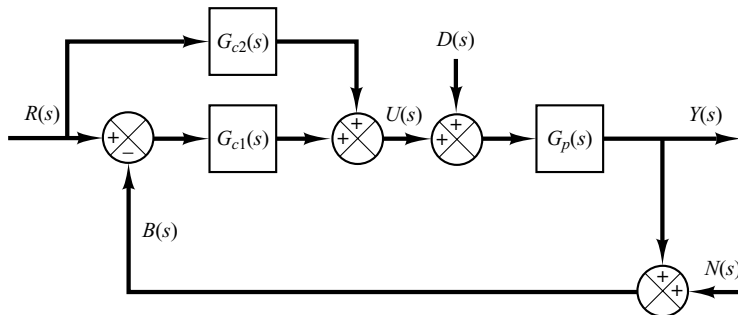


Figure 8–66
Two-degrees-of-
freedom control
system.



2. The response to the unit-step reference input has a maximum overshoot of 25% or less, and the settling time is 1 sec or less.
3. The steady-state errors in following ramp reference input and acceleration reference input are zero.

Solution. The closed-loop transfer functions for the disturbance input and reference input are given, respectively, by

$$\frac{Y(s)}{D(s)} = \frac{G_p(s)}{1 + G_{c1}(s)G_p(s)}$$

$$\frac{Y(s)}{R(s)} = \frac{[G_{c1}(s) + G_{c2}(s)]G_p(s)}{1 + G_{c1}(s)G_p(s)}$$

Let us assume that $G_{c1}(s)$ is a PID controller and has the following form:

$$G_{c1}(s) = \frac{K(s + a)^2}{s}$$

The characteristic equation for the system is

$$1 + G_{c1}(s)G_p(s) = 1 + \frac{K(s + a)^2}{s} \frac{100}{s(s + 1)}$$

Notice that the open-loop poles are located at $s = 0$ (a double pole) and $s = -1$. The zeros are located at $s = -a$ (a double zero).

In what follows, we shall use the root-locus approach to determine the values of a and K . Let us choose the dominant closed-loop poles at $s = -5 \pm j5$. Then, the angle deficiency at the desired closed-loop pole at $s = -5 + j5$ is

$$-135^\circ - 135^\circ - 128.66^\circ + 180^\circ = -218.66^\circ$$

The double zero at $s = -a$ must contribute 218.66° . (Each zero must contribute 109.33° .) By a simple calculation, we find

$$a = -3.2460$$

The controller $G_{c1}(s)$ is then determined as

$$G_{c1}(s) = \frac{K(s + 3.2460)^2}{s}$$

The constant K must be determined by use of the magnitude condition. This condition is

$$|G_{c1}(s)G_p(s)|_{s=-5+j5} = 1$$

Since

$$G_{c1}(s)G_p(s) = \frac{K(s + 3.2460)^2}{s} \frac{100}{s(s + 1)}$$

we obtain

$$\begin{aligned} K &= \left| \frac{s^2(s + 1)}{100(s + 3.2460)^2} \right|_{s=-5+j5} \\ &= 0.11403 \end{aligned}$$

The controller $G_{c1}(s)$ thus becomes

$$\begin{aligned} G_{c1}(s) &= \frac{0.11403(s + 3.2460)^2}{s} \\ &= \frac{0.11403s^2 + 0.74028s + 1.20148}{s} \\ &= 0.74028 + \frac{1.20148}{s} + 0.11403s \end{aligned} \quad (8-17)$$

Then, the closed-loop transfer function $Y(s)/D(s)$ is obtained as follows:

$$\begin{aligned} \frac{Y(s)}{D(s)} &= \frac{G_p(s)}{1 + G_{c1}(s)G_p(s)} \\ &= \frac{100}{s(s + 1)} \\ &= \frac{100}{1 + \frac{0.11403(s + 3.2460)^2}{s} \frac{100}{s(s + 1)}} \\ &= \frac{100s}{s^3 + 12.403s^2 + 74.028s + 120.148} \end{aligned}$$

The response curve when $D(s)$ is a unit-step disturbance is shown in Figure 8–67.

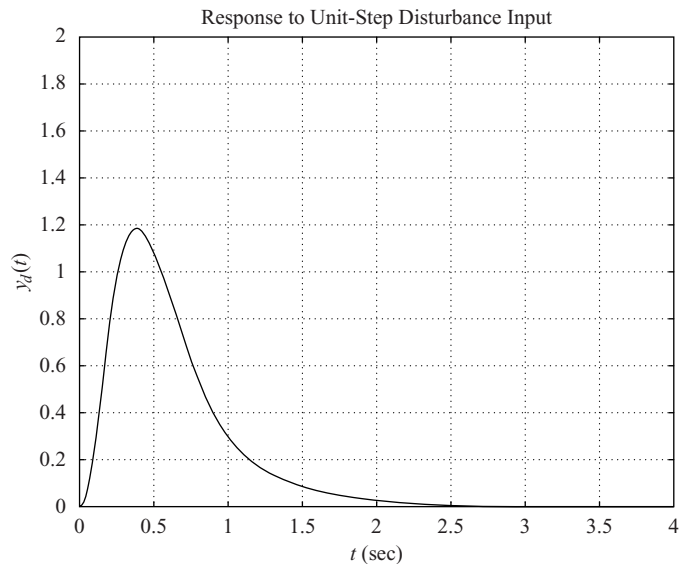


Figure 8–67
Response to unit-step disturbance input.

Next, we consider the responses to reference inputs. The closed-loop transfer function $Y(s)/R(s)$ is

$$\frac{Y(s)}{R(s)} = \frac{[G_{c1}(s) + G_{c2}(s)]G_p(s)}{1 + G_{c1}(s)G_p(s)}$$

Let us define

$$G_{c1}(s) + G_{c2}(s) = G_c(s)$$

Then

$$\begin{aligned} \frac{Y(s)}{R(s)} &= \frac{G_c(s)G_p(s)}{1 + G_{c1}(s)G_p(s)} \\ &= \frac{100sG_c(s)}{s^3 + 12.403s^2 + 74.028s + 120.148} \end{aligned}$$

To satisfy the requirements on the responses to the ramp reference input and acceleration reference input, we use the zero-placement approach. That is, we choose the numerator of $Y(s)/R(s)$ to be the sum of the last three terms of the denominator, or

$$100sG_c(s) = 12.403s^2 + 74.028s + 120.148$$

from which we get

$$\begin{aligned} G_c(s) &= \frac{0.12403s^2 + 0.74028s + 1.20148}{s} \\ &= 0.74028 + \frac{1.20148}{s} + 0.12403s \end{aligned} \quad (8-18)$$

Hence, the closed-loop transfer function $Y(s)/R(s)$ becomes as

$$\frac{Y(s)}{R(s)} = \frac{12.403s^2 + 74.028s + 120.148}{s^3 + 12.403s^2 + 74.028s + 120.148}$$

The response curves to the unit-step reference input, unit-ramp reference input, and unit-acceleration reference input are shown in Figures 8-68(a), (b), and (c), respectively. The maximum

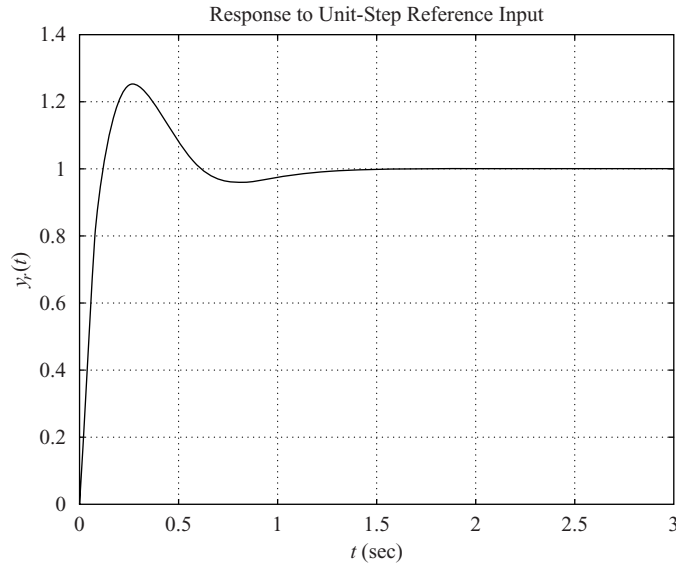
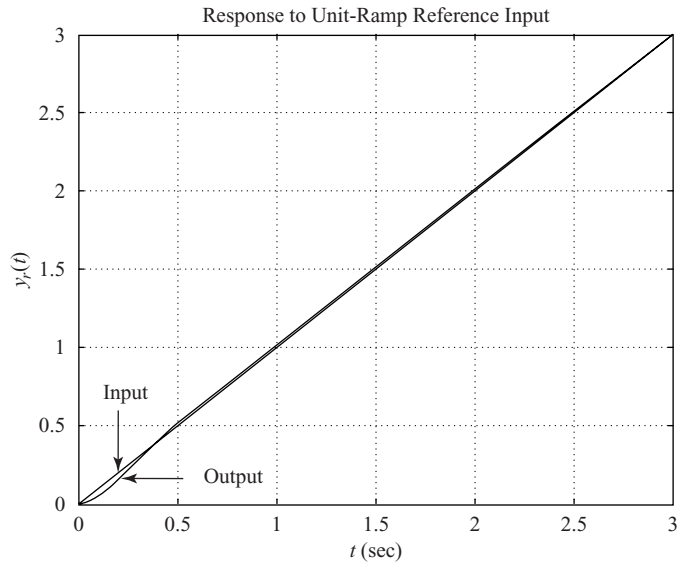


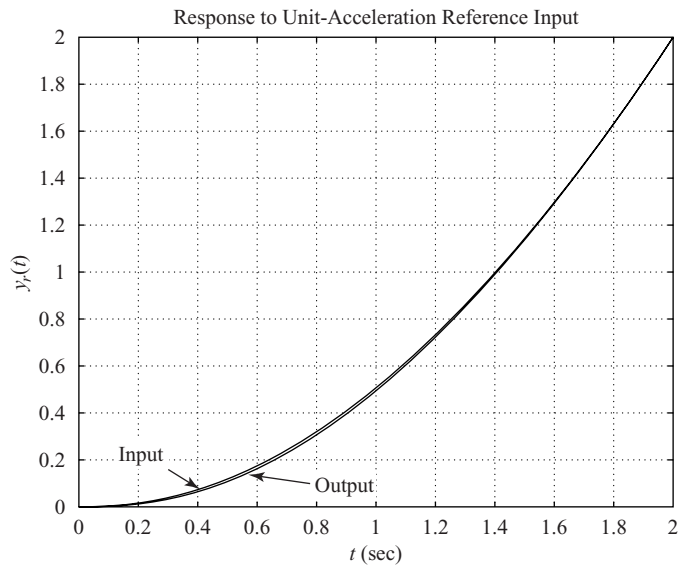
Figure 8-68

(a) Response to unit-step reference input; (b) response to unit-ramp reference input; (c) response to unit-acceleration reference input.

(a)



(b)



(c)

Figure 8–68
(continued)

overshoot in the unit-step response is approximately 25% and the settling time is approximately 1.2 sec. The steady-state errors in the ramp response and acceleration response are zero. Therefore, the designed controller $G_c(s)$ given by Equation (8–18) is satisfactory.

Finally, we determine $G_{c2}(s)$. Noting that

$$G_{c2}(s) = G_c(s) - G_{c1}(s)$$

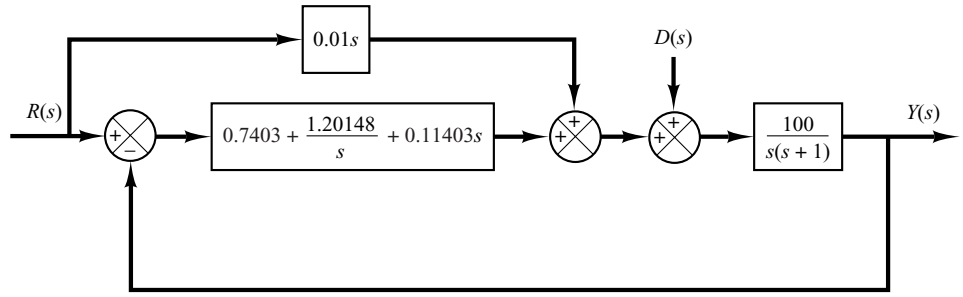


Figure 8–69
Block diagram of the designed system.

and from Equation (8–17)

$$G_{c1}(s) = 0.7403 + \frac{1.20148}{s} + 0.11403s$$

we obtain

$$\begin{aligned} G_{c2}(s) &= \left(0.7403 + \frac{1.20148}{s} + 0.12403s \right) \\ &\quad - \left(0.7403 + \frac{1.20148}{s} + 0.11403s \right) \\ &= 0.01s \end{aligned} \tag{8-19}$$

Equations (8–17) and (8–19) give the transfer functions of the controllers $G_{c1}(s)$ and $G_{c2}(s)$, respectively. The block diagram of the designed system is shown in Figure 8–69.

Note that if the maximum overshoot were much higher than 25% and/or the settling time were much larger than 1.2 sec, then we might assume a search region (such as $3 \leq a \leq 6$, $3 \leq b \leq 6$, and $6 \leq c \leq 12$) and use the computational method presented in Example 8–4 to find a set or sets of variables that would give the desired response to the unit-step reference input.

PROBLEMS

B–8–1. Consider the electronic PID controller shown in Figure 8–70. Determine the values of R_1 , R_2 , R_3 , R_4 , C_1 , and C_2 of the controller such that the transfer function $G_c(s) = E_o(s)/E_i(s)$ is

$$\begin{aligned} G_c(s) &= 39.42 \left(1 + \frac{1}{3.077s} + 0.7692s \right) \\ &= 30.3215 \frac{(s + 0.65)^2}{s} \end{aligned}$$

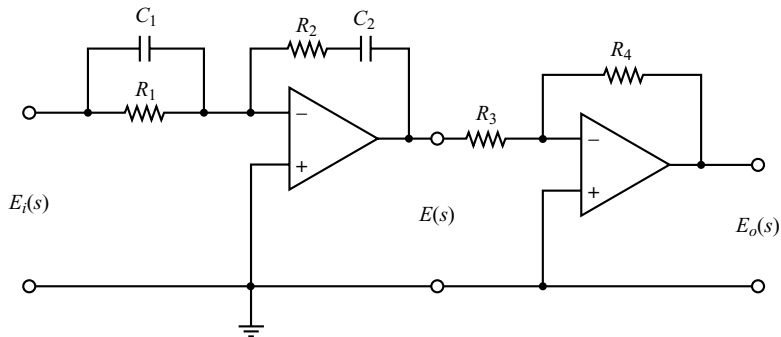


Figure 8–70
Electronic PID controller.

B-8-2. Consider the system shown in Figure 8-71. Assume that disturbances $D(s)$ enter the system as shown in the diagram. Determine parameters K , a , and b such that the response to the unit-step disturbance input and the response to the unit-step reference input satisfy the following specifications: The response to the step disturbance input should attenuate rapidly with no steady-state error, and the response to the step reference input exhibits a maximum overshoot of 20% or less and a settling time of 2 sec.

B-8-3. Show that the PID-controlled system shown in Figure 8-72(a) is equivalent to the I-PD-controlled system with feedforward control shown in Figure 8-72(b).

B-8-4. Consider the systems shown in Figures 8-73(a) and (b). The system shown in Figure 8-73(a) is the system designed in Example 8-1. The response to the unit-step reference input in the absence of the disturbance input is shown in Figure 8-10. The system shown in Figure 8-73(b) is the I-PD-controlled system using the same K_p , T_i , and T_d as the system shown in Figure 8-73(a).

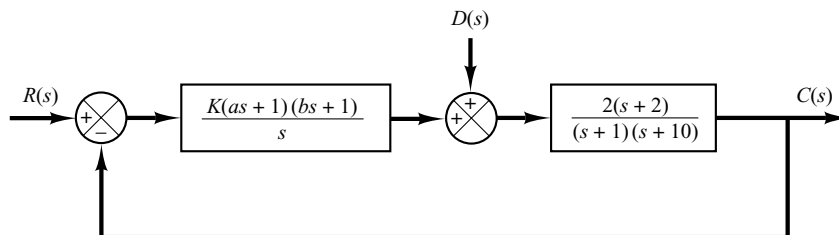
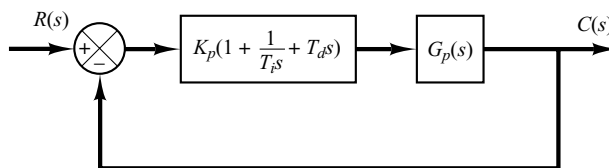
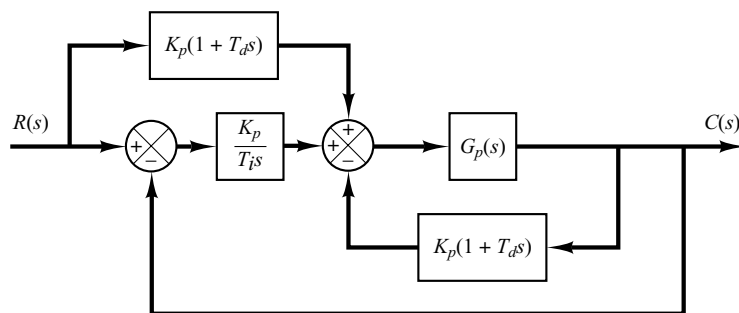


Figure 8-71
Control system.



(a)



(b)

Figure 8-72
(a) PID-controlled system; (b) I-PD-controlled system with feedforward control.

Obtain the response of the I-PD-controlled system to the unit-step reference input with MATLAB. Compare the unit-step response curves of the two systems.

B-8-5. Referring to Problem B-8-4, obtain the response of the PID-controlled system shown in Figure 8-73(a) to the unit-step disturbance input.

Show that for the disturbance input, the responses of the PID-controlled system shown in Figure 8-73(a) and of the I-PD-controlled system shown in Figure 8-73(b) are

exactly the same. [When considering $D(s)$ to be the input, assume that the reference input $R(s)$ is zero, and vice versa.] Also, compare the closed-loop transfer function $C(s)/R(s)$ of both systems.

B-8-6. Consider the system shown in Figure 8-74. This system is subjected to three input signals: the reference input, disturbance input, and noise input. Show that the characteristic equation of this system is the same regardless of which input signal is chosen as input.

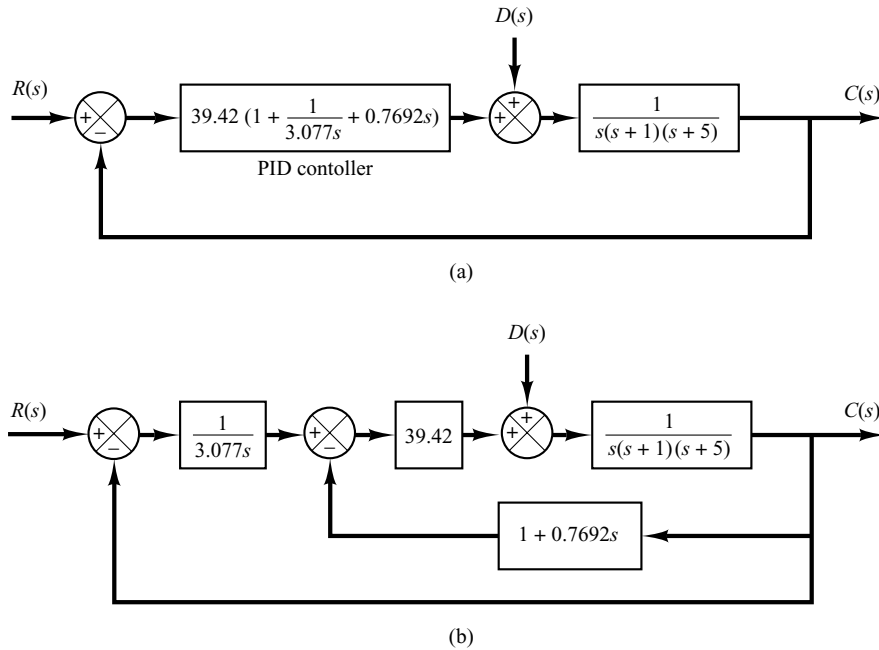


Figure 8-73
(a) PID-controlled system; (b) I-PD-controlled system.

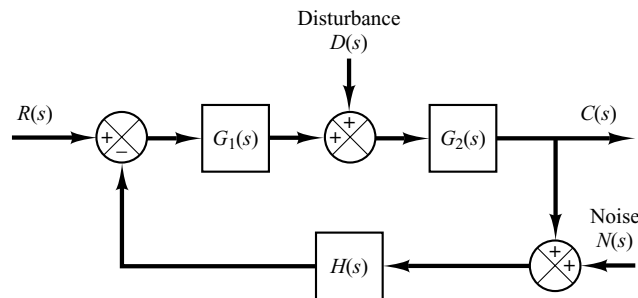


Figure 8-74
Control system.

