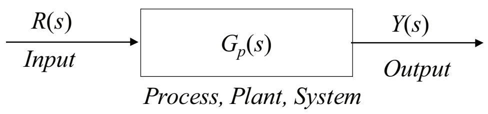
Lecture 3 : Sensitivity of Control Systems

System, Process, Plant

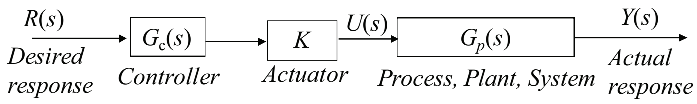
The input-output relationship represents the **cause-and-effect** relationship of the process, which in turn represents a processing of the input signal to provide an output signal variable..



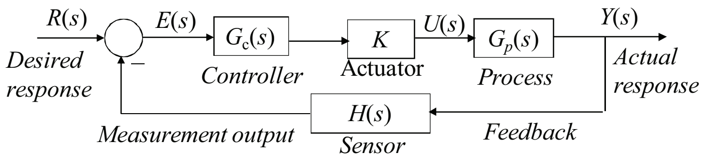
Control Systems

A control system is defined as an interconnection of components forming a system configuration that will provide a desired system response.
It achieves the desired response by two form of systems structures;

***a*) Open Loop Control System**



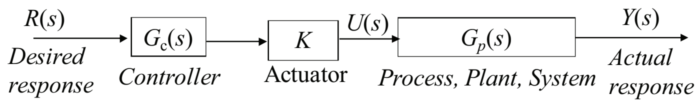
***b*) Closed Loop (Feedback) control system**



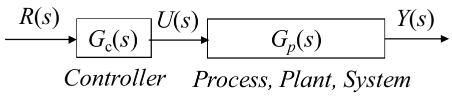
Because the desired system response, $R(s)$, is known, a signal proportional to the error, $E(s)$, between the desired, $R(s)$, and the actual response, $Y(s)$, is generated. $E(s) = R(s) - Y(s)$

Open Loop Systems

Open loop system: An open-loop control system utilizes an actuating device to control the process directly without using feedback.



For K=1



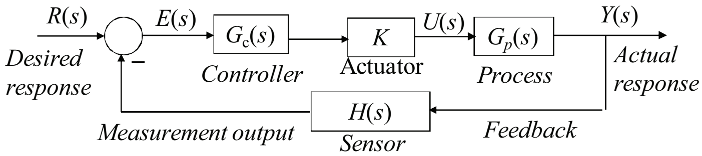
The system transfer function and output are;

$$\frac{Y(s)}{R(s)} = T(s) = G_c(s)G_p(s)$$

$$Y(s) = G_c(s)G_p(s)R(s)$$

Feedback (Closed Loop) Systems

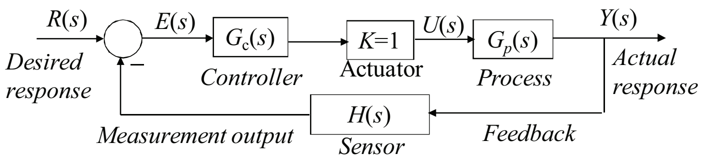
Feedback (Closed loop) system: The use of error (between the desired and the actual response) signal to control the process results in a closed-loop sequence of operations that is called a feedback system. This closed loop sequence of operations is shown in Figure.



The introduction of feedback to improve the control system is often necessary. It is interesting that this is also the case for systems in nature, such as biological and physiological systems; feedback is inherent in these systems. For example, the human heart rate control system is a feedback control system. An open-loop control system utilizes an actuating device to control the process directly without using feedback.

Feedback (Closed Loop) Systems

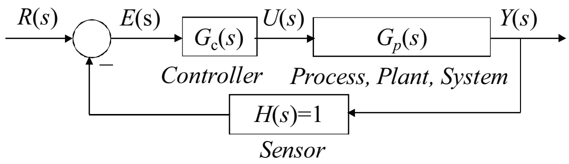
Feedback system: A closed-loop system uses a measurement of the output signal and a comparison with the desired output to generate an error signal that is used by the controller to adjust the actuator.



The system transfer function is;
$$\frac{Y(s)}{R(s)} = T(s) = \frac{G_c(s)G_p(s)}{1+G_c(s)G_p(s)H(s)}$$

The system output is;
$$Y(s) = \frac{G_c(s)G_p(s)}{1+G_c(s)G_p(s)H(s)} R(s)$$

Feedback (Closed Loop) Systems



If the sensor measures the output accurately, $H(s)=1$, then;

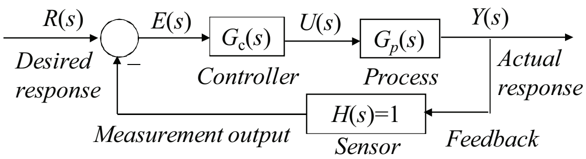
The system transfer function is; $$\frac{Y(s)}{R(s)} = T(s) = \frac{G_c(s)G_p(s)}{1+G_c(s)G_p(s)}$$

The system output is; $$Y(s) = \frac{G_c(s)G_p(s)}{1+G_c(s)G_p(s)} R(s)$$

if $G_c(s)\ G_p(s) >> 1$ for all complex frequencies of interest, the system output is

$$Y(s) \cong R(s)$$

Feedback (Closed Loop) Systems

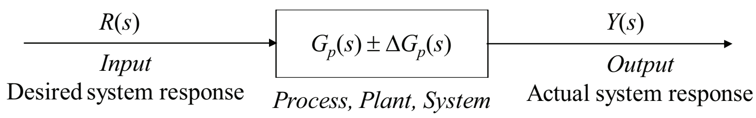


For $G_c(s)\ G_p(s) >> 1$ the system output is $Y(s) \cong R(s)$

The output is approximately equal to the input.
However, the condition $G_c(s)G_p(s) >> 1$ may cause the system response to be highly oscillatory and even unstable.
But the fact that increasing the magnitude of the loop gain $L(s)=G_c(s)G_p(s)$ reduces the effect of $G_p(s)$ on the output is an exceedingly useful result.
Therefore, one advantage of a feedback system is that the effect of the variation of the parameters of the process, $G_p(s)$, is reduced.

