ratio is a dimensionless quantity with maximum possible value is 0.5, so that $0 \leq \nu \leq 0.5$. For most solids, it has a value that is generally between 0.25 and 0.355.

$\nu = 0.1 - 0.16$ (concrete).

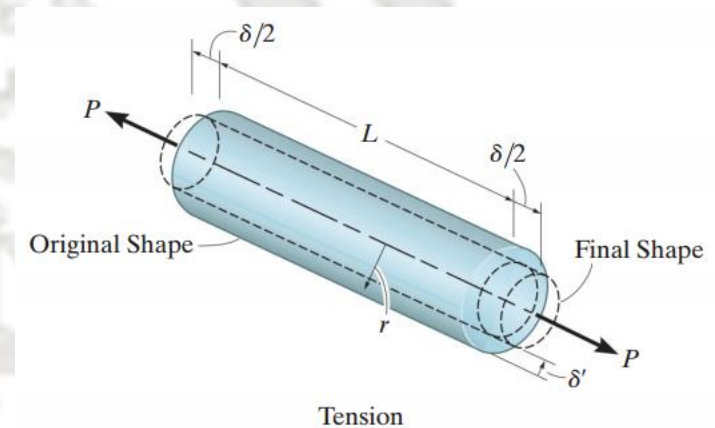
$\nu = 0.25$ (steel).

$\nu = 0.333$ (aluminum).

$\nu = 0.5$ (rubber).

Poisson's Ratio = Absolute (Lateral Strain / Axial Strain)

$$\nu = \left| -\frac{\epsilon_y}{\epsilon_x} \right| = \left| -\frac{\epsilon_z}{\epsilon_x} \right|$$



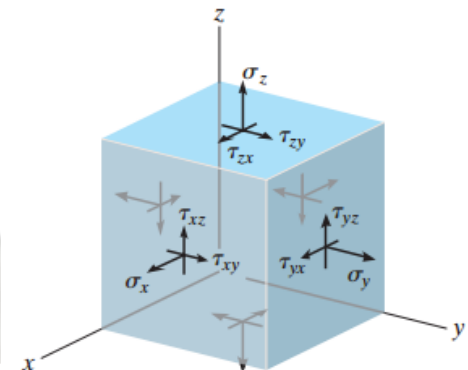
Generalized Hooke's Law

Consider an element of an isotropic material in the shape of a cube subjected to a triaxial tensile stress, as shown in the figure. By using the principle of superposition, the general Hooke's law can be written as:

$$\epsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} - \nu \frac{\sigma_z}{E}$$

$$\epsilon_y = \frac{\sigma_y}{E} - \nu \frac{\sigma_x}{E} - \nu \frac{\sigma_z}{E}$$

$$\epsilon_z = \frac{\sigma_z}{E} - \nu \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E}$$



EXAMPLE 2-17

A bar made of steel has the dimensions shown in the Figure. If an axial force of $P = 80 \text{ kN}$ is applied to the bar, determine the change in its length and the change in the dimensions of its cross section. The material behaves elastically. (Assume $E_{\text{steel}} = 200 \text{ GPa}$, $\nu_{\text{steel}} = 0.32$).

$$\sigma = \frac{P}{A} = \frac{80000}{100 \times 50} = 16 \text{ MPa}$$

$$\varepsilon_z = \frac{\sigma_z}{E} = \frac{16}{200000} = 8 \times 10^{-5}$$

$$\delta_z = \varepsilon_z \times L_z = 8 \times 10^{-5} \times 1500 = 0.12 \text{ mm}$$

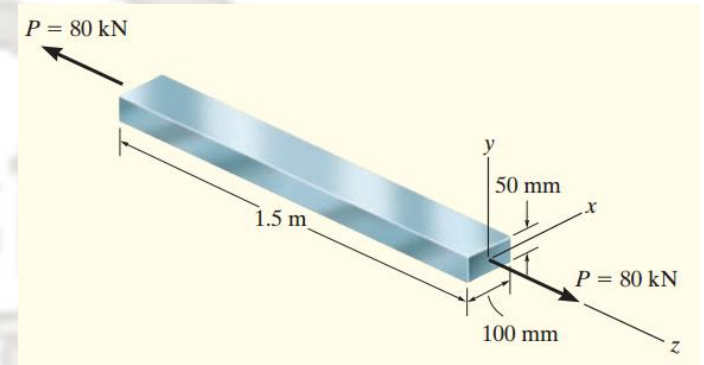
The lateral contraction strains in both the x and y directions are equal

$$\nu_{st} = -\frac{\varepsilon_x}{\varepsilon_z} = -\frac{\varepsilon_y}{\varepsilon_z}$$

$$\varepsilon_x = \varepsilon_y = -\nu_{st} \times \varepsilon_z = -0.32 \times 8 \times 10^{-5} = -2.56 \times 10^{-5}$$

$$\delta_x = \varepsilon_x \times L_x = -2.56 \times 10^{-5} \times 100 = -2.56 \times 10^{-3} \text{ mm}$$

$$\delta_y = \varepsilon_y \times L_y = -2.56 \times 10^{-5} \times 50 = -1.28 \times 10^{-3} \text{ mm}$$



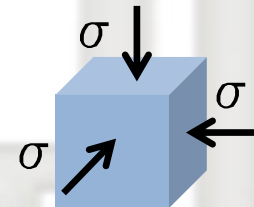
EXAMPLE 2-18

What material should be used in order to produce a cube has no change in its volume when it is subjected to a uniform pressure?

$$\sigma_x = \sigma_y = \sigma_z = \sigma$$

$$\varepsilon_x = 0 = -\frac{\sigma}{E} + \nu \frac{\sigma}{E} + \nu \frac{\sigma}{E} = -\frac{\sigma}{E} (1 - 2\nu)$$

$$\therefore -\frac{\sigma}{E} \neq 0 \quad \therefore (1 - 2\nu) = 0 \rightarrow \nu = \frac{1}{2}$$



EXAMPLE 2-19

A piece of a steel plate of (250 x 50 x 10)mm dimensions is subjected to a biaxial force system in x and y directions, as shown in the figure. Knowing that $E=200000$ MPa and $\nu=0.25$:

a) What is the change in the thickness.

b) To cause the same change in thickness as in (a) by P_x alone, what must be its magnitude?

$$\text{a) } \sigma_x = \frac{P_x}{A_x} = \frac{100000}{50 \times 10} = 200 \text{ MPa}$$

$$\sigma_y = \frac{P_y}{A_y} = \frac{200000}{250 \times 10} = 80 \text{ MPa}$$

$$\sigma_z = 0$$

$$\varepsilon_z = \frac{\sigma_z}{E} - \nu \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} = \frac{-\nu}{E} (\sigma_x + \sigma_y)$$

$$\varepsilon_z = \frac{-0.25}{200000} (200 + 80) = -0.00035$$

$$\delta_z = \varepsilon_z \times t = -0.00035 \times 10 = -0.0035$$

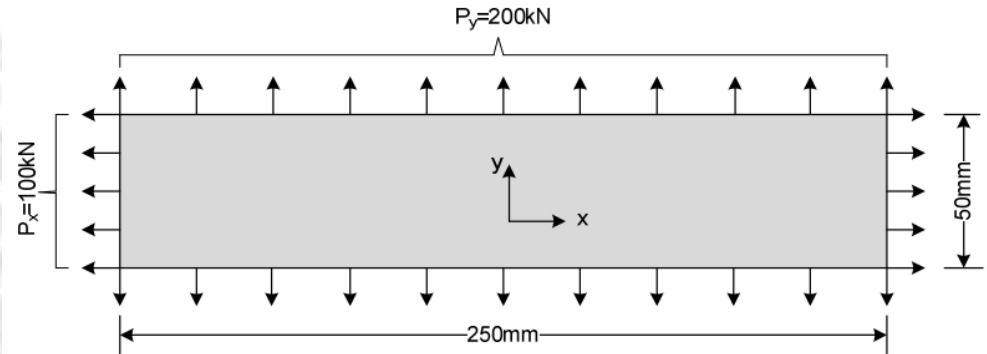
$$\text{b) } \sigma_z = \sigma_y = 0$$

$$\varepsilon_z = \frac{\sigma_z}{E} - \nu \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} = 0 - \nu \frac{\sigma_x}{E} - 0$$

$$-0.00035 = -0.25 \times \frac{\sigma_x}{200000}$$

$$\sigma_x = 280 \text{ MPa}$$

$$P_x = \sigma_x \times A_x = 280 \times 50 \times 10 = 140 \text{ kN}$$



EXAMPLE 2-20

Knowing that Poisson's ratio $\nu=0.25$, determine the magnitude of a single force acting only in the y-direction that would cause the same deformation in the y-direction as the initial forces.

$$\sigma_x = \frac{P_x}{A_x} = \frac{180000}{100 \times 75} = 24 \text{ MPa}$$

$$\sigma_y = \frac{P_y}{A_y} = \frac{200000}{50 \times 100} = 40 \text{ MPa}$$

$$\sigma_z = \frac{P_z}{A_z} = \frac{-240000}{50 \times 75} = -64 \text{ MPa}$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \nu \frac{\sigma_x}{E} - \nu \frac{\sigma_z}{E} = \frac{1}{E} \{40 - 0.25 \times 24 - 0.25 \times (-64)\}$$

$$\varepsilon_y = +\frac{50}{E}$$

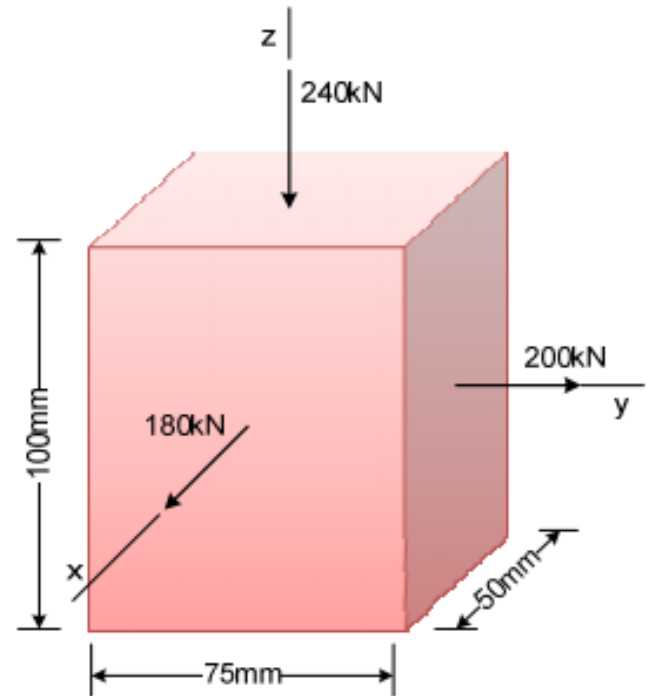
$$\sigma_x = \sigma_z = 0, \quad \sigma_y = ?$$

$$\varepsilon_y = +\frac{\sigma_y}{E}$$

$$\frac{50}{E} = \frac{\sigma_y}{E} \rightarrow \sigma_y = 50 \text{ MPa}$$

$$P = \sigma A$$

$$P = 50 \times 10^{-3} \times 50 \times 100 = 250 \text{ kN}$$



Shear Strain

Shearing forces cause shearing deformation. An element subject to shear does not change in length but undergoes a change in shape. The change in angle at the corner of an original rectangular element is called the **shear strain**. This angle is denoted by γ (gamma) and is always measured in radians (rad). For instance, the angles at the points q and s, which were $\pi/2$ before deformation, are reduced to $\pi/2 - \gamma$. At the same time, the angles at points p and r are increased to $\pi/2 + \gamma$.

The shear strain is expressed as:

$$\gamma = \frac{\delta_{sh}}{L}$$

The ratio of the shear stress τ and the shear strain γ is called the **modulus of rigidity** or **shear modulus of elasticity** and is denoted as G , in MPa. the following equation for Hooke's law in shear.

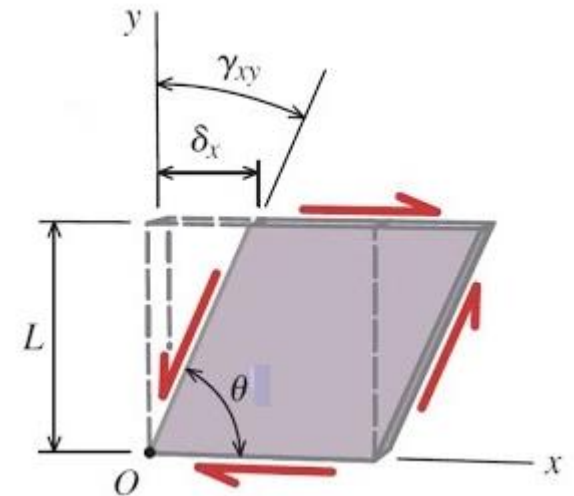
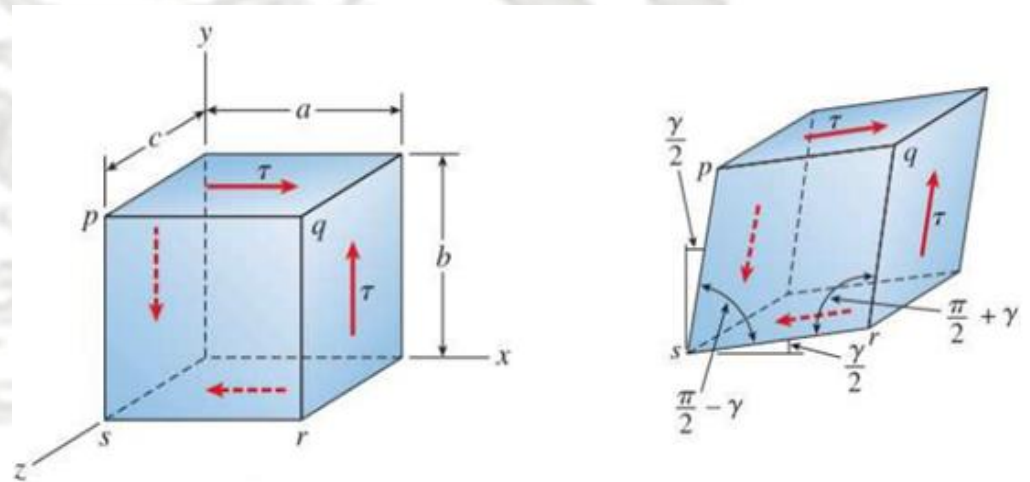
$$\tau = \gamma \times G$$

The relationship between the shearing deformation and the applied shearing force is:

$$\delta_{sh} = \frac{VL}{A_{sh} \times G} = \frac{\tau \times L}{G} \quad \text{where } V \text{ is the shearing force acting over an area } A_{sh}.$$

The **modulus of elasticity** in tension and **modulus of rigidity** in shear are related by the following equation:

$$G = \frac{E}{2(1+\nu)}$$



EXAMPLE 2-21

The right side CD of the plate shown in the Figure deforms by a uniform horizontal displacement of 2 mm. Determine:

- (a) the average normal strain along the diagonal AC.
- (b) the shear strain at E relative to the x, y axes.

(a) $AC = \sqrt{0.15^2 + 0.15^2} = 0.21213m$

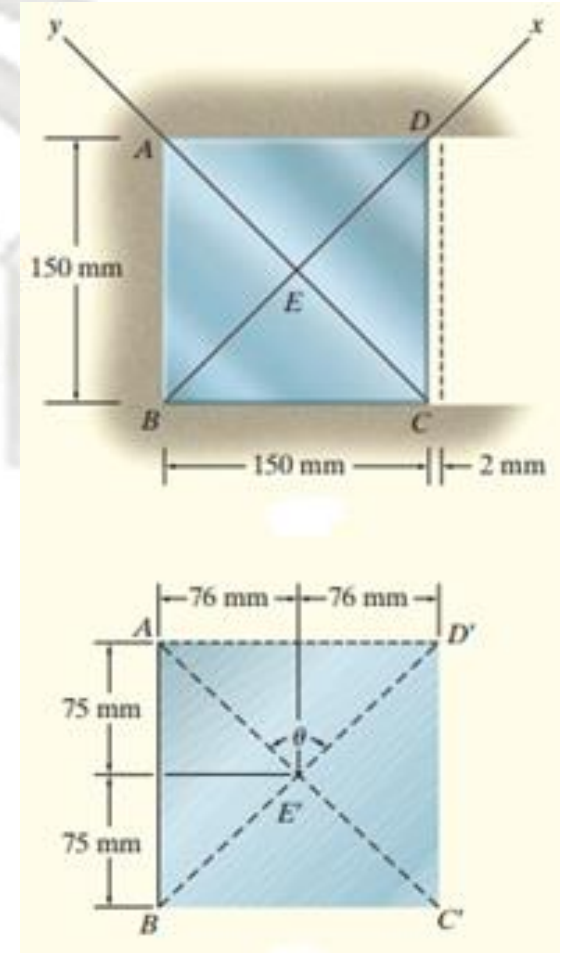
$$AC' = \sqrt{0.15^2 + 0.152^2} = 0.21355m$$

$$\varepsilon_{AC} = \frac{AC' - AC}{AC} = \frac{0.21355 - 0.21213}{0.21213} = 0.00669$$

(b) $\tan\left(\frac{\theta}{2}\right) = \frac{76}{75}$

$$\theta = 90.759^\circ = 90.759^\circ \times \frac{\pi}{180^\circ} = 1.58404rad$$

$$\gamma_{xy} = \frac{\pi}{2} - 1.58404 = -0.0132rad$$



EXAMPLE 2-22

A joint between two concrete slabs A and B is filled with a flexible epoxy that bonds to the concrete as shown in the Figure. The height of the joint is $h=100$ mm, its length is $L=1000$ mm, and its thickness is $t=12.5$ mm. Under the action of shear forces V , the slabs displace vertically through the distance $d=0.05$ mm relative to each other.

(a) What is the average shear strain in the epoxy?

(b) What is the magnitude of the forces V if the shear modulus of elasticity G for the epoxy is 965 MPa?

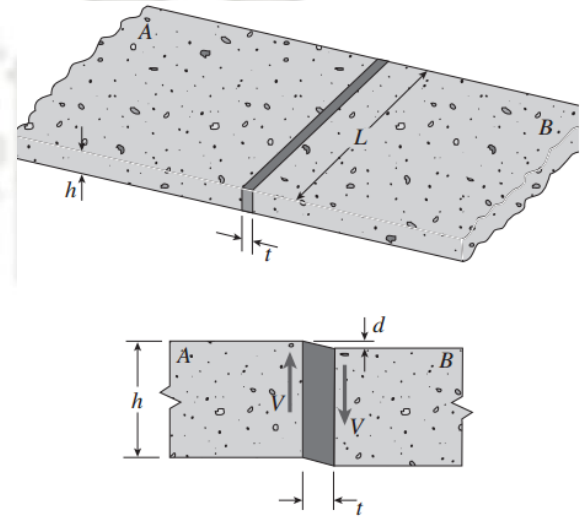
$$\gamma = \frac{d}{t} = \frac{0.05}{12.5} = 0.004$$

$$\tau = \gamma \times G$$

$$V = \tau \times h \times L$$

$$\therefore V = \gamma \times G \times h \times L$$

$$V = 0.004 \times 965 \times 100 \times 1000 \times 10^{-3} = 386 \text{ kN}$$



CHAPER THREE – AXIAL FORCE, SHEAR AND BENDING MOMENT

Introduction:

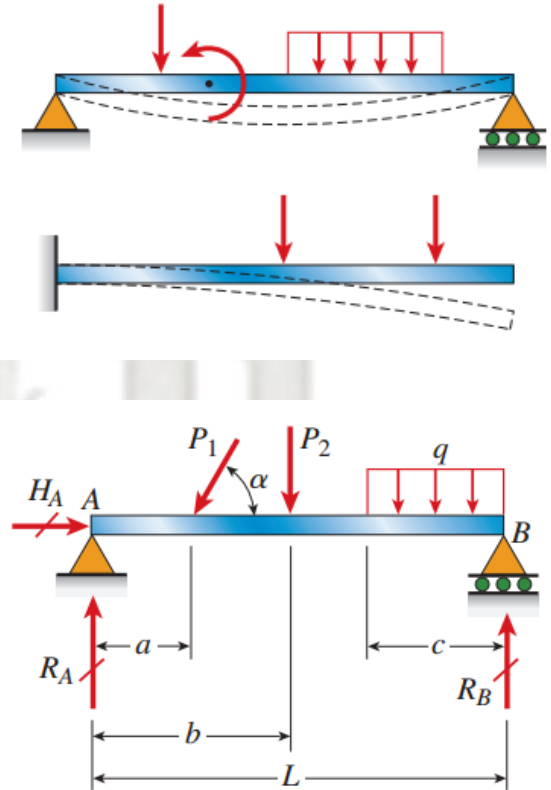
Structural members are usually classified according to the types of loads that they support. For instance, an axially loaded bar supports forces along the axis of the bar, and a bar in torsion supports torques along the axis. Members that are slender and support loadings that are applied perpendicular to their longitudinal axis are called **beams**.

Types of Beams, Load, and Reactions:

Beams are usually described by the manner in which they are supported.

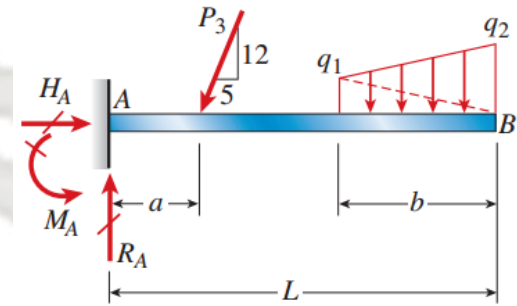
1- Simply supported beam:

A beam with a pin support at one end and a roller support at the other called a **simply supported beam** or a **simple beam**. The essential feature of a **pin support** is that it prevents translation at the end of a beam but does not prevent rotation. Thus, end A of the beam cannot move horizontally or vertically but the axis of the beam can rotate in the plane of the figure. Consequently, a pin support is capable of developing a force reaction with both horizontal and vertical components (H_A and R_A), but it cannot develop a moment reaction. At end B of the beam, the **roller support** prevents translation in the vertical direction but not in the horizontal direction; hence this support can resist a vertical force (R_B) but not a horizontal force. The axis of the beam is free to rotate at B just as it is at A.



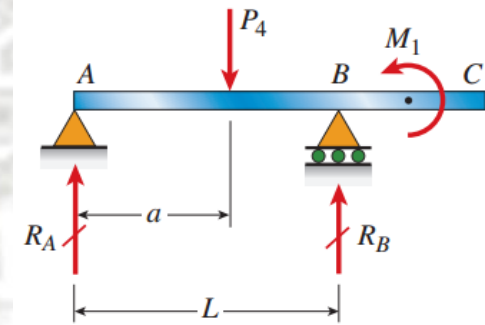
2- Cantilever beam :

The beam is fixed at one end and free at the other, is called a **cantilever beam**. At the **fixed support** (or clamped support) the beam can neither translate nor rotate, whereas at the free end it may do both. Consequently, both force and moment reactions may exist at the fixed support.



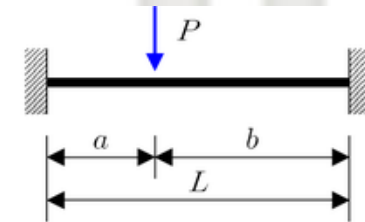
3- Over hanged beam:

This beam is simply supported at points A and B but it also projects beyond the support at B. The overhanging segment BC is similar to a cantilever beam except that the beam axis may rotate at point B.



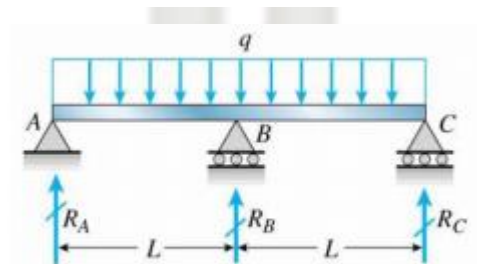
4- Fixed ended beam (clamped beam)

The beam is a statically indeterminate beam



5- Continuous beam

The beam is a statically indeterminate beam



Types of Loads:

1- Concentrated load:

When a load is applied over a very small area it may be idealized as a **concentrated load**, which is a single force. Also a concentrated moment M_1 acting on the overhanging beam as a single moment.

2- Distributed load:

When a load is spread along the axis of a beam, it is represented as a **distributed load**, such as the load q .

3- Linearly Varying Load:

A varying load has an intensity that changes with distance along the axis.

4- Couple:

The couple of moment M_1 acting on the overhanging beam.

Internal Forces: There are three types of internal forces in the plane as follows:

- **Axial force (P)**, which algebraically equals the summation of all the axial forces exist on one side of the section.
- **Shear force (V)**, which equals the algebraic summation of all the forces that exist perpendicularly on one side of the section.
- **Bending moment (M)**, which equals the algebraic summation of the moments caused by all the perpendicular forces affect one side of the section.

Sign convention:

Axial force (P)

Tensile (+)

Compressive (-)

Shear force (V)

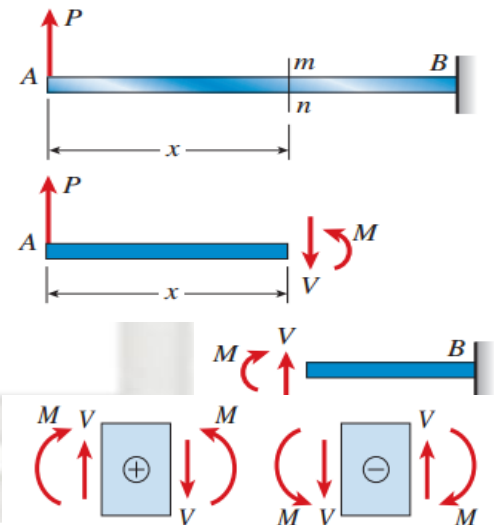
At the right-hand side of the segment, downward (+)

At the right-hand side of the segment, upward (-)

Bending moment (M)

If the moment causes concave, that moment should be considered (+)

If the moment causes convex, that moment should be considered (-)



EXAMPLE 3-1

Determine the shear force V and bending moment M at the midpoint C of the simple beam AB shown in the figure.

$$\sum M_A = 0$$

$$4R_B = 2 \times 2 \times 3 + 6 \times 1$$

$$R_B = 4.5 \text{ kN}$$

$$\sum M_B = 0$$

$$4R_A = 2 \times 2 \times 1 + 6 \times 3$$

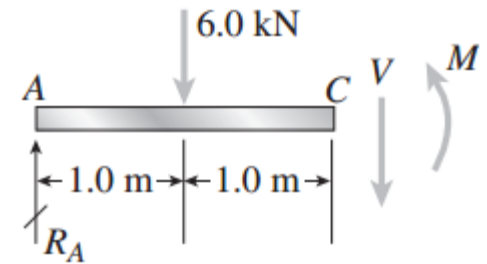
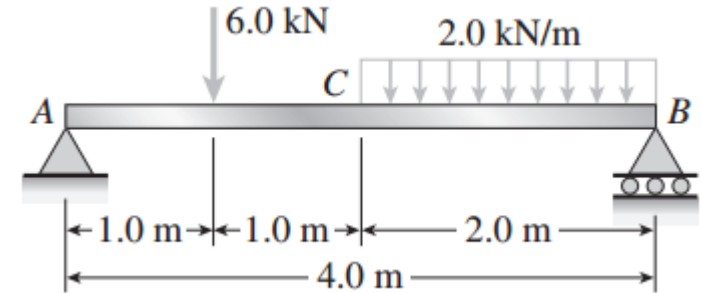
$$R_A = 5.5 \text{ kN}$$

$$+\uparrow \sum F_y = 0$$

$$V + 6 = 5.5 \rightarrow V = -0.5 \text{ kN}$$

$$\sum M_C = 0$$

$$6 \times 1 + M = 5.5 \times 2 \rightarrow M = 5 \text{ kN.m}$$



EXAMPLE 3-2

Calculate the shear force V and bending moment M at a cross section located 0.5 m from the fixed support of the cantilever beam AB shown in the figure.