B-8-7. Consider the system shown in Figure 8-75. Obtain the closed-loop transfer function $C(s)/R(s)$ for the reference input and the closed-loop transfer function $C(s)/D(s)$ for the disturbance input. When considering $R(s)$ as the input, assume that $D(s)$ is zero, and vice versa.

B-8-8. Consider the system shown in Figure 8-76(a), where K is an adjustable gain and $G(s)$ and $H(s)$ are fixed

components. The closed-loop transfer function for the disturbance is

$$\frac{C(s)}{D(s)} = \frac{1}{1 + KG(s)H(s)}$$

To minimize the effect of disturbances, the adjustable gain K should be chosen as large as possible.

Is this true for the system in Figure 8-76(b), too?

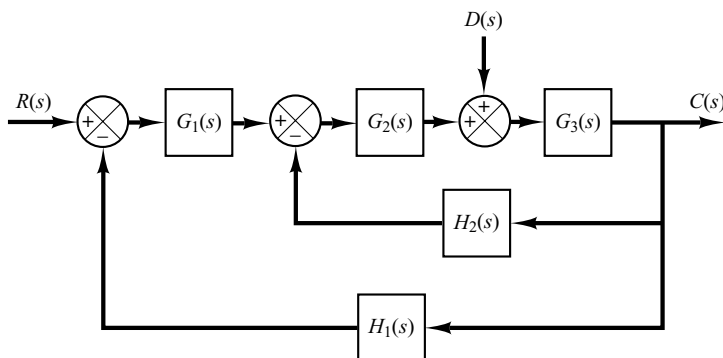


Figure 8-75
Control system.

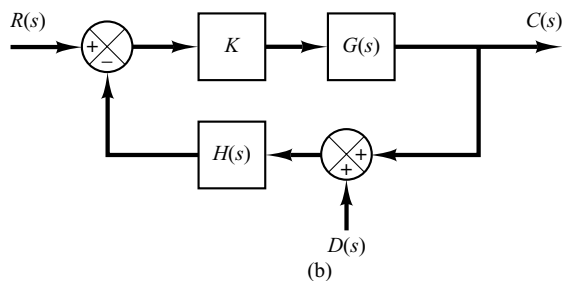
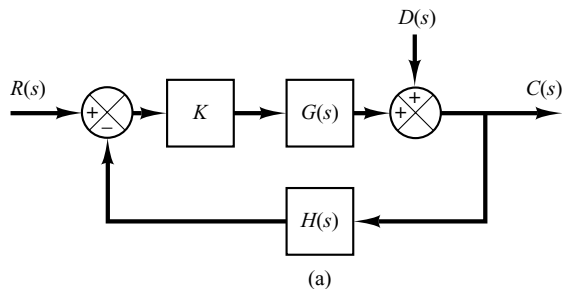


Figure 8-76
(a) Control system with disturbance entering in the feedforward path; (b) control system with disturbance entering in the feedback path.

B-8-9. Show that the control systems shown in Figures 8-77(a), (b), and (c) are two-degrees-of-freedom systems. In the diagrams, G_{c1} and G_{c2} are controllers and G_p is the plant.

B-8-10. Show that the control system shown in Figure 8-78 is a three-degrees-of-freedom system. The transfer functions G_{c1} , G_{c2} , and G_{c3} are controllers. The plant consists of transfer functions G_1 and G_2 .

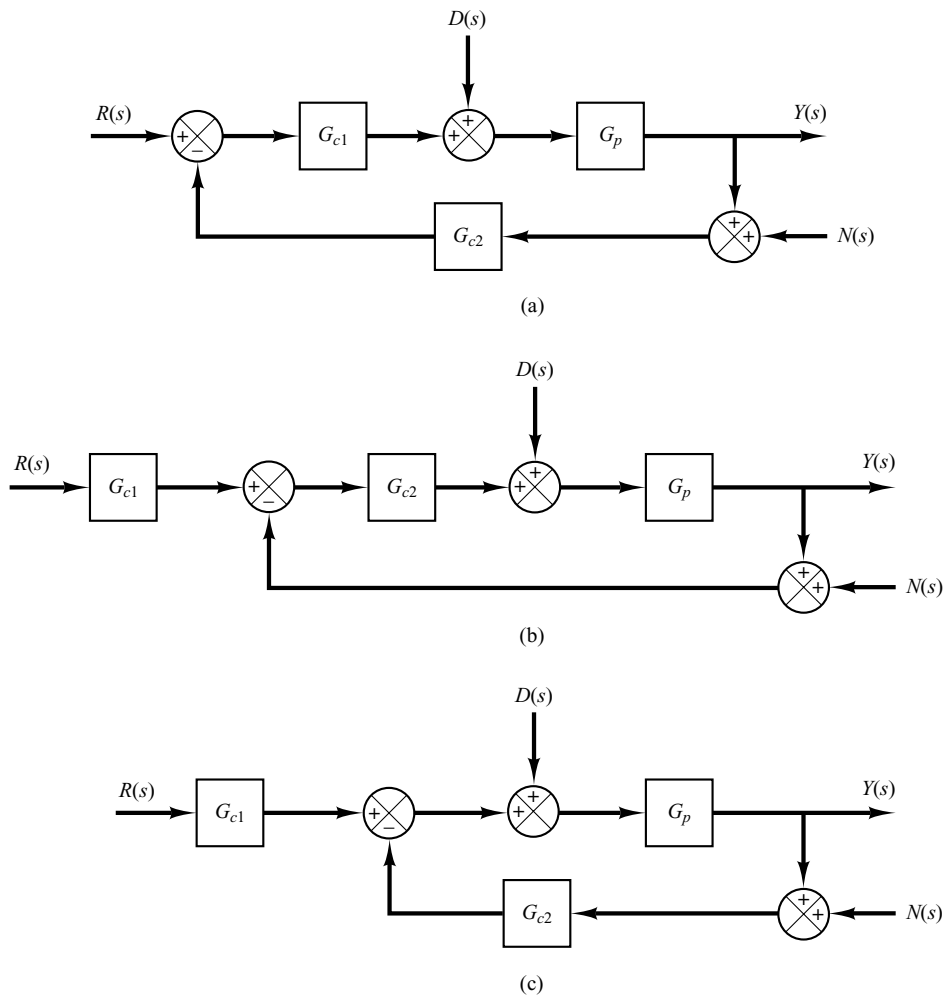


Figure 8-77
(a), (b), (c) Two degrees-of-freedom systems.

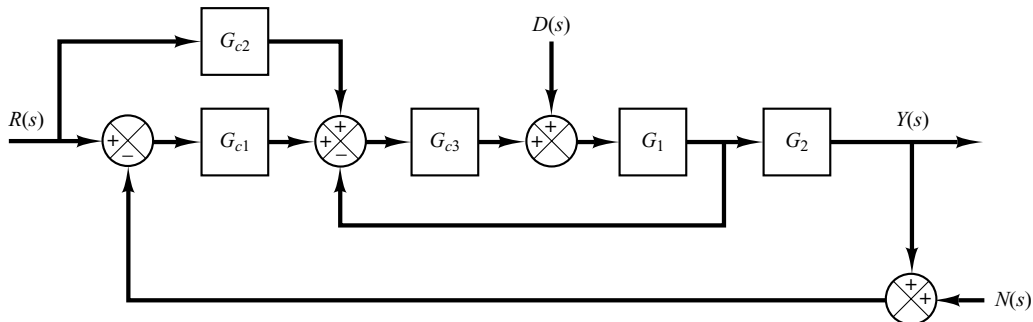


Figure 8-78
Three-degrees-of-freedom system.

B-8-11. Consider the control system shown in Figure 8-79. Assume that the PID controller is given by

$$G_c(s) = K \frac{(s + a)^2}{s}$$

It is desired that the unit-step response of the system exhibit the maximum overshoot of less than 10%, but more than 2% (to avoid an almost overdamped system), and the settling time be less than 2 sec.

Using the computational approach presented in Section 8-4, write a MATLAB program to determine the values of K and a that will satisfy the given specifications. Choose the search region to be

$$1 \leq K \leq 4, \quad 0.4 \leq a \leq 4$$

Choose the step size for K and a to be 0.05. Write the program such that the nested loops start with the highest values of K and a and step toward the lowest.

Using the first-found solution, plot the unit-step response curve.

B-8-12. Consider the same control system as treated in Problem B-8-11 (Figure 8-79). The PID controller is given by

$$G_c(s) = K \frac{(s + a)^2}{s}$$

It is desired to determine the values of K and a such that the unit-step response of the system exhibits the maximum

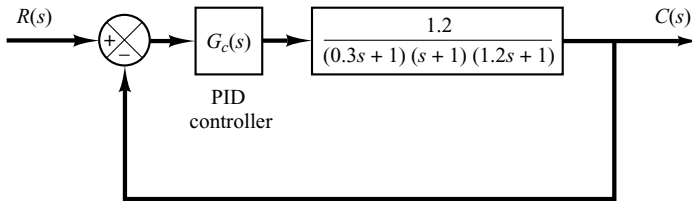


Figure 8-79
Control system.

overshoot of less than 8%, but more than 3%, and the settling time is less than 2 sec. Choose the search region to be

$$2 \leq K \leq 4, \quad 0.5 \leq a \leq 3$$

Choose the step size for K and a to be 0.05.

First, write a MATLAB program such that the nested loops in the program start with the highest values of K and a and step toward the lowest and the computation stops when a successful set of K and a is found for the first time.

Next, write a MATLAB program that will find all possible sets of K and a that will satisfy the given specifications.

Among multiple sets of K and a that satisfy the given specifications, determine the best choice. Then, plot the unit-step response curves of the system with the best choice of K and a .

B-8-13. Consider the two-degrees-of-freedom control system shown in Figure 8-80. The plant $G_p(s)$ is given by

$$G_p(s) = \frac{3(s + 5)}{s(s + 1)(s^2 + 4s + 13)}$$

Design controllers $G_{c1}(s)$ and $G_{c2}(s)$ such that the response to the unit-step disturbance input should have small amplitude and settle to zero quickly (in approximately 2 sec). The response to the unit-step reference input should be such that the maximum overshoot is 25% (or less) and the settling time is 2 sec. Also, the steady-state errors in the response to the ramp and acceleration reference inputs should be zero.

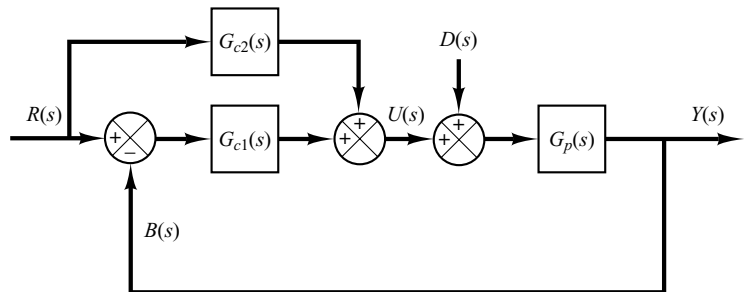


Figure 8-80
Two-degrees-of-freedom control system.

B-8-14. Consider the system shown in Figure 8-81. The plant $G_p(s)$ is given by

$$G_p(s) = \frac{2(s + 1)}{s(s + 3)(s + 5)}$$

Determine the controllers $G_{c1}(s)$ and $G_{c2}(s)$ such that, for the step disturbance input, the response shows a small amplitude and approaches zero quickly (in a matter of 1 to 2 sec). For the response to the unit-step reference input, it is desired that the maximum overshoot be 20% or less and the settling time 1 sec or less. For the ramp reference input and acceleration reference input, the steady-state errors should be zero.

B-8-15. Consider the two-degrees-of-freedom control system shown in Figure 8-82. Design controllers $G_{c1}(s)$ and $G_{c2}(s)$ such that the response to the step disturbance input shows a small amplitude and settles to zero quickly (in 1 to 2 sec) and the response to the step reference input exhibits 25% or less maximum overshoot and the settling time is less than 1 sec. The steady-state error in following the ramp reference input or acceleration reference input should be zero.

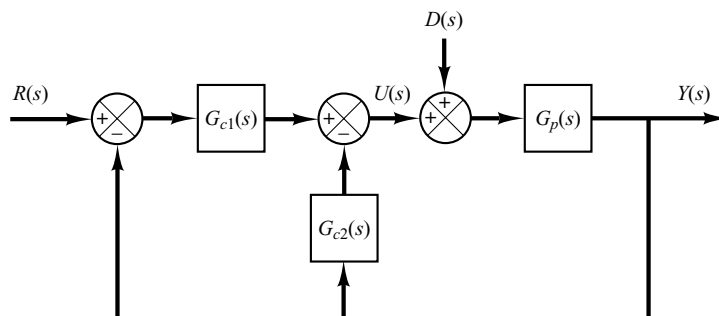


Figure 8-81
Two-degrees-of-freedom control system.

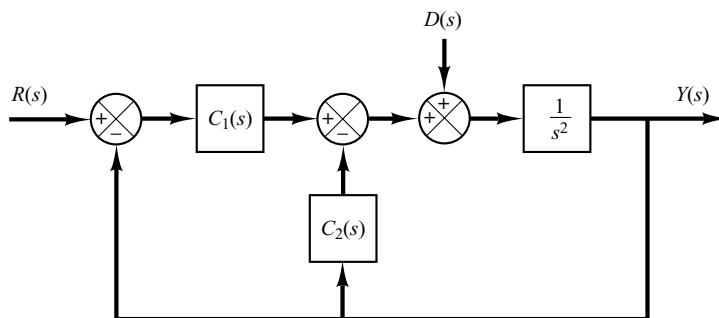


Figure 8-82
Two-degrees-of-freedom control system.



Control Systems Analysis in State Space

9-1 INTRODUCTION*

A modern complex system may have many inputs and many outputs, and these may be interrelated in a complicated manner. To analyze such a system, it is essential to reduce the complexity of the mathematical expressions, as well as to resort to computers for most of the tedious computations necessary in the analysis. The state-space approach to system analysis is best suited from this viewpoint.

While conventional control theory is based on the input–output relationship, or transfer function, modern control theory is based on the description of system equations in terms of n first-order differential equations, which may be combined into a first-order vector-matrix differential equation. The use of vector-matrix notation greatly simplifies the mathematical representation of systems of equations. The increase in the number of state variables, the number of inputs, or the number of outputs does not increase the complexity of the equations. In fact, the analysis of complicated multiple-input, multiple-output systems can be carried out by procedures that are only slightly more complicated than those required for the analysis of systems of first-order scalar differential equations.

This chapter and the next deal with the state-space analysis and design of control systems. Basic materials of state-space analysis, including the state-space representation of

* It is noted that in this book an asterisk used as a superscript of a matrix, such as \mathbf{A}^* , implies that it is a **conjugate transpose** of matrix \mathbf{A} . The conjugate transpose is the conjugate of the transpose of a matrix. For a real matrix (a matrix whose elements are all real), the conjugate transpose \mathbf{A}^* is the same as the transpose \mathbf{A}^T .

systems, controllability, and observability are presented in this chapter. Useful design methods based on state-feedback control are given in Chapter 10.

Outline of the Chapter. Section 9–1 has presented an introduction to state-space analysis of control systems. Section 9–2 deals with the state-space representation of transfer-function systems. Here we present various canonical forms of state-space equations. Section 9–3 discusses the transformation of system models (such as from transfer-function to state-space models, and vice versa) with MATLAB. Section 9–4 presents the solution of time-invariant state equations. Section 9–5 gives some useful results in vector-matrix analysis that are necessary in studying the state-space analysis of control systems. Section 9–6 discusses the controllability of control systems and Section 9–7 treats the observability of control systems.

9–2 STATE-SPACE REPRESENTATIONS OF TRANSFER-FUNCTION SYSTEMS

Many techniques are available for obtaining state-space representations of transfer-function systems. In Chapter 2 we presented a few such methods. This section presents state-space representations in the controllable, observable, diagonal, or Jordan canonical forms. (Methods for obtaining such state-space representations from transfer functions are discussed in detail in Problems **A–9–1** through **A–9–4**.)

State-Space Representations in Canonical Forms. Consider a system defined by

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} \dot{y} + a_n y = b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1} \dot{u} + b_n u \quad (9-1)$$

where u is the input and y is the output. This equation can also be written as

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} \quad (9-2)$$

In what follows we shall present state-space representations of the system defined by Equation (9–1) or (9–2) in controllable canonical form, observable canonical form, and diagonal (or Jordan) canonical form.

Controllable Canonical Form. The following state-space representation is called a controllable canonical form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} u \quad (9-3)$$

$$y = [b_n - a_n b_0 \quad b_{n-1} - a_{n-1} b_0 \quad \cdots \quad b_1 - a_1 b_0] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u \quad (9-4)$$

The controllable canonical form is important in discussing the pole-placement approach to control systems design.

Observable Canonical Form. The following state-space representation is called an observable canonical form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \cdot \\ \cdot \\ b_1 - a_1 b_0 \end{bmatrix} u \quad (9-5)$$

$$y = [0 \quad 0 \quad \cdots \quad 0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u \quad (9-6)$$

Note that the $n \times n$ state matrix of the state equation given by Equation (9-5) is the transpose of that of the state equation defined by Equation (9-3).

Diagonal Canonical Form. Consider the transfer-function system defined by Equation (9-2). Here we consider the case where the denominator polynomial involves only distinct roots. For the distinct-roots case, Equation (9-2) can be written as

$$\begin{aligned} \frac{Y(s)}{U(s)} &= \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{(s + p_1)(s + p_2) \cdots (s + p_n)} \\ &= b_0 + \frac{c_1}{s + p_1} + \frac{c_2}{s + p_2} + \cdots + \frac{c_n}{s + p_n} \end{aligned} \quad (9-7)$$

The diagonal canonical form of the state-space representation of this system is given by

EXAMPLE 9–1 Consider the system given by

$$\frac{Y(s)}{U(s)} = \frac{s + 3}{s^2 + 3s + 2}$$

Obtain state-space representations in the controllable canonical form, observable canonical form, and diagonal canonical form.

Controllable Canonical Form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [3 \quad 1] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Observable Canonical Form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [0 \quad 1] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Diagonal Canonical Form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [2 \quad -1] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Eigenvalues of an $n \times n$ Matrix \mathbf{A} . The eigenvalues of an $n \times n$ matrix \mathbf{A} are the roots of the characteristic equation

$$|\lambda \mathbf{I} - \mathbf{A}| = 0$$

The eigenvalues are also called the characteristic roots.

Consider, for example, the following matrix \mathbf{A} :

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

The characteristic equation is

$$\begin{aligned} |\lambda \mathbf{I} - \mathbf{A}| &= \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda + 6 \end{vmatrix} \\ &= \lambda^3 + 6\lambda^2 + 11\lambda + 6 \\ &= (\lambda + 1)(\lambda + 2)(\lambda + 3) = 0 \end{aligned}$$

The eigenvalues of \mathbf{A} are the roots of the characteristic equation, or -1 , -2 , and -3 .

Diagonalization of $n \times n$ Matrix. Note that if an $n \times n$ matrix \mathbf{A} with distinct eigenvalues is given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \quad (9-12)$$

the transformation $\mathbf{x} = \mathbf{Pz}$, where

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

$\lambda_1, \lambda_2, \dots, \lambda_n = n$ distinct eigenvalues of \mathbf{A}

will transform $\mathbf{P}^{-1}\mathbf{AP}$ into the diagonal matrix, or

$$\mathbf{P}^{-1}\mathbf{AP} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \lambda_n \\ 0 & & & & & \lambda_n \end{bmatrix}$$

If the matrix \mathbf{A} defined by Equation (9-12) involves multiple eigenvalues, then diagonalization is impossible. For example, if the 3×3 matrix \mathbf{A} , where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$$

has the eigenvalues $\lambda_1, \lambda_1, \lambda_3$, then the transformation $\mathbf{x} = \mathbf{Sz}$, where

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 1 \\ \lambda_1 & 1 & \lambda_3 \\ \lambda_1^2 & 2\lambda_1 & \lambda_3^2 \end{bmatrix}$$

will yield

$$\mathbf{S}^{-1}\mathbf{AS} = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

This is in the Jordan canonical form.

EXAMPLE 9-2 Consider the following state-space representation of a system.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} u \quad (9-13)$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (9-14)$$

Equations (9-13) and (9-14) can be put in a standard form as

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \quad (9-15)$$

$$y = \mathbf{Cx} \quad (9-16)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0 \quad 0]$$

The eigenvalues of matrix \mathbf{A} are

$$\lambda_1 = -1, \quad \lambda_2 = -2, \quad \lambda_3 = -3$$

Thus, three eigenvalues are distinct. If we define a set of new state variables z_1 , z_2 , and z_3 by the transformation

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

or

$$\mathbf{x} = \mathbf{Pz} \quad (9-17)$$

where

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \quad (9-18)$$

then, by substituting Equation (9-17) into Equation (9-15), we obtain

$$\mathbf{P}\dot{\mathbf{z}} = \mathbf{APz} + \mathbf{Bu}$$

By premultiplying both sides of this last equation by \mathbf{P}^{-1} , we get

$$\dot{\mathbf{z}} = \mathbf{P}^{-1}\mathbf{APz} + \mathbf{P}^{-1}\mathbf{Bu} \quad (9-19)$$

or

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} u$$

Simplifying gives

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} u \quad (9-20)$$

Equation (9-20) is also a state equation that describes the same system as defined by Equation (9-13).