Controlling Uncertain or Unmodelled Systems

Control system engineers are concerned with understanding and controlling segments of their environment, often called systems, to provide useful economic products for society. The twin goals of **understanding** and **controlling** are complementary because effective systems control requires that the systems be **understood** and **modeled**. Furthermore, control engineering must often consider the control of **poorly understood** systems such as chemical process systems.



Where $G_p(s)$ is the model of the system and $\pm\, \Delta G_p(s)$ uncertainty or unmodelled part of the system.

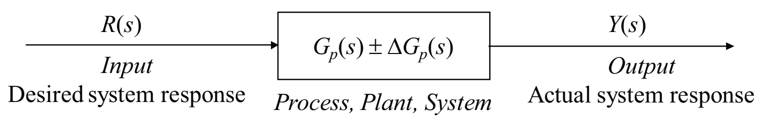
Sensitivity of Control Systems to Parameter Variations

The sensitivity of a control system to parameter variations is of prime importance since it is desirable to minimize the effects of parameter variations and uncertainties.

A process, represented by the transfer function $G_p(s)$, whatever its nature, is subject to

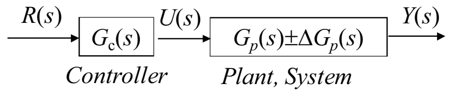
- a changing environment,
- aging,
- ignorance of the exact values of the process parameters, and
- other natural factors that affect a control process.

These can be represented by $\Delta G_p(s)$ where $|G_p(s)| >> |\Delta G_p(s)|$



Sensitivity of Open Loop Systems to Parameter Variations

If the system changed by $\Delta G_p(s)$ where $|G_p(s)| >> |\Delta G_p(s)|$ then the system becomes $G_p(s) \pm \Delta G_p(s)$;



For the open loop system the system transfer function and output are;

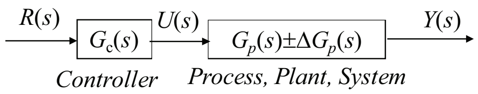
$$\frac{Y(s)}{R(s)} = T(s) = G_c(s)[G_p(s) \pm \Delta G_p(s)]$$

$$Y(s) \pm \Delta Y(s) = G_c(s)[G_p(s) \pm \Delta G_p(s)]R(s)$$

The system parameter changes, $\pm\Delta G_p(s)$, directly appears in the system output and is

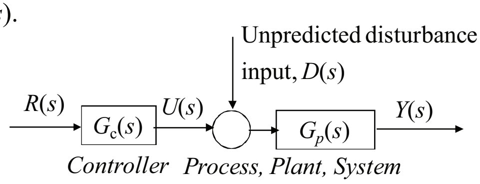
$$\Delta Y(s) = \pm G_c(s) \Delta G_p(s) R(s)$$

Sensitivity of Open Loop Systems to Parameter Variations



$$Y(s) \pm \Delta Y(s) = G_c(s)[G_p(s) \pm \Delta G_p(s)]R(s)$$

Consider the following open loop control system with the disturbance $D(s)$ and without $\pm\Delta G_p(s)$.

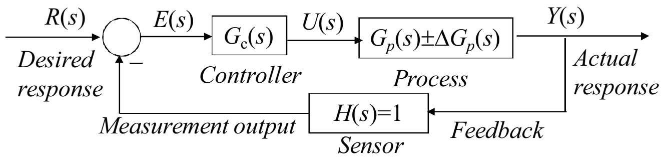


The output is; $$Y_R(s) + Y_D(s) = G_c(s)G_p(s)R(s) + G_p(s)D(s)$$

The disturbance, $D(s)$, directly influences the output, $Y(s)$.
The open loop control system is highly sensitive to disturbances, $D(s)$, and to changes $\pm\Delta G_p(s)$ in parameters of $G_p(s)$

Sensitivity of Feedback Systems to Parameter Variations

Suppose the system (or plant) $G_p(s)$ undergoes a change such that the true plant model is $G_p(s)\pm\Delta G_p(s)$ where $|G_p(s)|>>|\Delta G_p(s)|$. The change in the plant may be due to a changing external environment or natural aging, or it may just represent the uncertainty in certain plant parameters..