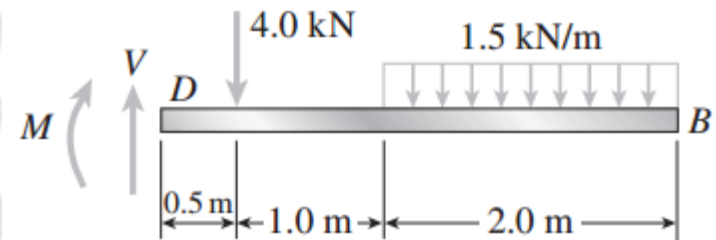
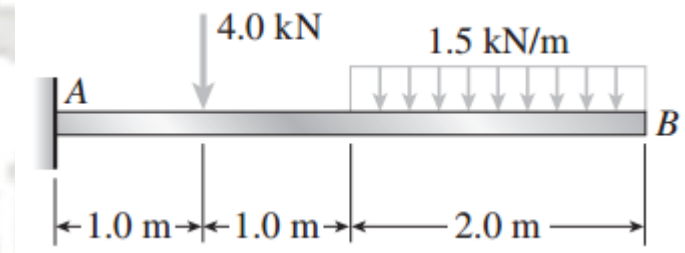
$$+\uparrow \sum F_y = 0$$

$$V = 4 + 1.5 \times 2 = 7 \text{ kN}$$

$$\sum M_D = 0$$

$$M + 4 \times 0.5 + 1.5 \times 2 \times 2.5 = 0$$

$$M = -9.5 \text{ kN.m}$$



EXAMPLE 3-3

Determine the shear force V and bending moment M at the midpoint of the beam with overhangs. Note that one load acts downward and the other upward.

$$\sum M_B = 0$$

$$R_A \times L = P(L + b) + Pb = PL + 2Pb$$

$$R_A = P\left(1 + \frac{2b}{L}\right) \uparrow$$

$$\sum M_A = 0$$

$$R_B = P\left(1 + \frac{2b}{L}\right) \downarrow$$

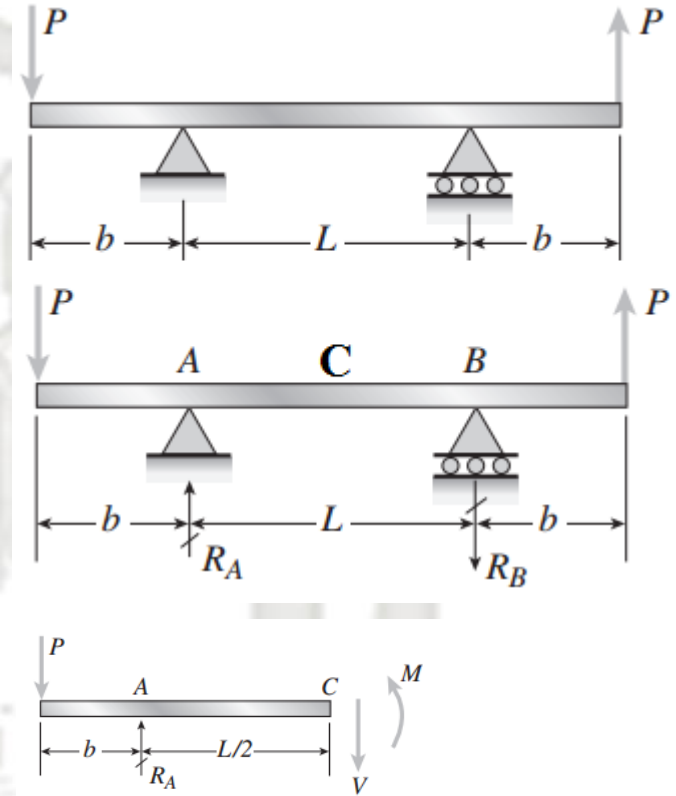
$$\sum F_y = 0$$

$$V = R_A - P = P\left(1 + \frac{2b}{L}\right) - P = \frac{2bP}{L}$$

$$\sum M_C = 0 \quad \text{C at mid span}$$

$$M = P\left(1 + \frac{2b}{L}\right)\left(\frac{L}{2}\right) - P\left(b + \frac{L}{2}\right)$$

$$M = \frac{PL}{2} + Pb - Pb - \frac{PL}{2} = 0$$



EXAMPLE 3-4

Calculate the reactions on the supports A and B

$$\sum F_x = 0$$

$$A_x = 10 \times \frac{3}{5} = 6kN$$

$$\sum M_A = 0$$

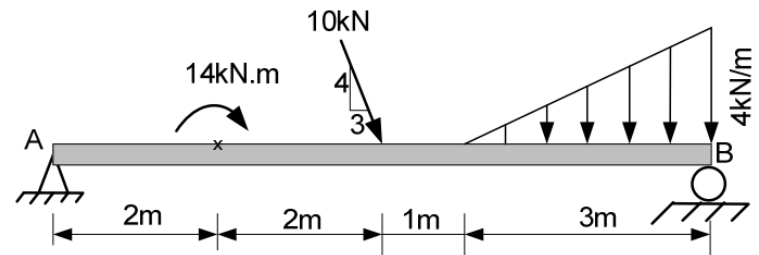
$$R_B \times 8 = \frac{1}{2} \times 3 \times 4 \times 7 + 10 \times \frac{4}{5} \times 4 + 14$$

$$R_B = 11kN$$

$$\sum F_y = 0$$

$$A_y + 11 = 8 + 6$$

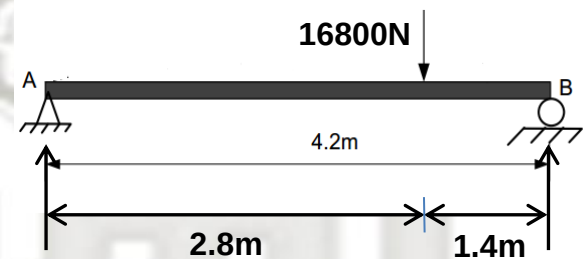
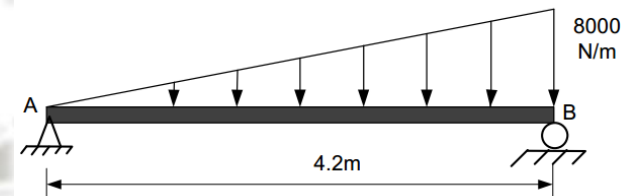
$$A_y = 3kN$$

**EXAMPLE 3-5**

Calculate the reactions on the supports A and B

$$R_A = 5600N$$

$$R_B = 11200N$$



EXAMPLE 3-6

Calculate the reactions on the supports A and B

The internal hinge always adds additional equilibrium equation to the three original equilibrium equations that is:

$$\sum M_{hinge} = 0$$

Part BC:

$$\sum M_c = 0$$

$$4 \times 1 = R_B \times 2 \Rightarrow R_B = 2 \text{ kN}$$

$$\sum F_y = 0$$

$$C_y + 2 = 4$$

$$C_y = 2 \text{ kN}$$

Part AC:

$$\sum F_y = 0$$

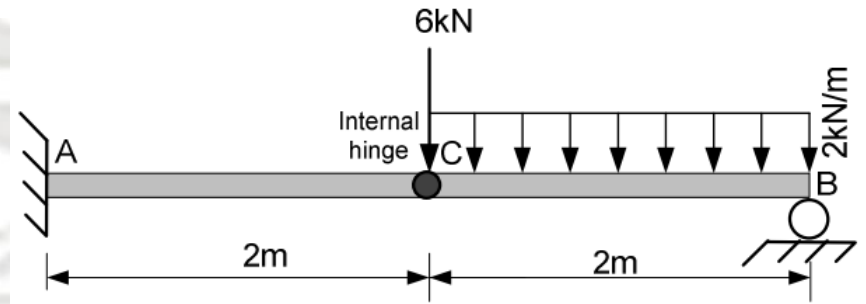
$$A_y = 6 + 2 = 8 \text{ kN}$$

$$\sum M_A = 0$$

$$M_A = 6 \times 2 + 2 \times 2 = 16 \text{ kN.m}$$

$$\sum F_x = 0$$

$$A_x = 0$$



Shear and Moment Diagrams by Equations:

EXAMPLE 3-7

Draw the shear and moment diagrams for the beam shown in the figure.

$$R_A = (3 \times 4) / 2 = 6 \text{ kN}$$

$$+\uparrow \sum F_y = 0$$

$$6 - 3x - V = 0 \rightarrow$$

$$V = 6 - 3x \rightarrow (1)$$

$$+\circlearrowleft \sum M = 0$$

$$-6x + 3x \times \frac{1}{2}x + M = 0$$

$$M = (6x - 1.5x^2) \rightarrow (2)$$

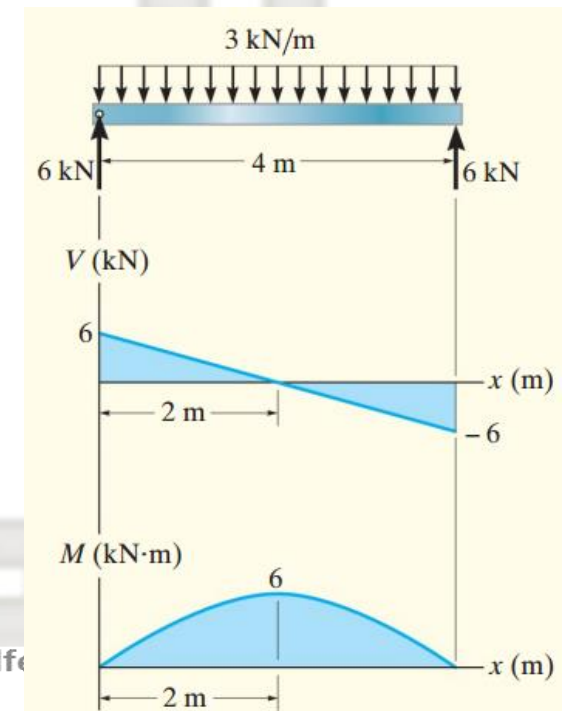
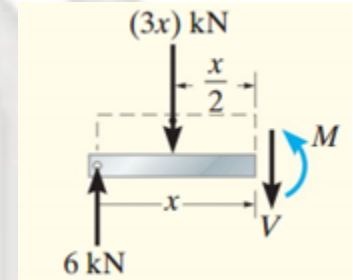
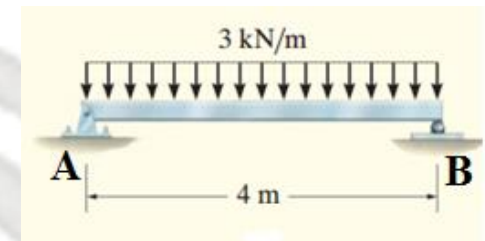
Shear and moments diagrams: The shear and moment diagrams are obtained by plotting Equations 1 and 2. The point of zero shear can be found from Equation 1.

$$V = 6 - 3x = 0$$

$$x = 2$$

From the moment diagram, this value of x represents the point on the beam where the maximum moment occurs. The slope $V = dM/dx = 0$.

$$M_{\max} = 6 \times 2 - 1.5 \times 2^2 = 6 \text{ kN}\cdot\text{m}$$



EXAMPLE 3-8

Draw the shear and moment diagrams for the beam shown in the figure.

$$+\uparrow \sum F_y = 0$$

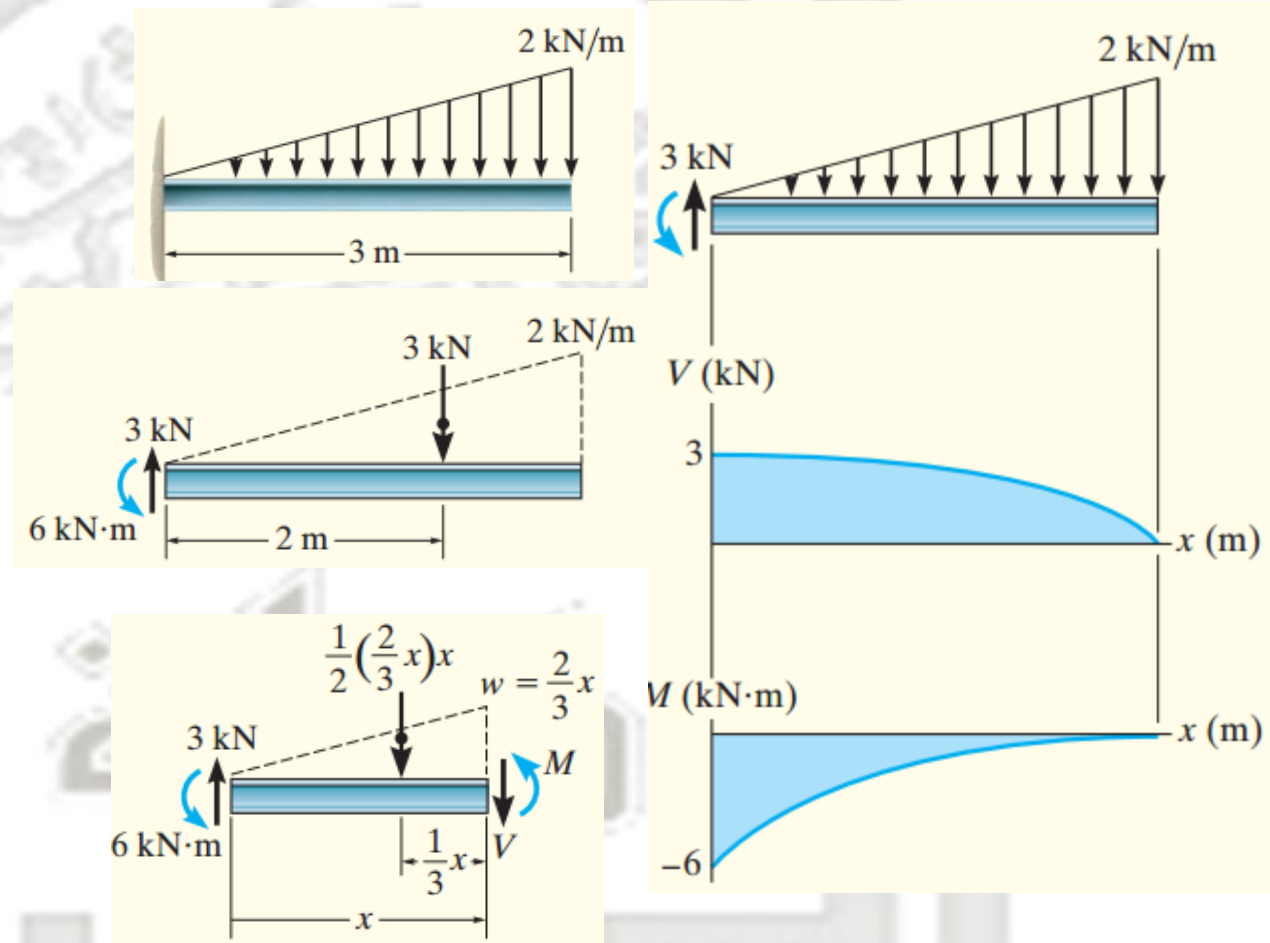
$$3 - \frac{1}{2} \left(\frac{2}{3} x \right) x - V = 0$$

$$V = 3 - \frac{1}{3} x^2$$

$$\sum M = 0$$

$$6 - 3x + \frac{1}{2} \left(\frac{2}{3} x \right) x \left(\frac{1}{3} x \right) + M = 0$$

$$M = -6 + 3x - \frac{1}{9} x^3$$



EXAMPLE 3-9

Draw the shear and moment diagrams for the beam shown in the figure.

$$0 \leq x_1 < 5:$$

$$+\uparrow \sum F_y = 0$$

$$5.75 - V = 0 \Rightarrow V = 5.75 \text{ kN}$$

$$\sum M = 0$$

$$-80 - 5.75x_1 + M = 0$$

$$M = 5.75x_1 + 80$$

$$5 < x_2 \leq 10:$$

$$+\uparrow \sum F_y = 0$$

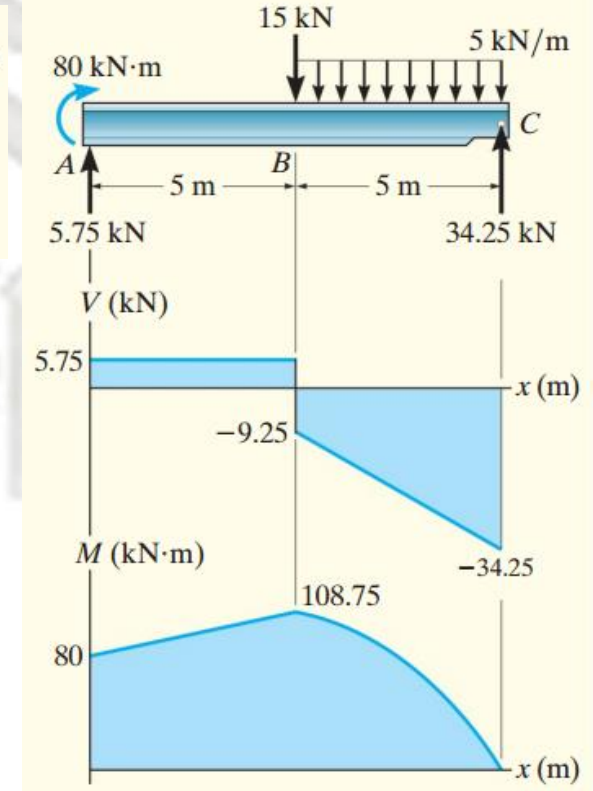
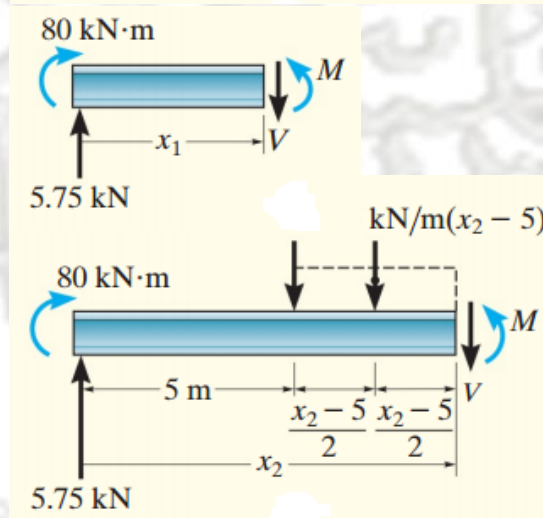
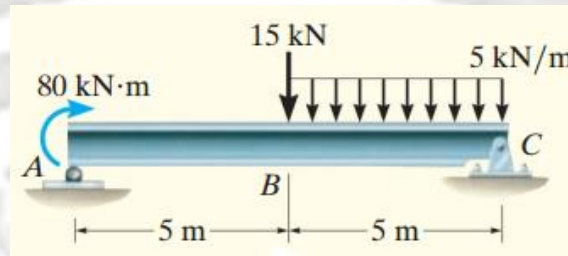
$$5.75 - 15 - 5(x_2 - 5) - V = 0$$

$$V = 15.75 - 5x_2$$

$$\sum M = 0$$

$$-80 - 5.75x_2 + 15(x_2 - 5) + 5(x_2 - 5)\left(\frac{x_2 - 5}{2}\right) + M = 0$$

$$M = -2.5x_2^2 + 15.75x_2 + 92.5$$



Shear and Moment Diagrams by Graphical Method:

Method based on two differential relations, one that exists between the distributed load and shear, and the other between the shear and moment is a simpler method for constructing the shear and moment diagrams. For region of distributed load:

$$+\uparrow \sum F_y = 0$$

$$V + w\Delta x - (V + \Delta V) = 0$$

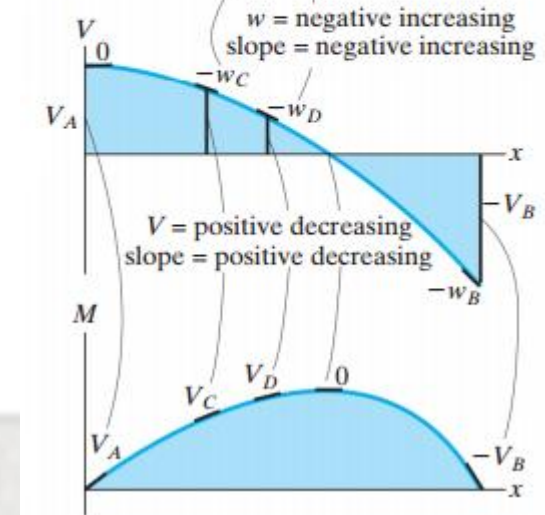
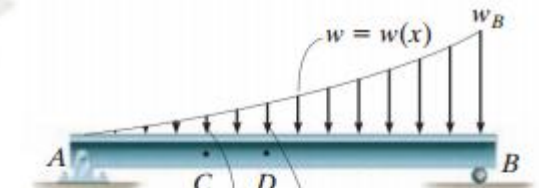
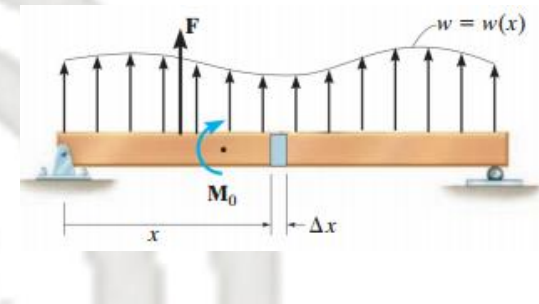
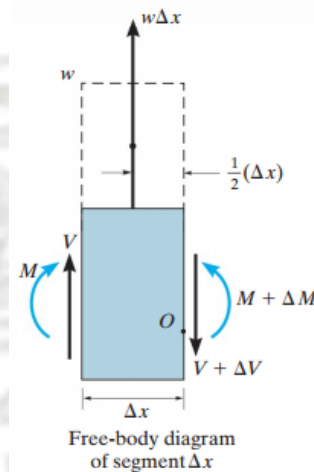
$$\Delta V = w\Delta x$$

$$\sum M = 0$$

$$-V\Delta x - M - w\Delta x\left(\frac{1}{2}\Delta x\right) + (M + \Delta M) = 0$$

$$\Delta M = V\Delta x + w\frac{1}{2}(\Delta x)^2$$

$$\Delta M = V\Delta x + w\frac{1}{2}(\Delta x)^2$$



Dividing by Δx and taking the limit as $\Delta x \rightarrow 0$, the above two equations become:

$$\frac{dV}{dx} = w \quad \text{and} \quad \frac{dM}{dx} = V \quad \text{or} \quad \frac{d^2 M}{dx^2} = \frac{dV}{dx} = w$$

slope of
shear diagram
at each point

distributed
load intensity
at each point

slope of
moment diagram
at each point

shear
at each
point

$$\text{Or: } \Delta V = \int w dx + C_1$$

$$\Delta M = \int V dx + C_2$$

change in
shear

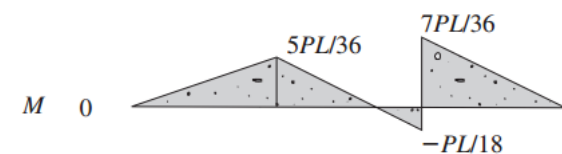
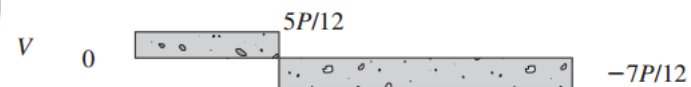
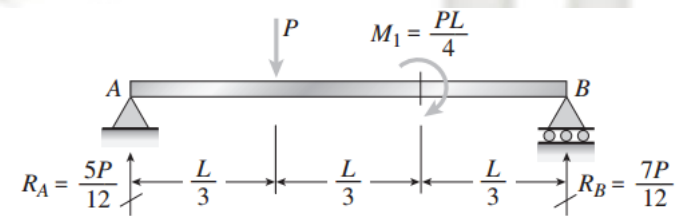
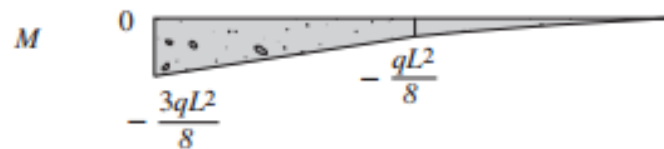
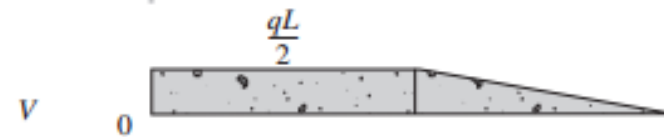
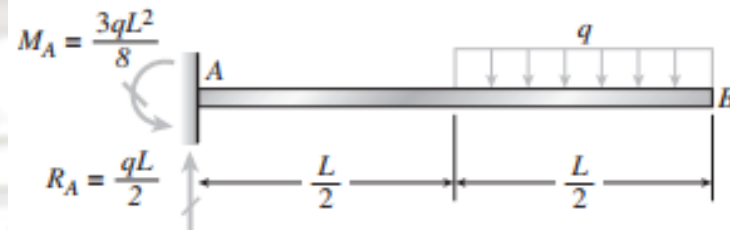
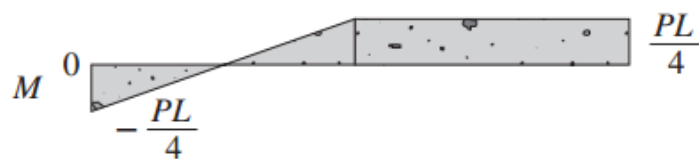
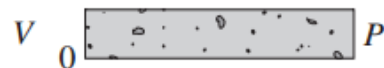
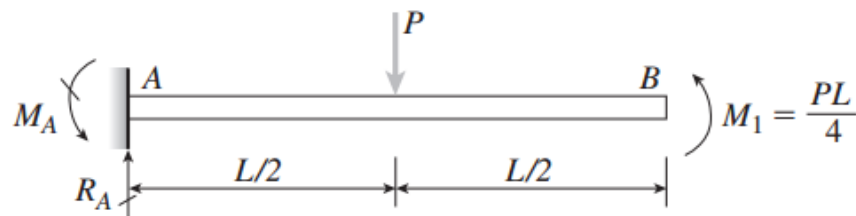
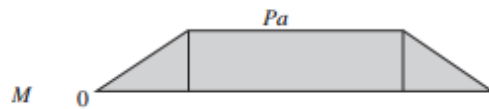
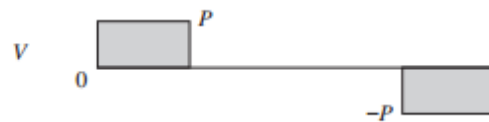
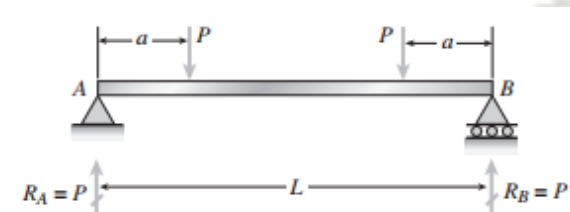
area under
distributed loading

change in
moment

area under
shear diagram

EXAMPLE 3-10

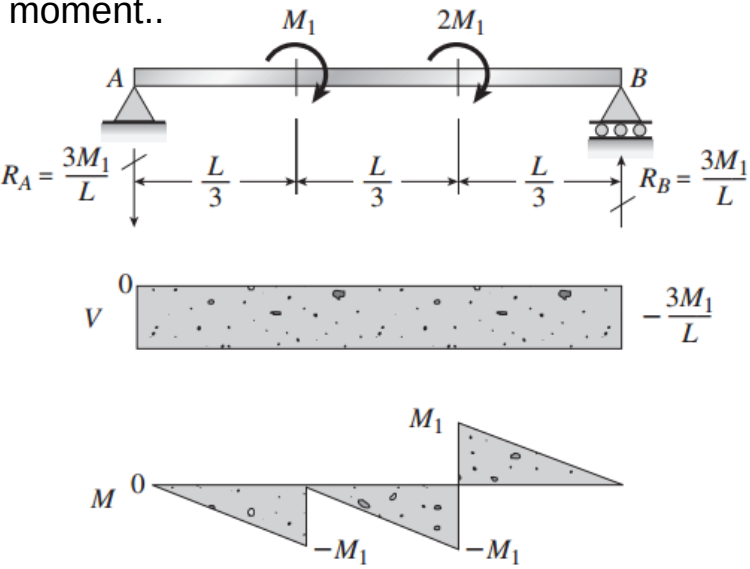
Draw the shear and moment diagrams for the beams shown in the figures.



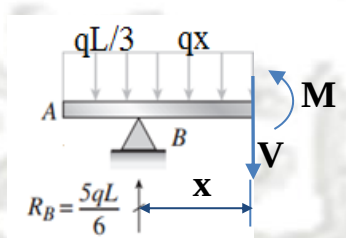
2nd

EXAMPLE 3-11

Draw the shear and moment diagrams for the beams shown in the figures and find the position of max. bending moment..