The output equation, Equation (9-16), is modified to

$$y = \mathbf{CPz}$$

or

$$\begin{aligned} y &= [1 \quad 0 \quad 0] \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \\ &= [1 \quad 1 \quad 1] \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \end{aligned} \quad (9-21)$$

Notice that the transformation matrix \mathbf{P} , defined by Equation (9-18), modifies the coefficient matrix of \mathbf{z} into the diagonal matrix. As is clearly seen from Equation (9-20), the three scalar state equations are uncoupled. Notice also that the diagonal elements of the matrix $\mathbf{P}^{-1}\mathbf{AP}$ in Equation (9-19) are identical with the three eigenvalues of \mathbf{A} . It is very important to note that the eigenvalues of \mathbf{A} and those of $\mathbf{P}^{-1}\mathbf{AP}$ are identical. We shall prove this for a general case in what follows.

Invariance of Eigenvalues. To prove the invariance of the eigenvalues under a linear transformation, we must show that the characteristic polynomials $|\lambda\mathbf{I} - \mathbf{A}|$ and $|\lambda\mathbf{I} - \mathbf{P}^{-1}\mathbf{AP}|$ are identical.

Since the determinant of a product is the product of the determinants, we obtain

$$\begin{aligned} |\lambda\mathbf{I} - \mathbf{P}^{-1}\mathbf{AP}| &= |\lambda\mathbf{P}^{-1}\mathbf{P} - \mathbf{P}^{-1}\mathbf{AP}| \\ &= |\mathbf{P}^{-1}(\lambda\mathbf{I} - \mathbf{A})\mathbf{P}| \\ &= |\mathbf{P}^{-1}||\lambda\mathbf{I} - \mathbf{A}||\mathbf{P}| \\ &= |\mathbf{P}^{-1}||\mathbf{P}||\lambda\mathbf{I} - \mathbf{A}| \end{aligned}$$

Noting that the product of the determinants $|\mathbf{P}^{-1}|$ and $|\mathbf{P}|$ is the determinant of the product $|\mathbf{P}^{-1}\mathbf{P}|$, we obtain

$$\begin{aligned} |\lambda\mathbf{I} - \mathbf{P}^{-1}\mathbf{AP}| &= |\mathbf{P}^{-1}\mathbf{P}||\lambda\mathbf{I} - \mathbf{A}| \\ &= |\lambda\mathbf{I} - \mathbf{A}| \end{aligned}$$

Thus, we have proved that the eigenvalues of \mathbf{A} are invariant under a linear transformation.

Nonuniqueness of a Set of State Variables. It has been stated that a set of state variables is not unique for a given system. Suppose that x_1, x_2, \dots, x_n are a set of state variables.

Then we may take as another set of state variables any set of functions

$$\begin{aligned}\hat{x}_1 &= X_1(x_1, x_2, \dots, x_n) \\ \hat{x}_2 &= X_2(x_1, x_2, \dots, x_n) \\ &\cdot \\ &\cdot \\ \hat{x}_n &= X_n(x_1, x_2, \dots, x_n)\end{aligned}$$

provided that, for every set of values $\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n$, there corresponds a unique set of values x_1, x_2, \dots, x_n , and vice versa. Thus, if \mathbf{x} is a state vector, then $\hat{\mathbf{x}}$, where

$$\hat{\mathbf{x}} = \mathbf{P}\mathbf{x}$$

is also a state vector, provided the matrix \mathbf{P} is nonsingular. Different state vectors convey the same information about the system behavior.

9-3 TRANSFORMATION OF SYSTEM MODELS WITH MATLAB

In this section we shall consider the transformation of the system model from transfer function to state space, and vice versa. We shall begin our discussion with the transformation from transfer function to state space.

Let us write the closed-loop transfer function as

$$\frac{Y(s)}{U(s)} = \frac{\text{numerator polynomial in } s}{\text{denominator polynomial in } s} = \frac{\text{num}}{\text{den}}$$

Once we have this transfer-function expression, the MATLAB command

$$[A, B, C, D] = \text{tf2ss}(\text{num}, \text{den})$$

will give a state-space representation. It is important to note that the state-space representation for any system is not unique. There are many (indeed, infinitely many) state-space representations for the same system. The MATLAB command gives one possible such state-space representation.

State-Space Formulation of Transfer-Function Systems. Consider the transfer-function system

$$\frac{Y(s)}{U(s)} = \frac{10s + 10}{s^3 + 6s^2 + 5s + 10} \quad (9-22)$$

There are many (again, infinitely many) possible state-space representations for this system. One possible state-space representation is

$$\begin{aligned}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -5 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 10 \\ -50 \end{bmatrix} u \\ y &= [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u\end{aligned}$$

Another possible state-space representation (among infinitely many alternatives) is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -6 & -5 & -10 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \quad (9-23)$$

$$y = [0 \quad 10 \quad 10] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u \quad (9-24)$$

MATLAB transforms the transfer function given by Equation (9-22) into the state-space representation given by Equations (9-23) and (9-24). For the example system considered here, MATLAB Program 9-1 will produce matrices **A**, **B**, **C**, and **D**.

MATLAB Program 9-1	
<pre> num = [10 10]; den = [1 6 5 10]; [A,B,C,D] = tf2ss(num,den) A = -6 -5 -10 1 0 0 0 1 0 B = 1 0 0 C = 0 10 10 D = 0 </pre>	

Transformation from State Space to Transfer Function. To obtain the transfer function from state-space equations, use the following command:

$$[\text{num}, \text{den}] = \text{ss2tf}(A, B, C, D, iu)$$

iu must be specified for systems with more than one input. For example, if the system has three inputs (*u*₁, *u*₂, *u*₃), then *iu* must be either 1, 2, or 3, where 1 implies *u*₁, 2 implies *u*₂, and 3 implies *u*₃.

If the system has only one input, then either

$$[\text{num}, \text{den}] = \text{ss2tf}(A, B, C, D)$$

or

$$[\text{num}, \text{den}] = \text{ss2tf}(A, B, C, D, 1)$$

may be used. (See Example 9-3 and MATLAB Program 9-2.)

For the case where the system has multiple inputs and multiple outputs, see Example 9–4.

EXAMPLE 9–3 Obtain the transfer function of the system defined by the following state-space equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5.008 & -25.1026 & -5.03247 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 25.04 \\ -121.005 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

MATLAB Program 9–2 will produce the transfer function for the given system. The transfer function obtained is given by

$$\frac{Y(s)}{U(s)} = \frac{25.04s + 5.008}{s^3 + 5.0325s^2 + 25.1026s + 5.008}$$

MATLAB Program 9–2

```
A = [0 1 0;0 0 1;-5.008 -25.1026 -5.03247];
B = [0;25.04;-121.005];
C = [1 0 0];
D = [0];
[num,den] = ss2tf(A,B,C,D)

num =
    0 -0.0000 25.0400 5.0080
den =
    1.0000 5.0325 25.1026 5.0080
% ***** The same result can be obtained by entering the following command *****

[num,den] = ss2tf(A,B,C,D,1)
num =
    0 -0.0000 25.0400 5.0080
den =
    1.0000 5.0325 25.1026 5.0080
```

EXAMPLE 9–4 Consider a system with multiple inputs and multiple outputs. When the system has more than one output, the command

$$[\text{NUM},\text{den}] = \text{ss2tf}(\text{A},\text{B},\text{C},\text{D},\text{i},\text{u})$$

produces transfer functions for all outputs to each input. (The numerator coefficients are returned to matrix NUM with as many rows as there are outputs.)

Consider the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -25 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

This system involves two inputs and two outputs. Four transfer functions are involved: $Y_1(s)/U_1(s)$, $Y_2(s)/U_1(s)$, $Y_1(s)/U_2(s)$, and $Y_2(s)/U_2(s)$. (When considering input u_1 , we assume that input u_2 is zero and vice versa.) See the output of MATLAB Program 9–3.

MATLAB Program 9–3	
A = [0 1;-25 -4];	
B = [1 1;0 1];	
C = [1 0;0 1];	
D = [0 0;0 0];	
[NUM,den] = ss2tf(A,B,C,D,1)	
NUM =	
0	1 4
0	0 -25
den =	
1	4 25
[NUM,den] = ss2tf(A,B,C,D,2)	
NUM =	
0	1.0000 5.0000
0	1.0000 -25.0000
den =	
1	4 25

This is the MATLAB representation of the following four transfer functions:

$$\frac{Y_1(s)}{U_1(s)} = \frac{s + 4}{s^2 + 4s + 25}, \quad \frac{Y_2(s)}{U_1(s)} = \frac{-25}{s^2 + 4s + 25}$$

$$\frac{Y_1(s)}{U_2(s)} = \frac{s + 5}{s^2 + 4s + 25}, \quad \frac{Y_2(s)}{U_2(s)} = \frac{s - 25}{s^2 + 4s + 25}$$

9-4 SOLVING THE TIME-INVARIANT STATE EQUATION

In this section, we shall obtain the general solution of the linear time-invariant state equation. We shall first consider the homogeneous case and then the nonhomogeneous case.

Solution of Homogeneous State Equations. Before we solve vector-matrix differential equations, let us review the solution of the scalar differential equation

$$\dot{x} = ax \quad (9-25)$$

In solving this equation, we may assume a solution $x(t)$ of the form

$$x(t) = b_0 + b_1t + b_2t^2 + \cdots + b_k t^k + \cdots \quad (9-26)$$

By substituting this assumed solution into Equation (9-25), we obtain

$$\begin{aligned} b_1 + 2b_2t + 3b_3t^2 + \cdots + kb_k t^{k-1} + \cdots \\ = a(b_0 + b_1t + b_2t^2 + \cdots + b_k t^k + \cdots) \end{aligned} \quad (9-27)$$

If the assumed solution is to be the true solution, Equation (9-27) must hold for any t . Hence, equating the coefficients of the equal powers of t , we obtain

$$\begin{aligned} b_1 &= ab_0 \\ b_2 &= \frac{1}{2} ab_1 = \frac{1}{2} a^2 b_0 \\ b_3 &= \frac{1}{3} ab_2 = \frac{1}{3 \times 2} a^3 b_0 \\ &\vdots \\ &\vdots \\ &\vdots \\ b_k &= \frac{1}{k!} a^k b_0 \end{aligned}$$

The value of b_0 is determined by substituting $t = 0$ into Equation (9-26), or

$$x(0) = b_0$$

Hence, the solution $x(t)$ can be written as

$$\begin{aligned} x(t) &= \left(1 + at + \frac{1}{2!} a^2 t^2 + \cdots + \frac{1}{k!} a^k t^k + \cdots \right) x(0) \\ &= e^{at} x(0) \end{aligned}$$

We shall now solve the vector-matrix differential equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad (9-28)$$

where $\mathbf{x} = n$ -vector

$\mathbf{A} = n \times n$ constant matrix

By analogy with the scalar case, we assume that the solution is in the form of a vector power series in t , or

$$\mathbf{x}(t) = \mathbf{b}_0 + \mathbf{b}_1 t + \mathbf{b}_2 t^2 + \cdots + \mathbf{b}_k t^k + \cdots \quad (9-29)$$

By substituting this assumed solution into Equation (9–28), we obtain

$$\begin{aligned} & \mathbf{b}_1 + 2\mathbf{b}_2t + 3\mathbf{b}_3t^2 + \cdots + k\mathbf{b}_kt^{k-1} + \cdots \\ & = \mathbf{A}(\mathbf{b}_0 + \mathbf{b}_1t + \mathbf{b}_2t^2 + \cdots + \mathbf{b}_kt^k + \cdots) \end{aligned} \quad (9-30)$$

If the assumed solution is to be the true solution, Equation (9–30) must hold for all t . Thus, by equating the coefficients of like powers of t on both sides of Equation (9–30), we obtain

$$\begin{aligned} \mathbf{b}_1 &= \mathbf{A}\mathbf{b}_0 \\ \mathbf{b}_2 &= \frac{1}{2}\mathbf{A}\mathbf{b}_1 = \frac{1}{2}\mathbf{A}^2\mathbf{b}_0 \\ \mathbf{b}_3 &= \frac{1}{3}\mathbf{A}\mathbf{b}_2 = \frac{1}{3 \times 2}\mathbf{A}^3\mathbf{b}_0 \\ &\vdots \\ &\vdots \\ &\vdots \\ \mathbf{b}_k &= \frac{1}{k!}\mathbf{A}^k\mathbf{b}_0 \end{aligned}$$

By substituting $t = 0$ into Equation (9–29), we obtain

$$\mathbf{x}(0) = \mathbf{b}_0$$

Thus, the solution $\mathbf{x}(t)$ can be written as

$$\mathbf{x}(t) = \left(\mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \cdots + \frac{1}{k!}\mathbf{A}^kt^k + \cdots \right) \mathbf{x}(0)$$

The expression in the parentheses on the right-hand side of this last equation is an $n \times n$ matrix. Because of its similarity to the infinite power series for a scalar exponential, we call it the matrix exponential and write

$$\mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \cdots + \frac{1}{k!}\mathbf{A}^kt^k + \cdots = e^{\mathbf{A}t}$$

In terms of the matrix exponential, the solution of Equation (9–28) can be written as

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) \quad (9-31)$$

Since the matrix exponential is very important in the state-space analysis of linear systems, we shall next examine its properties.

Matrix Exponential. It can be proved that the matrix exponential of an $n \times n$ matrix \mathbf{A} ,

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^kt^k}{k!}$$

converges absolutely for all finite t . (Hence, computer calculations for evaluating the elements of $e^{\mathbf{A}t}$ by using the series expansion can be easily carried out.)

Because of the convergence of the infinite series $\sum_{k=0}^{\infty} \mathbf{A}^k t^k / k!$, the series can be differentiated term by term to give

$$\begin{aligned} \frac{d}{dt} e^{\mathbf{A}t} &= \mathbf{A} + \mathbf{A}^2 t + \frac{\mathbf{A}^3 t^2}{2!} + \cdots + \frac{\mathbf{A}^k t^{k-1}}{(k-1)!} + \cdots \\ &= \mathbf{A} \left[\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \cdots + \frac{\mathbf{A}^{k-1} t^{k-1}}{(k-1)!} + \cdots \right] = \mathbf{A} e^{\mathbf{A}t} \\ &= \left[\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \cdots + \frac{\mathbf{A}^{k-1} t^{k-1}}{(k-1)!} + \cdots \right] \mathbf{A} = e^{\mathbf{A}t} \mathbf{A} \end{aligned}$$

The matrix exponential has the property that

$$e^{\mathbf{A}(t+s)} = e^{\mathbf{A}t} e^{\mathbf{A}s}$$

This can be proved as follows:

$$\begin{aligned} e^{\mathbf{A}t} e^{\mathbf{A}s} &= \left(\sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!} \right) \left(\sum_{k=0}^{\infty} \frac{\mathbf{A}^k s^k}{k!} \right) \\ &= \sum_{k=0}^{\infty} \mathbf{A}^k \left(\sum_{i=0}^{\infty} \frac{t^i s^{k-i}}{i! (k-i)!} \right) \\ &= \sum_{k=0}^{\infty} \mathbf{A}^k \frac{(t+s)^k}{k!} \\ &= e^{\mathbf{A}(t+s)} \end{aligned}$$

In particular, if $s = -t$, then

$$e^{\mathbf{A}t} e^{-\mathbf{A}t} = e^{-\mathbf{A}t} e^{\mathbf{A}t} = e^{\mathbf{A}(t-t)} = \mathbf{I}$$

Thus, the inverse of $e^{\mathbf{A}t}$ is $e^{-\mathbf{A}t}$. Since the inverse of $e^{\mathbf{A}t}$ always exists, $e^{\mathbf{A}t}$ is nonsingular.

It is very important to remember that

$$\begin{aligned} e^{(\mathbf{A}+\mathbf{B})t} &= e^{\mathbf{A}t} e^{\mathbf{B}t}, & \text{if } \mathbf{AB} = \mathbf{BA} \\ e^{(\mathbf{A}+\mathbf{B})t} &\neq e^{\mathbf{A}t} e^{\mathbf{B}t}, & \text{if } \mathbf{AB} \neq \mathbf{BA} \end{aligned}$$

To prove this, note that

$$\begin{aligned} e^{(\mathbf{A}+\mathbf{B})t} &= \mathbf{I} + (\mathbf{A} + \mathbf{B})t + \frac{(\mathbf{A} + \mathbf{B})^2}{2!} t^2 + \frac{(\mathbf{A} + \mathbf{B})^3}{3!} t^3 + \cdots \\ e^{\mathbf{A}t} e^{\mathbf{B}t} &= \left(\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + \cdots \right) \left(\mathbf{I} + \mathbf{B}t + \frac{\mathbf{B}^2 t^2}{2!} + \frac{\mathbf{B}^3 t^3}{3!} + \cdots \right) \\ &= \mathbf{I} + (\mathbf{A} + \mathbf{B})t + \frac{\mathbf{A}^2 t^2}{2!} + \mathbf{AB}t^2 + \frac{\mathbf{B}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} \\ &\quad + \frac{\mathbf{A}^2 \mathbf{B} t^3}{2!} + \frac{\mathbf{AB}^2 t^3}{2!} + \frac{\mathbf{B}^3 t^3}{3!} + \cdots \end{aligned}$$

Hence,

$$e^{(\mathbf{A}+\mathbf{B})t} - e^{\mathbf{A}t}e^{\mathbf{B}t} = \frac{\mathbf{BA} - \mathbf{AB}}{2!} t^2 + \frac{\mathbf{BA}^2 + \mathbf{ABA} + \mathbf{B}^2\mathbf{A} + \mathbf{BAB} - 2\mathbf{A}^2\mathbf{B} - 2\mathbf{AB}^2}{3!} t^3 + \dots$$

The difference between $e^{(\mathbf{A}+\mathbf{B})t}$ and $e^{\mathbf{A}t}e^{\mathbf{B}t}$ vanishes if \mathbf{A} and \mathbf{B} commute.

Laplace Transform Approach to the Solution of Homogeneous State Equations. Let us first consider the scalar case:

$$\dot{x} = ax \quad (9-32)$$

Taking the Laplace transform of Equation (9-32), we obtain

$$sX(s) - x(0) = aX(s) \quad (9-33)$$

where $X(s) = \mathcal{L}[x]$. Solving Equation (9-33) for $X(s)$ gives

$$X(s) = \frac{x(0)}{s-a} = (s-a)^{-1}x(0)$$

The inverse Laplace transform of this last equation gives the solution

$$x(t) = e^{at}x(0)$$

The foregoing approach to the solution of the homogeneous scalar differential equation can be extended to the homogeneous state equation:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \quad (9-34)$$

Taking the Laplace transform of both sides of Equation (9-34), we obtain

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s)$$

where $\mathbf{X}(s) = \mathcal{L}[\mathbf{x}]$. Hence,

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{x}(0)$$

Premultiplying both sides of this last equation by $(s\mathbf{I} - \mathbf{A})^{-1}$, we obtain

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0)$$

The inverse Laplace transform of $\mathbf{X}(s)$ gives the solution $\mathbf{x}(t)$. Thus,

$$\mathbf{x}(t) = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}]\mathbf{x}(0) \quad (9-35)$$

Note that

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\mathbf{I}}{s} + \frac{\mathbf{A}}{s^2} + \frac{\mathbf{A}^2}{s^3} + \dots$$

Hence, the inverse Laplace transform of $(s\mathbf{I} - \mathbf{A})^{-1}$ gives

$$\mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + \dots = e^{\mathbf{A}t} \quad (9-36)$$

(The inverse Laplace transform of a matrix is the matrix consisting of the inverse Laplace transforms of all elements.) From Equations (9–35) and (9–36), the solution of Equation (9–34) is obtained as

$$\mathbf{x}(t) = e^{\mathbf{A}t} \mathbf{x}(0)$$

The importance of Equation (9–36) lies in the fact that it provides a convenient means for finding the closed solution for the matrix exponential.

State-Transition Matrix. We can write the solution of the homogeneous state equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad (9-37)$$

as

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) \quad (9-38)$$

where $\mathbf{\Phi}(t)$ is an $n \times n$ matrix and is the unique solution of

$$\dot{\mathbf{\Phi}}(t) = \mathbf{A}\mathbf{\Phi}(t), \quad \mathbf{\Phi}(0) = \mathbf{I}$$

To verify this, note that

$$\mathbf{x}(0) = \mathbf{\Phi}(0)\mathbf{x}(0) = \mathbf{x}(0)$$

and

$$\dot{\mathbf{x}}(t) = \dot{\mathbf{\Phi}}(t)\mathbf{x}(0) = \mathbf{A}\mathbf{\Phi}(t)\mathbf{x}(0) = \mathbf{A}\mathbf{x}(t)$$

We thus confirm that Equation (9–38) is the solution of Equation (9–37).