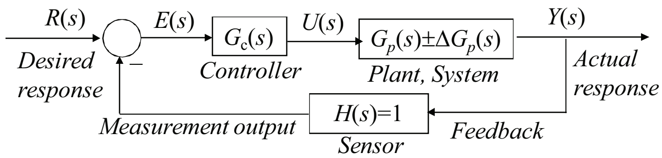


The system transfer function is; $\dfrac{Y(s)}{R(s)} = T(s) = \dfrac{G_c(s)[G_p(s)\pm\Delta G_p(s)]}{1+G_c(s)[G_p(s)\pm\Delta G_p(s)]}$

The system output is;

$$Y(s)\pm\Delta Y(s) = \frac{G_c(s)G_p(s)}{1+G_c(s)[G_p(s)\pm\Delta G_p(s)]}R(s) + \frac{G_c(s)\Delta G_p(s)}{1+G_c(s)[G_p(s)\pm\Delta G_p(s)]}R(s)$$

Sensitivity of Feedback Control Systems to Parameter Variations



$$Y(s)\pm\Delta Y(s) = \frac{G_c(s)G_p(s)}{1+G_c(s)[G_p(s)\pm\Delta G_p(s)]}R(s) \pm \frac{G_c(s)\Delta G_p(s)}{1+G_c(s)[G_p(s)\pm\Delta G_p(s)]}R(s)$$

For $|G_p(s)|>>|\Delta G_p(s)|$ and $G_c(s)G_p(s)>>1$ then the change appears in output is

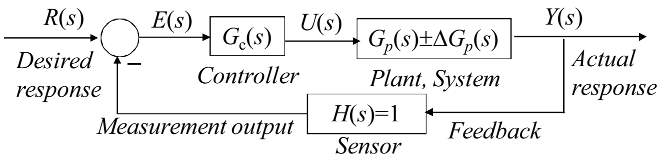
$$\Delta Y(s) = \pm\frac{G_c(s)\Delta G_p(s)}{1+G_c(s)[G_p(s)+\Delta G_p(s)]}R(s)$$

$$\cong 0$$

Again if $G_c(s)G_p(s)>>1$ for all complex frequencies of interest the system output is $\quad Y(s)\approx R(s)$

Sensitivity of Feedback Control Systems to Parameter Variations

Let study the effect on the tracking error $E(s)$ due to $\Delta G_p(s)$.

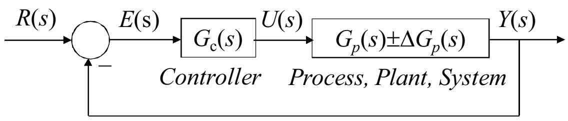


$$E(s) \pm \Delta E(s) = \frac{1}{1 + G_c(s)[G_p(s) \pm \Delta G_p(s)]} R(s)$$

Then the change in the tracking error is

$$\Delta E(s) = \frac{\mp G_c(s)\Delta G_p(s)}{[1 + G_c(s)G_p(s) \pm G_c(s)\Delta G_p(s)][1 + G_c(s)G_p(s)]} R(s)$$

Sensitivity of Feedback Control Systems to Parameter Variations



$$\Delta E(s) = \frac{\mp G_c(s)\Delta G_p(s)}{[1 + G_c(s)G_p(s) \pm G_c(s)\Delta G_p(s)][1 + G_c(s)G_p(s)]} R(s)$$

Since we usually find that $G_c(s)\ G_p(s) >> G_c(s)\Delta G_p(s)$ we have

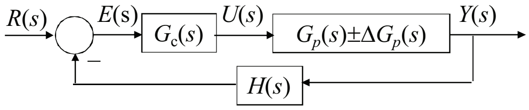
$$\Delta E(s) \approx \frac{\mp G_c(s)\Delta G_p(s)}{[1 + G_c(s)G_p(s)]^2} R(s) \quad \text{or} \quad \Delta E(s) \approx \frac{\mp G_c(s)\Delta G_p(s)}{[1 + L(s)]^2} R(s)$$

where $L(s) = G_c(s)G_p(s)$ is the loop transfer function.

Thus the tracking error is reduced by the factor $1 + L(s)$, which is generally greater than 1 over the range of frequencies of interest.

Sensitivity of Feedback Control Systems to Parameter Variations

$$\Delta E(s) \approx \frac{\mp G_c(s)\Delta G_p(s)}{[1+L(s)]^2} R(s)$$



where $L(s) = G_c(s)G_p(s)$

For large $L(s)$, then $1 + L(s) \approx L(s)$, and the approximate the change in the tracking error is

$$\Delta E(s) \approx \frac{\mp 1}{L(s)} \frac{\Delta G_p(s)}{G_p(s)} R(s)$$

Larger magnitude $L(s)$ translates into smaller changes in the tracking error (that is, reduced sensitivity to changes in $\Delta G_p(s)$ in the process).
Also, larger $L(s)$ implies smaller sensitivity, $S(s)$.
The question arises,
How do we define sensitivity $S(s)$ for the feedback system?

Since our goal is to reduce system sensitivity, it makes sense to formally define the sensitivity term.

Sensitivity of Control Systems to Parameter Variations

System sensitivity, $S(s)$, is the ratio of the change in input-output system transfer function, $T(s)$, to the change of a process transfer function, $G_p(s)$ (or parameter) for a small incremental change.

$$S_G^T(s) = \frac{\Delta T(s)/T(s)}{\Delta G_p(s)/G_p(s)}$$

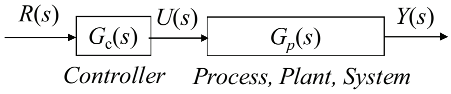
In the limit, for small incremental changes, it becomes;

$$S_G^T(s) = \frac{\partial T(s)/T(s)}{\partial G_p(s)/G_p(s)} = \frac{\partial T(s)}{\partial G_p(s)} \frac{G_p(s)}{T(s)}$$

$$= \frac{\partial \ln T(s)}{\partial \ln G_p(s)}$$

Sensitivity of Open Loop Systems

Consider again the following open-loop control system



The system transfer function is; $T(s) = G_c(s)G_p(s)$

The sensitivity of the open-loop system to changes in the plant $G_p(s)$ is

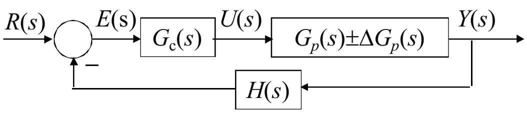
$$S_G^T = \frac{\partial T(s)}{\partial G_p(s)} \frac{G_p(s)}{T(s)} = G_c(s) \frac{G_p(s)}{G_c(s)G_p(s)} = 1$$

The sensitivity of the open-loop system to changes in the plant $G_p(s)$ is equal to 1. Any change in the plant, $G_p(s)$, or the disturbance, $D(s)$, directly influences the output, $Y(s)$.

