Maxwell's Equations
(Instantaneous and Phasor Forms)

Maxwell's Equations (instantaneous form)

$$\nabla \times \mathscr{E} = -\frac{\partial \mathscr{B}}{\partial t}$$

$$\nabla \times \mathscr{H} = \frac{\partial \mathscr{D}}{\partial t} + \mathscr{J}$$

$$\nabla \cdot \mathscr{D} = \rho_t$$

$$\nabla \cdot \mathscr{B} = 0$$

$\mathscr{E}, \mathscr{H}, \mathscr{D}, \mathscr{B}, \mathscr{J}$ - instantaneous vectors [$\mathscr{E} = \mathscr{E}(x,y,z,t)$, etc.]
ρ_t - instantaneous scalar

Maxwell's Equations (phasor form, time-harmonic form)

$$\nabla \times \boldsymbol{E} = -j\omega \boldsymbol{B}$$

$$\nabla \times \boldsymbol{H} = j\omega \boldsymbol{D} + \boldsymbol{J}$$

$$\nabla \cdot \boldsymbol{D} = \rho$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{D}, \boldsymbol{B}, \boldsymbol{J}$ - phasor vectors [$\boldsymbol{E} = \boldsymbol{E}(x,y,z)$, etc.]
ρ - phasor scalar

Relation of instantaneous quantities to phasor quantities ...

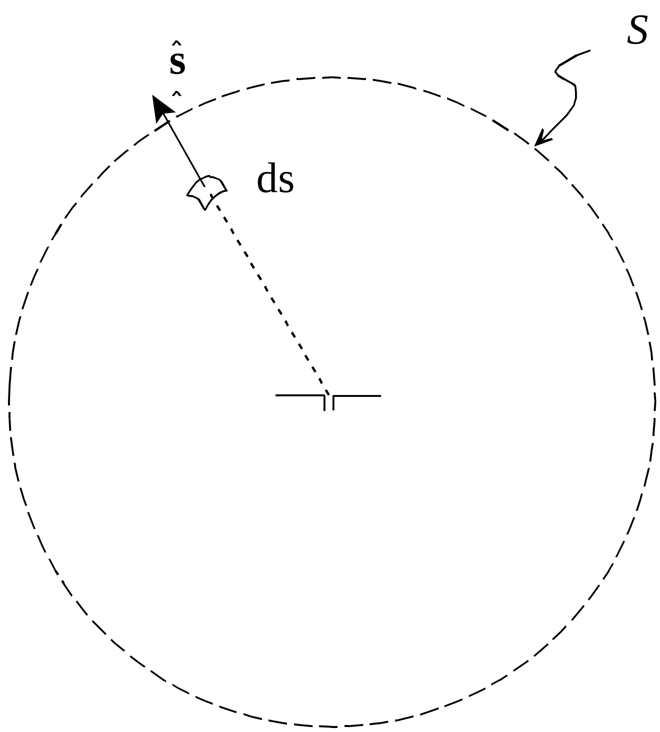
$\mathscr{E}(x,y,z,t) = \mathrm{Re}\{\boldsymbol{E}(x,y,z)e^{j\omega t}\}$, etc.

Average Power Radiated by an Antenna

To determine the average power radiated by an antenna, we start with the instantaneous Poynting vector $\mathscr{S}$ (vector power density) defined by

$$\mathscr{S} = \mathscr{E} \times \mathscr{H} \qquad (\text{V/m} \times \text{A/m} = \text{W/m}^2)$$

Assume the antenna is enclosed by some surface *S*.



The total instantaneous radiated power $\mathscr{P}_{rad}$ leaving the surface *S* is found by integrating the instantaneous Poynting vector over the surface.

$$\mathscr{P}_{rad} = \oint_S \mathscr{S} \cdot d\mathbf{s} = \oint_S (\mathscr{E} \times \mathscr{H}) \cdot d\mathbf{s} \qquad d\mathbf{s} = \hat{\mathbf{s}}\, ds$$

ds = differential surface
$\hat{\mathbf{s}}$ = unit vector normal to ds

For time-harmonic fields, the time average instantaneous Poynting vector (time average vector power density) is found by integrating the instantaneous Poynting vector over one period (T) and dividing by the period.

$$\boldsymbol{P}_{avg} = \frac{1}{T}\oint_T (\mathscr{E} \times \mathscr{H})\, dt$$

$$\mathscr{E} = \mathrm{Re}\{\boldsymbol{E}e^{j\omega t}\}$$

$$\mathscr{H} = \mathrm{Re}\{\boldsymbol{H}e^{j\omega t}\}$$

The instantaneous magnetic field may be rewritten as

$$\mathscr{H} = \mathrm{Re}\{\tfrac{1}{2}\,[\,\boldsymbol{H}e^{j\omega t} + \boldsymbol{H}^{*}e^{-j\omega t}\,]\}$$

which gives an instantaneous Poynting vector of

$$\mathscr{E} \times \mathscr{H} = \tfrac{1}{2}\,\mathrm{Re}\,\{\underbrace{[\boldsymbol{E} \times \boldsymbol{H}]e^{j2\omega t}}_{\substack{\text{time-harmonic} \\ \text{(integrates to zero over } T\text{)}}} + \underbrace{[\boldsymbol{E} \times \boldsymbol{H}^{*}]}_{\text{independent of time}}\}$$

and the time-average vector power density becomes

$$\boldsymbol{P}_{avg} = \frac{1}{2T}\mathrm{Re}\,[\boldsymbol{E} \times \boldsymbol{H}^{*}]\oint_T dt$$

$$= \tfrac{1}{2}\,\mathrm{Re}\,[\boldsymbol{E} \times \boldsymbol{H}^{*}]$$

The total time-average power radiated by the antenna (P_{rad}) is found by integrating the time-average power density over S.