From Equations (9–31), (9–35), and (9–38), we obtain

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}]$$

Note that

$$\mathbf{\Phi}^{-1}(t) = e^{-\mathbf{A}t} = \mathbf{\Phi}(-t)$$

From Equation (9–38), we see that the solution of Equation (9–37) is simply a transformation of the initial condition. Hence, the unique matrix $\mathbf{\Phi}(t)$ is called the state-transition matrix. The state-transition matrix contains all the information about the free motions of the system defined by Equation (9–37).

If the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of the matrix \mathbf{A} are distinct, then $\mathbf{\Phi}(t)$ will contain the n exponentials

$$e^{\lambda_1 t}, e^{\lambda_2 t}, \dots, e^{\lambda_n t}$$

In particular, if the matrix \mathbf{A} is diagonal, then

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \begin{bmatrix} e^{\lambda_1 t} & & & & 0 \\ & e^{\lambda_2 t} & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & e^{\lambda_n t} \end{bmatrix} \quad (\mathbf{A}: \text{diagonal})$$

If there is a multiplicity in the eigenvalues—for example, if the eigenvalues of \mathbf{A} are

$$\lambda_1, \lambda_1, \lambda_1, \lambda_4, \lambda_5, \dots, \lambda_n,$$

then $\Phi(t)$ will contain, in addition to the exponentials $e^{\lambda_1 t}, e^{\lambda_4 t}, e^{\lambda_5 t}, \dots, e^{\lambda_n t}$, terms like $t e^{\lambda_1 t}$ and $t^2 e^{\lambda_1 t}$.

Properties of State-Transition Matrices. We shall now summarize the important properties of the state-transition matrix $\Phi(t)$. For the time-invariant system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

for which

$$\Phi(t) = e^{\mathbf{A}t}$$

we have the following:

1. $\Phi(0) = e^{\mathbf{A}0} = \mathbf{I}$
2. $\Phi(t) = e^{\mathbf{A}t} = (e^{-\mathbf{A}t})^{-1} = [\Phi(-t)]^{-1}$ or $\Phi^{-1}(t) = \Phi(-t)$
3. $\Phi(t_1 + t_2) = e^{\mathbf{A}(t_1+t_2)} = e^{\mathbf{A}t_1} e^{\mathbf{A}t_2} = \Phi(t_1)\Phi(t_2) = \Phi(t_2)\Phi(t_1)$
4. $[\Phi(t)]^n = \Phi(nt)$
5. $\Phi(t_2 - t_1)\Phi(t_1 - t_0) = \Phi(t_2 - t_0) = \Phi(t_1 - t_0)\Phi(t_2 - t_1)$

EXAMPLE 9-5 Obtain the state-transition matrix $\Phi(t)$ of the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Obtain also the inverse of the state-transition matrix, $\Phi^{-1}(t)$.

For this system,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

The state-transition matrix $\Phi(t)$ is given by

$$\Phi(t) = e^{\mathbf{A}t} = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}]$$

Since

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

the inverse of $(s\mathbf{I} - \mathbf{A})$ is given by

$$\begin{aligned} (s\mathbf{I} - \mathbf{A})^{-1} &= \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \\ &= \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} \end{aligned}$$

Hence,

$$\begin{aligned}\Phi(t) &= e^{\mathbf{A}t} = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] \\ &= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}\end{aligned}$$

Noting that $\Phi^{-1}(t) = \Phi(-t)$, we obtain the inverse of the state-transition matrix as follows:

$$\Phi^{-1}(t) = e^{-\mathbf{A}t} = \begin{bmatrix} 2e^t - e^{2t} & e^t - e^{2t} \\ -2e^t + 2e^{2t} & -e^t + 2e^{2t} \end{bmatrix}$$

Solution of Nonhomogeneous State Equations. We shall begin by considering the scalar case

$$\dot{x} = ax + bu \quad (9-39)$$

Let us rewrite Equation (9-39) as

$$\dot{x} - ax = bu$$

Multiplying both sides of this equation by e^{-at} , we obtain

$$e^{-at}[\dot{x}(t) - ax(t)] = \frac{d}{dt}[e^{-at}x(t)] = e^{-at}bu(t)$$

Integrating this equation between 0 and t gives

$$e^{-at}x(t) - x(0) = \int_0^t e^{-a\tau}bu(\tau) d\tau$$

or

$$x(t) = e^{at}x(0) + e^{at} \int_0^t e^{-a\tau}bu(\tau) d\tau$$

The first term on the right-hand side is the response to the initial condition and the second term is the response to the input $u(t)$.

Let us now consider the nonhomogeneous state equation described by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (9-40)$$

where $\mathbf{x} = n$ -vector

$\mathbf{u} = r$ -vector

$\mathbf{A} = n \times n$ constant matrix

$\mathbf{B} = n \times r$ constant matrix

By writing Equation (9-40) as

$$\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) = \mathbf{B}\mathbf{u}(t)$$

and premultiplying both sides of this equation by $e^{-\mathbf{A}t}$, we obtain

$$e^{-\mathbf{A}t}[\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t)] = \frac{d}{dt}[e^{-\mathbf{A}t}\mathbf{x}(t)] = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t)$$

Integrating the preceding equation between 0 and t gives

$$e^{-\mathbf{A}t}\mathbf{x}(t) - \mathbf{x}(0) = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau) d\tau$$

or

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau) d\tau \quad (9-41)$$

Equation (9-41) can also be written as

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}(0) + \int_0^t \Phi(t-\tau)\mathbf{B}\mathbf{u}(\tau) d\tau \quad (9-42)$$

where $\Phi(t) = e^{\mathbf{A}t}$. Equation (9-41) or (9-42) is the solution of Equation (9-40). The solution $\mathbf{x}(t)$ is clearly the sum of a term consisting of the transition of the initial state and a term arising from the input vector.

Laplace Transform Approach to the Solution of Nonhomogeneous State Equations. The solution of the nonhomogeneous state equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

can also be obtained by the Laplace transform approach. The Laplace transform of this last equation yields

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

or

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)$$

Premultiplying both sides of this last equation by $(s\mathbf{I} - \mathbf{A})^{-1}$, we obtain

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

Using the relationship given by Equation (9-36) gives

$$\mathbf{X}(s) = \mathcal{L}[e^{\mathbf{A}t}]\mathbf{x}(0) + \mathcal{L}[e^{\mathbf{A}t}]\mathbf{B}\mathbf{U}(s)$$

The inverse Laplace transform of this last equation can be obtained by use of the convolution integral as follows:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau) d\tau$$

Solution in Terms of $\mathbf{x}(t_0)$. Thus far we have assumed the initial time to be zero. If, however, the initial time is given by t_0 instead of 0, then the solution to Equation (9-40) must be modified to

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau) d\tau \quad (9-43)$$

EXAMPLE 9–6 Obtain the time response of the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

where $u(t)$ is the unit-step function occurring at $t = 0$, or

$$u(t) = 1(t)$$

For this system,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The state-transition matrix $\Phi(t) = e^{\mathbf{A}t}$ was obtained in Example 9–5 as

$$\Phi(t) = e^{\mathbf{A}t} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

The response to the unit-step input is then obtained as

$$\mathbf{x}(t) = e^{\mathbf{A}t} \mathbf{x}(0) + \int_0^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} [1] d\tau$$

or

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

If the initial state is zero, or $\mathbf{x}(0) = \mathbf{0}$, then $\mathbf{x}(t)$ can be simplified to

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

9–5 SOME USEFUL RESULTS IN VECTOR-MATRIX ANALYSIS

In this section we present some useful results in vector-matrix analysis that we use in Section 9–6. Specifically, we present the Cayley–Hamilton theorem, the minimal polynomial, Sylvester’s interpolation method for calculating $e^{\mathbf{A}t}$, and the linear independence of vectors.

Cayley–Hamilton Theorem. The Cayley–Hamilton theorem is very useful in proving theorems involving matrix equations or solving problems involving matrix equations.

Consider an $n \times n$ matrix \mathbf{A} and its characteristic equation:

$$|\lambda \mathbf{I} - \mathbf{A}| = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1} \lambda + a_n = 0$$

The Cayley–Hamilton theorem states that the matrix \mathbf{A} satisfies its own characteristic equation, or that

$$\mathbf{A}^n + a_1 \mathbf{A}^{n-1} + \cdots + a_{n-1} \mathbf{A} + a_n \mathbf{I} = \mathbf{0} \quad (9-44)$$

To prove this theorem, note that $\text{adj}(\lambda \mathbf{I} - \mathbf{A})$ is a polynomial in λ of degree $n - 1$. That is,

$$\text{adj}(\lambda \mathbf{I} - \mathbf{A}) = \mathbf{B}_1 \lambda^{n-1} + \mathbf{B}_2 \lambda^{n-2} + \cdots + \mathbf{B}_{n-1} \lambda + \mathbf{B}_n$$

where $\mathbf{B}_1 = \mathbf{I}$. Since

$$(\lambda \mathbf{I} - \mathbf{A}) \operatorname{adj}(\lambda \mathbf{I} - \mathbf{A}) = [\operatorname{adj}(\lambda \mathbf{I} - \mathbf{A})](\lambda \mathbf{I} - \mathbf{A}) = |\lambda \mathbf{I} - \mathbf{A}| \mathbf{I}$$

we obtain

$$\begin{aligned} |\lambda \mathbf{I} - \mathbf{A}| \mathbf{I} &= \mathbf{I} \lambda^n + a_1 \mathbf{I} \lambda^{n-1} + \cdots + a_{n-1} \mathbf{I} \lambda + a_n \mathbf{I} \\ &= (\lambda \mathbf{I} - \mathbf{A})(\mathbf{B}_1 \lambda^{n-1} + \mathbf{B}_2 \lambda^{n-2} + \cdots + \mathbf{B}_{n-1} \lambda + \mathbf{B}_n) \\ &= (\mathbf{B}_1 \lambda^{n-1} + \mathbf{B}_2 \lambda^{n-2} + \cdots + \mathbf{B}_{n-1} \lambda + \mathbf{B}_n)(\lambda \mathbf{I} - \mathbf{A}) \end{aligned}$$

From this equation, we see that \mathbf{A} and \mathbf{B}_i ($i = 1, 2, \dots, n$) commute. Hence, the product of $(\lambda \mathbf{I} - \mathbf{A})$ and $\operatorname{adj}(\lambda \mathbf{I} - \mathbf{A})$ becomes zero if either of these is zero. If \mathbf{A} is substituted for λ in this last equation, then clearly $\lambda \mathbf{I} - \mathbf{A}$ becomes zero. Hence, we obtain

$$\mathbf{A}^n + a_1 \mathbf{A}^{n-1} + \cdots + a_{n-1} \mathbf{A} + a_n \mathbf{I} = \mathbf{0}$$

This proves the Cayley–Hamilton theorem, or Equation (9–44).

Minimal Polynomial. Referring to the Cayley–Hamilton theorem, every $n \times n$ matrix \mathbf{A} satisfies its own characteristic equation. The characteristic equation is not, however, necessarily the scalar equation of least degree that \mathbf{A} satisfies. The least-degree polynomial having \mathbf{A} as a root is called the *minimal polynomial*. That is, the minimal polynomial of an $n \times n$ matrix \mathbf{A} is defined as the polynomial $\phi(\lambda)$ of least degree,

$$\phi(\lambda) = \lambda^m + a_1 \lambda^{m-1} + \cdots + a_{m-1} \lambda + a_m, \quad m \leq n$$

such that $\phi(\mathbf{A}) = \mathbf{0}$, or

$$\phi(\mathbf{A}) = \mathbf{A}^m + a_1 \mathbf{A}^{m-1} + \cdots + a_{m-1} \mathbf{A} + a_m \mathbf{I} = \mathbf{0}$$

The minimal polynomial plays an important role in the computation of polynomials in an $n \times n$ matrix.

Let us suppose that $d(\lambda)$, a polynomial in λ , is the greatest common divisor of all the elements of $\operatorname{adj}(\lambda \mathbf{I} - \mathbf{A})$. We can show that if the coefficient of the highest-degree term in λ of $d(\lambda)$ is chosen as 1, then the minimal polynomial $\phi(\lambda)$ is given by

$$\phi(\lambda) = \frac{|\lambda \mathbf{I} - \mathbf{A}|}{d(\lambda)} \quad (9-45)$$

[See Problem **A–9–8** for the derivation of Equation (9–45).]

It is noted that the minimal polynomial $\phi(\lambda)$ of an $n \times n$ matrix \mathbf{A} can be determined by the following procedure:

1. Form $\operatorname{adj}(\lambda \mathbf{I} - \mathbf{A})$ and write the elements of $\operatorname{adj}(\lambda \mathbf{I} - \mathbf{A})$ as factored polynomials in λ .
2. Determine $d(\lambda)$ as the greatest common divisor of all the elements of $\operatorname{adj}(\lambda \mathbf{I} - \mathbf{A})$. Choose the coefficient of the highest-degree term in λ of $d(\lambda)$ to be 1. If there is no common divisor, $d(\lambda) = 1$.
3. The minimal polynomial $\phi(\lambda)$ is then given as $|\lambda \mathbf{I} - \mathbf{A}|$ divided by $d(\lambda)$.

Matrix Exponential $e^{\mathbf{A}t}$. In solving control engineering problems, it often becomes necessary to compute $e^{\mathbf{A}t}$. If matrix \mathbf{A} is given with all elements in numerical values, MATLAB provides a simple way to compute $e^{\mathbf{A}T}$, where T is a constant.

Aside from computational methods, several analytical methods are available for the computation of $e^{\mathbf{A}t}$. We shall present three methods here.

Computation of $e^{\mathbf{A}t}$: Method 1. If matrix \mathbf{A} can be transformed into a diagonal form, then $e^{\mathbf{A}t}$ can be given by

$$e^{\mathbf{A}t} = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1} = \mathbf{P} \begin{bmatrix} e^{\lambda_1 t} & & & & 0 \\ & e^{\lambda_2 t} & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & e^{\lambda_n t} \end{bmatrix} \mathbf{P}^{-1} \quad (9-46)$$

where \mathbf{P} is a diagonalizing matrix for \mathbf{A} . [For the derivation of Equation (9-46), see Problem A-9-11.]

If matrix \mathbf{A} can be transformed into a Jordan canonical form, then $e^{\mathbf{A}t}$ can be given by

$$e^{\mathbf{A}t} = \mathbf{S}e^{\mathbf{J}t}\mathbf{S}^{-1}$$

where \mathbf{S} is a transformation matrix that transforms matrix \mathbf{A} into a Jordan canonical form \mathbf{J} .

As an example, consider the following matrix \mathbf{A} :

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$$

The characteristic equation is

$$|\lambda\mathbf{I} - \mathbf{A}| = \lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)^3 = 0$$

Thus, matrix \mathbf{A} has a multiple eigenvalue of order 3 at $\lambda = 1$. It can be shown that matrix \mathbf{A} has a multiple eigenvector of order 3. The transformation matrix that will transform matrix \mathbf{A} into a Jordan canonical form can be given by

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

The inverse of matrix \mathbf{S} is

$$\mathbf{S}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

Then it can be seen that

$$\begin{aligned} \mathbf{S}^{-1}\mathbf{A}\mathbf{S} &= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{J} \end{aligned}$$

Noting that

$$e^{\mathbf{J}t} = \begin{bmatrix} e^t & te^t & \frac{1}{2}t^2e^t \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix}$$

we find

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{S}e^{\mathbf{J}t}\mathbf{S}^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} e^t & te^t & \frac{1}{2}t^2e^t \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^t - te^t + \frac{1}{2}t^2e^t & te^t - t^2e^t & \frac{1}{2}t^2e^t \\ \frac{1}{2}t^2e^t & e^t - te^t - t^2e^t & te^t + \frac{1}{2}t^2e^t \\ te^t + \frac{1}{2}t^2e^t & -3te^t - t^2e^t & e^t + 2te^t + \frac{1}{2}t^2e^t \end{bmatrix} \end{aligned}$$

Computation of $e^{\mathbf{A}t}$: Method 2. The second method of computing $e^{\mathbf{A}t}$ uses the Laplace transform approach. Referring to Equation (9–36), $e^{\mathbf{A}t}$ can be given as follows:

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}]$$

Thus, to obtain $e^{\mathbf{A}t}$, first invert the matrix $(s\mathbf{I} - \mathbf{A})$. This results in a matrix whose elements are rational functions of s . Then take the inverse Laplace transform of each element of the matrix.

EXAMPLE 9–7 Consider the following matrix \mathbf{A} :

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$$

Compute $e^{\mathbf{A}t}$ by use of the two analytical methods presented previously.

Method 1. The eigenvalues of \mathbf{A} are 0 and -2 ($\lambda_1 = 0$, $\lambda_2 = -2$). A necessary transformation matrix \mathbf{P} may be obtained as

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$$

Then, from Equation (9–46), $e^{\mathbf{A}t}$ is obtained as follows:

$$e^{\mathbf{A}t} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} e^0 & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2}(1 - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}$$

Method 2. Since

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 0 & s + 2 \end{bmatrix}$$

we obtain

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

Hence,

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] = \begin{bmatrix} 1 & \frac{1}{2}(1 - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}$$

Computation of $e^{\mathbf{A}t}$: Method 3. The third method is based on Sylvester's interpolation method. (For Sylvester's interpolation formula, see Problem **A-9-12**.) We shall first consider the case where the roots of the minimal polynomial $\phi(\lambda)$ of \mathbf{A} are distinct. Then we shall deal with the case of multiple roots.

Case 1: Minimal Polynomial of \mathbf{A} Involves Only Distinct Roots. We shall assume that the degree of the minimal polynomial of \mathbf{A} is m . By using Sylvester's interpolation formula, it can be shown that $e^{\mathbf{A}t}$ can be obtained by solving the following determinant equation:

$$\begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{m-1} & e^{\lambda_1 t} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{m-1} & e^{\lambda_2 t} \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 1 & \lambda_m & \lambda_m^2 & \cdots & \lambda_m^{m-1} & e^{\lambda_m t} \\ \mathbf{I} & \mathbf{A} & \mathbf{A}^2 & \cdots & \mathbf{A}^{m-1} & e^{\mathbf{A}t} \end{vmatrix} = \mathbf{0} \quad (9-47)$$

By solving Equation (9-47) for $e^{\mathbf{A}t}$, $e^{\mathbf{A}t}$ can be obtained in terms of the \mathbf{A}^k ($k = 0, 1, 2, \dots, m-1$) and the $e^{\lambda_i t}$ ($i = 1, 2, 3, \dots, m$). [Equation (9-47) may be expanded, for example, about the last column.]

Notice that solving Equation (9-47) for $e^{\mathbf{A}t}$ is the same as writing

$$e^{\mathbf{A}t} = \alpha_0(t)\mathbf{I} + \alpha_1(t)\mathbf{A} + \alpha_2(t)\mathbf{A}^2 + \cdots + \alpha_{m-1}(t)\mathbf{A}^{m-1} \quad (9-48)$$

and determining the $\alpha_k(t)$ ($k = 0, 1, 2, \dots, m-1$) by solving the following set of m equations for the $\alpha_k(t)$:

$$\begin{aligned} \alpha_0(t) + \alpha_1(t)\lambda_1 + \alpha_2(t)\lambda_1^2 + \cdots + \alpha_{m-1}(t)\lambda_1^{m-1} &= e^{\lambda_1 t} \\ \alpha_0(t) + \alpha_1(t)\lambda_2 + \alpha_2(t)\lambda_2^2 + \cdots + \alpha_{m-1}(t)\lambda_2^{m-1} &= e^{\lambda_2 t} \\ &\vdots \\ &\vdots \\ &\vdots \\ \alpha_0(t) + \alpha_1(t)\lambda_m + \alpha_2(t)\lambda_m^2 + \cdots + \alpha_{m-1}(t)\lambda_m^{m-1} &= e^{\lambda_m t} \end{aligned}$$

If \mathbf{A} is an $n \times n$ matrix and has distinct eigenvalues, then the number of $\alpha_k(t)$'s to be determined is $m = n$. If \mathbf{A} involves multiple eigenvalues, but its minimal polynomial has only simple roots, however, then the number m of $\alpha_k(t)$'s to be determined is less than n .

Case 2: Minimal Polynomial of \mathbf{A} Involves Multiple Roots. As an example, consider the case where the minimal polynomial of \mathbf{A} involves three equal roots ($\lambda_1 = \lambda_2 = \lambda_3$) and has other roots ($\lambda_4, \lambda_5, \dots, \lambda_m$) that are all distinct. By applying Sylvester's interpolation formula, it can be shown that $e^{\mathbf{A}t}$ can be obtained from the following determinant equation:

$$\begin{vmatrix}
0 & 0 & 1 & 3\lambda_1 & \cdots & \frac{(m-1)(m-2)}{2}\lambda_1^{m-3} & \frac{t^2}{2}e^{\lambda_1 t} \\
0 & 1 & 2\lambda_1 & 3\lambda_1^2 & \cdots & (m-1)\lambda_1^{m-2} & te^{\lambda_1 t} \\
1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 & \cdots & \lambda_1^{m-1} & e^{\lambda_1 t} \\
1 & \lambda_4 & \lambda_4^2 & \lambda_4^3 & \cdots & \lambda_4^{m-1} & e^{\lambda_4 t} \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
1 & \lambda_m & \lambda_m^2 & \lambda_m^3 & \cdots & \lambda_m^{m-1} & e^{\lambda_m t} \\
\mathbf{I} & \mathbf{A} & \mathbf{A}^2 & \mathbf{A}^3 & \cdots & \mathbf{A}^{m-1} & e^{\mathbf{A}t}
\end{vmatrix} = \mathbf{0} \quad (9-49)$$

Equation (9-49) can be solved for $e^{\mathbf{A}t}$ by expanding it about the last column.