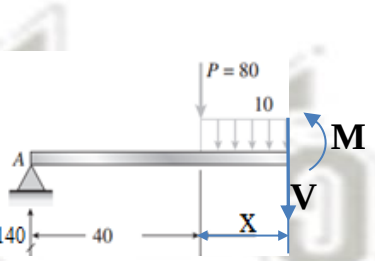
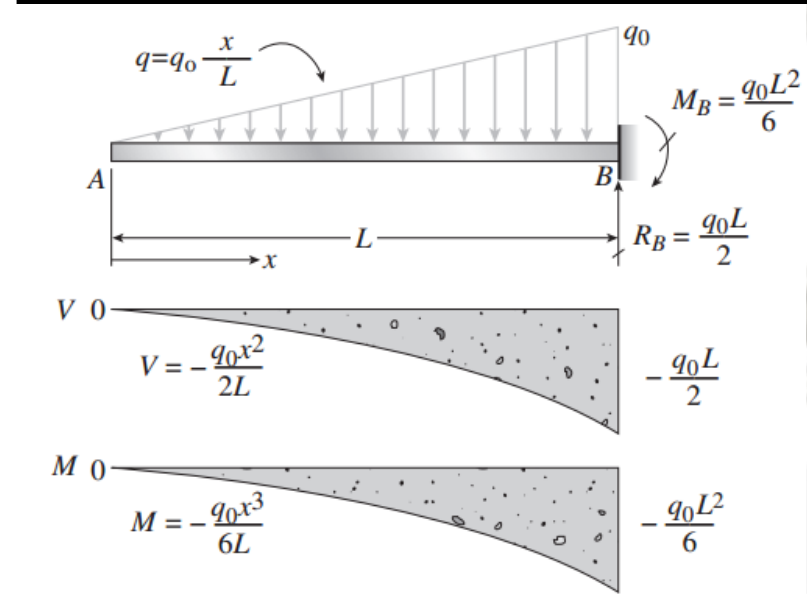
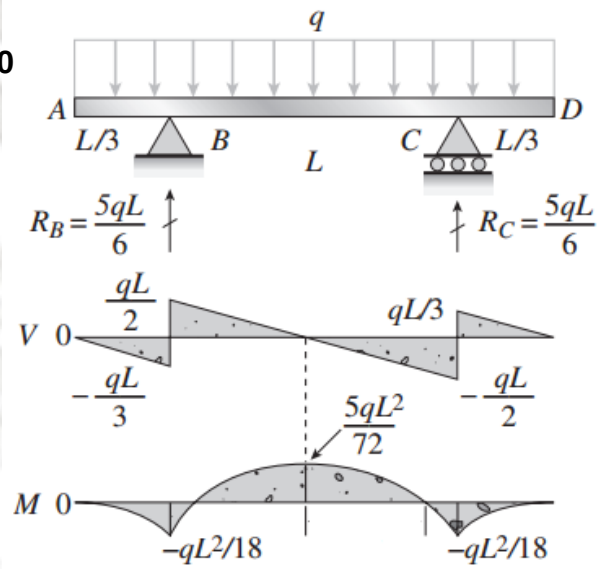
Maximum bending moment occurs when $V=0$ or in the interior support



$$\sum Fy = 0$$

$$\frac{qL}{3} + qx + V = \frac{5qL}{6}$$

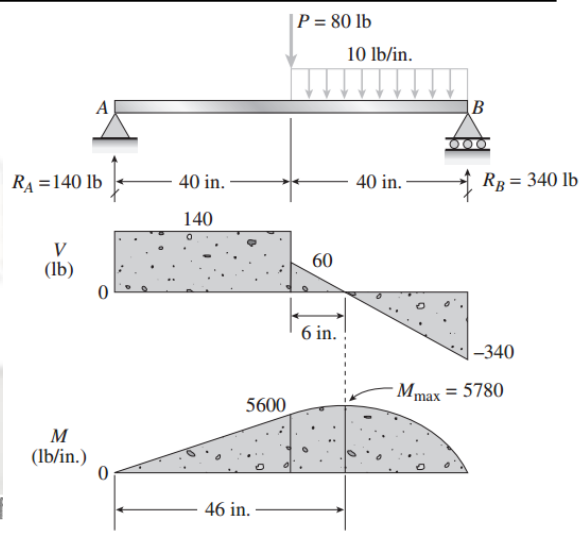
$$V = 0 \Rightarrow \frac{qL}{2} = qx \Rightarrow x = \frac{L}{2}$$



$$\sum Fy = 0$$

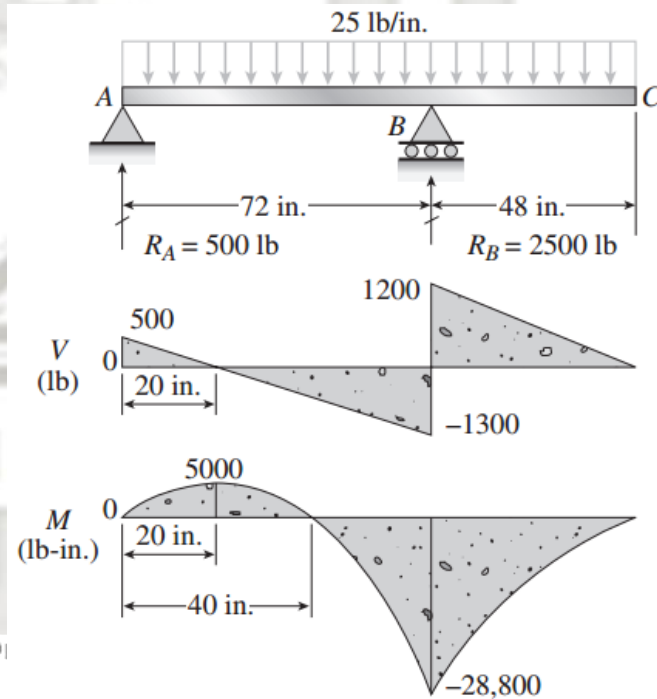
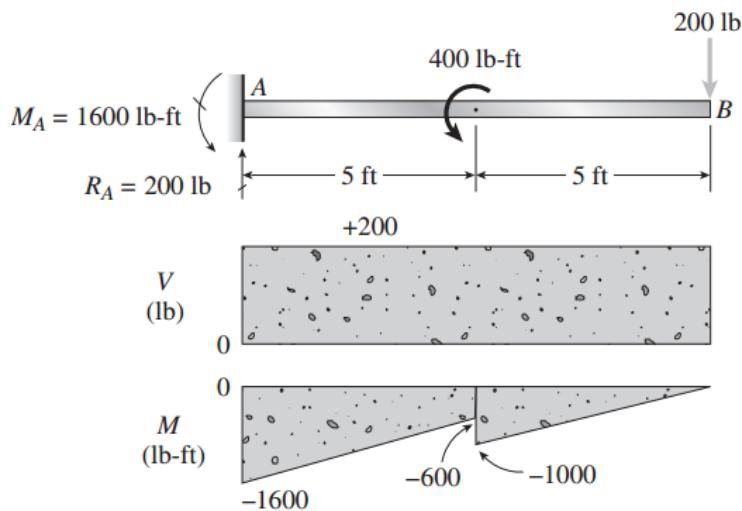
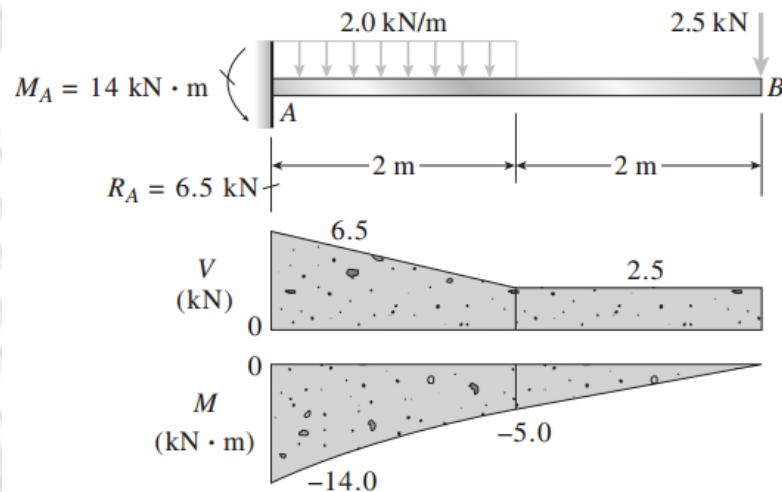
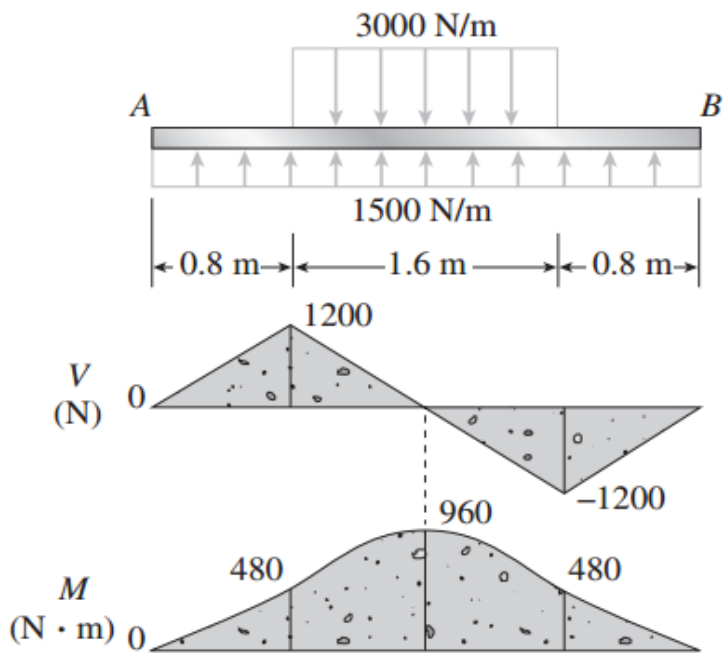
$$80 + 10x + V = 140$$

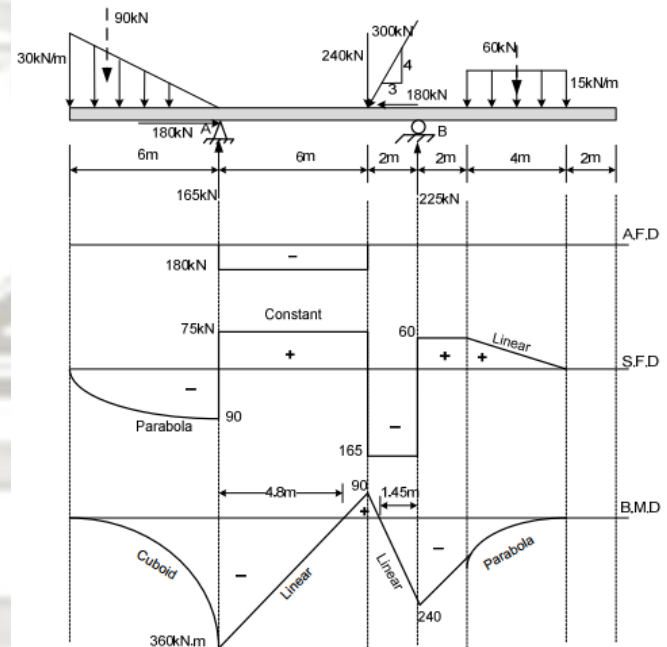
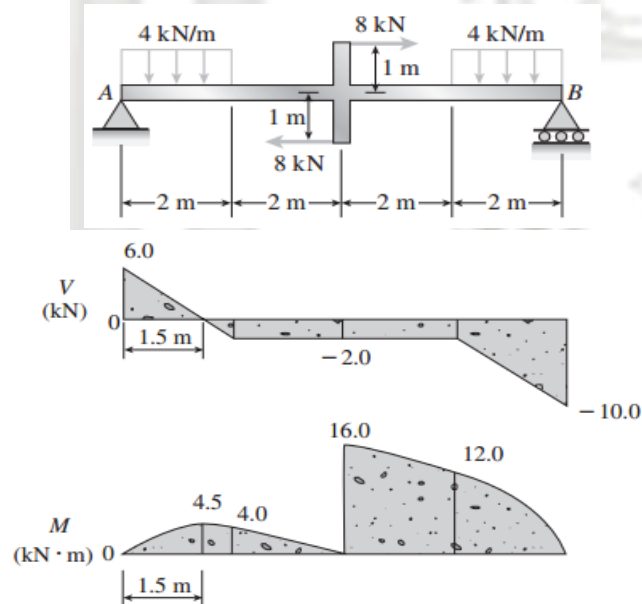
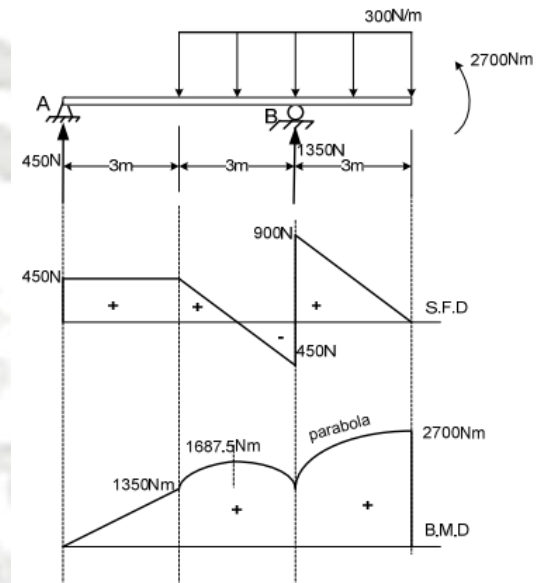
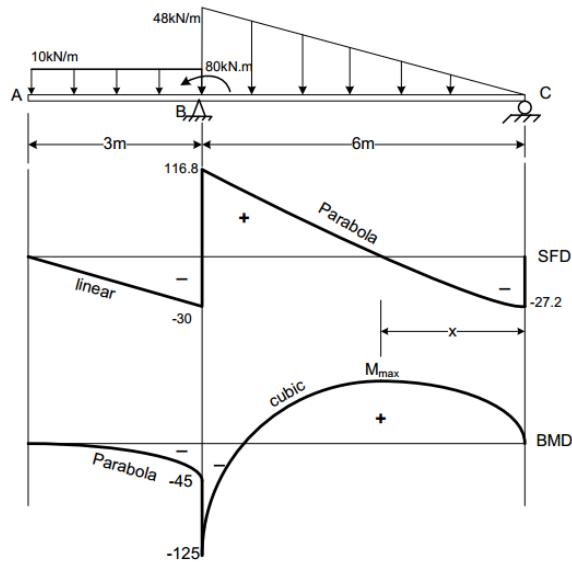
$$x = 6 \text{ in}$$



EXAMPLE 3-12

Draw the shear and moment diagrams for the beams shown in the figures and find the position of max. bending moment..





EXAMPLE 3-13

Draw the shear and moment diagrams for the beam shown in the figure and find the position of max. bending moment..

$$\sum F_y = 0$$

$$6 = 1.5x + \frac{4.5x^2}{8} + V$$

$$V = 0$$

$$6 = 1.5x + \frac{4.5x^2}{8}$$

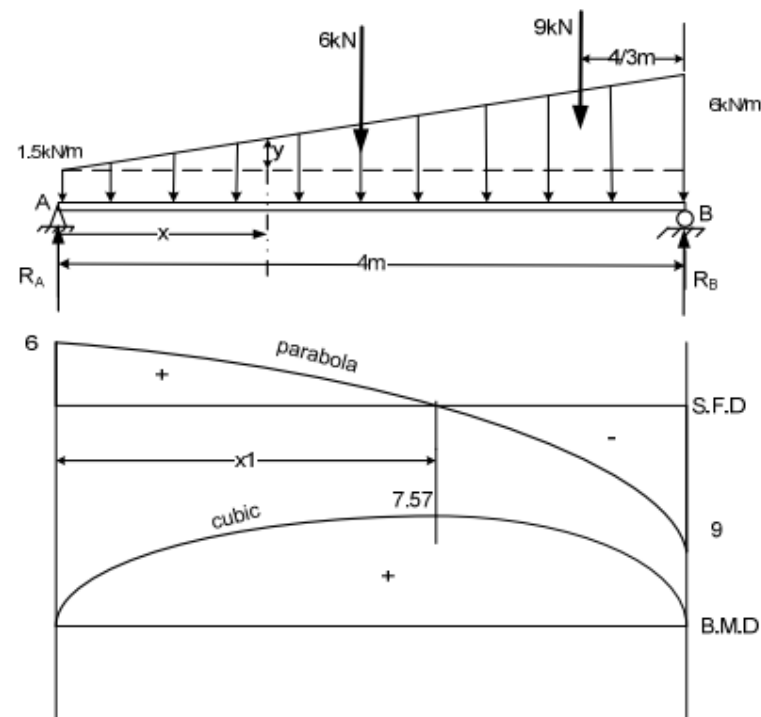
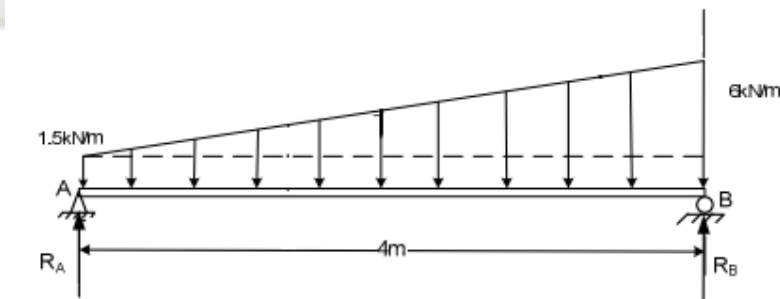
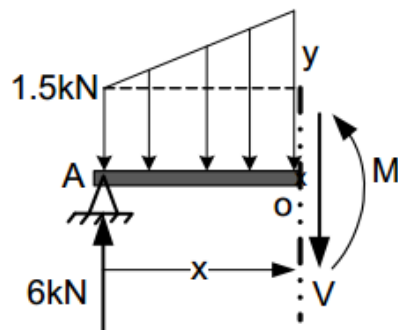
$$0.563x^2 + 1.5x - 6 = 0$$

$$x = \frac{-1.5 \pm \sqrt{(-1.5)^2 + 4 \times 0.563 \times 6}}{2 \times 0.563}$$

$$x = 2.2m$$

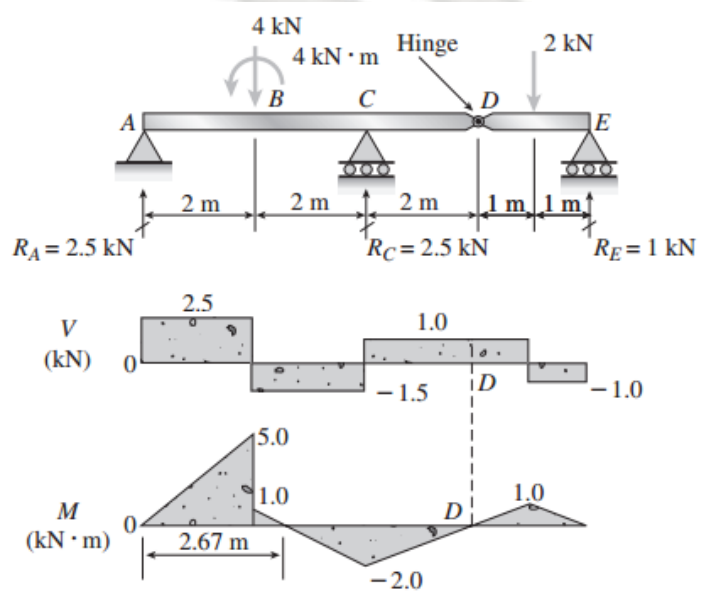
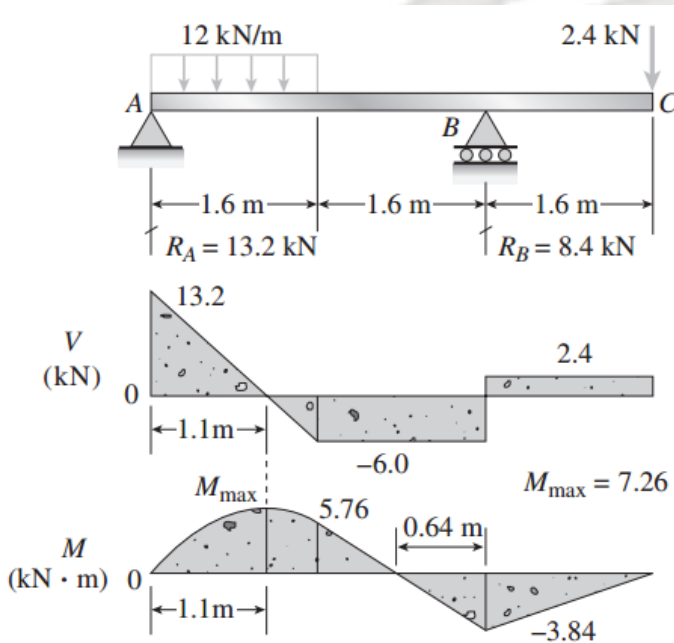
$$\sum M_o = 0$$

$$M + \frac{1.5x^2}{2} + \frac{4.5x}{4} \times \frac{x}{2} \times \frac{x}{3} - 6x = 0$$



EXAMPLE 3-14

Draw the shear and moment diagrams for the beam shown in the figure and find the position of max. bending moment..



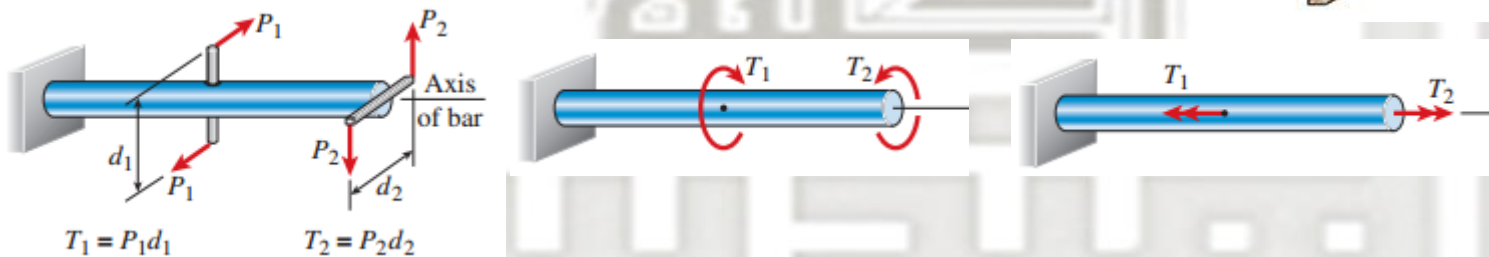
CHAPER FOUR – TORSION

Introduction:

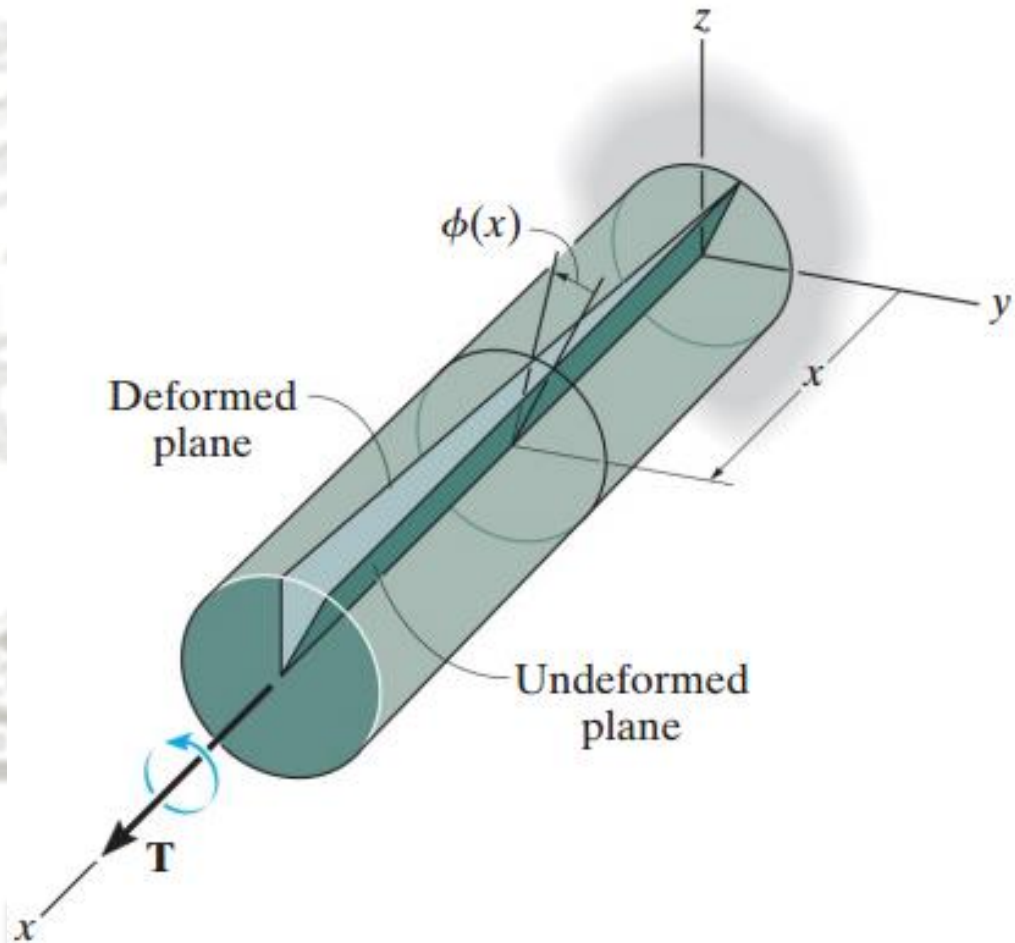
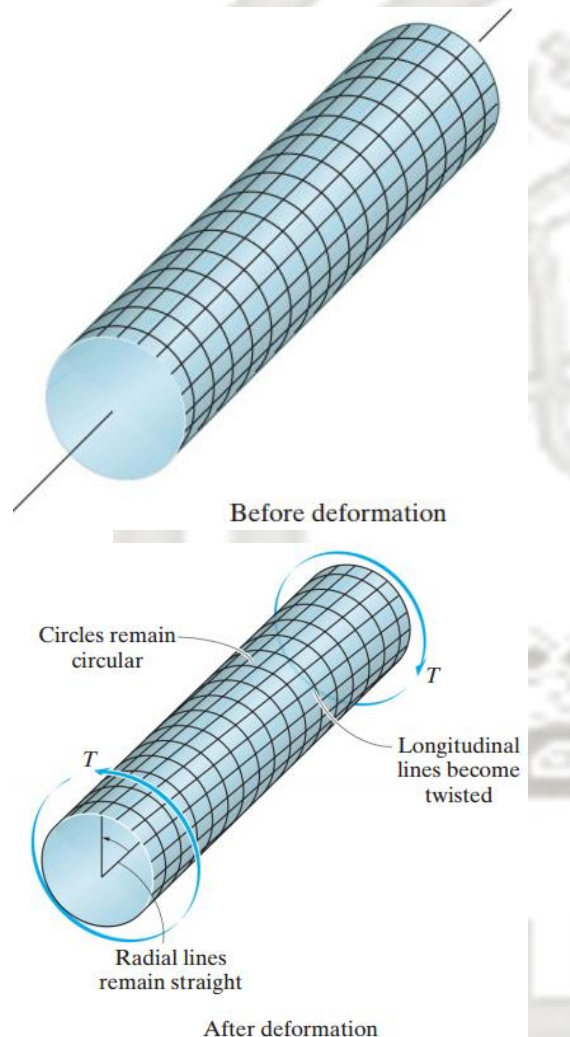
Torsion refers to the twisting of a straight bar when it is loaded by moments (or torques) that tend to produce rotation about the longitudinal axis of the bar. Its effect is of primary concern in the design of drive shafts used in vehicles and machinery, and for this reason it is important to be able to determine the stress and the deformation that occur in a shaft when it is subjected to torsional loads.

For instance, when you turn a screwdriver as shown in the figure, your hand applies a torque T to the handle and twists the rod of the screwdriver.