Sensitivity of Feedback Systems

The sensitivity of a control system to parameter variations is of prime importance. A primary advantage of a feedback feedback control system is its ability to reduce the system's sensitivity

Consider again the following feedback (closed-loop) control system

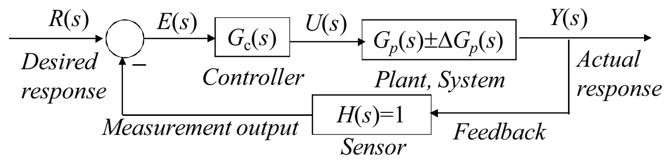


The system transfer function is; $T(s) = \dfrac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)H(s)}$

The sensitivity of feedback system to changes in the plant $G_p(s)$ is

$$S_G^T(s) = \frac{\partial T(s)}{\partial G_p(s)} \frac{G_p(s)}{T(s)} = \frac{1}{1 + G_c(s)G_p(s)H(s)}$$

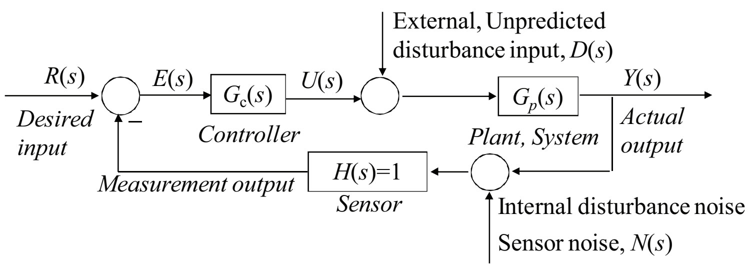
Sensitivity of Feedback Systems



$$S_G^T = \frac{\partial T(s)}{\partial G_p(s)}\frac{G_p(s)}{T(s)} = \frac{1}{1+G_c(s)G_p(s)H(s)}$$

We find that the sensitivity of the system may be reduced below that of the open loop system by increasing $L(s) = G_c(s)G_p(s)$ over the frequency range of interest.

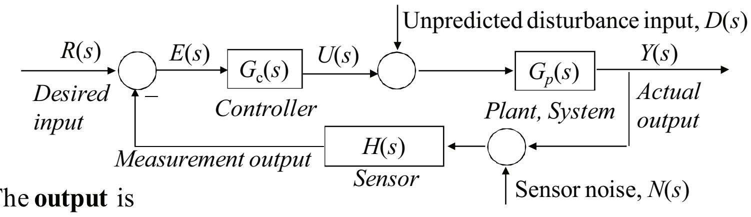
Error Signal Analysis of Feedback Systems



The closed-loop feedback control system has three inputs;
Desired input $R(s)$
External and unpredicted disturbance input $D(s)$
Internal disturbance noise $N(s)$
and one output, $Y(s)$.

Define the **tracking error** as $E(s) = R(s) - Y(s)$

Error Signal Analysis of Feedback Systems



The **output** is

$$Y(s) = \frac{G_c(s)G_p(s)}{1+G_c(s)G_p(s)H(s)}R(s) + \frac{G_p(s)}{1+G_c(s)G_p(s)H(s)}D(s) - \frac{G_c(s)G_p(s)H(s)}{1+G_c(s)G_p(s)H(s)}N(s)$$

$$= \frac{G_c(s)G_p(s)}{1+L(s)}R(s) + \frac{G_p(s)}{1+L(s)}D(s) - \frac{L(s)}{1+L(s)}N(s)$$

The the **tracking error** is

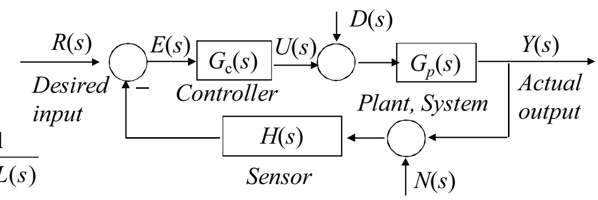
$$E(s) = \frac{1}{1+G_c(s)G_p(s)H(s)}R(s) - \frac{G_p(s)}{1+G_c(s)G_p(s)H(s)}D(s) - \frac{G_c(s)G_p(s)H(s)}{1+G_c(s)G_p(s)H(s)}N(s)$$

$$= \frac{1}{1+L(s)}R(s) - \frac{G_p(s)}{1+L(s)}D(s) - \frac{L(s)}{1+L(s)}N(s)$$

where $L(s) = G_c(s)G_p(s)H(s)$

Error Signal Analysis of Feedback Systems

$L(s) = G_c(s)G_p(s)H(s)$



The **sensitivity function** is $S(s) = \dfrac{1}{1+L(s)}$

The **complementary sensitivity function** is $C(s) = \dfrac{L(s)}{1+L(s)}$

In terms of the functions $S(s)$ and $C(s)$ the tracking error is

$$E(s) = S(s)R(s) - S(s)G_p(s)D(s) - C(s)N(s)$$

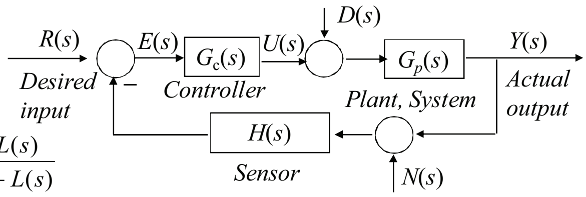
For a given $G_p(s)$, if we want to minimize the tracking error, $E(s)$ we want both $S(s)$ and $C(s)$ to be small. Notice that $S(s)$ and $C(s)$ are both functions of the controller, $G_c(s)$, which the control design engineer must select. However, the following special relationship between $S(s)$ and $C(s)$ holds.