It is noted that, just as in case 1, solving Equation (9-49) for $e^{\mathbf{A}t}$ is the same as writing

$$e^{\mathbf{A}t} = \alpha_0(t)\mathbf{I} + \alpha_1(t)\mathbf{A} + \alpha_2(t)\mathbf{A}^2 + \cdots + \alpha_{m-1}(t)\mathbf{A}^{m-1} \quad (9-50)$$

and determining the $\alpha_k(t)$'s ($k = 0, 1, 2, \dots, m-1$) from

$$\begin{aligned}
\alpha_2(t) + 3\alpha_3(t)\lambda_1 + \cdots + \frac{(m-1)(m-2)}{2}\alpha_{m-1}(t)\lambda_1^{m-3} &= \frac{t^2}{2}e^{\lambda_1 t} \\
\alpha_1(t) + 2\alpha_2(t)\lambda_1 + 3\alpha_3(t)\lambda_1^2 + \cdots + (m-1)\alpha_{m-1}(t)\lambda_1^{m-2} &= te^{\lambda_1 t} \\
\alpha_0(t) + \alpha_1(t)\lambda_1 + \alpha_2(t)\lambda_1^2 + \cdots + \alpha_{m-1}(t)\lambda_1^{m-1} &= e^{\lambda_1 t} \\
\alpha_0(t) + \alpha_1(t)\lambda_4 + \alpha_2(t)\lambda_4^2 + \cdots + \alpha_{m-1}(t)\lambda_4^{m-1} &= e^{\lambda_4 t} \\
&\cdot \\
&\cdot \\
&\cdot \\
\alpha_0(t) + \alpha_1(t)\lambda_m + \alpha_2(t)\lambda_m^2 + \cdots + \alpha_{m-1}(t)\lambda_m^{m-1} &= e^{\lambda_m t}
\end{aligned}$$

The extension to other cases where, for example, there are two or more sets of multiple roots will be apparent. Note that if the minimal polynomial of \mathbf{A} is not found, it is possible to substitute the characteristic polynomial for the minimal polynomial. The number of computations may, of course, be increased.

EXAMPLE 9-8 Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$$

Compute $e^{\mathbf{A}t}$ using Sylvester's interpolation formula.

From Equation (9-47), we get

$$\begin{vmatrix}
1 & \lambda_1 & e^{\lambda_1 t} \\
1 & \lambda_2 & e^{\lambda_2 t} \\
\mathbf{I} & \mathbf{A} & e^{\mathbf{A}t}
\end{vmatrix} = \mathbf{0}$$

Substituting 0 for λ_1 and -2 for λ_2 in this last equation, we obtain

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & -2 & e^{-2t} \\ \mathbf{I} & \mathbf{A} & e^{\mathbf{A}t} \end{vmatrix} = \mathbf{0}$$

Expanding the determinant, we obtain

$$-2e^{\mathbf{A}t} + \mathbf{A} + 2\mathbf{I} - \mathbf{A}e^{-2t} = \mathbf{0}$$

or

$$\begin{aligned} e^{\mathbf{A}t} &= \frac{1}{2}(\mathbf{A} + 2\mathbf{I} - \mathbf{A}e^{-2t}) \\ &= \frac{1}{2} \left\{ \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} e^{-2t} \right\} \\ &= \begin{bmatrix} 1 & \frac{1}{2}(1 - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix} \end{aligned}$$

An alternative approach is to use Equation (9-48). We first determine $\alpha_0(t)$ and $\alpha_1(t)$ from

$$\alpha_0(t) + \alpha_1(t)\lambda_1 = e^{\lambda_1 t}$$

$$\alpha_0(t) + \alpha_1(t)\lambda_2 = e^{\lambda_2 t}$$

Since $\lambda_1 = 0$ and $\lambda_2 = -2$, the last two equations become

$$\alpha_0(t) = 1$$

$$\alpha_0(t) - 2\alpha_1(t) = e^{-2t}$$

Solving for $\alpha_0(t)$ and $\alpha_1(t)$ gives

$$\alpha_0(t) = 1, \quad \alpha_1(t) = \frac{1}{2}(1 - e^{-2t})$$

Then $e^{\mathbf{A}t}$ can be written as

$$e^{\mathbf{A}t} = \alpha_0(t)\mathbf{I} + \alpha_1(t)\mathbf{A} = \mathbf{I} + \frac{1}{2}(1 - e^{-2t})\mathbf{A} = \begin{bmatrix} 1 & \frac{1}{2}(1 - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}$$

Linear Independence of Vectors. The vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ are said to be linearly independent if

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \dots + c_n\mathbf{x}_n = \mathbf{0}$$

where c_1, c_2, \dots, c_n are constants, implies that

$$c_1 = c_2 = \dots = c_n = 0$$

Conversely, the vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ are said to be linearly dependent if and only if \mathbf{x}_i can be expressed as a linear combination of \mathbf{x}_j ($j = 1, 2, \dots, n; j \neq i$), or

$$\mathbf{x}_i = \sum_{\substack{j=1 \\ j \neq i}}^n c_j \mathbf{x}_j$$

for some set of constants c_j . This means that if \mathbf{x}_i can be expressed as a linear combination of the other vectors in the set, it is linearly dependent on them or it is not an independent member of the set.

EXAMPLE 9–9 The vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

are linearly dependent since

$$\mathbf{x}_1 + \mathbf{x}_2 - \mathbf{x}_3 = \mathbf{0}$$

The vectors

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

are linearly independent since

$$c_1 \mathbf{y}_1 + c_2 \mathbf{y}_2 + c_3 \mathbf{y}_3 = \mathbf{0}$$

implies that

$$c_1 = c_2 = c_3 = 0$$

Note that if an $n \times n$ matrix is nonsingular (that is, the matrix is of rank n or the determinant is nonzero) then n column (or row) vectors are linearly independent. If the $n \times n$ matrix is singular (that is, the rank of the matrix is less than n or the determinant is zero), then n column (or row) vectors are linearly dependent. To demonstrate this, notice that

$$\begin{aligned} [\mathbf{x}_1 \mid \mathbf{x}_2 \mid \mathbf{x}_3] &= \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 2 \\ 3 & 1 & 4 \end{bmatrix} = \text{singular} \\ [\mathbf{y}_1 \mid \mathbf{y}_2 \mid \mathbf{y}_3] &= \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 2 \\ 3 & 1 & 2 \end{bmatrix} = \text{nonsingular} \end{aligned}$$

9–6 CONTROLLABILITY

Controllability and Observability. A system is said to be controllable at time t_0 if it is possible by means of an unconstrained control vector to transfer the system from any initial state $\mathbf{x}(t_0)$ to any other state in a finite interval of time.

A system is said to be observable at time t_0 if, with the system in state $\mathbf{x}(t_0)$, it is possible to determine this state from the observation of the output over a finite time interval.

The concepts of controllability and observability were introduced by Kalman. They play an important role in the design of control systems in state space. In fact, the conditions of controllability and observability may govern the existence of a complete solution to the control system design problem. The solution to this problem may not

exist if the system considered is not controllable. Although most physical systems are controllable and observable, corresponding mathematical models may not possess the property of controllability and observability. Then it is necessary to know the conditions under which a system is controllable and observable. This section deals with controllability and the next section discusses observability.

In what follows, we shall first derive the condition for complete state controllability. Then we derive alternative forms of the condition for complete state controllability followed by discussions of complete output controllability. Finally, we present the concept of stabilizability.

Complete State Controllability of Continuous-Time Systems. Consider the continuous-time system.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \quad (9-51)$$

where \mathbf{x} = state vector (n -vector)

u = control signal (scalar)

\mathbf{A} = $n \times n$ matrix

\mathbf{B} = $n \times 1$ matrix

The system described by Equation (9-51) is said to be state controllable at $t = t_0$ if it is possible to construct an unconstrained control signal that will transfer an initial state to any final state in a finite time interval $t_0 \leq t \leq t_1$. If every state is controllable, then the system is said to be completely state controllable.

We shall now derive the condition for complete state controllability. Without loss of generality, we can assume that the final state is the origin of the state space and that the initial time is zero, or $t_0 = 0$.

The solution of Equation (9-51) is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau) d\tau$$

Applying the definition of complete state controllability just given, we have

$$\mathbf{x}(t_1) = \mathbf{0} = e^{\mathbf{A}t_1}\mathbf{x}(0) + \int_0^{t_1} e^{\mathbf{A}(t_1-\tau)}\mathbf{B}u(\tau) d\tau$$

or

$$\mathbf{x}(0) = - \int_0^{t_1} e^{-\mathbf{A}\tau}\mathbf{B}u(\tau) d\tau \quad (9-52)$$

Referring to Equation (9-48) or (9-50), $e^{-\mathbf{A}\tau}$ can be written

$$e^{-\mathbf{A}\tau} = \sum_{k=0}^{n-1} \alpha_k(\tau)\mathbf{A}^k \quad (9-53)$$

Substituting Equation (9-53) into Equation (9-52) gives

$$\mathbf{x}(0) = - \sum_{k=0}^{n-1} \mathbf{A}^k \mathbf{B} \int_0^{t_1} \alpha_k(\tau)u(\tau) d\tau \quad (9-54)$$

Let us put

$$\int_0^{t_1} \alpha_k(\tau) u(\tau) d\tau = \beta_k$$

Then Equation (9-54) becomes

$$\begin{aligned} \mathbf{x}(0) &= -\sum_{k=0}^{n-1} \mathbf{A}^k \mathbf{B} \beta_k \\ &= -[\mathbf{B} \mid \mathbf{A}\mathbf{B} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}] \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix} \end{aligned} \quad (9-55)$$

If the system is completely state controllable, then, given any initial state $\mathbf{x}(0)$, Equation (9-55) must be satisfied. This requires that the rank of the $n \times n$ matrix

$$[\mathbf{B} \mid \mathbf{A}\mathbf{B} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}]$$

be n .

From this analysis, we can state the condition for complete state controllability as follows: The system given by Equation (9-51) is completely state controllable if and only if the vectors $\mathbf{B}, \mathbf{A}\mathbf{B}, \dots, \mathbf{A}^{n-1}\mathbf{B}$ are linearly independent, or the $n \times n$ matrix

$$[\mathbf{B} \mid \mathbf{A}\mathbf{B} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}]$$

is of rank n .

The result just obtained can be extended to the case where the control vector \mathbf{u} is r -dimensional. If the system is described by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

where \mathbf{u} is an r -vector, then it can be proved that the condition for complete state controllability is that the $n \times nr$ matrix

$$[\mathbf{B} \mid \mathbf{A}\mathbf{B} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}]$$

be of rank n , or contain n linearly independent column vectors. The matrix

$$[\mathbf{B} \mid \mathbf{A}\mathbf{B} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}]$$

is commonly called the *controllability matrix*.

EXAMPLE 9-10 Consider the system given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

Since

$$[\mathbf{B} \mid \mathbf{A}\mathbf{B}] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \text{singular}$$

the system is not completely state controllable.

EXAMPLE 9–11 Consider the system given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [u]$$

For this case,

$$[\mathbf{B} \ \mathbf{AB}] = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = \text{nonsingular}$$

The system is therefore completely state controllable.

Alternative Form of the Condition for Complete State Controllability. Consider the system defined by

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \quad (9-56)$$

where \mathbf{x} = state vector (n -vector)

\mathbf{u} = control vector (r -vector)

\mathbf{A} = $n \times n$ matrix

\mathbf{B} = $n \times r$ matrix

If the eigenvectors of \mathbf{A} are distinct, then it is possible to find a transformation matrix \mathbf{P} such that

$$\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \ddots & \\ 0 & & & & & \lambda_n \end{bmatrix}$$

Note that if the eigenvalues of \mathbf{A} are distinct, then the eigenvectors of \mathbf{A} are distinct; however, the converse is not true. For example, an $n \times n$ real symmetric matrix having multiple eigenvalues has n distinct eigenvectors. Note also that each column of the \mathbf{P} matrix is an eigenvector of \mathbf{A} associated with λ_i ($i = 1, 2, \dots, n$).

Let us define

$$\mathbf{x} = \mathbf{Pz} \quad (9-57)$$

Substituting Equation (9-57) into Equation (9-56), we obtain

$$\dot{\mathbf{z}} = \mathbf{P}^{-1}\mathbf{APz} + \mathbf{P}^{-1}\mathbf{Bu} \quad (9-58)$$

By defining

$$\mathbf{P}^{-1}\mathbf{B} = \mathbf{F} = (f_{ij})$$

no two Jordan blocks in \mathbf{J} of Equation (9–60) are associated with the same eigenvalues, (2) the elements of any row of $\mathbf{S}^{-1}\mathbf{B}$ that correspond to the last row of each Jordan block are not all zero, and (3) the elements of each row of $\mathbf{S}^{-1}\mathbf{B}$ that correspond to distinct eigenvalues are not all zero.

EXAMPLE 9–12 The following systems are completely state controllable:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 & & 0 \\ & 0 & -2 & 1 & \\ & 0 & 0 & -2 & \\ & & & & -5 & 1 \\ 0 & & & & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 3 & 0 \\ 0 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

The following systems are not completely state controllable:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 0 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 & & 0 \\ & 0 & -2 & 1 & \\ & 0 & 0 & -2 & \\ & & & & -5 & 1 \\ 0 & & & & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 1 \\ 3 \\ 0 \end{bmatrix} u$$

Condition for Complete State Controllability in the s Plane. The condition for complete state controllability can be stated in terms of transfer functions or transfer matrices.

It can be proved that a necessary and sufficient condition for complete state controllability is that no cancellation occur in the transfer function or transfer matrix. If cancellation occurs, the system cannot be controlled in the direction of the canceled mode.

EXAMPLE 9–13 Consider the following transfer function:

$$\frac{X(s)}{U(s)} = \frac{s + 2.5}{(s + 2.5)(s - 1)}$$

Clearly, cancellation of the factor $(s + 2.5)$ occurs in the numerator and denominator of this transfer function. (Thus one degree of freedom is lost.) Because of this cancellation, this system is not completely state controllable.

The same conclusion can be obtained by writing this transfer function in the form of a state equation. A state-space representation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2.5 & -1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

Since

$$[\mathbf{B} \mid \mathbf{AB}] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

the rank of the matrix $[\mathbf{B} \mid \mathbf{AB}]$ is 1. Therefore, we arrive at the same conclusion: The system is not completely state controllable.

Output Controllability. In the practical design of a control system, we may want to control the output rather than the state of the system. Complete state controllability is neither necessary nor sufficient for controlling the output of the system. For this reason, it is desirable to define separately complete output controllability.

Consider the system described by

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \quad (9-61)$$

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du} \quad (9-62)$$

where \mathbf{x} = state vector (n -vector)

\mathbf{u} = control vector (r -vector)

\mathbf{y} = output vector (m -vector)

\mathbf{A} = $n \times n$ matrix

\mathbf{B} = $n \times r$ matrix

\mathbf{C} = $m \times n$ matrix

\mathbf{D} = $m \times r$ matrix

The system described by Equations (9-61) and (9-62) is said to be completely output controllable if it is possible to construct an unconstrained control vector $\mathbf{u}(t)$ that will transfer any given initial output $\mathbf{y}(t_0)$ to any final output $\mathbf{y}(t_1)$ in a finite time interval $t_0 \leq t \leq t_1$.

It can be proved that the condition for complete output controllability is as follows: The system described by Equations (9-61) and (9-62) is completely output controllable if and only if the $m \times (n + 1)r$ matrix

$$[\mathbf{CB} \mid \mathbf{CAB} \mid \mathbf{CA}^2\mathbf{B} \mid \cdots \mid \mathbf{CA}^{n-1}\mathbf{B} \mid \mathbf{D}]$$

is of rank m . (For a proof, see Problem A-9-16.) Note that the presence of the \mathbf{Du} term in Equation (9-62) always helps to establish output controllability.

Uncontrollable System. An uncontrollable system has a subsystem that is physically disconnected from the input.

Stabilizability. For a partially controllable system, if the uncontrollable modes are stable and the unstable modes are controllable, the system is said to be stabilizable. For example, the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

is not state controllable. The stable mode that corresponds to the eigenvalue of -1 is not controllable. The unstable mode that corresponds to the eigenvalue of 1 is controllable. Such a system can be made stable by the use of a suitable feedback. Thus this system is stabilizable.

9-7 OBSERVABILITY

In this section we discuss the observability of linear systems. Consider the unforced system described by the following equations:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad (9-63)$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} \quad (9-64)$$

where \mathbf{x} = state vector (n -vector)

\mathbf{y} = output vector (m -vector)

\mathbf{A} = $n \times n$ matrix

\mathbf{C} = $m \times n$ matrix

The system is said to be completely observable if every state $\mathbf{x}(t_0)$ can be determined from the observation of $\mathbf{y}(t)$ over a finite time interval, $t_0 \leq t \leq t_1$. The system is, therefore, completely observable if every transition of the state eventually affects every element of the output vector. The concept of observability is useful in solving the problem of reconstructing unmeasurable state variables from measurable variables in the minimum possible length of time. In this section we treat only linear, time-invariant systems. Therefore, without loss of generality, we can assume that $t_0 = 0$.

The concept of observability is very important because, in practice, the difficulty encountered with state feedback control is that some of the state variables are not accessible for direct measurement, with the result that it becomes necessary to estimate the unmeasurable state variables in order to construct the control signals. It will be shown in Section 10-5 that such estimates of state variables are possible if and only if the system is completely observable.

In discussing observability conditions, we consider the unforced system as given by Equations (9-63) and (9-64). The reason for this is as follows: If the system is described by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

then

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau) d\tau$$

and $\mathbf{y}(t)$ is

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{C} \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau) d\tau + \mathbf{D}\mathbf{u}$$

Since the matrices \mathbf{A} , \mathbf{B} , \mathbf{C} , and \mathbf{D} are known and $\mathbf{u}(t)$ is also known, the last two terms on the right-hand side of this last equation are known quantities. Therefore, they may be subtracted from the observed value of $\mathbf{y}(t)$. Hence, for investigating a necessary and sufficient condition for complete observability, it suffices to consider the system described by Equations (9-63) and (9-64).

Complete Observability of Continuous-Time Systems. Consider the system described by Equations (9-63) and (9-64). The output vector $\mathbf{y}(t)$ is

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0)$$

Referring to Equation (9-48) or (9-50), we have

$$e^{\mathbf{A}t} = \sum_{k=0}^{n-1} \alpha_k(t) \mathbf{A}^k$$

where n is the degree of the characteristic polynomial. [Note that Equations (9-48) and (9-50) with m replaced by n can be derived using the characteristic polynomial.]

Hence, we obtain

$$\mathbf{y}(t) = \sum_{k=0}^{n-1} \alpha_k(t) \mathbf{C}\mathbf{A}^k\mathbf{x}(0)$$

or

$$\mathbf{y}(t) = \alpha_0(t) \mathbf{C}\mathbf{x}(0) + \alpha_1(t) \mathbf{C}\mathbf{A}\mathbf{x}(0) + \cdots + \alpha_{n-1}(t) \mathbf{C}\mathbf{A}^{n-1}\mathbf{x}(0) \quad (9-65)$$

If the system is completely observable, then, given the output $\mathbf{y}(t)$ over a time interval $0 \leq t \leq t_1$, $\mathbf{x}(0)$ is uniquely determined from Equation (9-65). It can be shown that this requires the rank of the $nm \times n$ matrix

$$\begin{bmatrix} \mathbf{C} \\ \cdots \\ \mathbf{C}\mathbf{A} \\ \cdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix}$$

to be n . (See Problem **A-9-19** for the derivation of this condition.)

From this analysis, we can state the condition for complete observability as follows: The system described by Equations (9-63) and (9-64) is completely observable if and only if the $n \times nm$ matrix

$$[\mathbf{C}^* \mid \mathbf{A}^*\mathbf{C}^* \mid \cdots \mid (\mathbf{A}^*)^{n-1}\mathbf{C}^*]$$

is of rank n or has n linearly independent column vectors. This matrix is called the *observability matrix*.

EXAMPLE 9–14 Consider the system described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Is this system controllable and observable?

Since the rank of the matrix

$$[\mathbf{B} \ \vdots \ \mathbf{AB}] = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

is 2, the system is completely state controllable.

For output controllability, let us find the rank of the matrix $[\mathbf{CB} \ \vdots \ \mathbf{CAB}]$. Since

$$[\mathbf{CB} \ \vdots \ \mathbf{CAB}] = [0 \quad 1]$$

the rank of this matrix is 1. Hence, the system is completely output controllable.