An idealized case of torsional loading is pictured in the figure which shows a straight bar supported at one end and loaded by two pairs of equal and opposite forces. The first pair consists of the forces P_1 acting near the midpoint of the bar and the second pair consists of the forces P_2 acting at the end. Each pair of forces forms a **couple** that tends to twist the bar about its longitudinal axis. The **moment of a couple** is equal to the product of one of the forces and the perpendicular distance between the lines of action of the forces; thus, the first couple has a moment $T_1 = P_1 d_1$ and the second has a moment $T_2 = P_2 d_2$. The SI unit for moment is the newton meter (N.m).



When the torque is applied, the longitudinal grid lines originally marked on the shaft tend to distort into a helix, that intersects the circles at equal angles. Also, all the cross sections of the shaft will remain flat and they do not warp or bulge in or out. The radial lines remain straight and rotate during this deformation. Provided the angle of twist is *small*, then the length of the shaft and its radius will remain practically unchanged.



The angle of twist $\phi(x)$ increases as x increases.

Torsion Formula of Circular Sections

Assumptions:

1. A plane section before twist remains plane after twist.
2. The distribution of shear strain (γ) through the section is linear.
3. The material of the body is linear elastic.

Arc length $ab = r\phi$

And $ab = L\gamma$

$$\gamma_{\max} = \frac{r\phi}{L}$$

And $\tau_{\max} = G\gamma_{\max} = G \frac{r\phi}{L}$

Torque = stress x area x arm

$$dA = 2\pi\rho d\rho, \quad (\rho = r)$$

For a bar in **pure torsion**, the total angle of twist ϕ , equal to the rate of twist times the length of the bar that is:

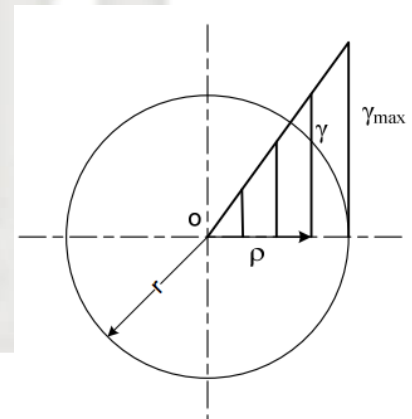
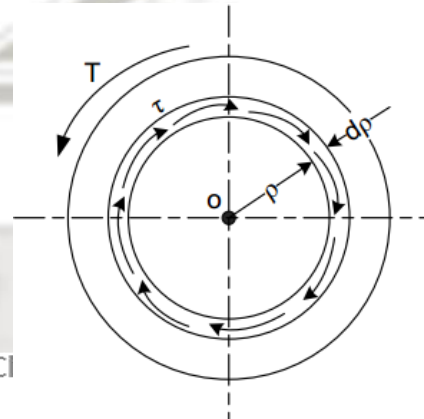
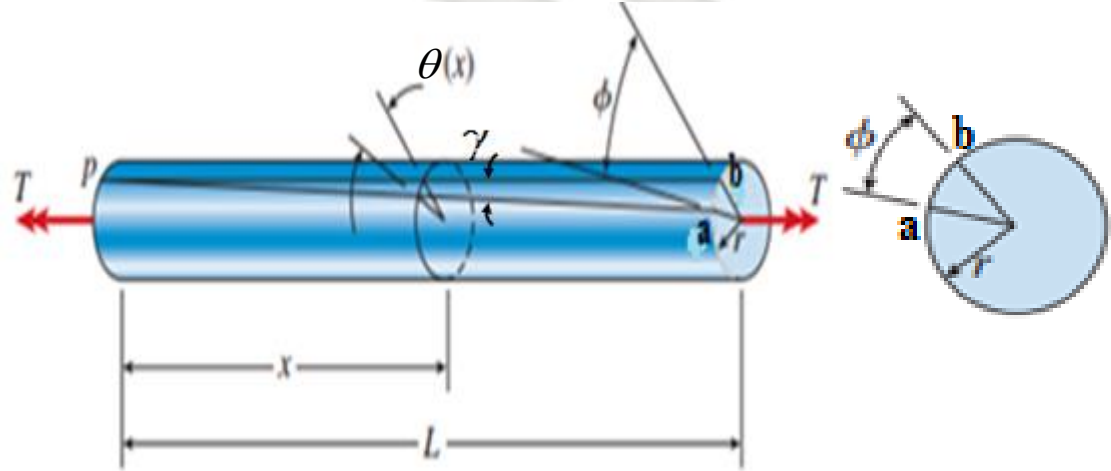
$$\phi = \theta \times L$$

$$T = \int_{\text{area}} 2\pi\rho \cdot \tau \cdot \rho d\rho = \int_{\text{area}} 2\pi\rho \cdot G\rho\theta \cdot \rho d\rho$$

$$T = 2\pi G\theta \int_0^r \rho^3 d\rho = 2\pi G\theta \frac{r^4}{4}$$

$$T = \frac{\pi}{2} G\theta r^4$$

Mechanics of Materials – 2nd Cl



Polar Moment of Inertia

$$J = \int_{\text{area}} \rho^2 dA$$

$$J = \int_0^r 2\pi\rho \cdot \rho^2 d\rho$$

$$J = 2\pi \int_0^r \rho^3 d\rho$$

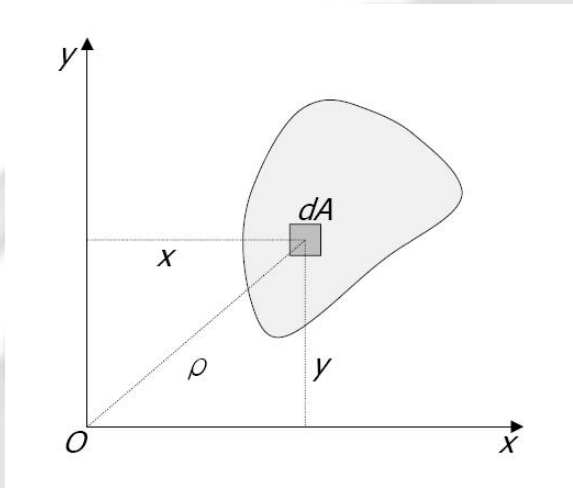
$$J = 2\pi \frac{r^4}{4} = \frac{\pi r^4}{2}$$

$$T = JG\theta$$

$$\theta = \frac{\phi}{L}$$

$$\phi = \theta L = \frac{TL}{GJ}$$

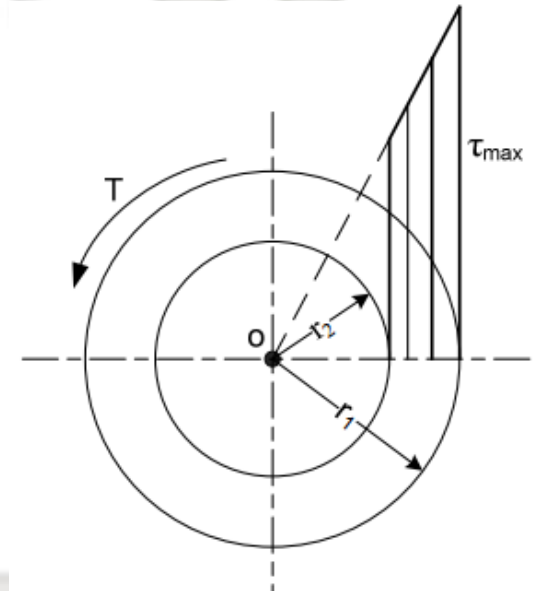
$$\tau_{\max} = \frac{Gr}{L} \frac{TL}{GJ} = \frac{Tr}{J}$$



Hollow Circular Sections

$$J = \frac{\pi}{2} (r_1^4 - r_2^4)$$

$$\tau_{\max} = \frac{Tr_1}{J}$$



EXAMPLE 4-1

A plastic bar of diameter $d=50$ mm is to be twisted by torques T as shown in the figure until the angle of rotation between the ends of the bar is 5.0° . If the allowable shear strain in the plastic is 0.012 rad, what is the minimum permissible length of the bar?

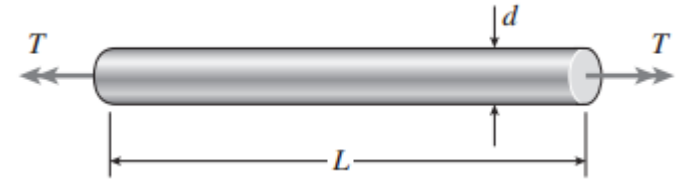
$$d = 50\text{mm}$$

$$\phi = 5^\circ = 5 \times \frac{\pi}{180} = 0.08727\text{rad}$$

$$\gamma_{\text{allow.}} = 0.012\text{rad}$$

$$\gamma_{\text{max}} = \frac{r\phi}{L} = \frac{d \times \phi}{2L}$$

$$L_{\text{min}} = \frac{d \times \phi}{2\gamma_{\text{allow}}} = \frac{50 \times 0.08727}{2 \times 0.012} = 182\text{mm}$$

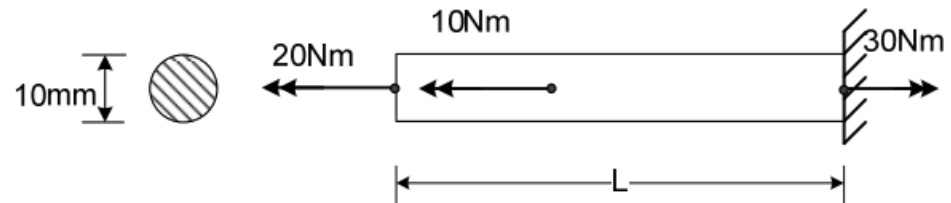


EXAMPLE 4-2

Determine the maximum shearing stress occurs in the circular shaft AC.

$$J = \frac{\pi \times r^4}{2} = \frac{\pi \times 5^4}{2} = 981.7\text{mm}^4$$

$$\tau_{\text{max}} = \frac{Tr}{J} = \frac{30 \times 10^3 \times 5}{981.7} = 152.8\text{MPa}$$



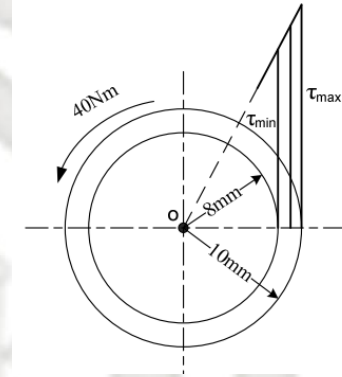
EXAMPLE 4-3

A circular hollow shaft with an outside diameter of 20mm and inside diameter of 16mm is subjected to a torque of 40Nm. Determine the torsional shear stress at the outside surface and inside surface of the shaft.

$$J = \frac{\pi}{2}(r_1^4 - r_2^4) = \frac{\pi}{2}(10^4 - 8^4) = 9274 \text{ mm}^4$$

$$\tau_{\max} = \tau_{\text{out}} = \frac{Tr_1}{J} = \frac{40 \times 10^3 \times 10}{9274} = 43.13 \text{ MPa}$$

$$\tau_{\min} = \tau_{\text{in}} = \frac{Tr_2}{J} = \frac{40 \times 10^3 \times 8}{9274} = 34.51 \text{ MPa}$$



EXAMPLE 4-4

The pipe shown in the figure has an inner radius of 40 mm and an outer radius of 50 mm. If its end is tightened against the support at A using the torque wrench, determine the shear stress developed in the material at the inner and outer walls along the central portion of the pipe.

$$\sum M_x = 0$$

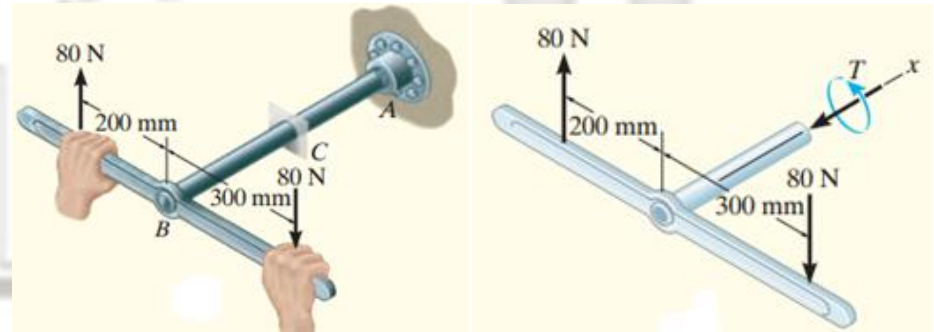
$$80 \times 0.3 + 80 \times 0.2 - T = 0$$

$$T = 40 \text{ N.m}$$

$$J = \frac{\pi}{2}(r_1^4 - r_2^4) = \frac{\pi}{2}(50^4 - 40^4) = 5796238 \text{ mm}^4$$

$$\tau_{\text{out}} = \frac{Tr_1}{J} = \frac{40 \times 10^3 \times 50}{5796238} = 0.345 \text{ MPa}$$

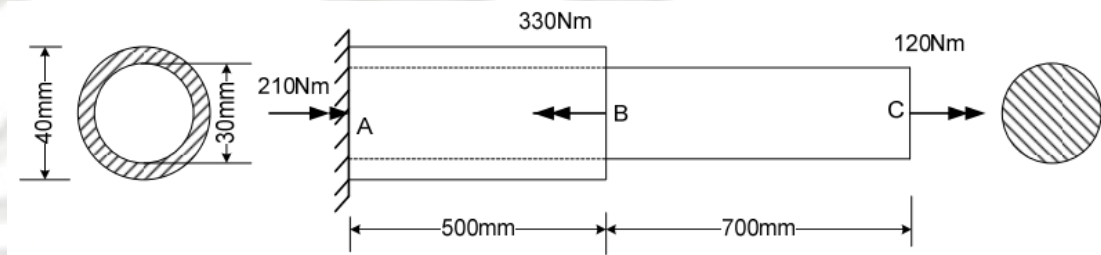
$$\tau_{\text{in}} = \frac{Tr_2}{J} = \frac{40 \times 10^3 \times 40}{5796238} = 0.276 \text{ MPa}$$



EXAMPLE 4-5

The steel shaft shown below is subjected to two concentrated torques at B and C. If the shear modulus of the steel material G is 80GPa, determine:

1. The angle of twist at the free end (total angle of twist).
2. The maximum shear strain in the shaft.



$$J_{AB} = \frac{\pi}{2} (r_1^4 - r_2^4) = \frac{\pi}{2} (20^4 - 15^4) = 171806 \text{ mm}^4$$

$$J_{BC} = \frac{\pi}{2} r^4 = \frac{\pi}{2} \times 15^4 = 79521.5 \text{ mm}^4$$

$$\phi_C = \phi_{AB} + \phi_{BC} = \left(\frac{TL}{GJ} \right)_{AB} + \left(\frac{TL}{GJ} \right)_{BC}$$

$$\phi_C = -\frac{210 \times 10^3 \times 500}{80 \times 10^3 \times 171806} + \frac{120 \times 10^3 \times 700}{80 \times 10^3 \times 79521.5} = 0.00556 \text{ rad.}$$

$$\tau_{\max} = \left(\frac{Tr}{J} \right)_{AB} = \frac{210 \times 10^3 \times 20}{171806} = 24.45 \text{ MPa}$$

Check the shear stress at BC:

$$\tau_{\max} = \left(\frac{Tr}{J} \right)_{BC} = \frac{120 \times 10^3 \times 15}{79521.5} = 22.64 \text{ MPa}$$

$$\gamma_{\max} = \frac{\tau_{\max}}{G} = \frac{24.45}{80000} = 0.0003 \text{ rad.}$$

EXAMPLE 4-6

The two solid steel shafts shown in figure are coupled together using the meshed gears. Determine the angle of twist of end A of shaft AB when the torque $T = 45 \text{ N}\cdot\text{m}$ is applied. Shaft DC is fixed at D. Each shaft has a diameter of 20 mm and $G = 80 \text{ GPa}$.

Summing moments along the axis of shaft AB yields the tangential reaction between the gears:

$$F = \frac{T_A}{L} = \frac{45}{0.15} = 300 \text{ N}$$

$$T_D = 300 \times 0.075 = 22.5 \text{ N}\cdot\text{m}$$

$$J = \frac{\pi}{2} \times 10^4 = 15708 \text{ mm}^4$$

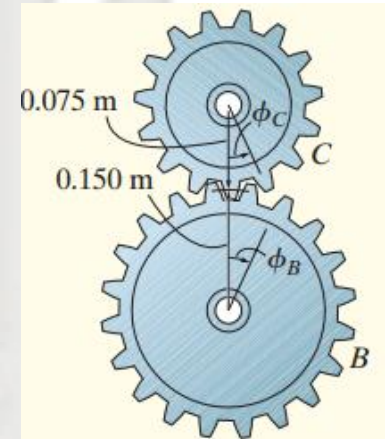
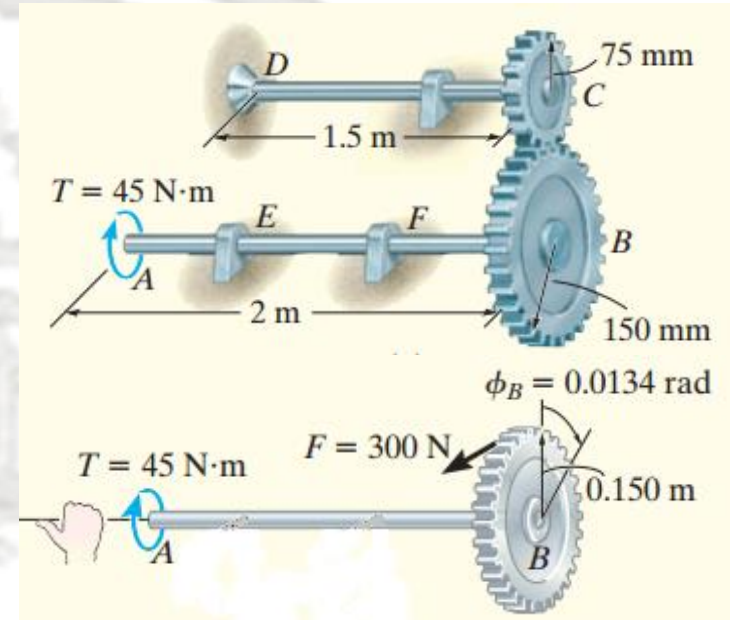
$$\phi_c = \left(\frac{TL}{GJ}\right)_{DC} = \frac{22.5 \times 10^3 \times 1500}{80 \times 10^3 \times 15708} = 0.0269 \text{ rad.}$$

$$\therefore \phi_B \times 0.15 = 0.0269 \times 0.075 \phi_{AB}$$

$$\phi_B = 0.0134 \text{ rad.}$$

$$\phi_{AB} = \left(\frac{TL}{GJ}\right)_{AB} = \frac{45 \times 10^3 \times 2000}{80 \times 10^3 \times 15708} = 0.0716 \text{ rad.}$$

$$\therefore \phi_A = \phi_B + \phi_{AB} = 0.0134 + 0.0716 = 0.085 \text{ rad}$$

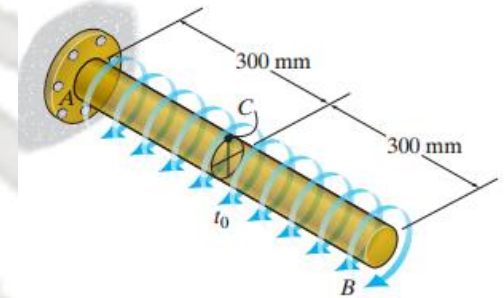


EXAMPLE 4-7

If the 40-mm-diameter rod is subjected to a uniform distributed torque of $t_0 = 1.5 \text{ kN.m/m}$, determine the shear stress developed at point C .

$$J = \frac{\pi}{2} r^4 = \frac{\pi}{2} \times 20^4 = 251327.4 \text{ mm}^4$$

$$\tau_{\max} = \frac{Tr}{J} = \frac{1.5 \times 0.3 \times 10^6 \times 20}{251327.4} = 35.8 \text{ MPa}$$



EXAMPLE 4-8

If the rod in Ex. 4-7 is subjected to a uniform distributed torque of $t_0 = 1.5 \text{ kN.m/m}$, determine the rod's minimum required diameter d if the material has an allowable shear stress of $\tau_{\text{allow}} = 75 \text{ MPa}$.

$$\tau_{\text{allow}} = \frac{T_{\max} r}{J} \Rightarrow J = \frac{T_{\max} r}{\tau_{\text{allow}}}$$

$$J = \frac{1.5 \times 0.6 \times 10^6 \times r}{75} = 12000r$$

$$J = \frac{\pi}{2} r^4 \Rightarrow 12000r = \frac{\pi}{2} r^4$$

$$r^3 = 7639.43 \Rightarrow r = 19.694 \text{ mm}$$

$$\therefore d = 39.4 \text{ mm}$$

EXAMPLE 4-9

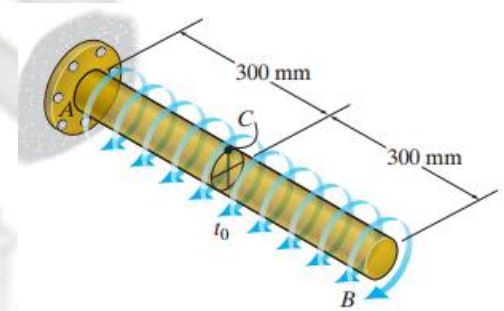
If the 40-mm-diameter rod is made from a material having an allowable shear stress of $\tau_{\text{allow}} = 75 \text{ MPa}$, determine the maximum allowable intensity of the uniform distributed torque.

$$J = \frac{\pi}{2} r^4 = \frac{\pi}{2} 20^4 = 251327.4$$

$$\tau_{\text{allow}} = \frac{T_{\text{max}} r}{J} \Rightarrow T_{\text{max}} = \frac{\tau_{\text{allow}} \times J}{r} = \frac{\pi}{2} r^3 \tau_{\text{allow}}$$

$$T_{\text{max}} = \frac{\pi}{2} \times 20^3 \times 75 \times 10^{-6} = 0.942 \text{ kN.m}$$

$$t_o = \frac{0.942}{0.6} = 1.57 \text{ kN.m/m}$$



EXAMPLE 4-10

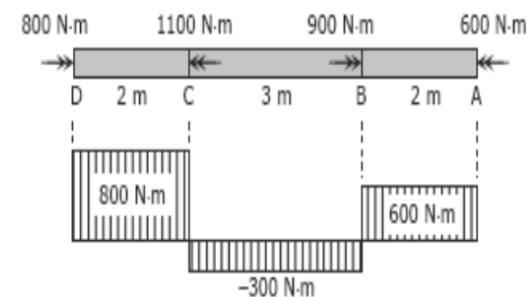
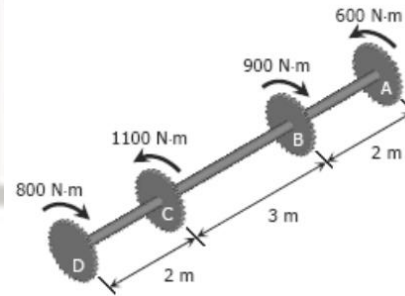
An aluminum shaft with a constant diameter of 50 mm is loaded by torques applied to gears attached to it as shown in figure. Using $G = 28 \text{ GPa}$, determine the relative angle of twist of gear D relative to gear A.

$$J = \frac{\pi}{2} r^4 = \frac{\pi}{2} 25^4 = 613592.3 \text{ mm}^4$$

$$\phi = \sum \frac{TL}{GJ}$$

$$\phi = \frac{(800 \times 2000 - 300 \times 3000 + 600 \times 2000) \times 10^3}{613592.3 \times 28000}$$

$$\phi = 0.1106$$



EXAMPLE 4-11

A compound shaft consisting of a steel segment and an aluminum segment is acted upon by two torques as shown in the figure. Determine the maximum permissible value of T subject to the following conditions: $\tau_{st} = 83 \text{ MPa}$, $\tau_{al} = 55 \text{ MPa}$, and the angle of rotation of the free end is limited to 6° . For steel, $G = 83 \text{ GPa}$ and for aluminum, $G = 28 \text{ GPa}$.

Based on the maximum shearing stress:

$$\tau_{st} = \frac{T_{\max} r}{J} = \frac{2T_{\max}}{\pi \times r^3}$$

$$83 = \frac{2 \times 3T}{\pi \times 25^3}$$

$$T = 679042.16 \text{ N.mm} \Rightarrow 679.04 \text{ N.m}$$

$$\tau_{al} = \frac{Tr}{J} = \frac{2T}{\pi \times r^3}$$

$$55 = \frac{2 \times T}{\pi \times 20^3}$$

$$T = 691150.38 \text{ N.mm} \Rightarrow 691.15 \text{ N.m}$$

Based on the maximum angle of twist:

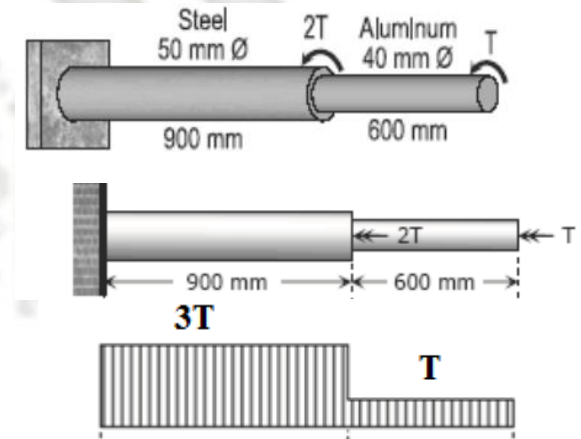
$$\phi = \phi_{st} + \phi_{al} = \left(\frac{TL}{GJ}\right)_{st} + \left(\frac{TL}{GJ}\right)_{al}$$

$$6 \times \frac{\pi}{180} = \frac{3T \times 900}{\frac{\pi}{2} \times 25^4 \times 83000} + \frac{T \times 600}{\frac{\pi}{2} \times 20^4 \times 28000}$$

$$T = 757316.32 \text{ N.mm} = 757.32 \text{ N.m}$$

$$\therefore T = 679.04 \text{ N.m}$$

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EXAMPLE 4-12

A solid steel shaft is loaded as shown in figure. Using $G = 83 \text{ GPa}$, determine the required diameter of the shaft if the shearing stress is limited to 60 MPa and the angle of rotation at the free end is not to exceed 4 deg .