$$S(s) + C(s) = 1$$

Error Signal Analysis of Feedback Systems

$L(s) = G_c(s)G_p(s)H(s)$

$S(s) = \dfrac{1}{1+L(s)}$ $\quad C(s) = \dfrac{L(s)}{1+L(s)}$



The tracking error is $E(s) = S(s)R(s) - S(s)G_p(s)D(s) - C(s)N(s)$

For a given $G_p(s)$, to minimize the tracking error, $E(s)$, both $S(s)$ and $C(s)$ to be small. Since $S(s)$ and $C(s)$ are both functions of the controller, $G_c(s)$, and the following special relationship between $S(s)$ and $C(s)$ holds.

$$S(s) + C(s) = 1$$

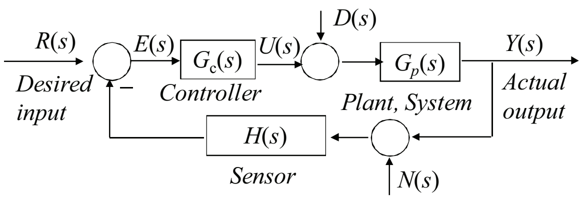
We cannot simultaneously make $S(s)$ and $C(s)$ small. Obviously, design compromises must be made.

Error Signal Analysis of Feedback Systems

$L(s) = G_c(s)G_p(s)H(s)$

$S(s) = \dfrac{1}{1+L(s)}$

$C(s) = \dfrac{L(s)}{1+L(s)}$



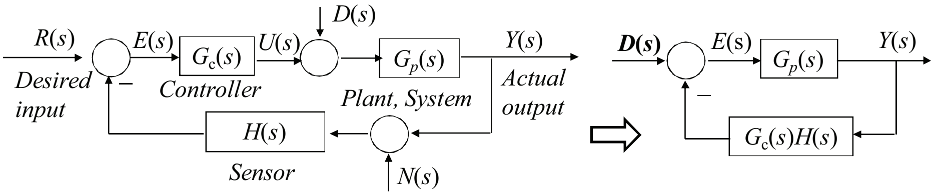
$$E(s) = S(s)R(s) - S(s)G_p(s)D(s) - C(s)N(s)$$

$$S(s) + C(s) = 1$$

To analyze the tracking error equation, we need to understand what it means for a transfer function to be "large" or to be "small." The discussion of magnitude of a transfer function is the subject of frequency response methods, Let describe the magnitude of the loop gain $L(s)$ by considering the magnitude $|L(j\omega)|$ over the range of frequencies, ω, of interest.

Disturbance rejection in the Feedback Control Systems

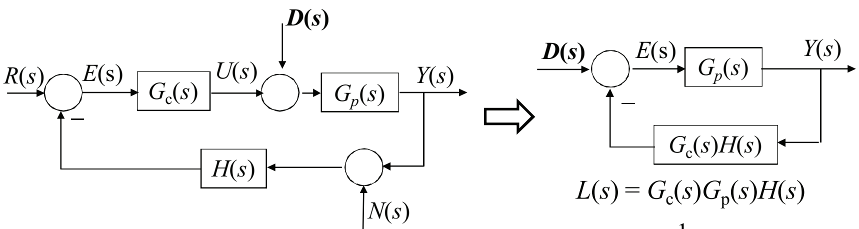
Disturbance Rejection, *D*(*s*): When $R(s) = N(s) = 0$, the system is



The tracking error equation is $E(s) = -S(s)G_p(s)D(s) = \dfrac{G_p(s)}{1 + L(s)}D(s)$

For a fixed $G_p(s)$ and a given $D(s)$, as the loop gain $L(s)$ increases, the effect of $D(s)$ on the tracking error decreases. In other words, the sensitivity function $S(s)$ is small when the loop gain is large. We say that large loop gain leads to good disturbance rejection. More precisely, for good disturbance rejection, we require a large loop gain over the frequencies of interest associated with the expected disturbance signals.

Disturbance rejection, *D*(*s*), in the Feedback Control Systems



$L(s) = G_c(s)G_p(s)H(s)$

$S(s) = \dfrac{1}{1 + L(s)}$

The tracking error equation is

$$E(s) = -S(s)G_p(s)D(s) = \frac{G_p(s)}{1 + L(s)}D(s)$$

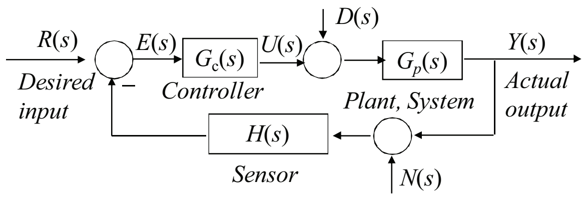
In practice, the disturbance signals $D(s)$ are often low frequency. When that is the case, we say that we want the loop gain to be large at low frequencies. This is equivalent to stating that we want to design the controller $G_c(s)$ so that the sensitivity function $S(s)$ is small at low frequencies.