To test the observability condition, examine the rank of $[\mathbf{C}^* \ \vdots \ \mathbf{A}^*\mathbf{C}^*]$. Since

$$[\mathbf{C}^* \ \vdots \ \mathbf{A}^*\mathbf{C}^*] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

the rank of $[\mathbf{C}^* \ \vdots \ \mathbf{A}^*\mathbf{C}^*]$ is 2. Hence, the system is completely observable.

Conditions for Complete Observability in the s Plane. The conditions for complete observability can also be stated in terms of transfer functions or transfer matrices. The necessary and sufficient conditions for complete observability is that no cancellation occur in the transfer function or transfer matrix. If cancellation occurs, the canceled mode cannot be observed in the output.

EXAMPLE 9–15 Show that the following system is not completely observable:

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$

$$y = \mathbf{Cx}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [4 \quad 5 \quad 1]$$

Note that the control function u does not affect the complete observability of the system. To examine complete observability, we may simply set $u = 0$. For this system, we have

$$[\mathbf{C}^* \ \vdots \ \mathbf{A}^*\mathbf{C}^* \ \vdots \ (\mathbf{A}^*)^2\mathbf{C}^*] = \begin{bmatrix} 4 & -6 & 6 \\ 5 & -7 & 5 \\ 1 & -1 & -1 \end{bmatrix}$$

Note that

$$\begin{vmatrix} 4 & -6 & 6 \\ 5 & -7 & 5 \\ 1 & -1 & -1 \end{vmatrix} = 0$$

Hence, the rank of the matrix $[\mathbf{C}^* \mid \mathbf{A}^*\mathbf{C}^* \mid (\mathbf{A}^*)^2\mathbf{C}^*]$ is less than 3. Therefore, the system is not completely observable.

In fact, in this system, cancellation occurs in the transfer function of the system. The transfer function between $X_1(s)$ and $U(s)$ is

$$\frac{X_1(s)}{U(s)} = \frac{1}{(s+1)(s+2)(s+3)}$$

and the transfer function between $Y(s)$ and $X_1(s)$ is

$$\frac{Y(s)}{X_1(s)} = (s+1)(s+4)$$

Therefore, the transfer function between the output $Y(s)$ and the input $U(s)$ is

$$\frac{Y(s)}{U(s)} = \frac{(s+1)(s+4)}{(s+1)(s+2)(s+3)}$$

Clearly, the two factors $(s+1)$ cancel each other. This means that there are nonzero initial states $\mathbf{x}(0)$, which cannot be determined from the measurement of $y(t)$.

Comments. The transfer function has no cancellation if and only if the system is completely state controllable and completely observable. This means that the canceled transfer function does not carry along all the information characterizing the dynamic system.

Alternative Form of the Condition for Complete Observability. Consider the system described by Equations (9-63) and (9-64), rewritten

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{9-66}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} \tag{9-67}$$

Suppose that the transformation matrix \mathbf{P} transforms \mathbf{A} into a diagonal matrix, or

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$$

where \mathbf{D} is a diagonal matrix. Let us define

$$\mathbf{x} = \mathbf{P}\mathbf{z}$$

Then Equations (9-66) and (9-67) can be written

$$\dot{\mathbf{z}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{z} = \mathbf{D}\mathbf{z}$$

$$\mathbf{y} = \mathbf{C}\mathbf{P}\mathbf{z}$$

Hence,

$$\mathbf{y}(t) = \mathbf{C}\mathbf{P}e^{\mathbf{D}t}\mathbf{z}(0)$$

or

$$\mathbf{y}(t) = \mathbf{CP} \begin{bmatrix} e^{\lambda_1 t} & & & & 0 \\ & e^{\lambda_2 t} & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & e^{\lambda_n t} \end{bmatrix} \mathbf{z}(0) = \mathbf{CP} \begin{bmatrix} e^{\lambda_1 t} z_1(0) \\ e^{\lambda_2 t} z_2(0) \\ \vdots \\ \vdots \\ e^{\lambda_n t} z_n(0) \end{bmatrix}$$

The system is completely observable if none of the columns of the $m \times n$ matrix \mathbf{CP} consists of all zero elements. This is because, if the i th column of \mathbf{CP} consists of all zero elements, then the state variable $z_i(0)$ will not appear in the output equation and therefore cannot be determined from observation of $\mathbf{y}(t)$. Thus, $\mathbf{x}(0)$, which is related to $\mathbf{z}(0)$ by the nonsingular matrix \mathbf{P} , cannot be determined. (Remember that this test applies only if the matrix $\mathbf{P}^{-1}\mathbf{AP}$ is in diagonal form.)

If the matrix \mathbf{A} cannot be transformed into a diagonal matrix, then by use of a suitable transformation matrix \mathbf{S} , we can transform \mathbf{A} into a Jordan canonical form, or

$$\mathbf{S}^{-1}\mathbf{AS} = \mathbf{J}$$

where \mathbf{J} is in the Jordan canonical form.

Let us define

$$\mathbf{x} = \mathbf{S}\mathbf{z}$$

Then Equations (9-66) and (9-67) can be written

$$\dot{\mathbf{z}} = \mathbf{S}^{-1}\mathbf{AS}\mathbf{z} = \mathbf{J}\mathbf{z}$$

$$\mathbf{y} = \mathbf{C}\mathbf{S}\mathbf{z}$$

Hence,

$$\mathbf{y}(t) = \mathbf{C}\mathbf{S}e^{\mathbf{J}t}\mathbf{z}(0)$$

The system is completely observable if (1) no two Jordan blocks in \mathbf{J} are associated with the same eigenvalues, (2) no columns of \mathbf{CS} that correspond to the first row of each Jordan block consist of zero elements, and (3) no columns of \mathbf{CS} that correspond to distinct eigenvalues consist of zero elements.

To clarify condition (2), in Example 9-16 we have encircled by dashed lines the columns of \mathbf{CS} that correspond to the first row of each Jordan block.

EXAMPLE 9-16 The following systems are completely observable.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad y = [1 \quad 3] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \boxed{3} & 0 & 0 \\ \boxed{4} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & & 0 \\ 0 & 2 & 1 & & \\ 0 & 0 & 2 & & \\ \hline & & & -3 & 1 \\ 0 & & & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \boxed{1} & 1 & 1 & \boxed{0} & 0 \\ \boxed{0} & 1 & 1 & \boxed{1} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

The following systems are not completely observable.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad y = [0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & & 0 \\ 0 & 2 & 1 & & \\ 0 & 0 & 2 & & \\ \hline & & & -3 & 1 \\ 0 & & & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

Principle of Duality. We shall now discuss the relationship between controllability and observability. We shall introduce the principle of duality, due to Kalman, to clarify apparent analogies between controllability and observability.

Consider the system S_1 described by

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$

$$\mathbf{y} = \mathbf{Cx}$$

where \mathbf{x} = state vector (n -vector)

\mathbf{u} = control vector (r -vector)

\mathbf{y} = output vector (m -vector)

\mathbf{A} = $n \times n$ matrix

\mathbf{B} = $n \times r$ matrix

\mathbf{C} = $m \times n$ matrix

and the dual system S_2 defined by

$$\dot{\mathbf{z}} = \mathbf{A}^*\mathbf{z} + \mathbf{C}^*\mathbf{v}$$

$$\mathbf{n} = \mathbf{B}^*\mathbf{z}$$

where \mathbf{z} = state vector (n -vector)

\mathbf{v} = control vector (m -vector)

\mathbf{n} = output vector (r -vector)

\mathbf{A}^* = conjugate transpose of \mathbf{A}

\mathbf{B}^* = conjugate transpose of \mathbf{B}

\mathbf{C}^* = conjugate transpose of \mathbf{C}

The principle of duality states that the system S_1 is completely state controllable (observable) if and only if system S_2 is completely observable (state controllable).

To verify this principle, let us write down the necessary and sufficient conditions for complete state controllability and complete observability for systems S_1 and S_2 .

For system S_1 :

1. A necessary and sufficient condition for complete state controllability is that the rank of the $n \times nr$ matrix

$$[\mathbf{B} \mid \mathbf{AB} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}]$$

be n .

2. A necessary and sufficient condition for complete observability is that the rank of the $n \times nm$ matrix

$$[\mathbf{C}^* \mid \mathbf{A}^*\mathbf{C}^* \mid \cdots \mid (\mathbf{A}^*)^{n-1}\mathbf{C}^*]$$

be n .

For system S_2 :

1. A necessary and sufficient condition for complete state controllability is that the rank of the $n \times nm$ matrix

$$[\mathbf{C}^* \mid \mathbf{A}^*\mathbf{C}^* \mid \cdots \mid (\mathbf{A}^*)^{n-1}\mathbf{C}^*]$$

be n .

2. A necessary and sufficient condition for complete observability is that the rank of the $n \times nr$ matrix

$$[\mathbf{B} \mid \mathbf{AB} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}]$$

be n .

By comparing these conditions, the truth of this principle is apparent. By use of this principle, the observability of a given system can be checked by testing the state controllability of its dual.

Detectability. For a partially observable system, if the unobservable modes are stable and the observable modes are unstable, the system is said to be detectable. Note that the concept of detectability is dual to the concept of stabilizability.

EXAMPLE PROBLEMS AND SOLUTIONS

- A-9-1.** Consider the transfer function system defined by Equation (9-2), rewritten

$$\frac{Y(s)}{U(s)} = \frac{b_0s^n + b_1s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1s^{n-1} + \cdots + a_{n-1}s + a_n} \quad (9-68)$$

Derive the following controllable canonical form of the state-space representation for this transfer-function system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} u \quad (9-69)$$

$$y = [b_n - a_n b_0 \quad b_{n-1} - a_{n-1} b_0 \quad \cdots \quad b_1 - a_1 b_0] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u \quad (9-70)$$

Solution. Equation (9-68) can be written as

$$\frac{Y(s)}{U(s)} = b_0 + \frac{(b_1 - a_1 b_0)s^{n-1} + \cdots + (b_{n-1} - a_{n-1} b_0)s + (b_n - a_n b_0)}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

which can be modified to

$$Y(s) = b_0 U(s) + \hat{Y}(s) \quad (9-71)$$

where

$$\hat{Y}(s) = \frac{(b_1 - a_1 b_0)s^{n-1} + \cdots + (b_{n-1} - a_{n-1} b_0)s + (b_n - a_n b_0)}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} U(s)$$

Let us rewrite this last equation in the following form:

$$\begin{aligned} & \frac{\hat{Y}(s)}{(b_1 - a_1 b_0)s^{n-1} + \cdots + (b_{n-1} - a_{n-1} b_0)s + (b_n - a_n b_0)} \\ &= \frac{U(s)}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} = Q(s) \end{aligned}$$

From this last equation, the following two equations may be obtained:

$$s^n Q(s) = -a_1 s^{n-1} Q(s) - \cdots - a_{n-1} s Q(s) - a_n Q(s) + U(s) \quad (9-72)$$

$$\begin{aligned} \hat{Y}(s) &= (b_1 - a_1 b_0)s^{n-1} Q(s) + \cdots + (b_{n-1} - a_{n-1} b_0)s Q(s) \\ &+ (b_n - a_n b_0) Q(s) \end{aligned} \quad (9-73)$$

Now define state variables as follows:

$$\begin{aligned} X_1(s) &= Q(s) \\ X_2(s) &= sQ(s) \\ &\vdots \\ &\vdots \\ X_{n-1}(s) &= s^{n-2} Q(s) \\ X_n(s) &= s^{n-1} Q(s) \end{aligned}$$

Then, clearly,

$$\begin{aligned} sX_1(s) &= X_2(s) \\ sX_2(s) &= X_3(s) \\ &\vdots \\ &\vdots \\ sX_{n-1}(s) &= X_n(s) \end{aligned}$$

which may be rewritten as

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ &\vdots \\ \dot{x}_{n-1} &= x_n\end{aligned}\tag{9-74}$$

Noting that $s^n Q(s) = sX_n(s)$, we can rewrite Equation (9-72) as

$$sX_n(s) = -a_1 X_n(s) - \cdots - a_{n-1} X_2(s) - a_n X_1(s) + U(s)$$

or

$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \cdots - a_1 x_n + u\tag{9-75}$$

Also, from Equations (9-71) and (9-73), we obtain

$$\begin{aligned}Y(s) &= b_0 U(s) + (b_1 - a_1 b_0) s^{n-1} Q(s) + \cdots + (b_{n-1} - a_{n-1} b_0) s Q(s) \\ &\quad + (b_n - a_n b_0) Q(s) \\ &= b_0 U(s) + (b_1 - a_1 b_0) X_n(s) + \cdots + (b_{n-1} - a_{n-1} b_0) X_2(s) \\ &\quad + (b_n - a_n b_0) X_1(s)\end{aligned}$$

The inverse Laplace transform of this output equation becomes

$$y = (b_n - a_n b_0) x_1 + (b_{n-1} - a_{n-1} b_0) x_2 + \cdots + (b_1 - a_1 b_0) x_n + b_0 u\tag{9-76}$$

Combining Equations (9-74) and (9-75) into one vector-matrix differential equation, we obtain Equation (9-69). Equation (9-76) can be rewritten as given by Equation (9-70). Equations (9-69) and (9-70) are said to be in the controllable canonical form. Figure 9-1 shows the block diagram representation of the system defined by Equations (9-69) and (9-70).

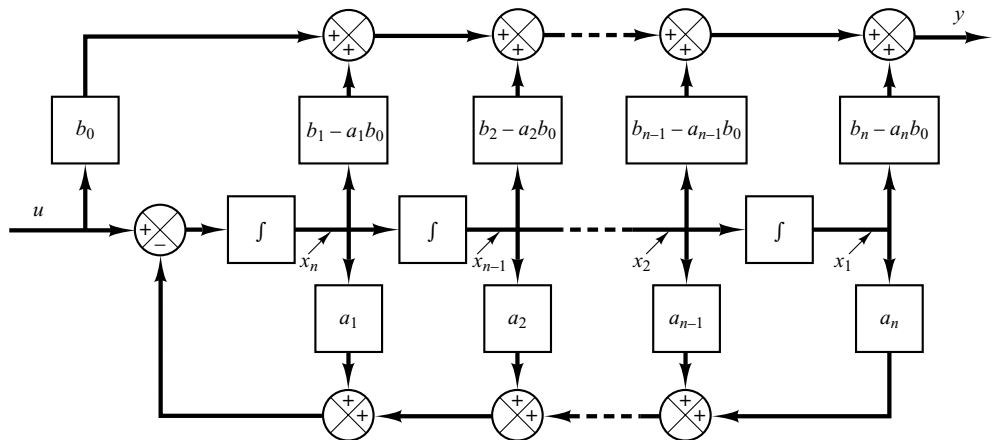


Figure 9-1
Block diagram representation of the system defined by Equations (9-69) and (9-70) (controllable canonical form).

A-9-2. Consider the following transfer-function system:

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} \quad (9-77)$$

Derive the following observable canonical form of the state-space representation for this transfer-function system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ \vdots \\ b_1 - a_1 b_0 \end{bmatrix} u \quad (9-78)$$

$$y = [0 \quad 0 \quad \cdots \quad 0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u \quad (9-79)$$

Solution. Equation (9-77) can be modified into the following form:

$$\begin{aligned} s^n[Y(s) - b_0 U(s)] + s^{n-1}[a_1 Y(s) - b_1 U(s)] + \cdots \\ + s[a_{n-1} Y(s) - b_{n-1} U(s)] + a_n Y(s) - b_n U(s) = 0 \end{aligned}$$

By dividing the entire equation by s^n and rearranging, we obtain

$$\begin{aligned} Y(s) = b_0 U(s) + \frac{1}{s} [b_1 U(s) - a_1 Y(s)] + \cdots \\ + \frac{1}{s^{n-1}} [b_{n-1} U(s) - a_{n-1} Y(s)] + \frac{1}{s^n} [b_n U(s) - a_n Y(s)] \end{aligned} \quad (9-80)$$

Now define state variables as follows:

$$\begin{aligned} X_n(s) &= \frac{1}{s} [b_1 U(s) - a_1 Y(s) + X_{n-1}(s)] \\ X_{n-1}(s) &= \frac{1}{s} [b_2 U(s) - a_2 Y(s) + X_{n-2}(s)] \\ &\vdots \\ &\vdots \\ X_2(s) &= \frac{1}{s} [b_{n-1} U(s) - a_{n-1} Y(s) + X_1(s)] \\ X_1(s) &= \frac{1}{s} [b_n U(s) - a_n Y(s)] \end{aligned} \quad (9-81)$$

Then Equation (9–80) can be written as

$$Y(s) = b_0U(s) + X_n(s) \quad (9-82)$$

By substituting Equation (9–82) into Equation (9–81) and multiplying both sides of the equations by s , we obtain

$$\begin{aligned} sX_n(s) &= X_{n-1}(s) - a_1X_n(s) + (b_1 - a_1b_0)U(s) \\ sX_{n-1}(s) &= X_{n-2}(s) - a_2X_n(s) + (b_2 - a_2b_0)U(s) \\ &\vdots \\ sX_2(s) &= X_1(s) - a_{n-1}X_n(s) + (b_{n-1} - a_{n-1}b_0)U(s) \\ sX_1(s) &= -a_nX_n(s) + (b_n - a_nb_0)U(s) \end{aligned}$$

Taking the inverse Laplace transforms of the preceding n equations and writing them in the reverse order, we get

$$\begin{aligned} \dot{x}_1 &= -a_nx_n + (b_n - a_nb_0)u \\ \dot{x}_2 &= x_1 - a_{n-1}x_n + (b_{n-1} - a_{n-1}b_0)u \\ &\vdots \\ \dot{x}_{n-1} &= x_{n-2} - a_2x_n + (b_2 - a_2b_0)u \\ \dot{x}_n &= x_{n-1} - a_1x_n + (b_1 - a_1b_0)u \end{aligned}$$

Also, the inverse Laplace transform of Equation (9–82) gives

$$y = x_n + b_0u$$

Rewriting the state and output equations in the standard vector-matrix forms gives Equations (9–78) and (9–79). Figure 9–2 shows a block diagram representation of the system defined by Equations (9–78) and (9–79).

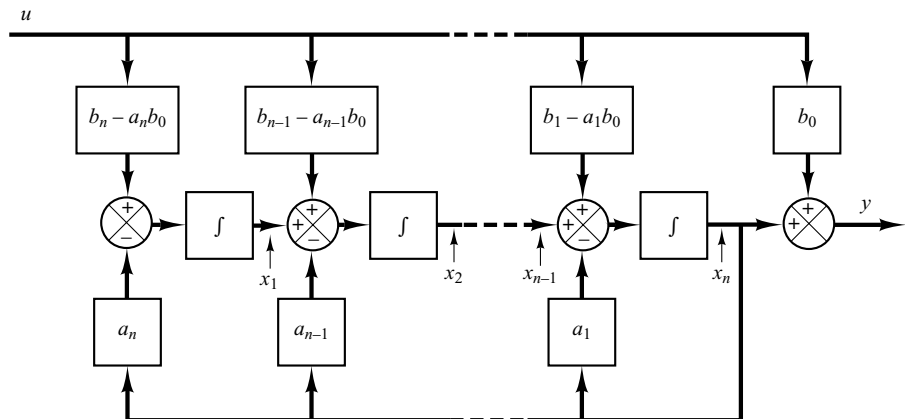


Figure 9–2
Block diagram representation of the system defined by Equations (9–78) and (9–79) (observable canonical form).

A-9-3. Consider the transfer-function system defined by

$$\begin{aligned} \frac{Y(s)}{U(s)} &= \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{(s + p_1)(s + p_2) \cdots (s + p_n)} \\ &= b_0 + \frac{c_1}{s + p_1} + \frac{c_2}{s + p_2} + \cdots + \frac{c_n}{s + p_n} \end{aligned} \quad (9-83)$$

where $p_i \neq p_j$. Derive the state-space representation of this system in the following diagonal canonical form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & & & 0 \\ & -p_2 & & \\ & & \ddots & \\ & & & -p_n \\ 0 & & & & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u \quad (9-84)$$

$$y = [c_1 \quad c_2 \quad \cdots \quad c_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u \quad (9-85)$$

Solution. Equation (9-83) may be written as

$$Y(s) = b_0 U(s) + \frac{c_1}{s + p_1} U(s) + \frac{c_2}{s + p_2} U(s) + \cdots + \frac{c_n}{s + p_n} U(s) \quad (9-86)$$

Define the state variables as follows:

$$\begin{aligned} X_1(s) &= \frac{1}{s + p_1} U(s) \\ X_2(s) &= \frac{1}{s + p_2} U(s) \\ &\vdots \\ &\vdots \\ &\vdots \\ X_n(s) &= \frac{1}{s + p_n} U(s) \end{aligned}$$

which may be rewritten as

$$\begin{aligned} sX_1(s) &= -p_1 X_1(s) + U(s) \\ sX_2(s) &= -p_2 X_2(s) + U(s) \\ &\vdots \\ &\vdots \\ &\vdots \\ sX_n(s) &= -p_n X_n(s) + U(s) \end{aligned}$$

The inverse Laplace transforms of these equations give

$$\begin{aligned} \dot{x}_1 &= -p_1 x_1 + u \\ \dot{x}_2 &= -p_2 x_2 + u \\ &\vdots \\ &\vdots \\ \dot{x}_n &= -p_n x_n + u \end{aligned} \tag{9-87}$$

These n equations make up a state equation.