$$P_{rad} = \oint_S \boldsymbol{P}_{avg}\cdot d\mathbf{s} = \tfrac{1}{2}\,\mathrm{Re}\oint_S [\boldsymbol{E} \times \boldsymbol{H}^{*}]\cdot d\mathbf{s}$$

Radiation Intensity

Radiation Intensity - radiated power per solid angle (radiated power normalized to a unit sphere).

$$P_{rad} = \oint_S \boldsymbol{P}_{avg} \cdot d\mathbf{s}$$

In the far field, the radiation electric and magnetic fields vary as $1/r$ and the direction of the vector power density ($\boldsymbol{P}_{avg}$) is radially outward. If we assume that the surface S is a sphere of radius r, then the integral for the total time-average radiated power becomes

$$\boldsymbol{P}_{avg} = P_{avg}\,\hat{\boldsymbol{r}}$$

$$d\boldsymbol{s} = \hat{\boldsymbol{s}}\,ds = \hat{\boldsymbol{r}}\,r^2 \sin\theta\, d\theta\, d\phi$$

$$P_{rad} = \int_0^{2\pi}\int_0^{\pi} P_{avg}\, r^2 \sin\theta\, d\theta\, d\phi$$

If we defined $P_{avg} r^2 = U(\theta,\phi)$ as the radiation intensity, then

$$P_{rad} = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi) \sin\theta\, d\theta\, d\phi = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\, d\Omega$$

where $d\Omega = \sin\theta d\theta d\phi$ defines the differential solid angle. The units on the radiation intensity are defined as watts per unit solid angle. The average radiation intensity is found by dividing the radiation intensity by the area of the unit sphere (4π) which gives

$$U_{avg} = \frac{\int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\, d\Omega}{4\pi} = \frac{P_{rad}}{4\pi}$$

The average radiation intensity for a given antenna represents the radiation intensity of a point source producing the same amount of radiated power as the antenna.

Directivity

Directivity (D) - the ratio of the radiation intensity in a given direction from the antenna to the radiation intensity averaged over all directions.

$$D(\theta,\phi) = \frac{U(\theta,\phi)}{U_{avg}} = \frac{4\pi\, U(\theta,\phi)}{P_{rad}}$$

The directivity of an isotropic radiator is $D(\theta,\phi) = 1$.

The maximum directivity is defined as $[D(\theta,\phi)]_{max} = D_o$.

The directivity range for any antenna is $0 \leq D(\theta,\phi) \leq D_o$.

Directivity in dB

$$D(\theta,\phi)\ \ [dB] = 10\log_{10} D(\theta,\phi)$$

Directivity in terms of Beam Solid Angle

We may define the radiation intensity as

$$U(\theta,\phi) = B_o\, F(\theta,\phi)$$

where B_o is a constant and $F(\theta,\phi)$ is the radiation intensity pattern function. The directivity then becomes

$$D(\theta,\phi) = 4\pi\, \frac{U(\theta,\phi)}{P_{rad}} = 4\pi\, B_o\, \frac{F(\theta,\phi)}{P_{rad}}$$

and the radiated power is

$$P_{rad} = \int_0^{2\pi}\!\!\int_0^{\pi} U(\theta,\phi) \sin\theta\, d\theta\, d\phi = B_o \int_0^{2\pi}\!\!\int_0^{\pi} F(\theta,\phi) \sin\theta\, d\theta\, d\phi$$

Inserting the expression for P_{rad} into the directivity expression yields

$$D(\theta,\phi) = 4\pi \frac{F(\theta,\phi)}{\int_0^{2\pi}\int_0^{\pi} F(\theta,\phi)\sin\theta\, d\theta\, d\phi}$$

The maximum directivity is

$$D_o = [D(\theta,\phi)]_{\max} = 4\pi \frac{[F(\theta,\phi)]_{\max}}{\int_0^{2\pi}\int_0^{\pi} F(\theta,\phi)\sin\theta\, d\theta\, d\phi} = \frac{4\pi}{\Omega_A}$$

where the term Ω_A in the previous equation is defined as the *beam solid angle* and is defined by

$$\Omega_A = \frac{\int_0^{2\pi}\int_0^{\pi} F(\theta,\phi)\sin\theta\, d\theta\, d\phi}{[F(\theta,\phi)]_{\max}} = \int_0^{2\pi}\int_0^{\pi} F_n(\theta,\phi)\sin\theta\, d\theta\, d\phi$$

$$F_n(\theta,\phi) = \frac{F(\theta,\phi)}{[F(\theta,\phi)]_{\max}}$$

Beam Solid Angle - the solid angle through which all of the antenna power would flow if the radiation intensity were $[U(\theta,\phi)]_{\max}$ for all angles in Ω_A.