Disturbance rejection, *D*(*s*), in the Feedback Control Systems

$L(s) = G_c(s)G_p(s)H(s)$

$$S(s) = \frac{1}{1+L(s)}$$

$$C(s) = \frac{L(s)}{1+L(s)}$$



$$S(s) + C(s) = 1$$

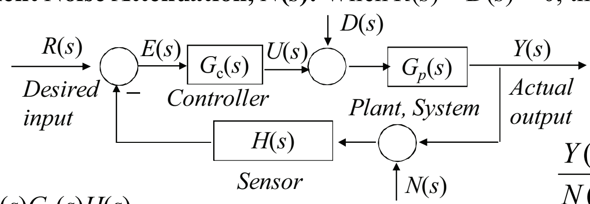
$$E(s) = -S(s)G_p(s)D(s) = \frac{G_p(s)}{1+L(s)}D(s)$$

Disturbance Rejection, *D*(*s*): Considering the tracking error in $E(s)$, it is evident that, for a given $G_p(s)$, to reduce the influence of the disturbance, $D(s)$, on the tracking error, $E(s)$, we desire the magnitude $|L(j\omega)|$ over the range of frequencies, ω, that characterize the disturbances.

That way, the transfer function $G_p(s)/(1 + L(s))$ will be small, thereby reducing the influence of $D(s)$, Since $L(s) = G_c(s)G_p(s)H(s)$ this implies that we need to design the controller $G_c(s)$ to have a large magnitude $|G_c(j\omega)|$.

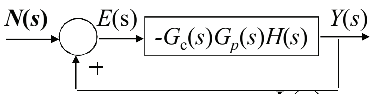
Measurement noise attenuation in the Feedback Control Systems

Measurement Noise Attenuation, *N*(*s*): When $R(s) = D(s) = 0$, the system is



$$\frac{Y(s)}{N(s)} = \frac{-L(s)}{1+L(s)}$$

$L(s) = G_c(s)G_p(s)H(s)$



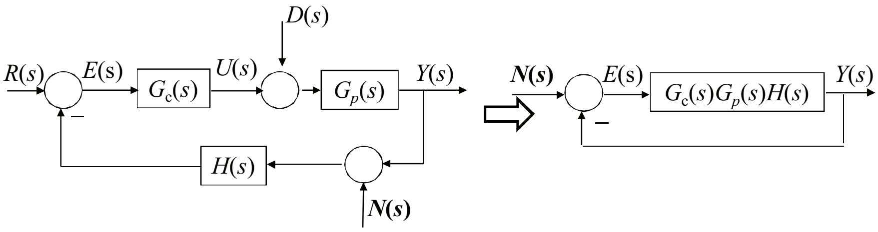
$$C(s) = \frac{L(s)}{1+L(s)}$$

The tracking error equation is $E(s) = C(s)N(s) = \dfrac{L(s)}{1+L(s)}N(s)$

As the loop gain $|L(j\omega)|$ decreases, the effect of $N(s)$ on the tracking error decreases. In other words, the complementary sensitivity function $C(s)$ is small when the loop gain $|L(j\omega)|$ is small. If we design $G_c(s)$such that $|L(j\omega)| << 1$, then the noise is attenuated because

$$C(s) \approx L(s)$$

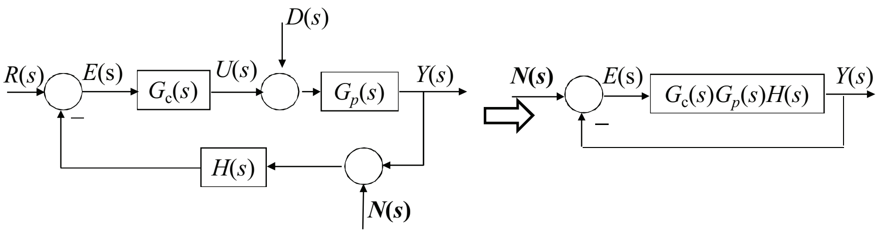
Measurement noise attenuation, *N*(*s*), in the Feedback Control Systems



The tracking error equation is $E(s) = C(s)N(s) = \dfrac{L(s)}{1+L(s)}N(s)$

We say that small loop gain, $|L(j\omega)| << 1$, leads to good noise attenuation. More precisely, for effective measurement noise attenuation, we need a small loop gain over the frequencies associated with the expected noise signals.

Measurement noise attenuation, *N*(*s*), in the Feedback Control Systems



The tracking error equation is $E(s) = C(s)N(s) = \dfrac{L(s)}{1+L(s)}N(s)$

In practice, measurement noise signals are often high frequency. Thus we want the loop gain, $|L(j\omega)|$, to be low at high frequencies. This is equivalent to a small complementary sensitivity function $C(s)$ at high frequencies.