Based on the maximum allowable shearing stress:

$$\tau_{\max} = \frac{T_{\max} r}{J} = \frac{2T_{\max}}{\pi \times r^3}$$

$$60 = \frac{2 \times 450 \times 10^3}{\pi \times r^3}$$

$$r = 16.838 \text{ mm} \Rightarrow D = 33.677 \text{ mm}$$

$$60 = \frac{2 \times 1200 \times 10^3}{\pi \times r^3}$$

$$r = 23.35 \text{ mm} \Rightarrow D = 46.7 \text{ mm}$$

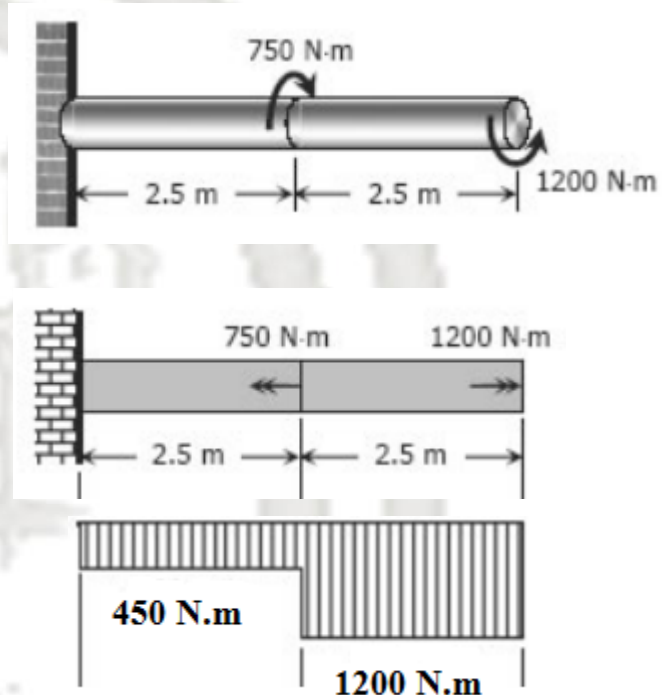
Based on the maximum angle of twist:

$$\phi = \sum \frac{TL}{GJ}$$

$$4 \times \frac{\pi}{180} = \frac{(450 + 1200) \times 10^3 \times 2500}{\frac{\pi}{2} \times r^4 \times 83000}$$

$$r = 25.94 \text{ mm} \Rightarrow D = 51.89 \text{ mm}$$

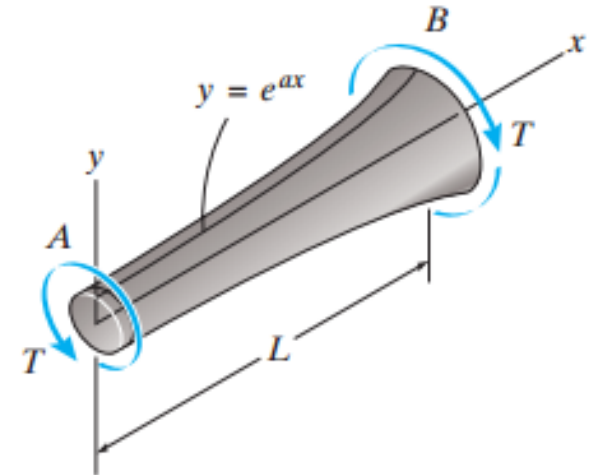
$$\therefore D = 51.89 \text{ mm}$$



EXAMPLE 4-13

The contour of the surface of the shaft is defined by the equation $y = e^{ax}$, where a is a constant. If the shaft is subjected to a torque T at its ends, determine the angle of twist of end A with respect to end B. The shear modulus is G .

$$\begin{aligned}\phi &= \int \frac{T dx}{J(x)G} \quad \text{where, } J(x) = \frac{\pi}{2} (e^{ax})^4 \\ &= \frac{2T}{\pi G} \int_0^L \frac{dx}{e^{4ax}} = \frac{2T}{\pi G} \left(-\frac{1}{4a e^{4ax}} \right) \bigg|_0^L \\ &= \frac{2T}{\pi G} \left(-\frac{1}{4a e^{4aL}} + \frac{1}{4a} \right) = \frac{T}{2a\pi G} \left(\frac{e^{4aL} - 1}{e^{4aL}} \right) \\ &= \frac{T}{2a\pi G} (1 - e^{-4aL})\end{aligned}$$



EXAMPLE 4-14

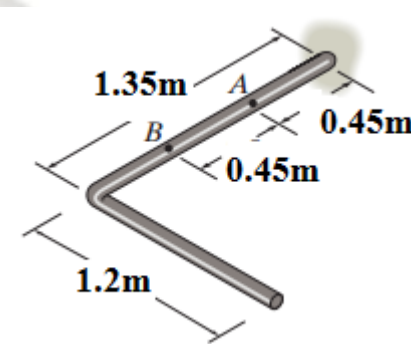
The rod has a diameter of 25mm and a weight of 15 N/m. Determine the maximum torsional stress in the rod at a section located at B due to the rod's weight

$$\sum M_x = 0$$

$$T_B = 15 \times 1.2 \times 0.6 = 10.8 \text{ N.m}$$

$$J = \frac{\pi}{2} r^4 = \frac{\pi}{2} 12.5^4 = 38.35 \text{ mm}^4$$

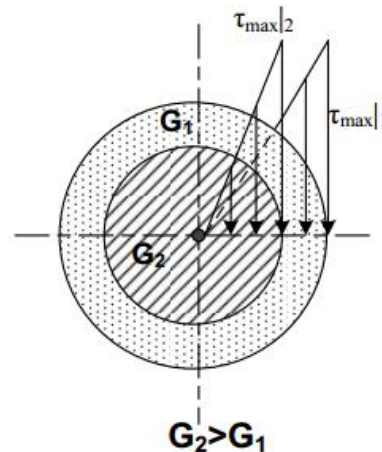
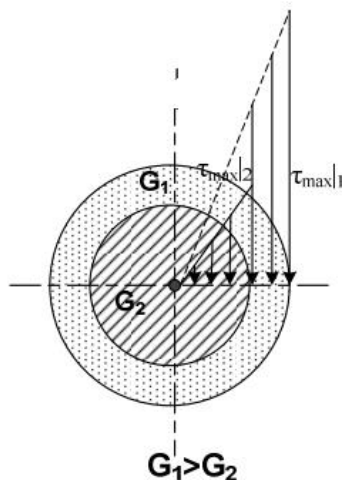
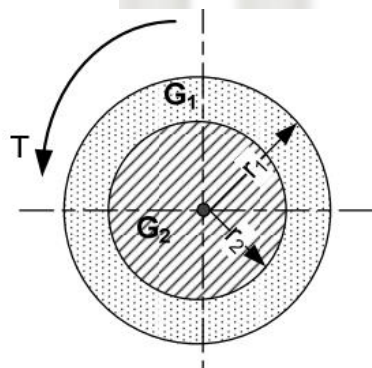
$$\tau_{\max} = \frac{Tr}{J} = \frac{10.8 \times 10^3 \times 12.5}{38.35} = 3.52 \text{ MPa}$$



Statically Indeterminate Problems

Equilibrium Equation: $T_1 + T_2 = T \rightarrow (1)$

Compatibility Equation: $\phi_1 = \phi_2 \rightarrow (2)$



EXAMPLE 4-15

The shaft shown in figure is made from a steel tube, which is bonded to a brass core. If a torque of $T = 340$ N.m is applied at its end, plot the shear-stress distribution along a radial line on its cross section. Take $G_{st} = 78600$ MPa and $G_{br} = 35850$.

$$T_{st} + T_{br} = 340 \rightarrow (1)$$

$$\phi = \phi_{st} = \phi_{br} \Rightarrow \left(\frac{TL}{GJ}\right)_{st} = \left(\frac{TL}{GJ}\right)_{br}$$

$$\frac{T_{st} \times L}{\frac{\pi}{2} \times (26^4 - 13^4) \times 78600} = \frac{T_{br} \times L}{\frac{\pi}{2} \times 13^4 \times 35850}$$

$$T_{st} = \frac{T_{br} \times (26^4 - 13^4) \times 78600}{13^4 \times 35850} \Rightarrow T_{st} = 32.887 T_{br} \rightarrow (2)$$

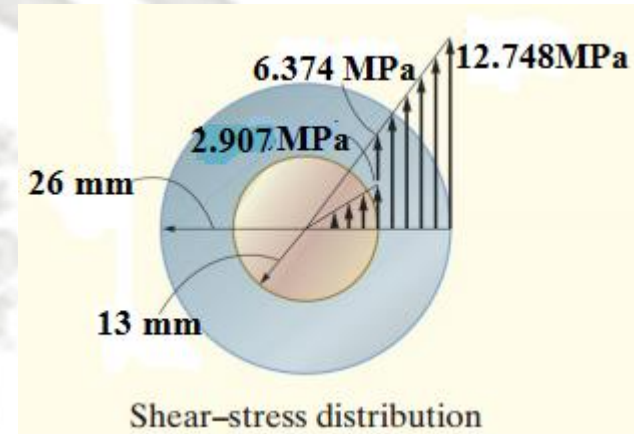
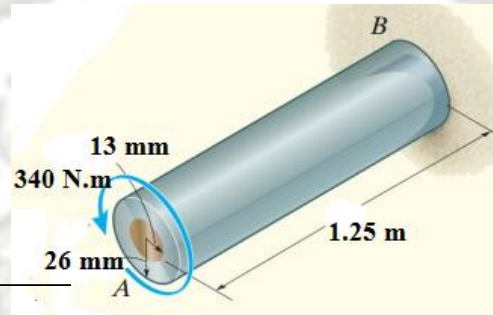
$$\therefore T_{br} = 10.033 \text{ N.m}$$

$$T_{st} = 329.966 \text{ N.m}$$

$$(\tau_{br})_{\max} = \frac{T_{br} r}{J} = \frac{2T_{\max}}{\pi \times r^3} = \frac{2 \times 10.033 \times 10^3}{\pi \times 13^3} = 2.907 \text{ MPa}$$

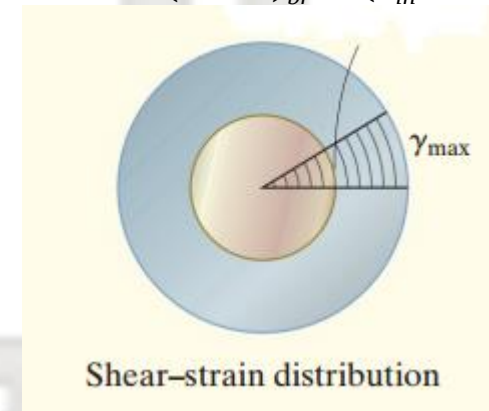
$$(\tau_{st})_{\max} = \frac{T_{st} r_2}{J} = \frac{T_{st} r_2}{\frac{\pi}{2} (r_2^4 - r_1^4)} = \frac{329.966 \times 10^3 \times 26}{\frac{\pi}{2} (26^4 - 13^4)} = 12.748 \text{ MPa}$$

$$(\tau_{st})_{\min} = \frac{T_{st} r_1}{J} = \frac{T_{st} r_1}{\frac{\pi}{2} (r_2^4 - r_1^4)} = \frac{329.966 \times 10^3 \times 13}{\frac{\pi}{2} (26^4 - 13^4)} = 6.374 \text{ MPa}$$



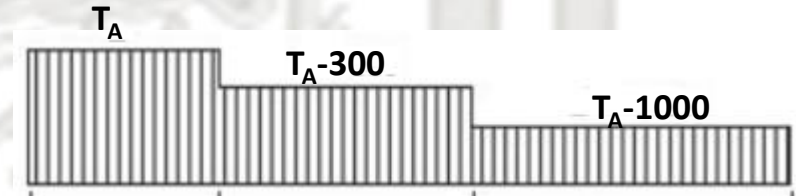
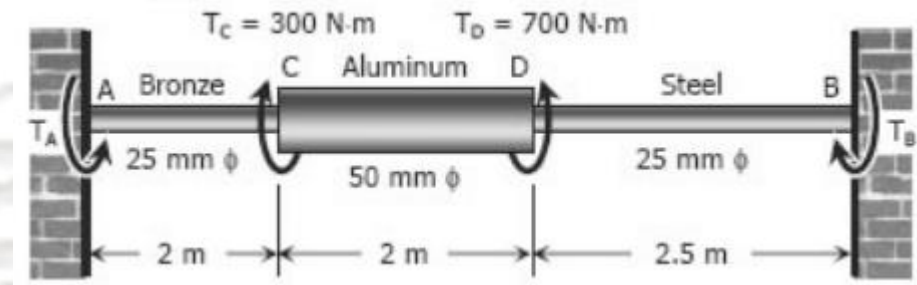
$$\gamma_{br} = (\gamma_{st})_{\min}$$

$$(\tau \times G)_{br} = (\tau_{in} \times G)_{st}$$



EXAMPLE 4-16

A shaft composed of segments AC, CD, and DB is fastened to rigid supports and loaded as shown in figure. For bronze, $G = 35 \text{ GPa}$; aluminum, $G = 28 \text{ GPa}$, and for steel, $G = 83 \text{ GPa}$. Determine the maximum shearing stress developed in each segment.



$$\phi_{A/B} = 0$$

$$\sum \left(\frac{TL}{GJ} \right)_{A/B} = 0$$

$$\frac{T_A \times 10^3 \times 2000}{\frac{\pi}{2} 12.5^4 \times 35000} + \frac{(T_A - 300) \times 10^3 \times 2000}{\frac{\pi}{2} 25^4 \times 28000} + \frac{(T_A - 1000) \times 10^3 \times 2500}{\frac{\pi}{2} 12.5^4 \times 83000} = 0$$

$$\frac{2T_A}{12.5^4 \times 35} + \frac{2(T_A - 300)}{25^4 \times 28} + \frac{2.5(T_A - 1000)}{12.5^4 \times 83} = 0$$

$$\frac{16T_A}{35} + \frac{(T_A - 300)}{28} + \frac{20(T_A - 1000)}{83} = 0$$

$$\frac{16}{35}T_A + \frac{1}{28}T_A - \frac{25}{7} + \frac{20}{83}T_A - \frac{20000}{83} = 0$$

$$T_A = 342.97 \text{ N}\cdot\text{m}$$

$$\sum M_x = 0$$

$$T_A + T_B = 300 + 700$$

$$342.97 + T_B = 1000 \rightarrow T_B = 657.03 \text{ N.m}$$

$$T_{br} = 342.97 \text{ N.m}, T_{al} = 342.97 - 300 = 42.97 \text{ N.m}, T_{st} = 342.97 - 1000 = -657.03 \text{ N.m}$$

$$\tau_{\max} = \frac{Tr}{J} = \frac{2T}{\pi \times r^3}$$

$$\tau_{br} = \frac{2 \times 342.97 \times 10^3}{\pi \times 12.5^3} = 111.79 \text{ MPa}$$

$$\tau_{al} = \frac{2 \times 42.97 \times 10^3}{\pi \times 25^3} = 1.75 \text{ MPa}$$

$$\tau_{st} = \frac{2 \times 657.03 \times 10^3}{\pi \times 12.5^3} = 214.16 \text{ MPa}$$

EXAMPLE 4-17

The two shafts are made of steel. Each has a diameter of 25 mm and they are connected using the gears fixed to their ends. If a torque of 500 N·m is applied to the gear at E as shown, determine the reactions at A and B and the angle of twist at point E. ($G=75\text{GPa}$)

$$T_A + F \times 0.1 - 500 = 0$$

$$T_B - F \times 0.05 = 0$$

$$T_A + 2T_B - 500 = 0 \rightarrow (1)$$

$$0.1\phi_E = 0.05\phi_F$$

$$\phi_E = 0.5\phi_F$$

$$\frac{T_A \times 1.5}{GJ} = 0.5 \times \frac{T_B \times 0.75}{GJ}$$

$$T_A = 0.25T_B$$

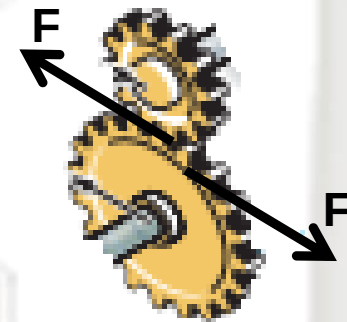
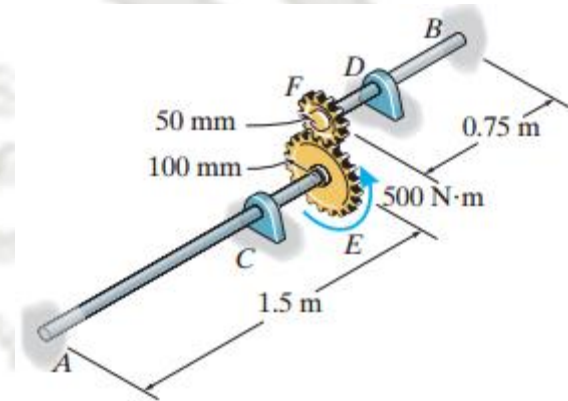
$$T_A = 55.56 \text{ N.m}$$

$$T_B = 222.22 \text{ N.m}$$

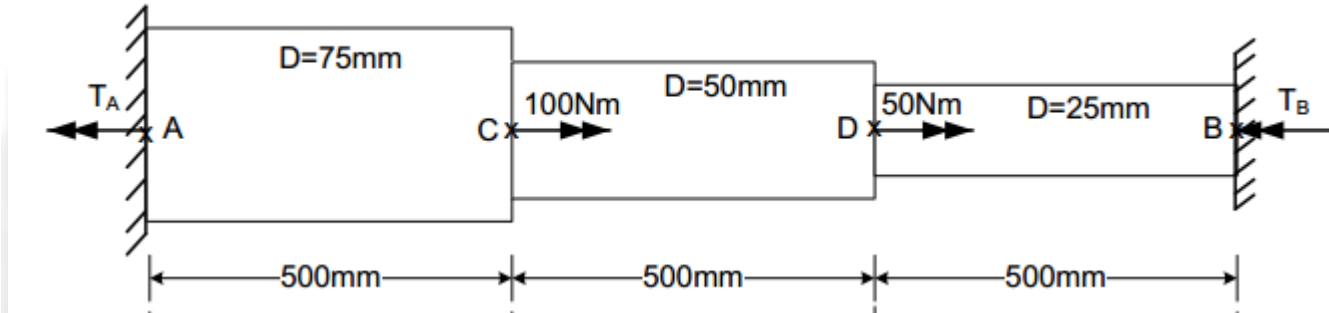
$$\phi_E = \frac{T_A \times L}{GJ}$$

$$\phi_E = \frac{55.56 \times 10^3 \times 1500}{\frac{\pi}{2} \times 12.5^4 \times 75000}$$

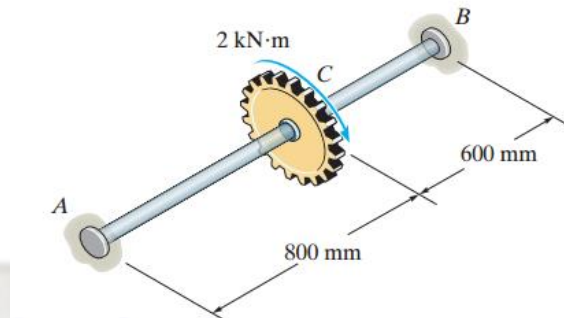
$$\phi_E = 0.02897 \text{ rad.} = 1.66^\circ$$



H.W: A stepped solid circular steel shaft has the shape shown in the figure below and is having $G=80 \times 10^3 \text{ MPa}$. The region AC is having $D=75 \text{ mm}$ region CD having $D=50 \text{ mm}$, and region BD having $D=25 \text{ mm}$. Determine the maximum shearing stress occurs in the shaft as well as the angle of twist at C where a torsional load of 100 N m is applied. Ends A and B are rigidly clamped.

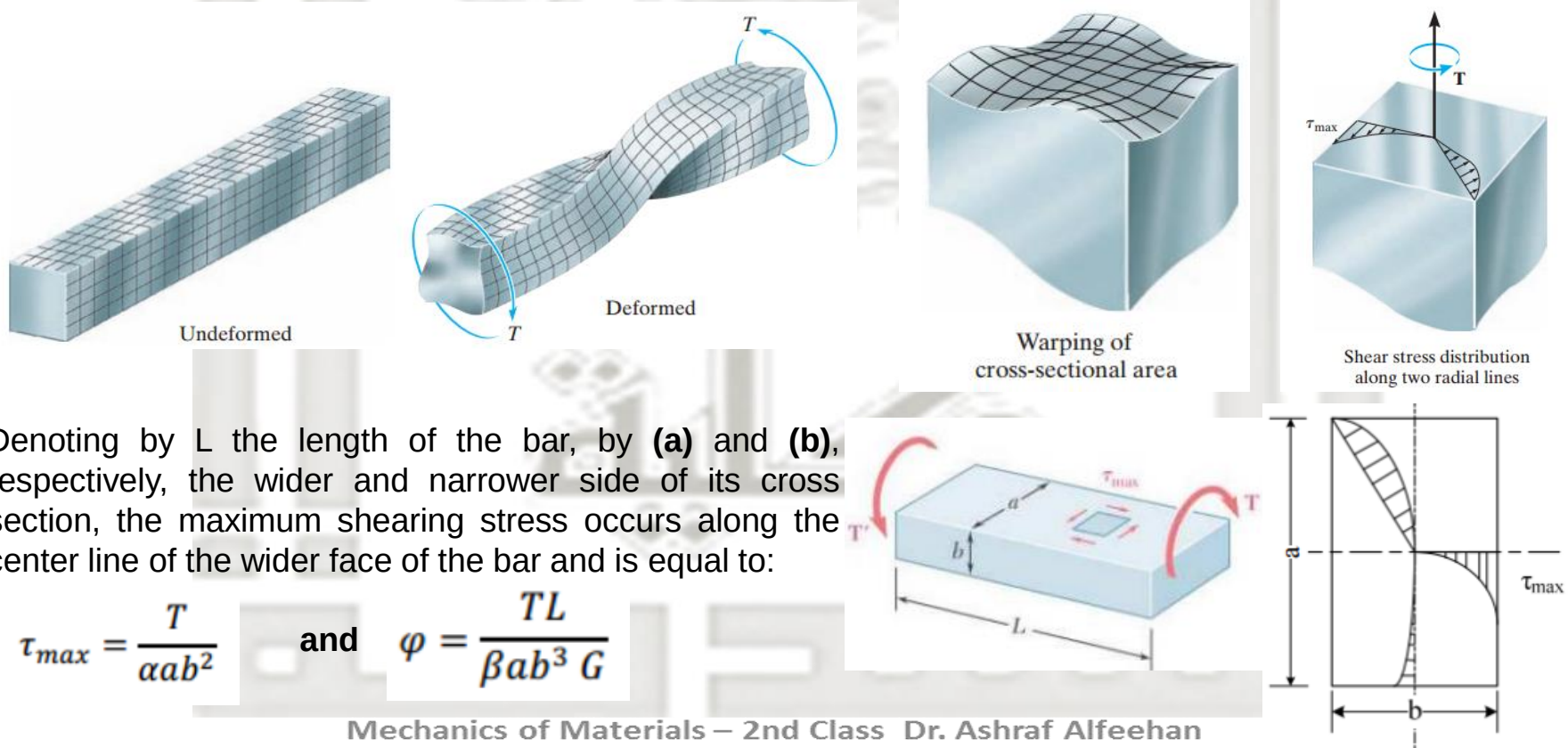


H.W: The shaft shown in the figure has a diameter of 40 mm , and is fixed at its ends A and B. If it is subjected to the torque, determine the maximum shear stress in regions AC and CB.



Torsion of Noncircular Members

It was demonstrated that when a torque is applied to a shaft having a circular cross section the shear strains vary linearly from zero at its center to a maximum at its outer surface. Furthermore, due to uniformity, the cross sections do not deform, but rather remain plane after the shaft has twisted. Shafts that have a noncircular cross section will **bulge or warp** when the shaft is twisted. Evidence of this can be seen from the way grid lines deform on a shaft having a square cross section. Because the shear-stress distributions along two radial lines of the shaft are different, the shear strains they create will **warp** the cross section. The corner points of the shaft must be subjected to zero shear stress and therefore zero shear strain. The reason for this can be shown by considering an element of material located at one of these corner points.



Denoting by L the length of the bar, by **(a)** and **(b)**, respectively, the wider and narrower side of its cross section, the maximum shearing stress occurs along the center line of the wider face of the bar and is equal to:

$$\tau_{max} = \frac{T}{\alpha ab^2} \quad \text{and} \quad \phi = \frac{TL}{\beta ab^3 G}$$

The coefficients α and β depend only upon the ratio a/b and are given in the table shown below for a number of values of that ratio.

a/b	α	β
1.0	0.208	0.141
1.2	0.219	0.166
1.5	0.231	0.196
2	0.249	0.229
2.5	0.258	0.249
3	0.267	0.263
4	0.282	0.281
5	0.291	0.291
10	0.312	0.312
∞	0.333	0.333

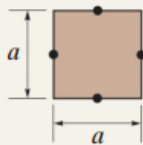
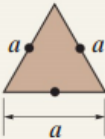
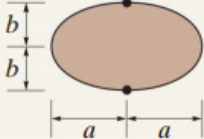
For very thin sections ($a/b \gg 10$), $\alpha = \beta = 1/3$. Therefore, the maximum shearing stress of the rectangular bar is written as:

$$\tau_{max} = \frac{3T}{ab^2}$$

and

$$\phi = \frac{3TL}{ab^3 G}$$

Using the theory of elasticity, the table below provides the results of the analysis for square cross sections, along with those for shafts having triangular and elliptical cross sections and gives the formulas for the angle of twist of each shaft. In all cases, the maximum shear stress occurs **at a point on the edge of the cross section that is closest to the center axis of the shaft**. By extending these results, it can be shown that the most efficient shaft has a circular cross section, since it is subjected to both a smaller maximum shear stress and a smaller angle of twist than one having the same cross-sectional area, but not circular, and subjected to the same torque.

Shape of cross section	τ_{max}	ϕ
Square 	$\frac{4.81 T}{a^3}$	$\frac{7.10 TL}{a^4 G}$
Equilateral triangle 	$\frac{20 T}{a^3}$	$\frac{46 TL}{a^4 G}$
Ellipse 	$\frac{2 T}{\pi ab^2}$	$\frac{(a^2 + b^2)TL}{\pi a^3 b^3 G}$

EXAMPLE 4-18

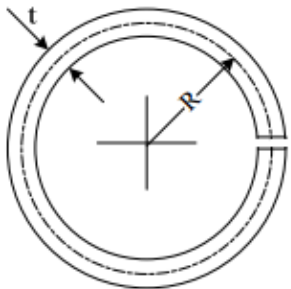
Compare the angle of twist and maximum shear stress for three members of length (L) having a square section, a rectangular section and a circular section of same area. All members are subjected to same torque (T). The circular section is of 100mm diameter, and the rectangular section is 25mm wide.