In terms of the state variables $X_1(s), X_2(s), \dots, X_n(s)$, Equation (9-86) can be written as

$$Y(s) = b_0 U(s) + c_1 X_1(s) + c_2 X_2(s) + \dots + c_n X_n(s)$$

The inverse Laplace transform of this last equation is

$$y = c_1 x_1 + c_2 x_2 + \dots + c_n x_n + b_0 u \tag{9-88}$$

which is the output equation.

Equation (9-87) can be put in the vector-matrix equation as given by Equation (9-84). Equation (9-88) can be put in the form of Equation (9-85).

Figure 9-3 shows a block diagram representation of the system defined by Equations (9-84) and (9-85).

It is noted that if we choose the state variables as

$$\hat{X}_1(s) = \frac{c_1}{s + p_1} U(s)$$

$$\hat{X}_2(s) = \frac{c_2}{s + p_2} U(s)$$

⋮

$$\hat{X}_n(s) = \frac{c_n}{s + p_n} U(s)$$

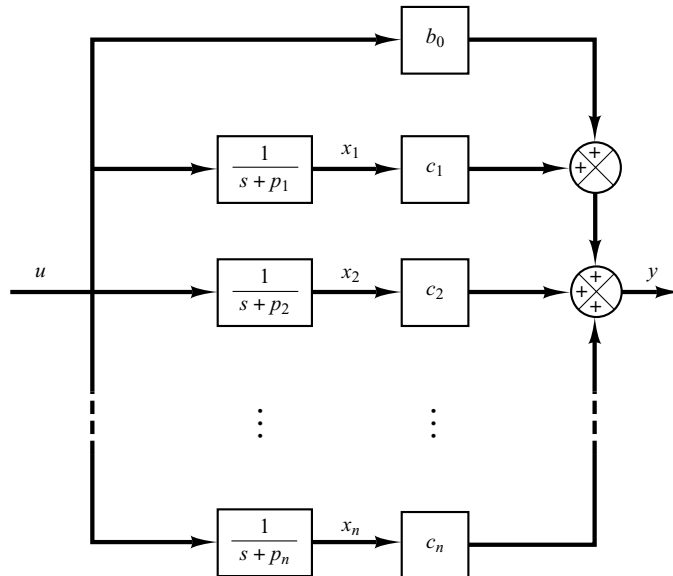


Figure 9-3
Block diagram representation of the system defined by Equations (9-84) and (9-85) (diagonal canonical form).

then we get a slightly different state-space representation. This choice of state variables gives

$$\begin{aligned} s\hat{X}_1(s) &= -p_1\hat{X}_1(s) + c_1U(s) \\ s\hat{X}_2(s) &= -p_2\hat{X}_2(s) + c_2U(s) \\ &\vdots \\ &\vdots \\ s\hat{X}_n(s) &= -p_n\hat{X}_n(s) + c_nU(s) \end{aligned}$$

from which we obtain

$$\begin{aligned} \dot{\hat{x}}_1 &= -p_1\hat{x}_1 + c_1u \\ \dot{\hat{x}}_2 &= -p_2\hat{x}_2 + c_2u \\ &\vdots \\ &\vdots \\ \dot{\hat{x}}_n &= -p_n\hat{x}_n + c_nu \end{aligned} \tag{9-89}$$

Referring to Equation (9-86), the output equation becomes

$$Y(s) = b_0U(s) + \hat{X}_1(s) + \hat{X}_2(s) + \cdots + \hat{X}_n(s)$$

from which we get

$$y = \hat{x}_1 + \hat{x}_2 + \cdots + \hat{x}_n + b_0u \tag{9-90}$$

Equations (9-89) and (9-90) give the following state-space representation for the system:

$$\begin{aligned} \begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \vdots \\ \vdots \\ \dot{\hat{x}}_n \end{bmatrix} &= \begin{bmatrix} -p_1 & & & 0 \\ & -p_2 & & \\ & & \ddots & \\ & & & \ddots & \\ 0 & & & & -p_n \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \vdots \\ \hat{x}_n \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ \vdots \\ c_n \end{bmatrix} u \\ y &= [1 \quad 1 \quad \cdots \quad 1] \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \vdots \\ \hat{x}_n \end{bmatrix} + b_0u \end{aligned}$$

A-9-4. Consider the system defined by

$$\frac{Y(s)}{U(s)} = \frac{b_0s^n + b_1s^{n-1} + \cdots + b_{n-1}s + b_n}{(s + p_1)^3(s + p_4)(s + p_5)\cdots(s + p_n)} \tag{9-91}$$

where the system involves a triple pole at $s = -p_1$. (We assume that, except for the first three p_i 's being equal, the p_i 's are different from one another.) Obtain the Jordan canonical form of the state-space representation for this system.

Solution. The partial-fraction expansion of Equation (9–91) becomes

$$\frac{Y(s)}{U(s)} = b_0 + \frac{c_1}{(s + p_1)^3} + \frac{c_2}{(s + p_1)^2} + \frac{c_3}{s + p_1} + \frac{c_4}{s + p_4} + \cdots + \frac{c_n}{s + p_n}$$

which may be written as

$$\begin{aligned} Y(s) = & b_0 U(s) + \frac{c_1}{(s + p_1)^3} U(s) + \frac{c_2}{(s + p_1)^2} U(s) \\ & + \frac{c_3}{s + p_1} U(s) + \frac{c_4}{s + p_4} U(s) + \cdots + \frac{c_n}{s + p_n} U(s) \end{aligned} \quad (9-92)$$

Define

$$X_1(s) = \frac{1}{(s + p_1)^3} U(s)$$

$$X_2(s) = \frac{1}{(s + p_1)^2} U(s)$$

$$X_3(s) = \frac{1}{s + p_1} U(s)$$

$$X_4(s) = \frac{1}{s + p_4} U(s)$$

⋮

$$X_n(s) = \frac{1}{s + p_n} U(s)$$

Notice that the following relationships exist among $X_1(s)$, $X_2(s)$, and $X_3(s)$:

$$\frac{X_1(s)}{X_2(s)} = \frac{1}{s + p_1}$$

$$\frac{X_2(s)}{X_3(s)} = \frac{1}{s + p_1}$$

Then, from the preceding definition of the state variables and the preceding relationships, we obtain

$$sX_1(s) = -p_1 X_1(s) + X_2(s)$$

$$sX_2(s) = -p_1 X_2(s) + X_3(s)$$

$$sX_3(s) = -p_1 X_3(s) + U(s)$$

$$sX_4(s) = -p_4 X_4(s) + U(s)$$

⋮

$$sX_n(s) = -p_n X_n(s) + U(s)$$

The inverse Laplace transforms of the preceding n equations give

$$\begin{aligned}\dot{x}_1 &= -p_1x_1 + x_2 \\ \dot{x}_2 &= -p_1x_2 + x_3 \\ \dot{x}_3 &= -p_1x_3 + u \\ \dot{x}_4 &= -p_4x_4 + u \\ &\vdots \\ &\vdots \\ &\vdots \\ \dot{x}_n &= -p_nx_n + u\end{aligned}$$

The output equation, Equation (9-92), can be rewritten as

$$Y(s) = b_0U(s) + c_1X_1(s) + c_2X_2(s) + c_3X_3(s) + c_4X_4(s) + \cdots + c_nX_n(s)$$

The inverse Laplace transform of this output equation is

$$y = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + \cdots + c_nx_n + b_0u$$

Thus, the state-space representation of the system for the case when the denominator polynomial involves a triple root $-p_1$ can be given as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \vdots \\ \vdots \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -p_1 & 1 & \cdot & & \cdot \\ 0 & 0 & -p_1 & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & -p_4 & & \\ \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & & \cdot \\ \cdot & & \cdot & & & \cdot \\ 0 & \cdots & 0 & 0 & & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} u \quad (9-93)$$

$$y = [c_1 \quad c_2 \quad \cdots \quad c_n] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0u \quad (9-94)$$

The state-space representation in the form given by Equations (9-93) and (9-94) is said to be in the Jordan canonical form. Figure 9-4 shows a block diagram representation of the system given by Equations (9-93) and (9-94).

A-9-5. Consider the transfer-function system

$$\frac{Y(s)}{U(s)} = \frac{25.04s + 5.008}{s^3 + 5.03247s^2 + 25.1026s + 5.008}$$

Obtain a state-space representation of this system with MATLAB.

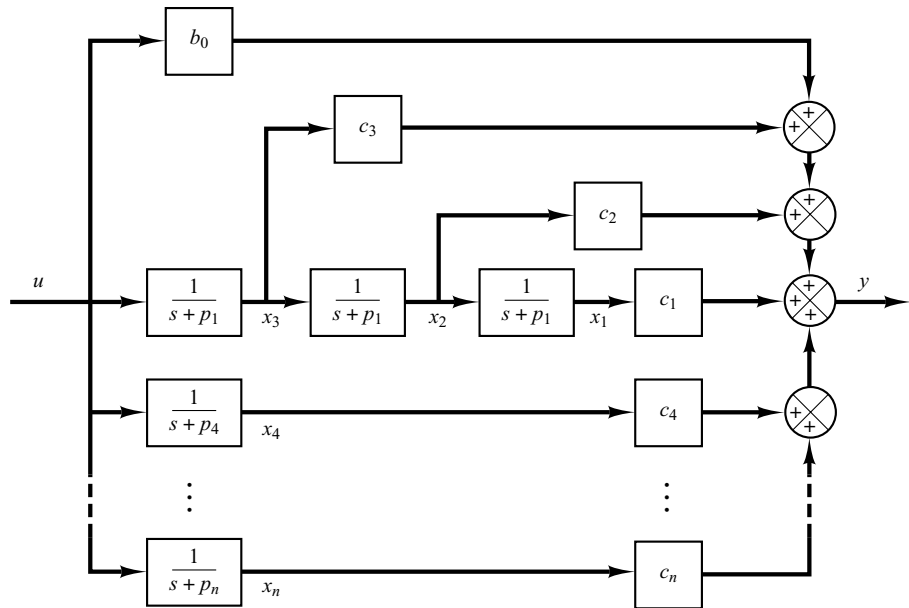


Figure 9-4
Block diagram representation of the system defined by Equations (9-93) and (9-94) (Jordan canonical form).

Solution. MATLAB command

$$[A,B,C,D] = \text{tf2ss}(\text{num},\text{den})$$

will produce a state-space representation for the system. See MATLAB Program 9-4.

MATLAB Program 9-4	
num =	[25.04 5.008];
den =	[1 5.03247 25.1026 5.008];
[A,B,C,D] =	tf2ss(num,den)
A =	
	-5.0325 -25.1026 -5.0080
	1.0000 0 0
	0 1.0000 0
B =	
	1
	0
	0
C =	
	0 25.0400 5.0080
D =	
	0

This is the MATLAB representation of the following state-space equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -5.0325 & -25.1026 & -5.008 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [0 \quad 25.04 \quad 5.008] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u$$

A-9-6. Consider the system defined by

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$

where \mathbf{x} = state vector (n -vector)

\mathbf{u} = control vector (r -vector)

\mathbf{A} = $n \times n$ constant matrix

\mathbf{B} = $n \times r$ constant matrix

Obtain the response of the system to each of the following inputs:

- (a) The r components of \mathbf{u} are impulse functions of various magnitudes.
- (b) The r components of \mathbf{u} are step functions of various magnitudes.
- (c) The r components of \mathbf{u} are ramp functions of various magnitudes.

Solution.

(a) *Impulse response:* Referring to Equation (9-43), the solution to the given state equation is

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^t e^{\mathbf{A}(t-\tau)}\mathbf{Bu}(\tau) d\tau$$

Substituting $t_0 = 0^-$ into this solution, we obtain

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0^-) + \int_{0^-}^t e^{\mathbf{A}(t-\tau)}\mathbf{Bu}(\tau) d\tau$$

Let us write the impulse input $\mathbf{u}(t)$ as

$$\mathbf{u}(t) = \delta(t)\mathbf{w}$$

where \mathbf{w} is a vector whose components are the magnitudes of r impulse functions applied at $t = 0$. The solution of the state equation when the impulse input $\delta(t)\mathbf{w}$ is given at $t = 0$ is

$$\begin{aligned} \mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0^-) + \int_{0^-}^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\delta(\tau)\mathbf{w} d\tau \\ &= e^{\mathbf{A}t}\mathbf{x}(0^-) + e^{\mathbf{A}t}\mathbf{B}\mathbf{w} \end{aligned} \quad (9-95)$$

(b) *Step response:* Let us write the step input $\mathbf{u}(t)$ as

$$\mathbf{u}(t) = \mathbf{k}$$

where \mathbf{k} is a vector whose components are the magnitudes of r step functions applied at $t = 0$. The solution to the step input at $t = 0$ is given by

$$\begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{k} d\tau \\ &= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\left[\int_0^t \left(\mathbf{I} - \mathbf{A}\tau + \frac{\mathbf{A}^2\tau^2}{2!} - \dots\right) d\tau\right]\mathbf{B}\mathbf{k} \\ &= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\left(\mathbf{I}t - \frac{\mathbf{A}t^2}{2!} + \frac{\mathbf{A}^2t^3}{3!} - \dots\right)\mathbf{B}\mathbf{k}\end{aligned}$$

If \mathbf{A} is nonsingular, then this last equation can be simplified to give

$$\begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}[-(\mathbf{A}^{-1})(e^{-\mathbf{A}t} - \mathbf{I})]\mathbf{B}\mathbf{k} \\ &= e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{A}^{-1}(e^{\mathbf{A}t} - \mathbf{I})\mathbf{B}\mathbf{k}\end{aligned}\quad (9-96)$$

(c) *Ramp response:* Let us write the ramp input $\mathbf{u}(t)$ as

$$\mathbf{u}(t) = t\mathbf{v}$$

where \mathbf{v} is a vector whose components are magnitudes of ramp functions applied at $t = 0$. The solution to the ramp input $t\mathbf{v}$ given at $t = 0$ is

$$\begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\tau\mathbf{v} d\tau \\ &= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}\tau} d\tau\mathbf{B}\mathbf{v} \\ &= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\left(\frac{\mathbf{I}}{2}t^2 - \frac{2\mathbf{A}}{3!}t^3 + \frac{3\mathbf{A}^2}{4!}t^4 - \frac{4\mathbf{A}^3}{5!}t^5 + \dots\right)\mathbf{B}\mathbf{v}\end{aligned}$$

If \mathbf{A} is nonsingular, then this last equation can be simplified to give

$$\begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + (\mathbf{A}^{-2})(e^{\mathbf{A}t} - \mathbf{I} - \mathbf{A}t)\mathbf{B}\mathbf{v} \\ &= e^{\mathbf{A}t}\mathbf{x}(0) + [\mathbf{A}^{-2}(e^{\mathbf{A}t} - \mathbf{I}) - \mathbf{A}^{-1}t]\mathbf{B}\mathbf{v}\end{aligned}\quad (9-97)$$

A-9-7. Obtain the response $y(t)$ of the following system:

$$\begin{aligned}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -1 & -0.5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u, & \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ y &= [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\end{aligned}$$

where $u(t)$ is the unit-step input occurring at $t = 0$, or

$$u(t) = 1(t)$$

Solution. For this system

$$\mathbf{A} = \begin{bmatrix} -1 & -0.5 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

The state transition matrix $\Phi(t) = e^{\mathbf{A}t}$ can be obtained as follows:

$$\Phi(t) = e^{\mathbf{A}t} = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}]$$

Since

$$\begin{aligned}(s\mathbf{I} - \mathbf{A})^{-1} &= \begin{bmatrix} s+1 & 0.5 \\ -1 & s \end{bmatrix}^{-1} = \frac{1}{s^2 + s + 0.5} \begin{bmatrix} s & -0.5 \\ 1 & s+1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{s+0.5-0.5}{(s+0.5)^2 + 0.5^2} & \frac{-0.5}{(s+0.5)^2 + 0.5^2} \\ \frac{1}{(s+0.5)^2 + 0.5^2} & \frac{s+0.5+0.5}{(s+0.5)^2 + 0.5^2} \end{bmatrix}\end{aligned}$$

we have

$$\begin{aligned}\Phi(t) &= e^{\mathbf{A}t} = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] \\ &= \begin{bmatrix} e^{-0.5t}(\cos 0.5t - \sin 0.5t) & -e^{-0.5t} \sin 0.5t \\ 2e^{-0.5t} \sin 0.5t & e^{-0.5t}(\cos 0.5t + \sin 0.5t) \end{bmatrix}\end{aligned}$$

Since $\mathbf{x}(0) = \mathbf{0}$ and $k = 1$, referring to Equation (9-96), we have

$$\begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{A}^{-1}(e^{\mathbf{A}t} - \mathbf{I})\mathbf{B}k \\ &= \mathbf{A}^{-1}(e^{\mathbf{A}t} - \mathbf{I})\mathbf{B} \\ &= \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} 0.5e^{-0.5t}(\cos 0.5t - \sin 0.5t) - 0.5 \\ e^{-0.5t} \sin 0.5t \end{bmatrix} \\ &= \begin{bmatrix} e^{-0.5t} \sin 0.5t \\ -e^{-0.5t}(\cos 0.5t + \sin 0.5t) + 1 \end{bmatrix}\end{aligned}$$

Hence, the output $y(t)$ can be given by

$$y(t) = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 = e^{-0.5t} \sin 0.5t$$

A-9-8. The Cayley–Hamilton theorem states that every $n \times n$ matrix \mathbf{A} satisfies its own characteristic equation. The characteristic equation is not, however, necessarily the scalar equation of least degree that \mathbf{A} satisfies. The least-degree polynomial having \mathbf{A} as a root is called the *minimal polynomial*. That is, the minimal polynomial of an $n \times n$ matrix \mathbf{A} is defined as the polynomial $\phi(\lambda)$ of least degree,

$$\phi(\lambda) = \lambda^m + a_1\lambda^{m-1} + \cdots + a_{m-1}\lambda + a_m, \quad m \leq n$$

such that $\phi(\mathbf{A}) = \mathbf{0}$, or

$$\phi(\mathbf{A}) = \mathbf{A}^m + a_1\mathbf{A}^{m-1} + \cdots + a_{m-1}\mathbf{A} + a_m\mathbf{I} = \mathbf{0}$$

The minimal polynomial plays an important role in the computation of polynomials in an $n \times n$ matrix.

Let us suppose that $d(\lambda)$, a polynomial in λ , is the greatest common divisor of all the elements of $\text{adj}(\lambda\mathbf{I} - \mathbf{A})$. Show that, if the coefficient of the highest-degree term in λ of $d(\lambda)$ is chosen as 1, then the minimal polynomial $\phi(\lambda)$ is given by

$$\phi(\lambda) = \left| \frac{\lambda\mathbf{I} - \mathbf{A}}{d(\lambda)} \right|$$

Solution. By assumption, the greatest common divisor of the matrix $\text{adj}(\lambda\mathbf{I} - \mathbf{A})$ is $d(\lambda)$. Therefore,

$$\text{adj}(\lambda\mathbf{I} - \mathbf{A}) = d(\lambda)\mathbf{B}(\lambda)$$

where the greatest common divisor of the n^2 elements (which are functions of λ) of $\mathbf{B}(\lambda)$ is unity. Since

$$(\lambda \mathbf{I} - \mathbf{A}) \text{adj}(\lambda \mathbf{I} - \mathbf{A}) = |\lambda \mathbf{I} - \mathbf{A}| \mathbf{I}$$

we obtain

$$d(\lambda)(\lambda \mathbf{I} - \mathbf{A})\mathbf{B}(\lambda) = |\lambda \mathbf{I} - \mathbf{A}| \mathbf{I} \quad (9-98)$$

from which we find that $|\lambda \mathbf{I} - \mathbf{A}|$ is divisible by $d(\lambda)$. Let us put

$$|\lambda \mathbf{I} - \mathbf{A}| = d(\lambda)\psi(\lambda) \quad (9-99)$$

Because the coefficient of the highest-degree term in λ of $d(\lambda)$ has been chosen to be 1, the coefficient of the highest-degree term in λ of $\psi(\lambda)$ is also 1. From Equations (9-98) and (9-99), we have

$$(\lambda \mathbf{I} - \mathbf{A})\mathbf{B}(\lambda) = \psi(\lambda)\mathbf{I}$$

Hence,

$$\psi(\mathbf{A}) = \mathbf{0}$$

Note that $\psi(\lambda)$ can be written as

$$\psi(\lambda) = g(\lambda)\phi(\lambda) + \alpha(\lambda)$$

where $\alpha(\lambda)$ is of lower degree than $\phi(\lambda)$. Since $\psi(\mathbf{A}) = \mathbf{0}$ and $\phi(\mathbf{A}) = \mathbf{0}$, we must have $\alpha(\mathbf{A}) = \mathbf{0}$. Also, since $\phi(\lambda)$ is the minimal polynomial, $\alpha(\lambda)$ must be identically zero, or

$$\psi(\lambda) = g(\lambda)\phi(\lambda)$$

Note that because $\phi(\mathbf{A}) = \mathbf{0}$, we can write

$$\phi(\lambda)\mathbf{I} = (\lambda \mathbf{I} - \mathbf{A})\mathbf{C}(\lambda)$$

Hence,

$$\psi(\lambda)\mathbf{I} = g(\lambda)\phi(\lambda)\mathbf{I} = g(\lambda)(\lambda \mathbf{I} - \mathbf{A})\mathbf{C}(\lambda)$$

Noting that $(\lambda \mathbf{I} - \mathbf{A})\mathbf{B}(\lambda) = \psi(\lambda)\mathbf{I}$, we obtain

$$\mathbf{B}(\lambda) = g(\lambda)\mathbf{C}(\lambda)$$

Since the greatest common divisor of the n^2 elements of $\mathbf{B}(\lambda)$ is unity, we have

$$g(\lambda) = 1$$

Therefore,

$$\psi(\lambda) = \phi(\lambda)$$

Then, from this last equation and Equation (9-99), we obtain

$$\phi(\lambda) = \frac{|\lambda \mathbf{I} - \mathbf{A}|}{d(\lambda)}$$

A-9-9. If an $n \times n$ matrix \mathbf{A} has n distinct eigenvalues, then the minimal polynomial of \mathbf{A} is identical to the characteristic polynomial. Also, if the multiple eigenvalues of \mathbf{A} are linked in a Jordan chain, the minimal polynomial and the characteristic polynomial are identical. If, however, the multiple eigenvalues of \mathbf{A} are not linked in a Jordan chain, the minimal polynomial is of lower degree than the characteristic polynomial.