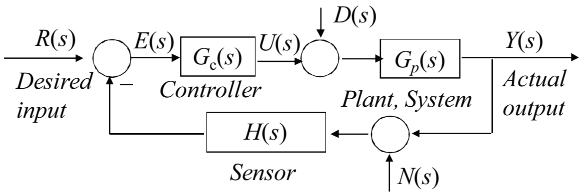
Measurement noise attenuation, $N(s)$, in the Feedback Control Systems

$L(s) = G_c(s)G_p(s)H(s)$

$S(s) = \dfrac{1}{1+L(s)}$

$C(s) = \dfrac{L(s)}{1+L(s)}$

$S(s) + C(s) = 1$



$$E(s) = C(s)N(s) = \frac{L(s)}{1+L(s)}N(s)$$

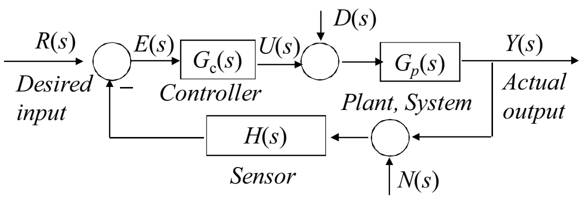
Measurement Noise Attenuation, $N(s)$: to attenuate the measurement noise, $N(s)$, and reduce the influence on the tracking error, $E(s)$, we desire the magnitude $|L(j\omega)|$ over the range of frequencies, ω, that characterize the measurement noise. The transfer function $L(s)/(1 + L(s))$ will be small, thereby reducing the influence of $N(s)$. Again, since $L(s) = G_c(s)G_p(s)H(s)$ that implies that we need to design the controller $G_c(s)$ to have a small magnitude $|G_c(j\omega)|$.

Error Signal Analysis of Feedback Systems

$L(s) = G_c(s)G_p(s)H(s)$

$S(s) = \dfrac{1}{1+L(s)}$

$C(s) = \dfrac{L(s)}{1+L(s)}$

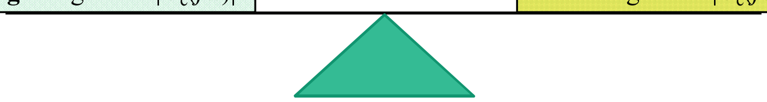


$$E(s) = S(s)R(s) - S(s)G_p(s)D(s) - C(s)N(s)$$

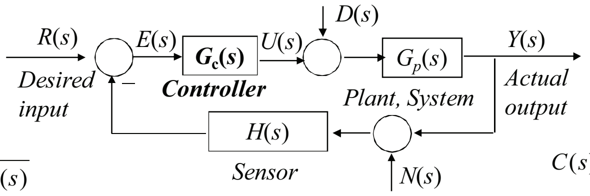
Disturbance Rejection, $D(s)$ needs to design the controller $G_c(s)$ to have a **large** magnitude $|G_c(j\omega)|$

$S(s) + C(s) = 1$

Noise Attenuation, $N(s)$ needs to design the controller $G_c(s)$ to have a **small** magnitude $|G_c(j\omega)|$



Error Signal Analysis of Feedback Systems



$$S(s) = \frac{1}{1+L(s)}$$

$$C(s) = \frac{L(s)}{1+L(s)}$$

Disturbance Rejection, *D*(*s*) needs to design the controller $G_c(s)$ to have a **large** magnitude $|G_c(j\omega)|$

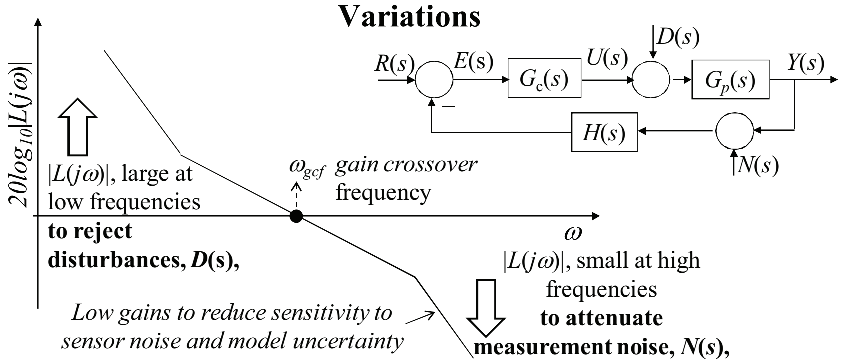
$$L(s) = G_c(s)G_p(s)H(s)$$

$$S(s) + C(s) = 1$$

Noise Attenuation, *N*(*s*) needs to design the controller $G_c(s)$ to have a **small** magnitude $|G_c(j\omega)|$

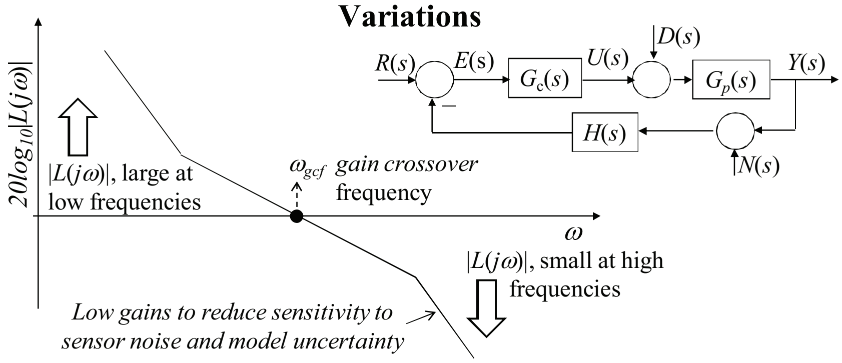
The separation of disturbances, ***D*(*s*)** (at low frequencies) and measurement noise, ***N*(*s*)** (at high frequencies) is very fortunate because it gives the control system designer a way to approach the design process: the controller, $G_c(s)$ should be high gain at low frequencies and low gain at high frequencies.