Square	Rectangular	Circular
$A=7854\text{mm}^2$	$A=7854\text{mm}^2$	$A=\pi/4(d)^2=$ $\pi/4(100)^2=7854\text{mm}^2$
$a=b=88.6\text{mm}$	$a=314\text{mm}$ $b=25\text{mm}$	$J=\pi/2(r)^4=\pi/2(50)^4=$ 9817500mm^4
$\varphi = \frac{TL}{\beta ab^3 G}$	$\varphi = \frac{3TL}{ab^3 G}$	$\varphi = \frac{TL}{GJ}$
$\varphi = \frac{TL}{0.141(88.6)^4 G}$	$\varphi = \frac{3TL}{314(25)^3 G}$	$\varphi = \frac{TL}{9817500 G}$
$\varphi = \frac{TL}{8696532 G}$	$\varphi = \frac{TL}{1633781 G}$	$\varphi = \frac{TL}{9817500 G}$
$\tau_{max} = \frac{T}{\alpha ab^2}$	$\tau_{max} = \frac{3T}{ab^2}$	$\tau_{max} = \frac{Tr}{J}$
$\tau_{max} = \frac{T}{0.208(88.6)^3}$	$\tau_{max} = \frac{3T}{314(25)^2}$	$\tau_{max} = \frac{T(50)}{9817500}$
$\tau_{max} = \frac{T}{144763}$	$\tau_{max} = \frac{T}{65351}$	$\tau_{max} = \frac{T}{196350}$

Summary: $(\tau_{max})_{rec.} > (\tau_{max})_{square} > (\tau_{max})_{circle}$

$\phi_{rec.} > \phi_{square} > \phi_{circle}$

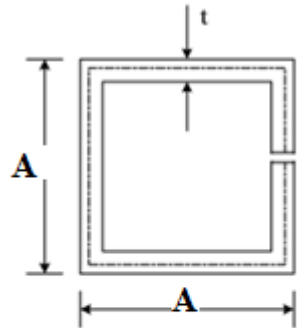
Thin-walled members with open cross sections:



Slotted circular pipe

$$a = 2\pi R$$

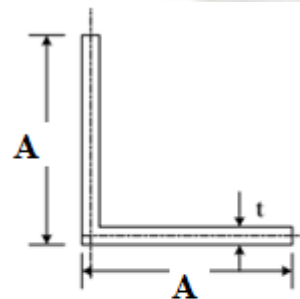
$$b = t$$



Slotted square tube

$$a = 4(A - t)$$

$$b = t$$



Angle

$$a = 2A - t$$

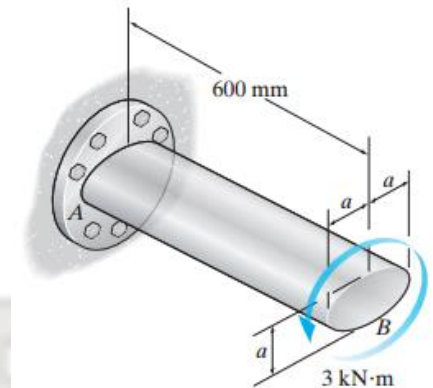
$$b = t$$

EXAMPLE 4-19

If the shaft is subjected to the torque of determine the maximum shear stress developed in the shaft. Also, find the angle of twist of end *B*. The shaft is made from A-36 steel. Set $a = 50$ mm.

$$\tau_{\max} = \frac{2T}{\pi ab^2} = \frac{2(3)(10^3)}{\pi(0.05)(0.025^2)} = 61.1 \text{ MPa}$$

$$\begin{aligned} \phi &= \frac{(a^2 + b^2)TL}{\pi a^3 b^3 G} \\ &= \frac{(0.05^2 + 0.025^2)(3)(10^3)(0.6)}{\pi(0.05^3)(0.025^3)(75)(10^9)} \\ &= 0.01222 \text{ rad} = 0.700^\circ \end{aligned}$$



EXAMPLE 4-20

For the thin-walled square shaft with a longitudinal slot shown in the figure below, determine the angle of twist at section 1-1 and section 2-2. Take $G=84 \text{ kN/mm}^2$

$$\phi = \frac{TL}{\beta ab^3 G}$$

$$a = 4(A - t) = 4(50 - 4) = 184 \text{ mm}$$

$$b = 4 \text{ mm}$$

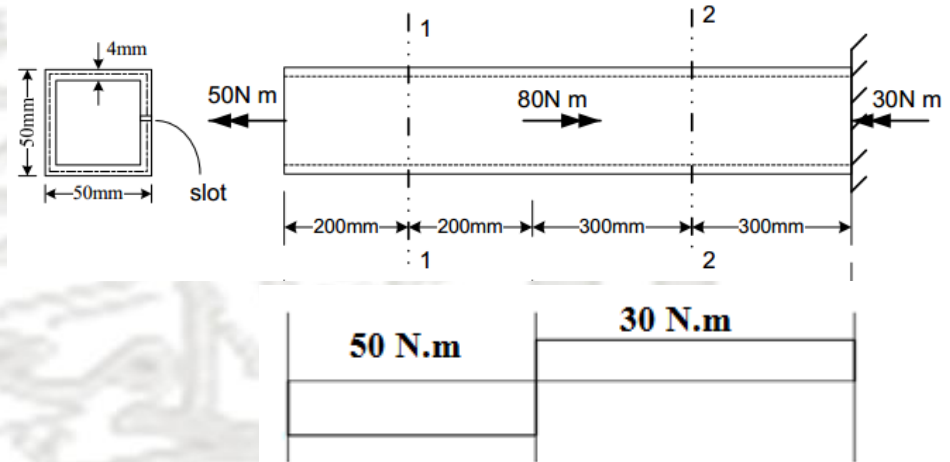
$$\frac{a}{b} = \frac{184}{4} = 46 > 10$$

$$\therefore \beta = \frac{1}{3}$$

$$\phi = \sum \frac{3TL}{ab^3 G}$$

$$\phi_{1-1} = \frac{3 \times 50 \times 10^3 \times 200}{184 \times 4^3 \times 84 \times 10^3} - \frac{3 \times 30 \times 10^3 \times 600}{184 \times 4^3 \times 84 \times 10^3} = -0.0243 \text{ rad.}$$

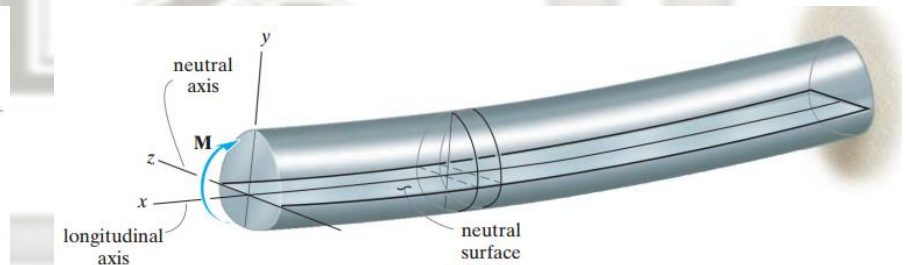
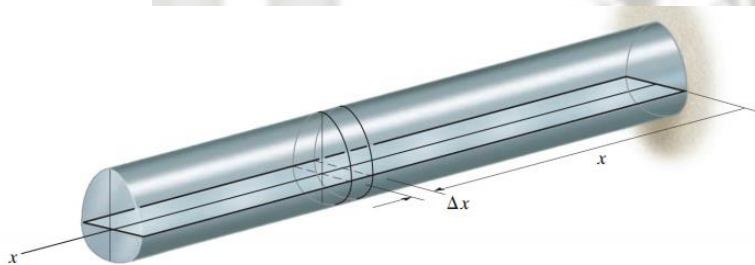
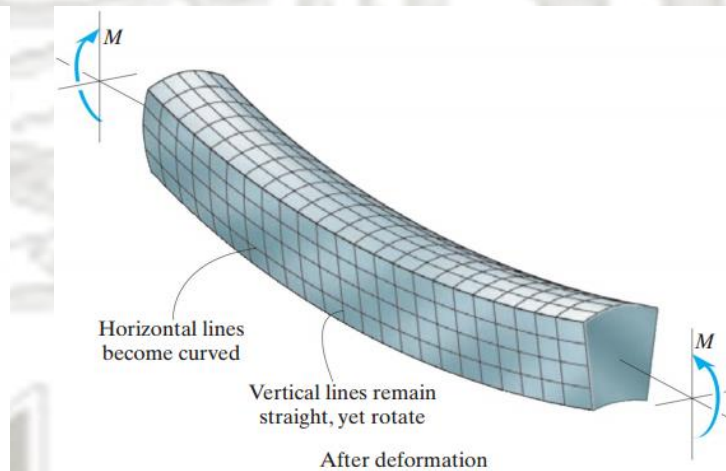
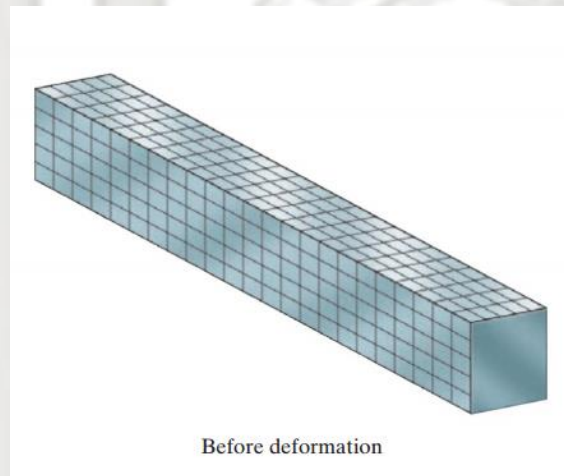
$$\phi_{2-2} = -\frac{3 \times 30 \times 10^3 \times 300}{184 \times 4^3 \times 84 \times 10^3} = -0.0273 \text{ rad}$$



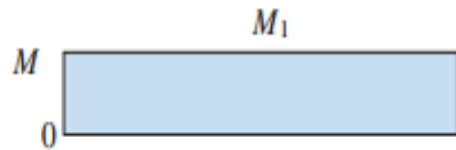
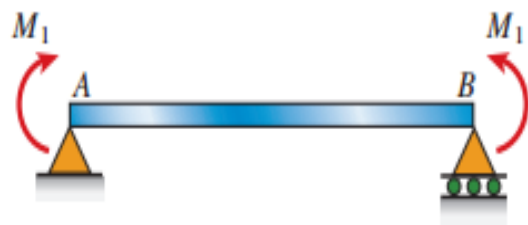
CHAPTER FIVE – PURE BENDING OF BEAMS

Introduction:

Consider the undeformed bar shown in the figure, which has a square cross section and is marked with horizontal and vertical grid lines. When a bending moment is applied, it tends to distort these lines. Here the horizontal lines become curved, while the vertical lines remain straight but undergo a rotation. The bending moment causes the material within the bottom portion of the bar to stretch and the material within the top portion to compress. Consequently, between these two regions there must be a surface, called the neutral surface, in which horizontal fibers of the material will not undergo a change in length. As noted, we will refer to the z axis that lies along the neutral surface as the neutral axis.



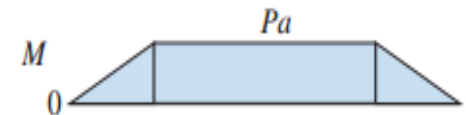
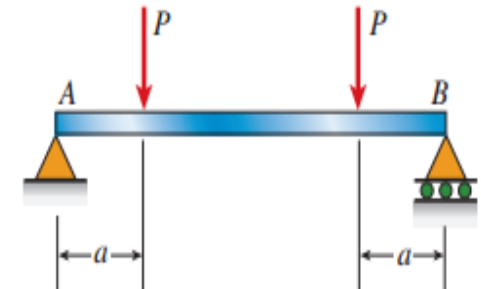
Pure bending refers to flexure of a beam under a constant bending moment. Therefore, pure bending occurs only in regions of a beam where the shear force is zero. However, **nonuniform bending** refers to flexure in the presence of shear forces, which means that the bending moment changes as we move along the axis of the beam.



Simple beam in pure bending



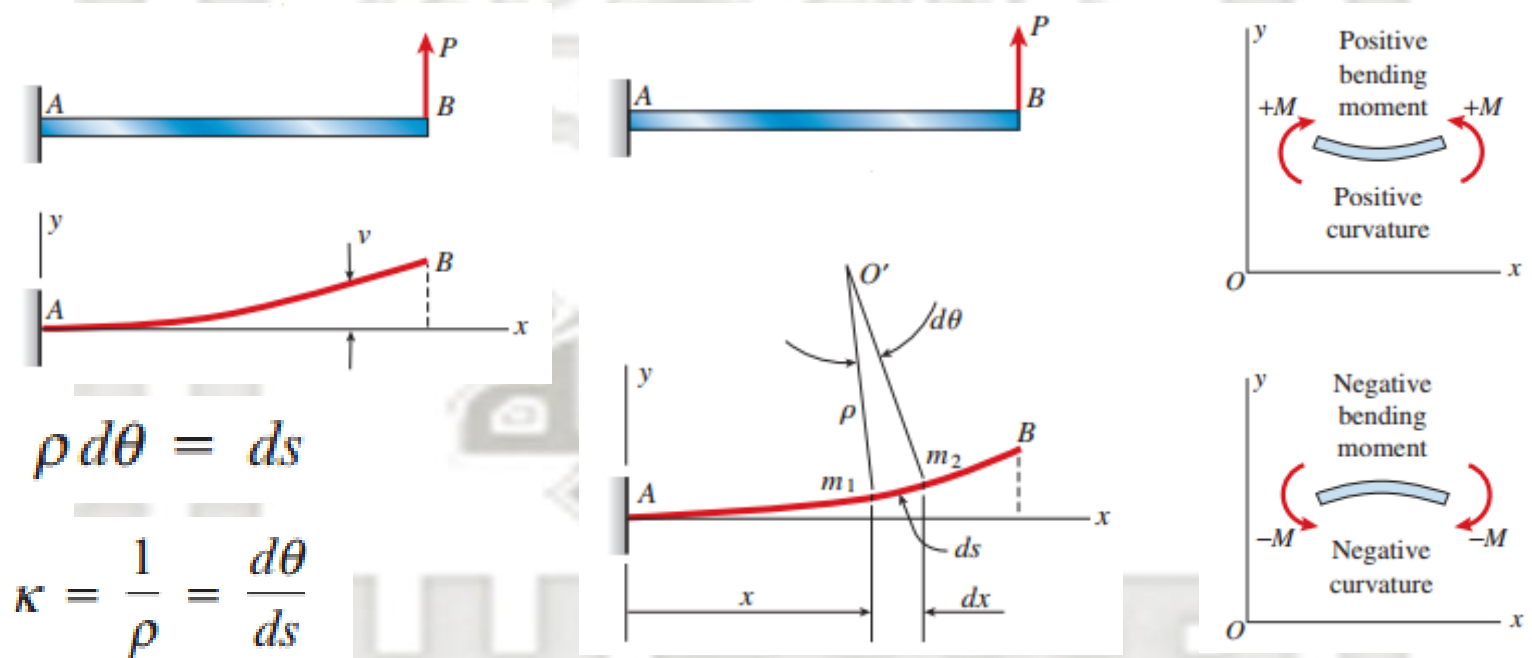
Cantilever beam in pure bending



Simple beam with central region in pure bending and end regions in nonuniform bending

Curvature of a Beam

When loads are applied to a beam, its longitudinal axis is deformed into a curve. The resulting strains and stresses in the beam are directly related to the **curvature** of the deflection curve. Curvature is a measure of how a beam is curved (bent). If the load on a beam is small, the beam will be nearly straight, the radius of curvature will be very large, and the curvature will be very small. If the load is increased, the amount of bending will increase and the radius of curvature will become smaller, and the curvature will become larger. To illustrate the idea of curvature, the two points (m_1 and m_2) are located on the deflection curve. Point m_1 is chosen at an arbitrary distance x from the y axis and point m_2 is located at small distance ds further along the curve. At each of these points, a line normal to the tangent to the deflection curve is drawn. These normals intersect at point O' , which is the **center of curvature** of the deflection curve. The distance ($O'm_1$) from the center of curvature to the curve is called the **radius of curvature ρ** (Greek letter rho), and the **curvature κ** (Greek letter kappa) is defined as the reciprocal of the radius of curvature. Therefore,



Bending Formula

Basic assumptions:

1. A plane section before bending remains plane after bending.
2. The material is linear elastic (i.e. follows Hooke's law).
3. The modulus of elasticity (E) is same in both tension and compression.

$$mp = \rho d\theta$$

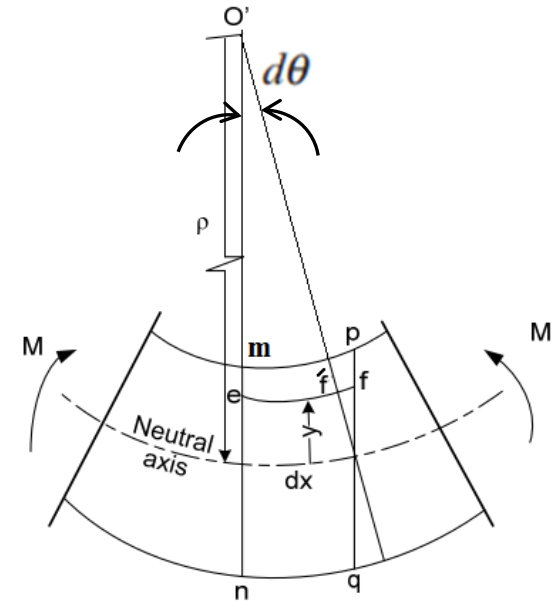
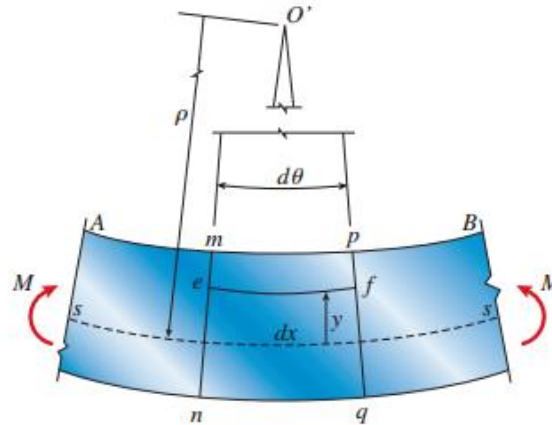
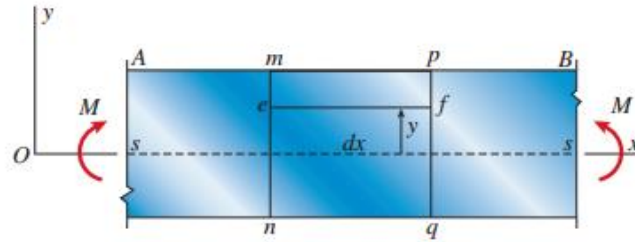
$$ff' = y d\theta$$

$$\epsilon = \frac{ff'}{mp} = \frac{y}{\rho}$$

Since the material is linear elastic:

$$\sigma = \frac{E y}{\rho} \dots \dots (1)$$

The normal bending stress is varying linearly from zero at NA to a maximum value at extreme fibers (mp and nq).



$$M = \int_{\text{area}} \sigma dA. y$$

$$M = \int_{\text{area}} \frac{E.y}{\rho} dA. y$$

$$M = \frac{E}{\rho} \int_{\text{area}} y^2 dA$$

$$M = \frac{E}{\rho} * I_{NA}$$

$$\rho = \frac{E I}{M} \dots \dots (2)$$

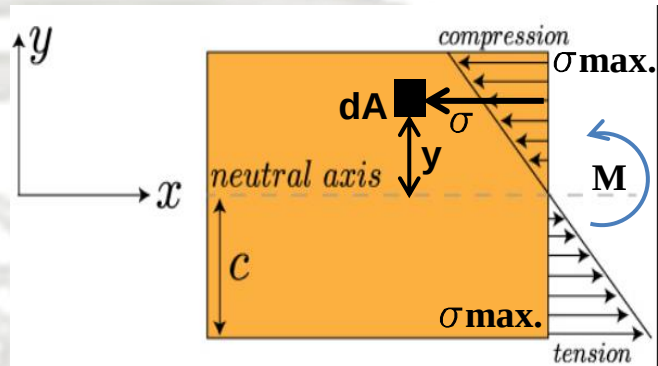
And from equation (1):

$$\rho = \frac{E y}{\sigma} \dots \dots (1)$$

$$\frac{E y}{\sigma} = \frac{E I}{M}$$

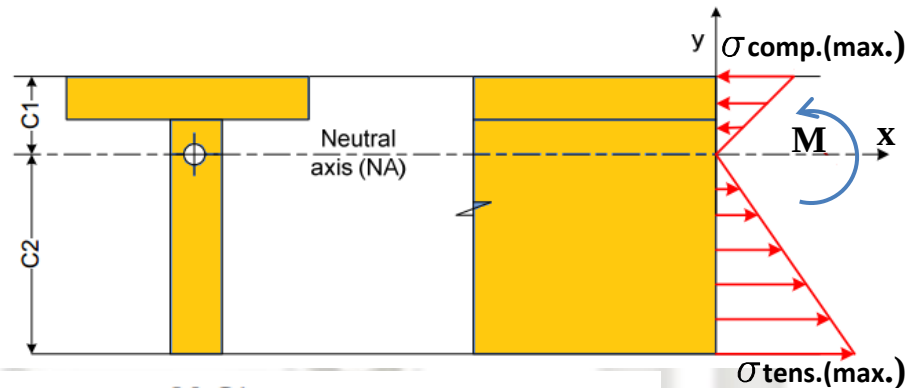
$$\sigma = \frac{M y}{I} \dots \dots \text{Bending formula}$$

For symmetrical sections about x-axis:



$$\sigma_{\max} = \frac{M C}{I} \dots \dots \text{Bending formula}$$

For unsymmetrical sections about x-axis:



$$\sigma_c)_{\max} = \frac{M C1}{I} \dots \dots \text{Bending formula}$$

$$\sigma_t)_{\max} = \frac{M C2}{I} \dots \dots \text{Bending formula}$$

Elastic Section Modulus

$$\sigma_{max} = \frac{M C}{I} = \frac{M}{S}$$

Where:

$$S = \frac{I}{C}$$

For a rectangular section:

$$I_{NA} = \frac{bh^3}{12}$$

$$S = \frac{I_{NA}}{C} = \frac{bh^2}{6}$$

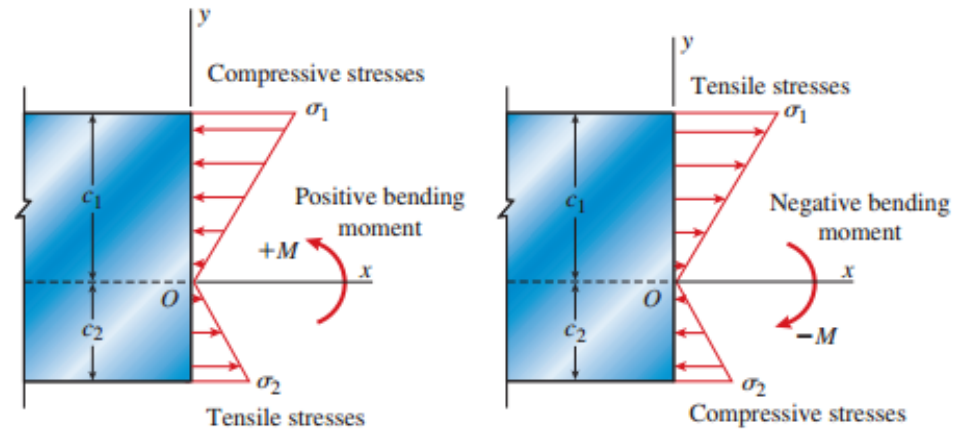
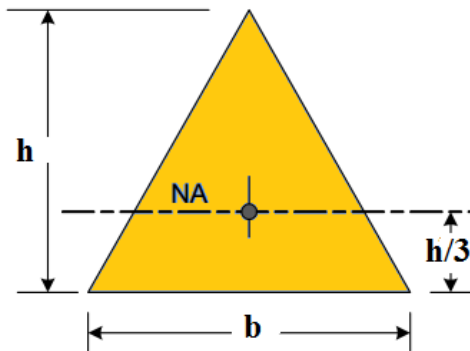
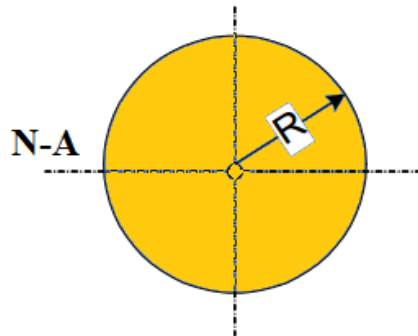
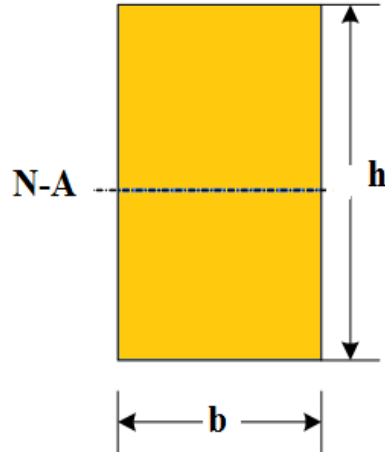
For a circular section:

$$I_{NA} = \frac{\pi}{4} R^4$$

$$S = \frac{I_{NA}}{C} = \frac{\pi}{4} R^3$$

Triangle

$$I_{NA} = \frac{bh^3}{36}$$

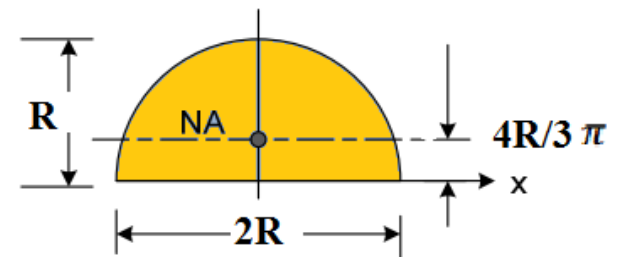


Semicircle

$$I_x = \frac{\pi}{8} R^4$$

$$I_x = I_{NA} + A \left(\frac{4R}{3\pi} \right)^2$$

$$I_{NA} = 0.11R^4$$



EXAMPLE 5-1

The cantilever beam, shown in the figure below, has 2.0m span and carries a uniformly distributed load of magnitude 2kN/m and a concentrated vertical load of 10kN. Determine the bending stress distribution at the critical section.

Solution: a critical section is the section where the maximum bending moment and consequently the maximum bending stress occurs. In order to find the location of N-A:

$$y_1 = \frac{\sum A y}{\sum A}$$

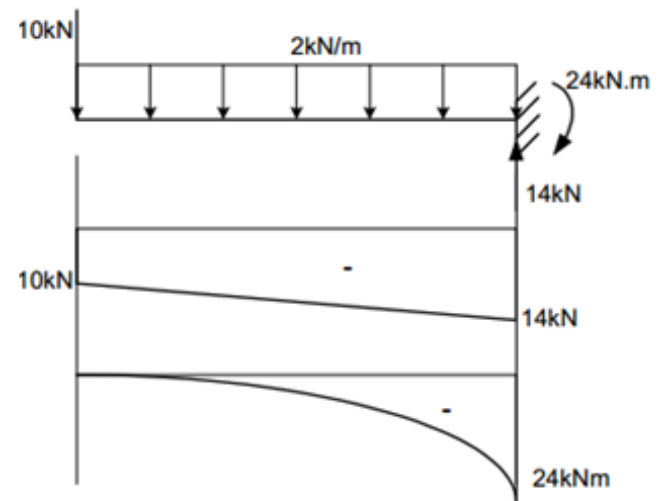
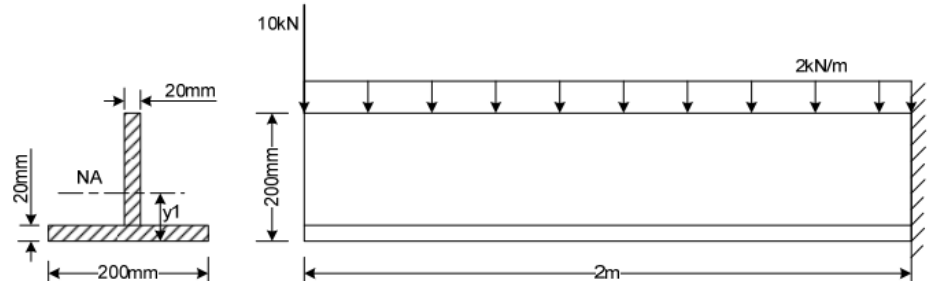
$$y_1 = \frac{200(20)(10) + 180(20)(110)}{200(20) + 180(20)} = 57.37\text{mm}$$

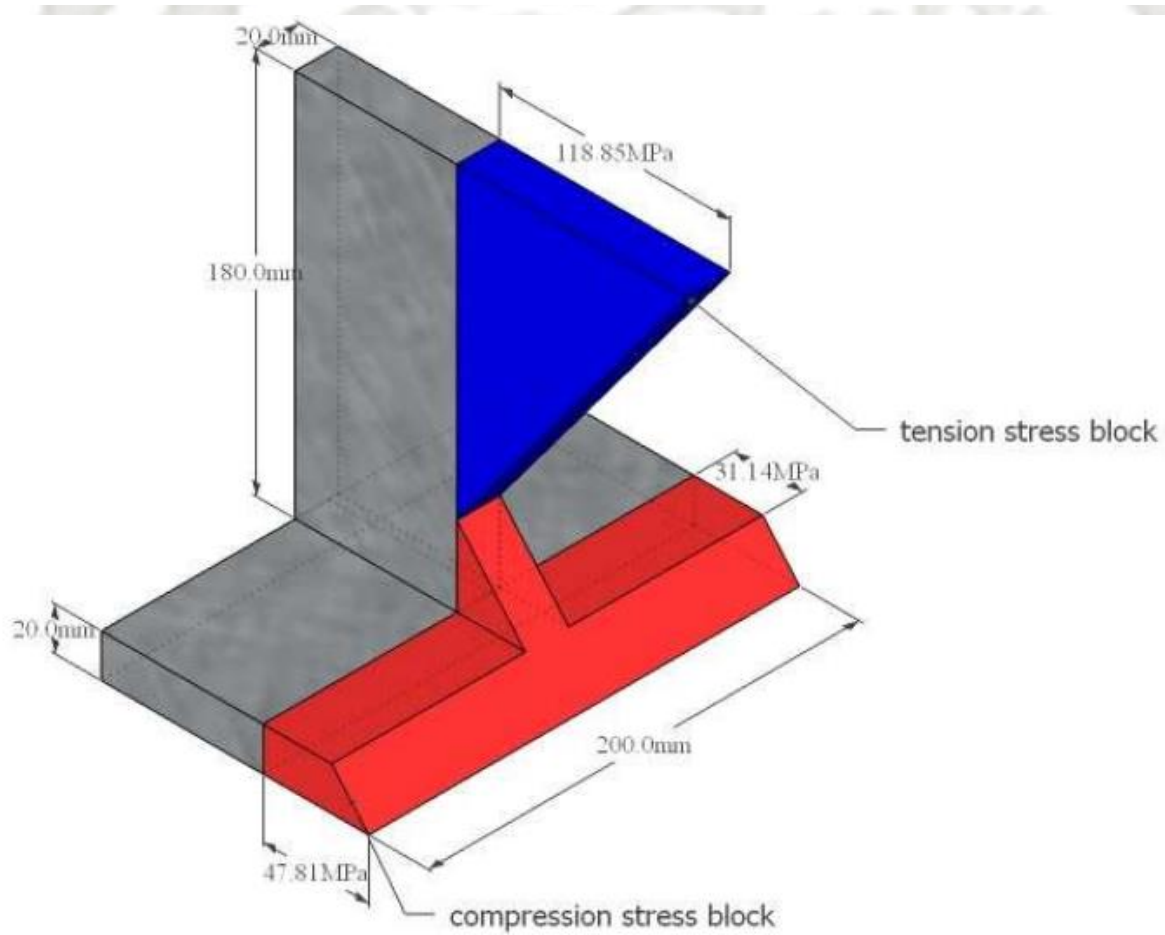
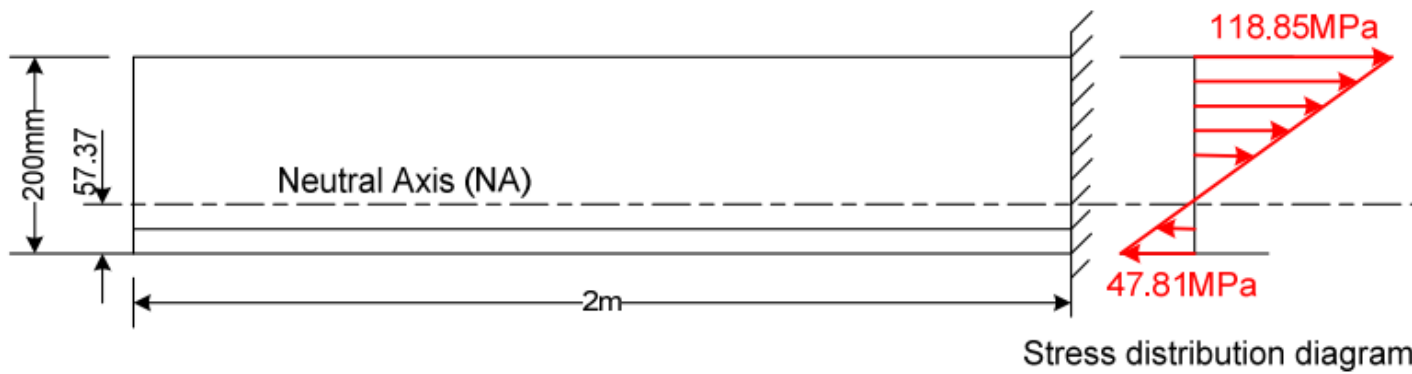
$$I_{NA} = \sum (I_o + Ad^2)$$

$$I_{NA} = \frac{200(20)^3}{12} + 200(20)(47.37)^2 + \frac{20(180)^3}{12} + 20(180)(110 - 57.37)^2 = 2.88 \times 10^7 \text{mm}^4$$

$$\sigma_t)_{max} = \frac{M C_1}{I} = \frac{24 \times 10^6 \text{ Nmm} \times (200 - 57.37)\text{mm}}{2.88 \times 10^7 \text{mm}^4} = 118.85 \text{MPa (Tension)}$$

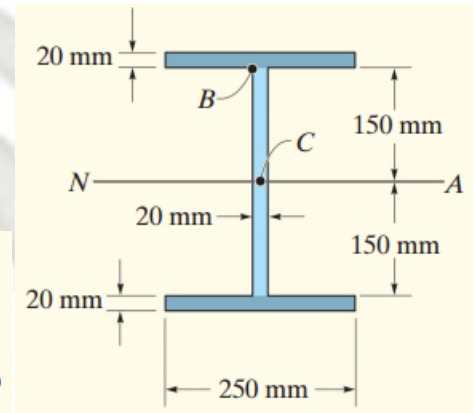
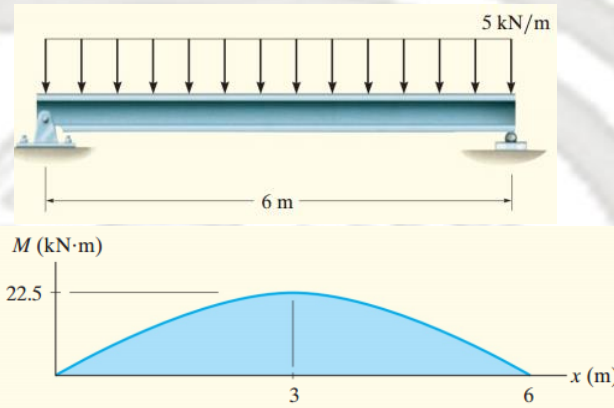
$$\sigma_c)_{max} = \frac{M C_2}{I} = \frac{24 \times 10^6 \text{ Nmm} \times (57.37)\text{mm}}{2.88 \times 10^7 \text{mm}^4} = 47.81 \text{MPa (Compression)}$$





EXAMPLE 5-2

The simply supported beam has the cross-sectional area as shown in the figure. Determine the absolute maximum bending stress in the beam and draw the stress distribution over the cross section at this location. Also, what is the stress at point B?

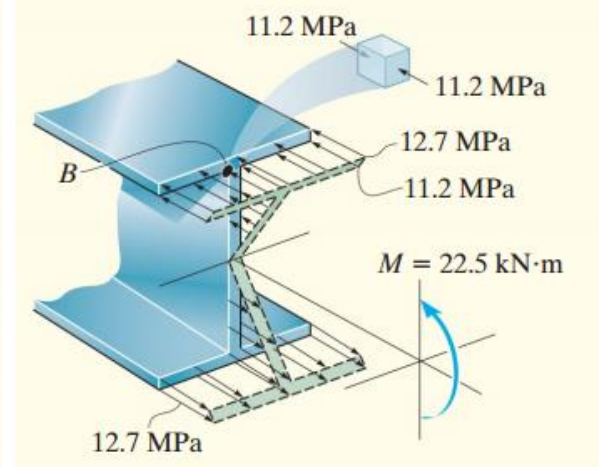


$$I = \Sigma(\bar{I} + Ad^2)$$

$$= 2 \left[\frac{1}{12} (0.25 \text{ m})(0.020 \text{ m})^3 + (0.25 \text{ m})(0.020 \text{ m})(0.160 \text{ m})^2 \right] + \left[\frac{1}{12} (0.020 \text{ m})(0.300 \text{ m})^3 \right]$$
$$= 301.3(10^{-6}) \text{ m}^4$$

$$\sigma_{\max} = \frac{Mc}{I}; \quad \sigma_{\max} = \frac{22.5(10^3) \text{ N} \cdot \text{m}(0.170 \text{ m})}{301.3(10^{-6}) \text{ m}^4(10^6)} = 12.7 \text{ MPa}$$

$$\sigma_B = -\frac{My_B}{I}; \quad \sigma_B = -\frac{22.5(10^3) \text{ N} \cdot \text{m}(0.150 \text{ m})}{301.3(10^{-6}) \text{ m}^4(10^6)} = -11.2 \text{ MPa}$$



EXAMPLE 5-3

Determine the maximum tensile and compressive bending stresses developed in the overhanging beam shown below.

Solution: from the bending moment diagram, there appear two critical sections because there are a maximum positive moment and a maximum negative moment. In order to find the location of NA, y_1 :

$$y_1 = \frac{\sum A y}{\sum A}$$

$$y_1 = \frac{400(600)(300) - [\pi(100)^2(300)] - 100(100)(550)}{400(600) - \pi(100)^2 - 100(100)} = 287.4\text{mm}$$

$$I_{NA} = \sum (I_o + Ad^2)$$

$$I_{NA} = \frac{400(600)^3}{12} + 400(600)(300 - 287.4)^2 - 2 \left[\frac{\pi(100)^4}{8} + \frac{\pi(100)^2}{2} (300 - 287.4)^2 \right] - \left[\frac{100(100)^3}{12} + 100(100)(550 - 287.4)^2 \right] = 6.46 \times 10^9 \text{mm}^4$$

Positive bending moment zone:

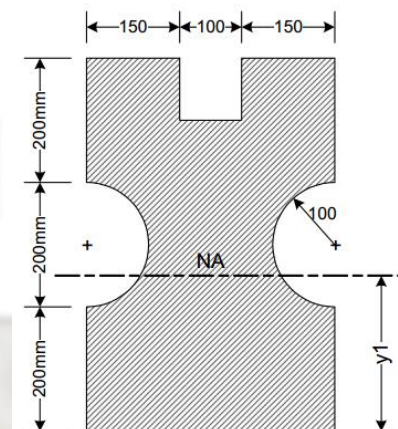
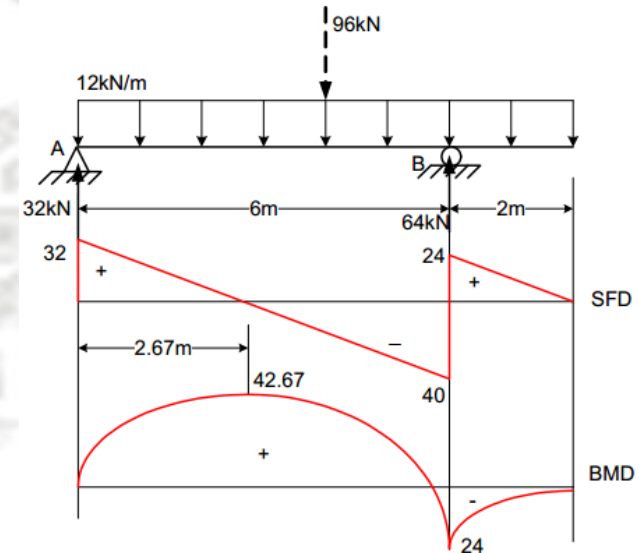
$$\sigma_c)_{max} = \frac{M C_1}{I} = \frac{42.67 \times 10^6 \text{ Nmm} \times (600 - 287.4)\text{mm}}{6.46 \times 10^9 \text{mm}^4} = 2\text{MPa}$$

$$\sigma_t)_{max} = \frac{M C_2}{I} = \frac{42.67 \times 10^6 \text{ Nmm} \times (287.4)\text{mm}}{6.46 \times 10^9 \text{mm}^4} = 1.9\text{MPa}$$

Negative bending moment zone:

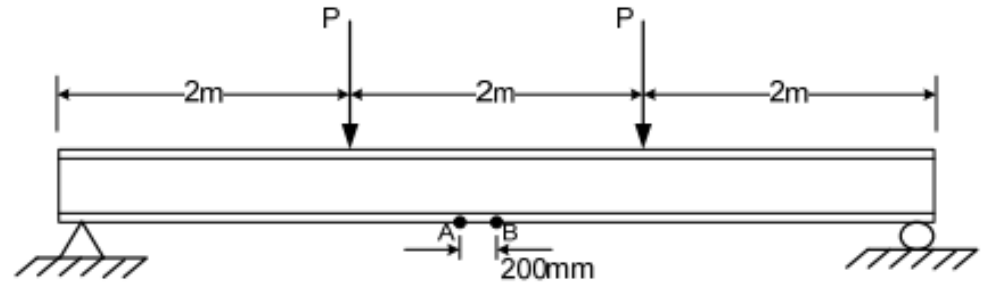
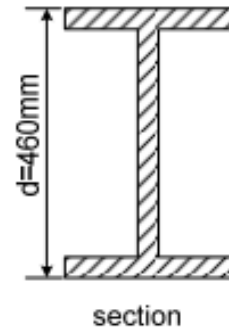
$$\sigma_c)_{max} = \frac{M C_2}{I} = \frac{24 \times 10^6 \text{ Nmm} \times (287.4)\text{mm}}{6.46 \times 10^9 \text{mm}^4} = 1.07\text{MPa}$$

$$\sigma_t)_{max} = \frac{M C_1}{I} = \frac{24 \times 10^6 \text{ Nmm} \times (600 - 287.4)\text{mm}}{6.46 \times 10^9 \text{mm}^4} = 1.16\text{MPa}$$



EXAMPLE 5-4

When two concentrated forces were applied at a W460 x 82 steel beam as shown in the figure, an elongation of 0.12mm was observed between the gauge points A and B. What was the magnitude of the applied forces? Take $E=200\text{GPa}$ and $I= 371 \times 10^6 \text{ mm}^4$.



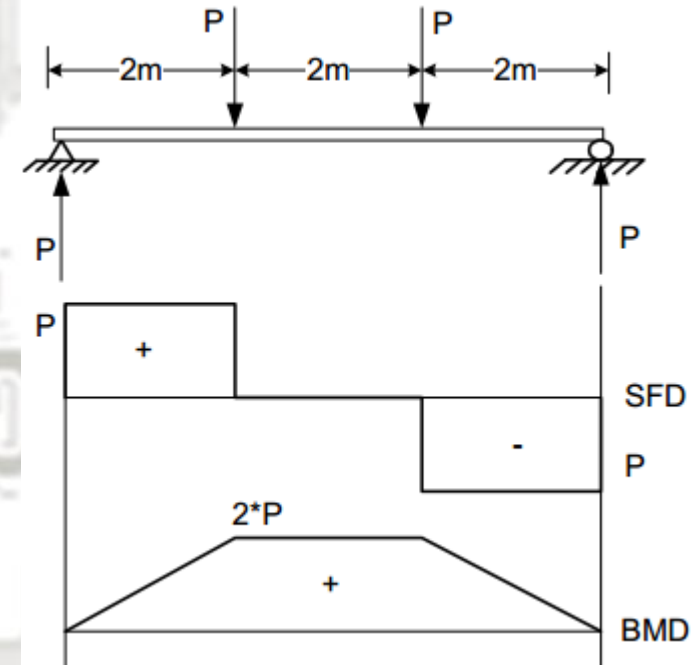
$$\varepsilon_{\text{bottom}} = \varepsilon_{AB} = \frac{\Delta_{AB}}{L_{AB}} = \frac{0.12\text{mm}}{200\text{mm}} = 0.0006$$

$$\sigma_{AB} = \varepsilon_{AB} * E = 0.0006 * 200 * 10^3 = 120\text{MPa}$$

Bending formula:

$$\sigma_{AB} = \frac{M * C}{I} \Rightarrow 120 = \frac{M * (230)\text{mm}}{371 * 10^6 \text{ mm}^4}$$

$$120 = \frac{(2000 * P) * (230)\text{mm}}{371 * 10^6 \text{ mm}^4} \Rightarrow P = 96783\text{N} = 96.8\text{kN}$$



EXAMPLE 5-5

A gray cast iron channel member as shown in the figure below acts as a horizontal beam. When vertical forces are applied to this beam the distance AB increases by 0.02mm and the distance CD decreases by 0.18mm. What the maximum normal stresses occur in the extreme fibers, and what is the sense of the moment? $E=100\text{GPa}$.

$$\varepsilon_{AB} = \frac{\Delta_{AB}}{L_{AB}} = \frac{0.02\text{mm}}{200\text{mm}} = +0.0001$$

$$\varepsilon_{CD} = \frac{\Delta_{CD}}{L_{CD}} = \frac{-0.18\text{mm}}{200\text{mm}} = -0.0009$$

The beam is subjected to a **negative** bending moment

$$\frac{\varepsilon_{AB} + \varepsilon_{CD}}{100} = \frac{\varepsilon_{CD}}{y}$$

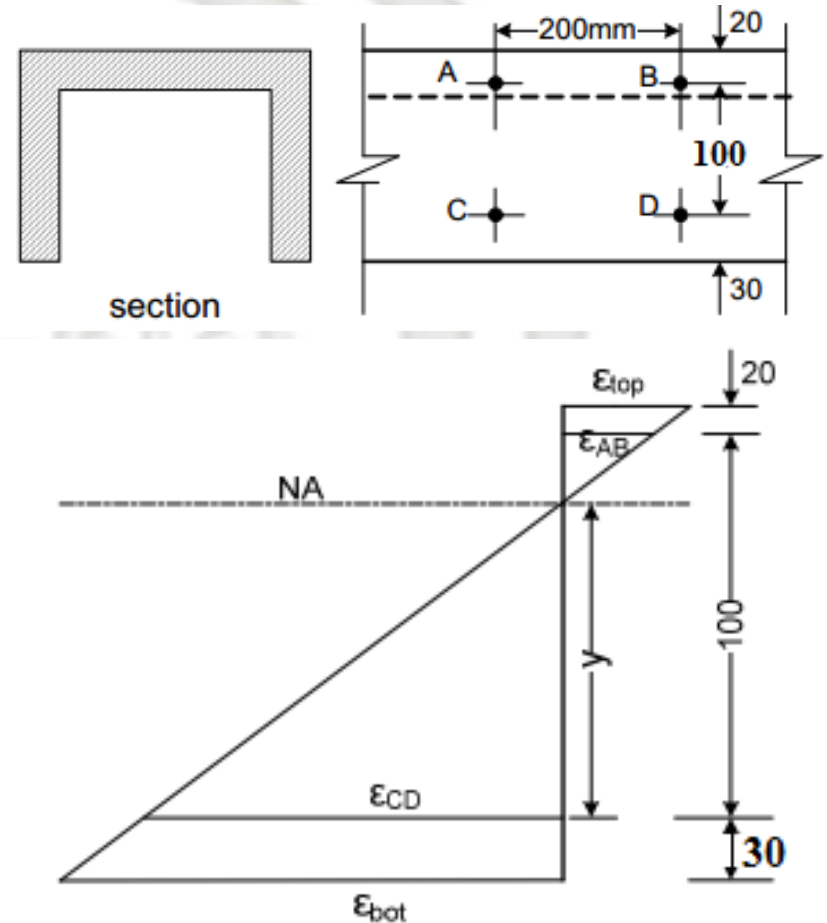
$$\frac{0.0001 + 0.0009}{100} = \frac{0.0009}{y} \Rightarrow y = 90\text{mm}$$

$$\frac{\varepsilon_{top}}{30} = \frac{\varepsilon_{AB}}{10} \Rightarrow \frac{\varepsilon_{top}}{30} = \frac{0.0001}{10} \Rightarrow \varepsilon_{top} = +0.0003 \text{ (tension)}$$

$$\sigma_{top} = \varepsilon_{top} * E = 0.0003 * 100000\text{MPa} = 30\text{MPa (tension)}$$

$$\frac{\varepsilon_{bot}}{120} = \frac{\varepsilon_{CD}}{90} \Rightarrow \frac{\varepsilon_{bot}}{120} = \frac{0.0009}{90} \Rightarrow \varepsilon_{bot} = -0.0012 \text{ (compression)}$$

$$\sigma_{bot} = \varepsilon_{bot} * E = -0.0012 * 100000\text{MPa} = 120\text{MPa (compression)}$$



EXAMPLE 5-6

If the overhanging beam is made of wood having the allowable tensile stress $(\sigma_{allow})_t = 4 \text{ MPa}$ and allowable compressive stress $(\sigma_{allow})_c = 5 \text{ MPa}$. Determine the maximum concentrated force **P** that can be applied at the free end.

$$\bar{y} = \frac{\sum \tilde{y}A}{\sum A} = \frac{2[0.1(0.2)(0.025)] + 0.2125(0.025)(0.15)}{2(0.2)(0.025) + 0.025(0.15)} = 0.13068 \text{ m}$$

$$I = \sum I_o + Ad^2$$
$$= 2\left[\frac{1}{12}(0.025)(0.2^3) + 0.025(0.2)(0.13068 - 0.1)^2\right]$$

$$+ \frac{1}{12}(0.15)(0.025^3) + 0.15(0.025)(0.2125 - 0.13068)^2$$
$$= 68.0457(10^{-6}) \text{ m}^4$$

Maximum Moment = P

For the top fiber

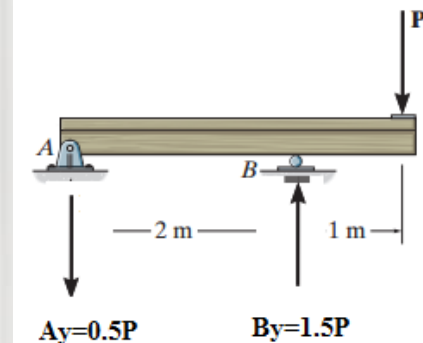
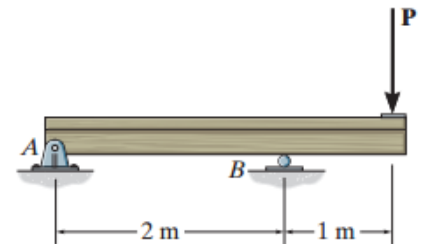
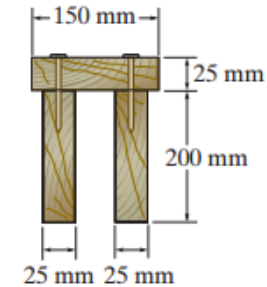
$$(\sigma_{allow})_t = \frac{M_{max}y}{I} \Rightarrow 4(10^6) = \frac{P(0.225 - 0.13068)}{68.0457(10^{-6})}$$

$$P = 2885.79 \text{ N} = 2.89 \text{ kN}$$

For the bottom fiber,

$$(\sigma_{allow})_c = \frac{M_{max}c}{I} \Rightarrow 5(10^6) = \frac{P(0.13068)}{68.0457(10^{-6})}$$

$$P = 2603.49 \text{ N} = 2.60 \text{ kN (controls)}$$



EXAMPLE 5-7

A cantilever beam with a rectangular cross section has a longitudinal hole drilled throughout its length as shown in the figure. Find the bending stresses at the top of the beam, at the top of the hole, and at the bottom of the beam.

$$M = PL = (600 \text{ N})(0.4 \text{ m}) = 240 \text{ N} \cdot \text{m}$$

A_1 = area of rectangle

$$= (25 \text{ mm})(50 \text{ mm}) = 1250 \text{ mm}^2$$

A_2 = area of hole

$$= \frac{\pi}{4} (10 \text{ mm})^2 = 78.54 \text{ mm}^2$$

A = area of cross section

$$= A_1 - A_2 = 1171.5 \text{ mm}^2$$

$$\sum A_i y_i = A_1(25 \text{ mm}) - A_2(37.5 \text{ mm}) = 28,305 \text{ mm}^3$$

$$\bar{y} = \frac{\sum A_i y_i}{A} = \frac{28,305 \text{ mm}^3}{1171.5 \text{ mm}^2} = 24.162 \text{ mm}$$

Distances to the centroid C :

$$c_2 = \bar{y} = 24.162 \text{ mm}$$

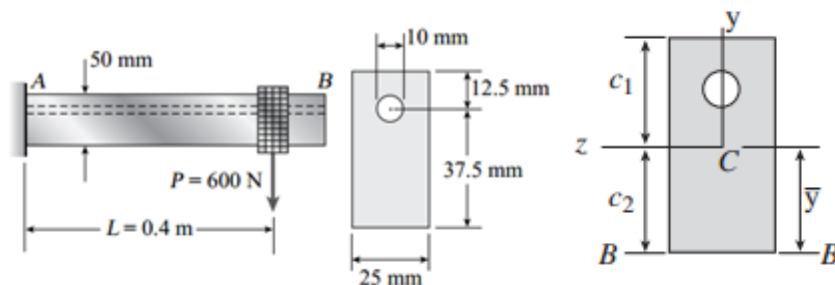
$$c_1 = 50 \text{ mm} - c_2 = 25.838 \text{ mm}$$

Rectangle:

$$I = I_o + Ad^2$$

$$= \frac{1}{12} (25)(50)^3 + (25)(50)(25 - 24.162)^2$$

$$= 260,420 + 878 = 261,300 \text{ mm}^4$$



Hole:

$$I_z = I_c + Ad^2 = \frac{\pi}{64} (10)^4 + (78.54)(37.5 - 24.162)^2$$
$$= 490.87 + 13,972 = 14,460 \text{ mm}^4$$

Cross-section:

$$I = 261,300 - 14,460 = 246,800 \text{ mm}^4$$

STRESS AT THE TOP OF THE BEAM

$$\sigma_1 = \frac{Mc_1}{I} = \frac{(240 \text{ N} \cdot \text{m})(25.838 \text{ mm})}{246,800 \text{ mm}^4} = 25.1 \text{ MPa}$$

STRESS AT THE TOP OF THE HOLE

$$\sigma_2 = \frac{My}{I} \quad y = c_1 - 7.5 \text{ mm} = 18.338 \text{ mm}$$

$$\sigma_2 = \frac{(240 \text{ N} \cdot \text{m})(18.338 \text{ mm})}{246,800 \text{ mm}^4} = 17.8 \text{ MPa}$$

STRESS AT THE BOTTOM OF THE BEAM

$$\sigma_3 = -\frac{Mc_2}{I} = -\frac{(240 \text{ N} \cdot \text{m})(24.162 \text{ mm})}{246,800 \text{ mm}^4} = -23.5 \text{ MPa}$$

EXAMPLE 5-8

A steel beam ABC is simply supported at A and B and has an overhang BC of length $L=150$ mm as shown in the figure. The beam supports a uniform load of intensity $q=4.0$ kN/m over its entire span AB and $1.5q$ over BC. The allowable bending stress in the steel is $\sigma_{\text{allow}}=60$ MPa, and its weight density is $\gamma=77.0$ kN/m³. Calculate the required width b .

$$L = 150 \text{ mm} \quad q = 4 \frac{\text{kN}}{\text{m}}$$

$$\sigma_a = 60 \text{ MPa} \quad \gamma = 77 \frac{\text{kN}}{\text{m}^3}$$

$$w = \gamma A \quad w = \gamma (2b^2)$$

$$M_{\text{max}} = (1.5q + w) \frac{L^2}{2}$$

$$M_{\text{max}} = \sigma_a S \quad S = \frac{2}{3} b^3$$

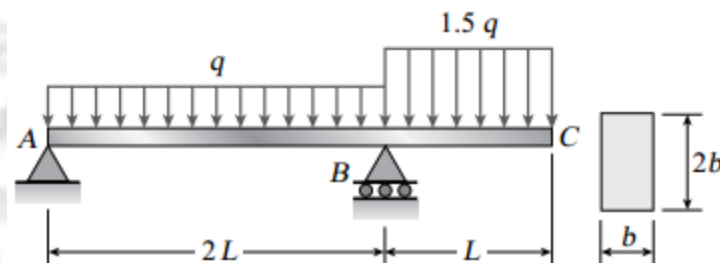
$$\text{and} \quad M_{\text{max}} = \sigma_a \left(\frac{2}{3} b^3 \right)$$

Equate $M_{\text{max}1}$ to $M_{\text{max}2}$ & solve for b_{min}

$$\left(\frac{2}{3} \sigma_a \right) b^3 - (\gamma L^2) b^2 - \frac{3}{4} q L^2 = 0$$

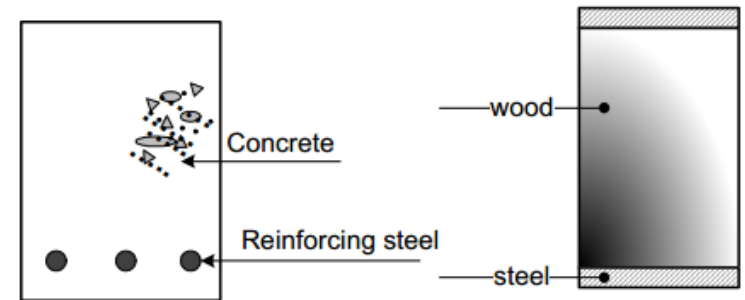
Insert numerical values, then solve for b

$$b_{\text{min}} = 11.92 \text{ mm}$$



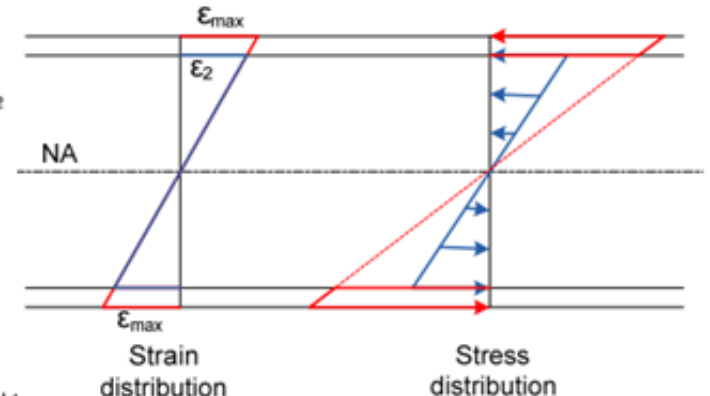
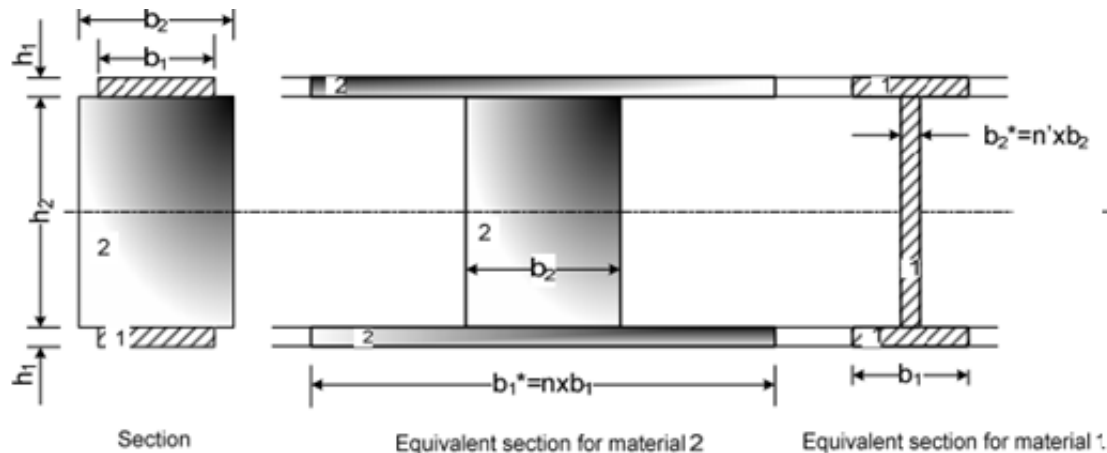
Beams of Two Materials (Composite Beams):

Beams constructed of two or more different materials are referred to as composite beams. Since the flexure formula was developed only for beams made of homogeneous material, this formula cannot be applied to directly determine the normal stress in a composite beam. It must be develop a method for modifying or “transforming” a composite beam’s cross section into one made of a single material.