Sensitivity of Closed Loop Control Systems to Parameter Variations



Fortunately, the apparent conflict between wanting to make $|G_c(j\omega)|$ **large to reject disturbances, *D*(s),** and the wanting to make $|G_c(j\omega)|$ **small to attenuate measurement noise, *N*(*s*),** can be **addressed in the design phase** by **making the loop gain, $|L(j\omega)|$, large at low frequencies** (generally associated with the frequency range of disturbances), and **making $|L(j\omega)|$, small at high frequencies** (generally associated with measurement noise).

Sensitivity of Closed Loop Control Systems to Parameter Variations

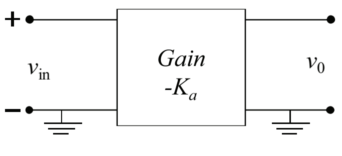


An important advantage of feedback control systems is the ability to reduce the effect of the variation of parameters of a control system by adding a feedback loop. The closed-loop system allows $G_p(s)$ to be less accurately specified, because the sensitivity to changes or errors in $G_p(s)$ is reduced by the loop gain $L(s)$.This benefit of closed-loop systems is a profound advantage for the electronic amplifiers of the communication industry. A simple example will illustrate the value of feedback for reducing sensitivity.

Sensitivity of Feedback Control Systems to Parameter Variations

Let study a simple example to illustrate the value of feedback for reducing sensitivity.

Open loop amplifier: An amplifier used in many applications has a gain $-K_a$, as shown in the following Figure. The output voltage is



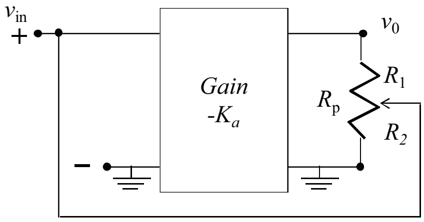
The output voltage is $v_0 = -K_a v_{in}$

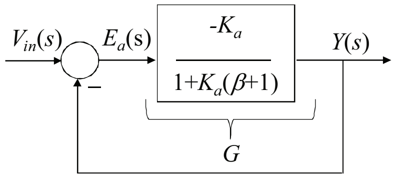
The transfer function is $T = -K_a$

The sensitivity to changes in the amplifier gain is $S_{K_a}^{T} = \frac{\partial T}{\partial K_a}\frac{K_a}{T} = 1$

Sensitivity of Feedback Control Systems to Parameter Variations

***Feedback amplifier*:** Often a potentiometer R_p *i*s added to achieve feedback *as* shown in Figure.





$R_p = R_1 + R_2$ and $\beta = \frac{R_2}{R_1}$

The closed-loop transfer function of the feedback amplifier is

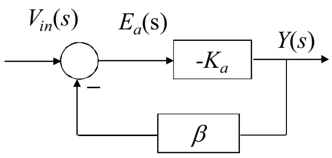
$$T = \frac{G}{1+G} = \frac{-K_a}{1+\beta K_a}$$

The sensitivity of the closed-loop feedback amplifier is

$$S_{K_a}^T = S_G^T S_{K_a}^G = \frac{1}{1+\beta K_a}$$

Sensitivity of Feedback Control Systems to Parameter Variations

***Feedback amplifier*:**



The closed-loop transfer function is $T = \frac{-K_a}{1+\beta K_a}$

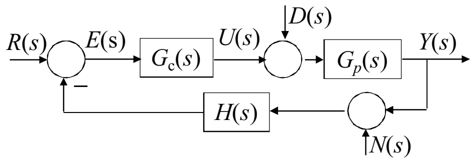
The sensitivity of the system to parameter variations is $S_{K_a}^T = S_G^T S_{K_a}^G = \frac{1}{1+\beta K_a}$

If K_a is large, the sensitivity is low. For example, if $K_a=10^4$ and $\beta=10$ then the sensitivity is

$$S_{K_a}^T = \frac{1}{1+\beta K_a} = \frac{1}{1+10^3}$$

The magnitude is one-thousandth of the magnitude of the open-loop amplifier !

Sensitivity of Closed Loop Control Systems to Parameter Variations



Often, we seek to determine $S_\alpha^T(s)$ where α is a parameter within the transfer function of a block $\boldsymbol{G_p(s)}$. Using the chain rule, we find that

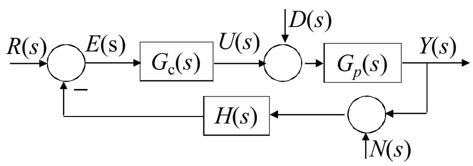
$$S_\alpha^T(s) = S_G^T S_\alpha^G$$

Very often, the transfer function of the system $T(s)$ is a fraction of the form

$$T(s,\alpha) = \frac{N(s,\alpha)}{D(s,\alpha)}$$

where α is a parameter that may be subject to variation due to the environment.

Sensitivity of Closed Loop Control Systems to Parameter Variations



Then we may obtain the sensitivity to α by rewriting Equation

$$S_G^T = \frac{\partial T(s)/T(s)}{\partial G(s)/G_p(s)} = \frac{\partial T(s)}{\partial G_p(s)}\frac{G(s)}{T(s)} = \frac{\partial \ln T(s)}{\partial \ln G_p(s)}$$

as

$$S_\alpha^T = \frac{\partial \ln T}{\partial \ln \alpha} = \left.\frac{\partial \ln N}{\partial \ln \alpha}\right|_{\alpha_0} - \left.\frac{\partial \ln D}{\partial \ln \alpha}\right|_{\alpha_0} = S_\alpha^N - S_\alpha^D$$

where α_0 is the nominal value of the parameter.