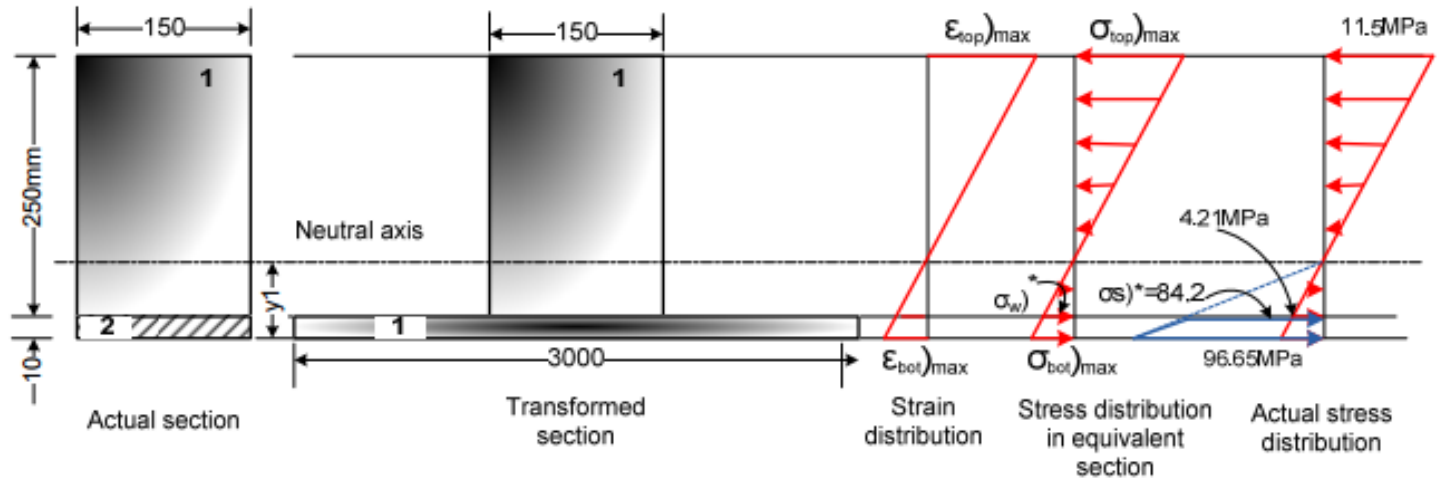
Stress Distribution in Composite Sections (Method of Transformed section):

$$\text{Modular ratio } n = \frac{E_1}{E_2} \quad \text{or} \quad n' = \frac{E_2}{E_1}$$



EXAMPLE 5-9

A composite beam is constructed from a wood beam (150mm x 250mm) and a steel reinforcing plate, as shown in the figure. If the beam is subjected to a positive bending moment of 30kN.m around its horizontal axis, determine the maximum stresses developed in steel and wood. Take $E_s=200\text{GPa}$ and $E_w=10\text{GPa}$



$$\text{Modular ratio } n = \frac{E_s}{E_w} = \frac{200}{10} = 20$$

$$b_2^* = n \cdot b_2 = 20(150) = 3000\text{mm}$$

$$y_1 = \frac{3000(10)(5) + 150(250)(135)}{3000(10) + 150(250)} = 77\text{mm}$$

$$I_{NA} = \frac{3000(10)^3}{12} + 3000(10)(72)^2 + \frac{150(250)^3}{12} + 150(250)(135 - 77)^2 = 4.78 \times 10^8 \text{mm}^4$$

The maximum stresses in the transformed wood section:

$$\sigma_w)_{max} = \frac{M C_1}{I_{NA}} = \frac{30 \times 10^6 \text{ N.m} \cdot (183)\text{mm}}{4.78 \times 10^8 \text{mm}^4} = 11.5\text{MPa (compression)}$$

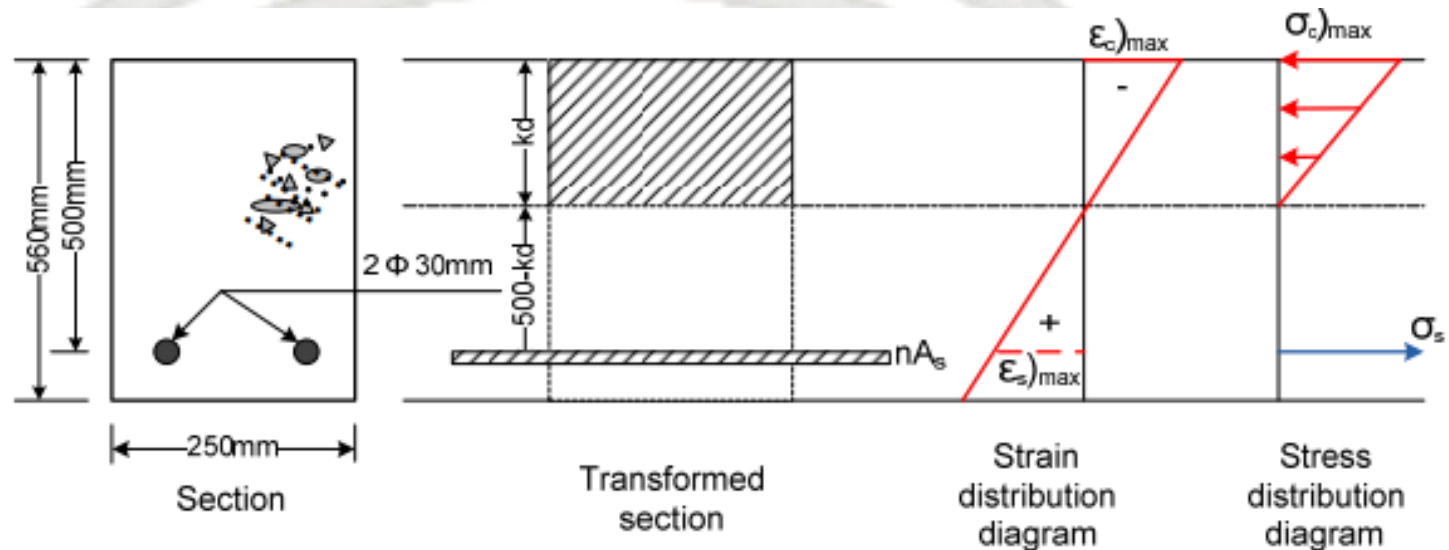
$$\sigma_{bott})_{max} = \sigma_w)_{bott} = \frac{M C_2}{I} = \frac{30 \times 10^6 \text{ N.m} \cdot (77)\text{mm}}{4.78 \times 10^8 \text{mm}^4} = 4.83\text{MPa (tension)}$$

$$\sigma_w^* = \frac{M y}{I} = \frac{30 \times 10^6 \text{ N.m} \cdot (67)\text{mm}}{4.78 \times 10^8 \text{mm}^4} = 4.21\text{MPa (tension)}$$

Note: At the contact plane the ratio of the stress in the steel to the stress in the wood is ($n = \frac{E_s}{E_w} = \frac{200}{10} = 20$). Although the strains in the steel and wood are equal at the contact plane, the stresses are different because of the different moduli (E_s and E_w).

EXAMPLE 5-10

Determine the maximum stress in concrete and steel for the reinforced concrete beam of the section shown below. The section is subjected to a positive bending moment of 70kN.m. Assume the modular ratio $n=15$. Compare the maximum stresses with the allowable stresses in both concrete and steel.



Solution: As the concrete under tension stresses is very weak, it will crack below the neutral surface (for the positive moment) and the steel rods will carry the entire tensile load, while the upper part of the concrete beam will carry the compressive load. That is to say, we consider the concrete area in the tension zone is negligible.

$$A_s = 2 * \frac{\pi}{4} (30)^2 = 1413.7 \text{mm}^2$$

$$n.A_s = 15 * 1413.7 = 21206 \text{mm}^2$$

To find the location of the NA:

The first moment of the compression area around the NA = the first moment of the tension area around the NA.

$$250 * kd * kd/2 = 21206 * (500 - kd)$$

$$kd^2 + 169.65kd - 84823 = 0$$

$$kd = 218.5 \text{ mm}$$

$$I_{NA} = \frac{250(218.5)^3}{3} + 21206(500 - 218.5)^2 = 2.55 * 10^9 \text{ mm}^4$$

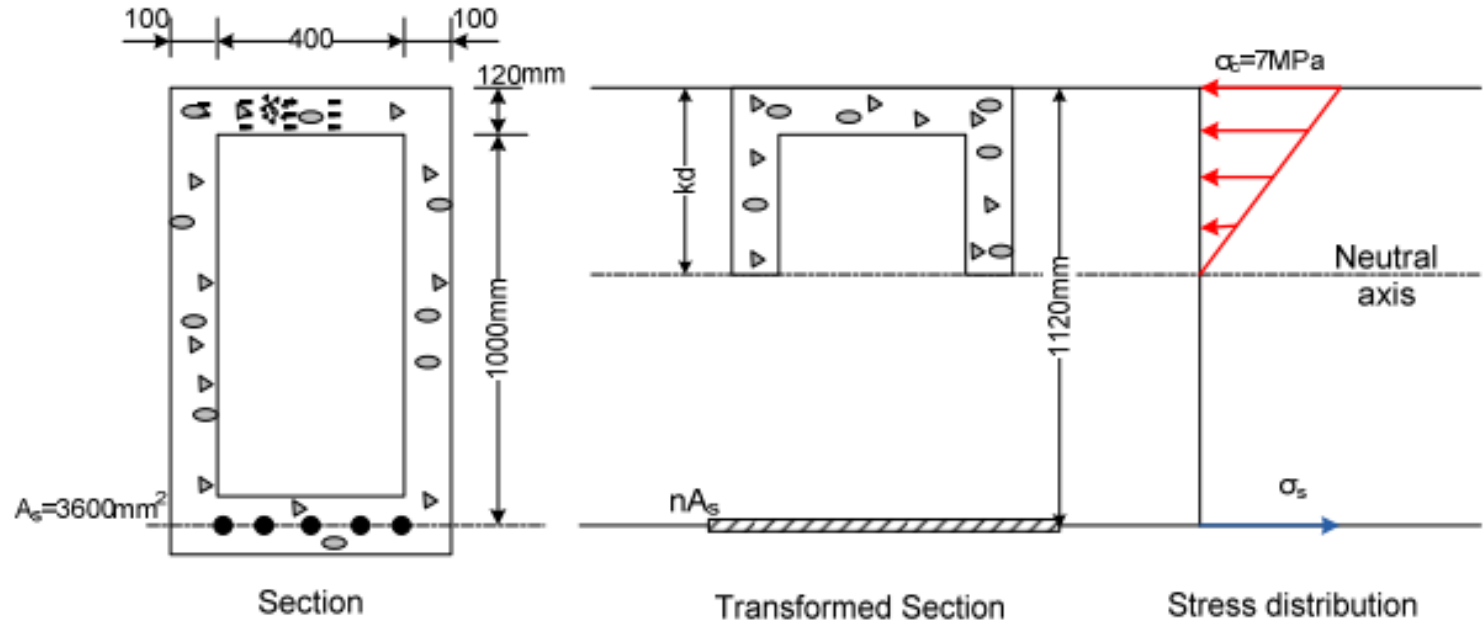
$$\begin{aligned} \sigma_c)_{max} &= \frac{M C1}{I_{NA}} = \frac{70 * 10^6 \text{ N.mm} * (218.5) \text{ mm}}{2.55 * 10^9 \text{ mm}^4} = 6.0 \text{ MPa (compression)} < 0.45 f'_c \\ &= 0.45(21) = 9.45 \text{ MPa} \end{aligned}$$

$$\sigma_s = n * \frac{M C2}{I_{NA}} = 15 * \frac{70 * 10^6 \text{ N.mm} * (281.5) \text{ mm}}{2.55 * 10^9 \text{ mm}^4} = 115.9 \text{ MPa} < f_s)_{allow} = 140 \text{ MPa}$$



EXAMPLE 5-11

A hollow rectangular reinforced concrete section is cast as shown in the figure below. The modular ratio $n=10$, and the maximum compressive stress in the concrete caused by positive bending moment is 7MPa. Determine the stress in the steel and the bending moment applied to the section.



Solution: We need first to transform the actual composite section to a concrete transformed section.

$$nA_s = 10 * 3600 = 36000 \text{ mm}^2$$

Now, we need to find the location of the neutral axis for the transformed section:

$$kd = \frac{2 * 100 * kd * \frac{kd}{2} + 400 * 120 * 60 + 36000 * 1120}{2 * 100 * kd + 400 * 120 + 36000}$$

$$kd = \frac{100kd^2 + 43200000}{200kd + 48000 + 36000}$$

$$kd^2 + 840kd - 432000 = 0$$

$$kd = \frac{-840 + \sqrt{840^2 + 4(432000)}}{2} = 360mm$$

$$I_{NA} = \frac{100 * (360)^3}{3} * 2 + \frac{400 * (120)^3}{12} + 400(120)(300)^2 + 36000(760)^2$$

$$I_{NA} = 3.1 * 10^9 + 5.76 * 10^7 + 4.32 * 10^9 + 2.08 * 10^{10} = 2.83 * 10^{10} mm^4$$

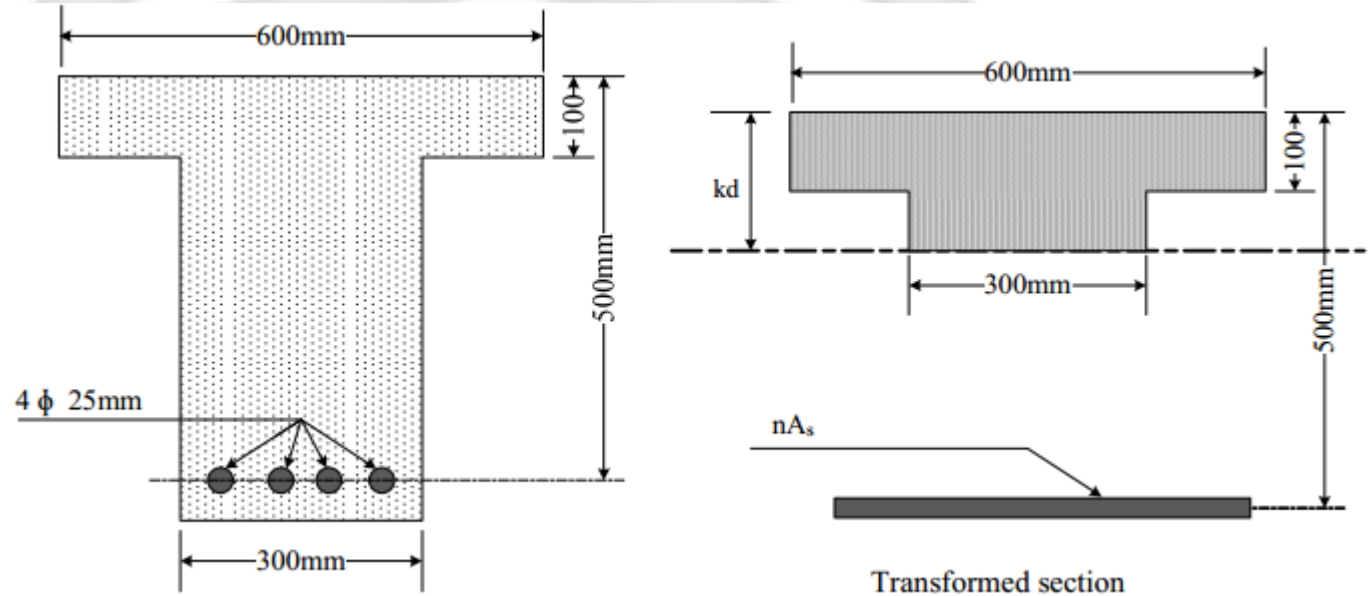
$$\sigma_{c)_{max}} = \frac{M * kd}{I_{NA}} \Rightarrow 7 \frac{N}{mm^2} = \frac{M * 360mm}{2.83 * 10^{10} mm^4} \Rightarrow M = 550 kN.m$$

$$\sigma_s = n * \frac{M * (1120 - kd)}{I_{NA}} = 10 * \frac{550 * 10^6 N.mm * (1120 - 360)mm}{2.83 * 10^{10} mm^4} = 147.7 MPa$$



EXAMPLE 5-12

The reinforced concrete section of the T-shape shown in figure is subjected to a positive bending moment of 150kN.m. If the modular ratio $n=10$, determine the maximum compressive stress in the concrete and the stress in the steel.



$$A_s = \text{No. of bars} * \frac{\pi}{4} d^2 = 4 * \frac{\pi}{4} (25)^2 = 1963.5 \text{ mm}^2$$

$$\text{Area of transformed steel to concrete} = n * A_s = 10 * 1963.5 = 19635 \text{ mm}^2$$

After making the transformed section (as shown in the figure above), obtain the position of N.A.
First moment of compression area about N.A = First moment of tension area about N.A.

$$600 * 100 * (kd - 50) + 300(kd - 100) * \frac{(kd - 100)}{2} = 19635(500 - kd)$$

$$150kd^2 + 300(kd - 100) * \frac{(kd - 100)}{2} = 19635(500 - kd)$$

$$150kd^2 + 49635kd - 11317500 = 0$$

$$kd^2 + 330.9kd - 75450 = 0$$

$$kd = \frac{-330.9 + \sqrt{330.9^2 + 4(75450)}}{2} = 155mm$$

$$I_{NA} = \frac{600 * (100)^3}{12} + 600 * 100 * (155 - 50)^2 + \frac{300 * (55)^3}{3} + 19635(500 - 155)^2$$

$$I_{NA} = 5 * 10^7 + 66 * 10^7 + 1.7 * 10^7 + 233 * 10^7 = 306 * 10^7 mm^4$$

$$\sigma_{c)max} = \frac{M * kd}{I_{NA}} = \frac{150 * 10^6 Nmm * 155mm}{306 * 10^7 mm^4} = 7.6MPa$$

$$\sigma_s = n * \frac{M * (d - kd)}{I_{NA}} = 10 * \frac{150 * 10^6 Nmm * (500 - 155)mm}{306 * 10^7 mm^4} = 169MPa$$

EXAMPLE 5-13

The composite beam shown in the figure is made of wood and reinforced with a steel plate located on its bottom side. If the beam is subjected to a bending moment of $M = 2 \text{ kN}\cdot\text{m}$, determine the normal stress at points B and C. Take $E_w = 12 \text{ GPa}$ and $E_{st} = 200 \text{ GPa}$.

Solution: Although the choice is arbitrary, here we will transform the section into one made entirely of steel. Since steel has a greater stiffness than wood ($E_{st} > E_w$), the width of the wood is reduced to an equivalent width for steel. For this to be the case, $n = E_w / E_{st}$, so that:

$$b_{st} = nb_w = \frac{12 \text{ GPa}}{200 \text{ GPa}}(150 \text{ mm}) = 9 \text{ mm}$$

$$\bar{y} = \frac{\Sigma \bar{y}A}{\Sigma A} = \frac{[0.01 \text{ m}](0.02 \text{ m})(0.150 \text{ m}) + [0.095 \text{ m}](0.009 \text{ m})(0.150 \text{ m})}{0.02 \text{ m}(0.150 \text{ m}) + 0.009 \text{ m}(0.150 \text{ m})} = 0.03638 \text{ m}$$

The moment of inertia about the neutral axis is therefore

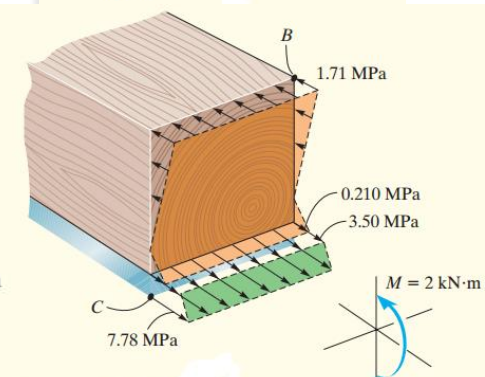
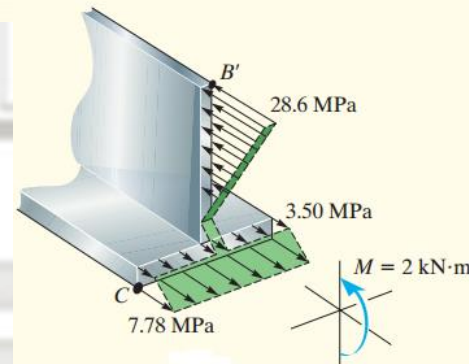
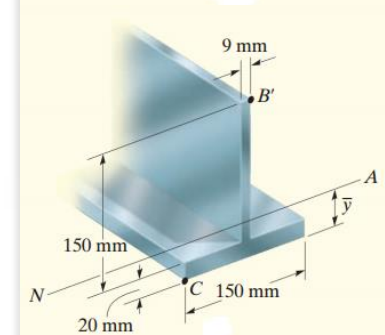
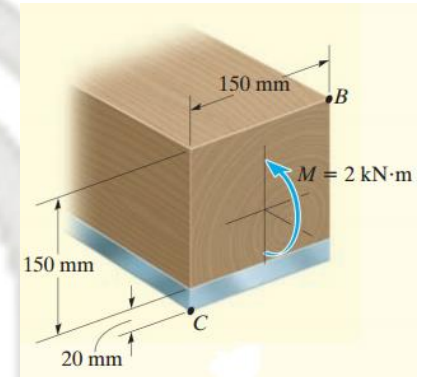
$$I_{NA} = \left[\frac{1}{12} (0.150 \text{ m})(0.02 \text{ m})^3 + (0.150 \text{ m})(0.02 \text{ m})(0.03638 \text{ m} - 0.01 \text{ m})^2 \right] + \left[\frac{1}{12} (0.009 \text{ m})(0.150 \text{ m})^3 + (0.009 \text{ m})(0.150 \text{ m})(0.095 \text{ m} - 0.03638 \text{ m})^2 \right]$$

$$= 9.358(10^{-6}) \text{ m}^4$$

$$\sigma_{B'} = \frac{2(10^3) \text{ N}\cdot\text{m}(0.170 \text{ m} - 0.03638 \text{ m})}{9.358(10^{-6}) \text{ m}^4 10^6} = 28.6 \text{ MPa}$$

$$\sigma_C = \frac{2(10^3) \text{ N}\cdot\text{m}(0.03638 \text{ m})}{9.358(10^{-6}) \text{ m}^4 10^6} = 7.78 \text{ MPa}$$

$$\sigma_B = n\sigma_{B'} = \frac{12 \text{ GPa}}{200 \text{ GPa}}(28.56 \text{ MPa}) = 1.71 \text{ MPa}$$



EXAMPLE 5-13

Find the maximum stress in the points 1,2,3,4,5, and 6 if the section subjected to positive bending moment of 20 kN.m. Take $E_{al} = 70 \text{ GPa}$, $E_{st} = 210 \text{ GPa}$ and $E_{co} = 140 \text{ GPa}$.

$$n_1 = \frac{E_{st}}{E_{al}} = \frac{210}{70} = 3$$

$$n_2 = \frac{E_{co}}{E_{al}} = \frac{140}{70} = 2$$

$$b_{st} = 600 \times 3 = 1800 \text{ mm}$$

$$b_{co} = 600 \times 2 = 1200 \text{ mm}$$

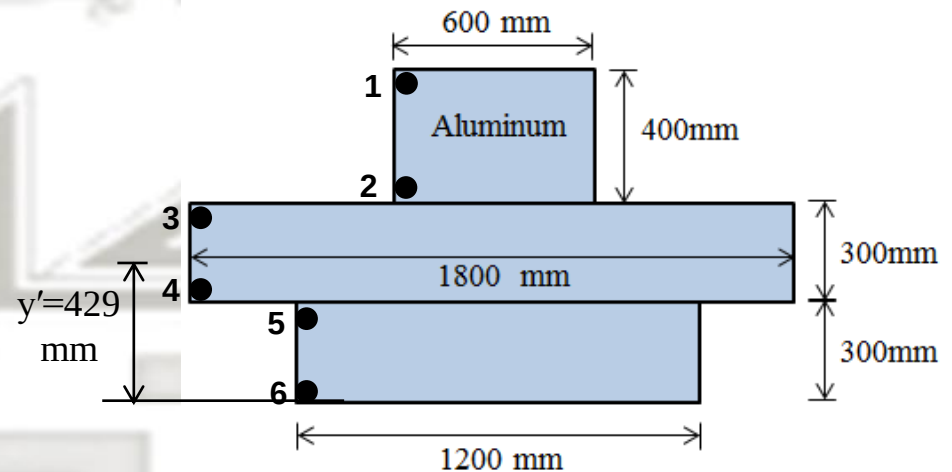
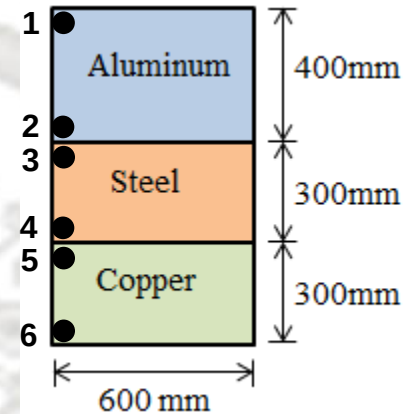
$$y' = \frac{600 \times 400 \times 800 + 1800 \times 300 \times 450 + 1200 \times 300 \times 150}{600 \times 400 + 1800 \times 300 + 1200 \times 300} = 429 \text{ mm}$$

$$I = \frac{600(400)^3}{12} + 600(400)(800 - 429)^2 +$$

$$\frac{1800(300)^3}{12} + 1800(300)(450 - 429)^2 +$$

$$+ \frac{1200(300)^3}{12} + 1200(300)(429 - 150)^2$$

$$I = 85.3 \times 10^9 \text{ mm}^4$$



$$\sigma_1 = \frac{20 \times 10^6 \times (1000 - 429)}{85.3 \times 10^9} = 0.134 \text{ MPa}$$

$$\sigma_2 = \frac{20 \times 10^6 \times (600 - 429)}{85.3 \times 10^9} = 0.04 \text{ MPa}$$

$$\sigma_3 = n_1 \times \frac{20 \times 10^6 \times (600 - 429)}{85.3 \times 10^9} = 0.12 \text{ MPa}$$

$$\sigma_4 = n_1 \times \frac{20 \times 10^6 \times (429 - 300)}{85.3 \times 10^9} = 0.0009 \text{ MPa}$$

$$\sigma_5 = n_2 \times \frac{20 \times 10^6 \times (429 - 300)}{85.3 \times 10^9} = 0.06 \text{ MPa}$$

$$\sigma_6 = n_2 \times \frac{20 \times 10^6 \times (429)}{85.3 \times 10^9} = 0.201 \text{ MPa}$$