Using the following matrices \mathbf{A} and \mathbf{B} as examples, verify the foregoing statements about the minimal polynomial when multiple eigenvalues are involved:

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

Solution. First, consider the matrix \mathbf{A} . The characteristic polynomial is given by

$$|\lambda \mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda - 2 & -1 & -4 \\ 0 & \lambda - 2 & 0 \\ 0 & -3 & \lambda - 1 \end{vmatrix} = (\lambda - 2)^2(\lambda - 1)$$

Thus, the eigenvalues of \mathbf{A} are 2, 2, and 1. It can be shown that the Jordan canonical form of \mathbf{A} is

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the multiple eigenvalues are linked in the Jordan chain as shown.

To determine the minimal polynomial, let us first obtain $\text{adj}(\lambda \mathbf{I} - \mathbf{A})$. It is given by

$$\text{adj}(\lambda \mathbf{I} - \mathbf{A}) = \begin{bmatrix} (\lambda - 2)(\lambda - 1) & (\lambda + 11) & 4(\lambda - 2) \\ 0 & (\lambda - 2)(\lambda - 1) & 0 \\ 0 & 3(\lambda - 2) & (\lambda - 2)^2 \end{bmatrix}$$

Notice that there is no common divisor of all the elements of $\text{adj}(\lambda \mathbf{I} - \mathbf{A})$. Hence, $d(\lambda) = 1$. Thus, the minimal polynomial $\phi(\lambda)$ is identical to the characteristic polynomial, or

$$\begin{aligned} \phi(\lambda) &= |\lambda \mathbf{I} - \mathbf{A}| = (\lambda - 2)^2(\lambda - 1) \\ &= \lambda^3 - 5\lambda^2 + 8\lambda - 4 \end{aligned}$$

A simple calculation proves that

$$\begin{aligned} &\mathbf{A}^3 - 5\mathbf{A}^2 + 8\mathbf{A} - 4\mathbf{I} \\ &= \begin{bmatrix} 8 & 72 & 28 \\ 0 & 8 & 0 \\ 0 & 21 & 1 \end{bmatrix} - 5 \begin{bmatrix} 4 & 16 & 12 \\ 0 & 4 & 0 \\ 0 & 9 & 1 \end{bmatrix} + 8 \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0} \end{aligned}$$

but

$$\begin{aligned} & \mathbf{A}^2 - 3\mathbf{A} + 2\mathbf{I} \\ &= \begin{bmatrix} 4 & 16 & 12 \\ 0 & 4 & 0 \\ 0 & 9 & 1 \end{bmatrix} - 3 \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 13 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \mathbf{0} \end{aligned}$$

Thus, we have shown that the minimal polynomial and the characteristic polynomial of this matrix \mathbf{A} are the same.

Next, consider the matrix \mathbf{B} . The characteristic polynomial is given by

$$|\lambda\mathbf{I} - \mathbf{B}| = \begin{vmatrix} \lambda - 2 & 0 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & -3 & \lambda - 1 \end{vmatrix} = (\lambda - 2)^2(\lambda - 1)$$

A simple computation reveals that matrix \mathbf{B} has three eigenvectors, and the Jordan canonical form of \mathbf{B} is given by

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the multiple eigenvalues are not linked. To obtain the minimal polynomial, we first compute $\text{adj}(\lambda\mathbf{I} - \mathbf{B})$:

$$\text{adj}(\lambda\mathbf{I} - \mathbf{B}) = \begin{bmatrix} (\lambda - 2)(\lambda - 1) & 0 & 0 \\ 0 & (\lambda - 2)(\lambda - 1) & 0 \\ 0 & 3(\lambda - 2) & (\lambda - 2)^2 \end{bmatrix}$$

from which it is evident that

$$d(\lambda) = \lambda - 2$$

Hence,

$$\phi(\lambda) = \frac{|\lambda\mathbf{I} - \mathbf{B}|}{d(\lambda)} = \frac{(\lambda - 2)^2(\lambda - 1)}{\lambda - 2} = \lambda^2 - 3\lambda + 2$$

As a check, let us compute $\phi(\mathbf{B})$:

$$\phi(\mathbf{B}) = \mathbf{B}^2 - 3\mathbf{B} + 2\mathbf{I} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 9 & 1 \end{bmatrix} - 3 \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0}$$

For the given matrix \mathbf{B} , the degree of the minimal polynomial is lower by 1 than that of the characteristic polynomial. As shown here, if the multiple eigenvalues of an $n \times n$ matrix are not linked in a Jordan chain, the minimal polynomial is of lower degree than the characteristic polynomial.

A-9-10. Show that by use of the minimal polynomial, the inverse of a nonsingular matrix \mathbf{A} can be expressed as a polynomial in \mathbf{A} with scalar coefficients as follows:

$$\mathbf{A}^{-1} = -\frac{1}{a_m}(\mathbf{A}^{m-1} + a_1\mathbf{A}^{m-2} + \cdots + a_{m-2}\mathbf{A} + a_{m-1}\mathbf{I}) \quad (9-100)$$

where a_1, a_2, \dots, a_m are coefficients of the minimal polynomial

$$\phi(\lambda) = \lambda^m + a_1\lambda^{m-1} + \cdots + a_{m-1}\lambda + a_m$$

Then obtain the inverse of the following matrix \mathbf{A} :

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & -2 \\ 1 & 0 & -3 \end{bmatrix}$$

Solution. For a nonsingular matrix \mathbf{A} , its minimal polynomial $\phi(\mathbf{A})$ can be written as

$$\phi(\mathbf{A}) = \mathbf{A}^m + a_1\mathbf{A}^{m-1} + \cdots + a_{m-1}\mathbf{A} + a_m\mathbf{I} = \mathbf{0}$$

where $a_m \neq 0$. Hence,

$$\mathbf{I} = -\frac{1}{a_m}(\mathbf{A}^m + a_1\mathbf{A}^{m-1} + \cdots + a_{m-2}\mathbf{A}^2 + a_{m-1}\mathbf{A})$$

Premultiplying by \mathbf{A}^{-1} , we obtain

$$\mathbf{A}^{-1} = -\frac{1}{a_m}(\mathbf{A}^{m-1} + a_1\mathbf{A}^{m-2} + \cdots + a_{m-2}\mathbf{A} + a_{m-1}\mathbf{I})$$

which is Equation (9-100).

For the given matrix \mathbf{A} , $\text{adj}(\lambda\mathbf{I} - \mathbf{A})$ can be given as

$$\text{adj}(\lambda\mathbf{I} - \mathbf{A}) = \begin{bmatrix} \lambda^2 + 4\lambda + 3 & 2\lambda + 6 & -4 \\ 3\lambda + 7 & \lambda^2 + 2\lambda - 3 & -2\lambda + 2 \\ \lambda + 1 & 2 & \lambda^2 - 7 \end{bmatrix}$$

Clearly, there is no common divisor $d(\lambda)$ of all elements of $\text{adj}(\lambda\mathbf{I} - \mathbf{A})$. Hence, $d(\lambda) = 1$. Consequently, the minimal polynomial $\phi(\lambda)$ is given by

$$\phi(\lambda) = \frac{|\lambda\mathbf{I} - \mathbf{A}|}{d(\lambda)} = |\lambda\mathbf{I} - \mathbf{A}|$$

Thus, the minimal polynomial $\phi(\lambda)$ is the same as the characteristic polynomial.

Since the characteristic polynomial is

$$|\lambda\mathbf{I} - \mathbf{A}| = \lambda^3 + 3\lambda^2 - 7\lambda - 17$$

we obtain

$$\phi(\lambda) = \lambda^3 + 3\lambda^2 - 7\lambda - 17$$

By identifying the coefficients a_i of the minimal polynomial (which is the same as the characteristic polynomial in this case), we have

$$a_1 = 3, \quad a_2 = -7, \quad a_3 = -17$$

The inverse of \mathbf{A} can then be obtained from Equation (9–100) as follows:

$$\begin{aligned} \mathbf{A}^{-1} &= -\frac{1}{a_3}(\mathbf{A}^2 + a_1\mathbf{A} + a_2\mathbf{I}) = \frac{1}{17}(\mathbf{A}^2 + 3\mathbf{A} - 7\mathbf{I}) \\ &= \frac{1}{17} \left\{ \begin{bmatrix} 7 & 0 & -4 \\ -2 & 7 & 8 \\ -2 & 2 & 9 \end{bmatrix} + 3 \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & -2 \\ 1 & 0 & -3 \end{bmatrix} - 7 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \\ &= \frac{1}{17} \begin{bmatrix} 3 & 6 & -4 \\ 7 & -3 & 2 \\ 1 & 2 & -7 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{17} & \frac{6}{17} & -\frac{4}{17} \\ \frac{7}{17} & -\frac{3}{17} & \frac{2}{17} \\ \frac{1}{17} & \frac{2}{17} & -\frac{7}{17} \end{bmatrix} \end{aligned}$$

A–9–11. Show that if matrix \mathbf{A} can be diagonalized, then

$$e^{\mathbf{A}t} = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1}$$

where \mathbf{P} is a diagonalizing transformation matrix that transforms \mathbf{A} into a diagonal matrix, or $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$, where \mathbf{D} is a diagonal matrix.

Show also that if matrix \mathbf{A} can be transformed into a Jordan canonical form, then

$$e^{\mathbf{A}t} = \mathbf{S}e^{\mathbf{J}t}\mathbf{S}^{-1}$$

where \mathbf{S} is a transformation matrix that transforms \mathbf{A} into a Jordan canonical form \mathbf{J} , or $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{J}$.

Solution. Consider the state equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

If a square matrix can be diagonalized, then a diagonalizing matrix (transformation matrix) exists and it can be obtained by a standard method. Let \mathbf{P} be a diagonalizing matrix for \mathbf{A} . Let us define

$$\mathbf{x} = \mathbf{P}\hat{\mathbf{x}}$$

Then

$$\dot{\hat{\mathbf{x}}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\hat{\mathbf{x}} = \mathbf{D}\hat{\mathbf{x}}$$

where \mathbf{D} is a diagonal matrix. The solution of this last equation is

$$\hat{\mathbf{x}}(t) = e^{\mathbf{D}t}\hat{\mathbf{x}}(0)$$

Hence,

$$\mathbf{x}(t) = \mathbf{P}\hat{\mathbf{x}}(t) = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1}\mathbf{x}(0)$$

Noting that $\mathbf{x}(t)$ can also be given by the equation

$$\mathbf{x}(t) = e^{\mathbf{A}t} \mathbf{x}(0)$$

we obtain $e^{\mathbf{A}t} = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1}$, or

$$e^{\mathbf{A}t} = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1} = \mathbf{P} \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} \mathbf{P}^{-1} \quad (9-101)$$

Next, we shall consider the case where matrix \mathbf{A} may be transformed into a Jordan canonical form. Consider again the state equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

First obtain a transformation matrix \mathbf{S} that will transform matrix \mathbf{A} into a Jordan canonical form so that

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{J}$$

where \mathbf{J} is a matrix in a Jordan canonical form. Now define

$$\mathbf{x} = \mathbf{S}\hat{\mathbf{x}}$$

Then

$$\dot{\hat{\mathbf{x}}} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}\hat{\mathbf{x}} = \mathbf{J}\hat{\mathbf{x}}$$

The solution of this last equation is

$$\hat{\mathbf{x}}(t) = e^{\mathbf{J}t}\hat{\mathbf{x}}(0)$$

Hence,

$$\mathbf{x}(t) = \mathbf{S}\hat{\mathbf{x}}(t) = \mathbf{S}e^{\mathbf{J}t}\mathbf{S}^{-1}\mathbf{x}(0)$$

Since the solution $\mathbf{x}(t)$ can also be given by the equation

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$$

we obtain

$$e^{\mathbf{A}t} = \mathbf{S}e^{\mathbf{J}t}\mathbf{S}^{-1}$$

Note that $e^{\mathbf{J}t}$ is a triangular matrix [which means that the elements below (or above, as the case may be) the principal diagonal line are zeros] whose elements are $e^{\lambda t}$, $te^{\lambda t}$, $\frac{1}{2}t^2e^{\lambda t}$, and so forth. For example, if matrix \mathbf{J} has the following Jordan canonical form:

$$\mathbf{J} = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

then

$$e^{\mathbf{J}t} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & \frac{1}{2}t^2e^{\lambda_1 t} \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & 0 & e^{\lambda_1 t} \end{bmatrix}$$

Similarly, if

$$\mathbf{J} = \begin{bmatrix} \lambda_1 & 1 & 0 & & & & & 0 \\ 0 & \lambda_1 & 1 & & & & & \\ 0 & 0 & \lambda_1 & & & & & \\ \hline & & & \lambda_4 & 1 & & & \\ & & & 0 & \lambda_4 & & & \\ \hline & & & & & & \lambda_6 & \\ 0 & & & & & & & \lambda_7 \end{bmatrix}$$

then

$$e^{\mathbf{J}t} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & \frac{1}{2}t^2 e^{\lambda_1 t} & & & & & 0 \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} & & & & & \\ 0 & 0 & e^{\lambda_1 t} & & & & & \\ \hline & & & e^{\lambda_4 t} & te^{\lambda_4 t} & & & \\ & & & 0 & e^{\lambda_4 t} & & & \\ \hline & & & & & & e^{\lambda_6 t} & 0 \\ 0 & & & & & & 0 & e^{\lambda_7 t} \end{bmatrix}$$

A-9-12. Consider the following polynomial in λ of degree $m - 1$, where we assume $\lambda_1, \lambda_2, \dots, \lambda_m$ to be distinct:

$$p_k(\lambda) = \frac{(\lambda - \lambda_1) \cdots (\lambda - \lambda_{k-1})(\lambda - \lambda_{k+1}) \cdots (\lambda - \lambda_m)}{(\lambda_k - \lambda_1) \cdots (\lambda_k - \lambda_{k-1})(\lambda_k - \lambda_{k+1}) \cdots (\lambda_k - \lambda_m)}$$

where $k = 1, 2, \dots, m$. Notice that

$$p_k(\lambda_i) = \begin{cases} 1, & \text{if } i = k \\ 0, & \text{if } i \neq k \end{cases}$$

Then the polynomial $f(\lambda)$ of degree $m - 1$,

$$\begin{aligned} f(\lambda) &= \sum_{k=1}^m f(\lambda_k) p_k(\lambda) \\ &= \sum_{k=1}^m f(\lambda_k) \frac{(\lambda - \lambda_1) \cdots (\lambda - \lambda_{k-1})(\lambda - \lambda_{k+1}) \cdots (\lambda - \lambda_m)}{(\lambda_k - \lambda_1) \cdots (\lambda_k - \lambda_{k-1})(\lambda_k - \lambda_{k+1}) \cdots (\lambda_k - \lambda_m)} \end{aligned}$$

takes on the values $f(\lambda_k)$ at the points λ_k . This last equation is commonly called *Lagrange's interpolation formula*. The polynomial $f(\lambda)$ of degree $m - 1$ is determined from m independent data $f(\lambda_1), f(\lambda_2), \dots, f(\lambda_m)$. That is, the polynomial $f(\lambda)$ passes through m points $f(\lambda_1), f(\lambda_2), \dots, f(\lambda_m)$. Since $f(\lambda)$ is a polynomial of degree $m - 1$, it is uniquely determined. Any other representations of the polynomial of degree $m - 1$ can be reduced to the Lagrange polynomial $f(\lambda)$.

Assuming that the eigenvalues of an $n \times n$ matrix \mathbf{A} are distinct, substitute \mathbf{A} for λ in the polynomial $p_k(\lambda)$. Then we get

$$p_k(\mathbf{A}) = \frac{(\mathbf{A} - \lambda_1 \mathbf{I}) \cdots (\mathbf{A} - \lambda_{k-1} \mathbf{I})(\mathbf{A} - \lambda_{k+1} \mathbf{I}) \cdots (\mathbf{A} - \lambda_m \mathbf{I})}{(\lambda_k - \lambda_1) \cdots (\lambda_k - \lambda_{k-1})(\lambda_k - \lambda_{k+1}) \cdots (\lambda_k - \lambda_m)}$$

Notice that $p_k(\mathbf{A})$ is a polynomial in \mathbf{A} of degree $m - 1$. Notice also that

$$p_k(\lambda_i \mathbf{I}) = \begin{cases} \mathbf{I}, & \text{if } i = k \\ \mathbf{0}, & \text{if } i \neq k \end{cases}$$

Now define

$$\begin{aligned} f(\mathbf{A}) &= \sum_{k=1}^m f(\lambda_k) p_k(\mathbf{A}) \\ &= \sum_{k=1}^m f(\lambda_k) \frac{(\mathbf{A} - \lambda_1 \mathbf{I}) \cdots (\mathbf{A} - \lambda_{k-1} \mathbf{I})(\mathbf{A} - \lambda_{k+1} \mathbf{I}) \cdots (\mathbf{A} - \lambda_m \mathbf{I})}{(\lambda_k - \lambda_1) \cdots (\lambda_k - \lambda_{k-1})(\lambda_k - \lambda_{k+1}) \cdots (\lambda_k - \lambda_m)} \end{aligned} \quad (9-102)$$

Equation (9-102) is known as Sylvester's interpolation formula. Equation (9-102) is equivalent to the following equation:

$$\begin{vmatrix} 1 & 1 & \cdots & 1 & \mathbf{I} \\ \lambda_1 & \lambda_2 & \cdots & \lambda_m & \mathbf{A} \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_m^2 & \mathbf{A}^2 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \lambda_1^{m-1} & \lambda_2^{m-1} & \cdots & \lambda_m^{m-1} & \mathbf{A}^{m-1} \\ f(\lambda_1) & f(\lambda_2) & \cdots & f(\lambda_m) & f(\mathbf{A}) \end{vmatrix} = \mathbf{0} \quad (9-103)$$

Equations (9-102) and (9-103) are frequently used for evaluating functions $f(\mathbf{A})$ of matrix \mathbf{A} —for example, $(\lambda \mathbf{I} - \mathbf{A})^{-1}$, $e^{\mathbf{A}t}$, and so forth. Note that Equation (9-103) can also be written as

$$\begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{m-1} & f(\lambda_1) \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{m-1} & f(\lambda_2) \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 1 & \lambda_m & \lambda_m^2 & \cdots & \lambda_m^{m-1} & f(\lambda_m) \\ \mathbf{I} & \mathbf{A} & \mathbf{A}^2 & \cdots & \mathbf{A}^{m-1} & f(\mathbf{A}) \end{vmatrix} = \mathbf{0} \quad (9-104)$$

Show that Equations (9-102) and (9-103) are equivalent. To simplify the arguments, assume that $m = 4$.

Solution. Equation (9–103), where $m = 4$, can be expanded as follows:

$$\begin{aligned} \Delta &= \begin{vmatrix} 1 & 1 & 1 & 1 & \mathbf{I} \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \mathbf{A} \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \mathbf{A}^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 & \mathbf{A}^3 \\ f(\lambda_1) & f(\lambda_2) & f(\lambda_3) & f(\lambda_4) & f(\mathbf{A}) \end{vmatrix} \\ &= f(\mathbf{A}) \begin{vmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 \end{vmatrix} - f(\lambda_4) \begin{vmatrix} 1 & 1 & 1 & \mathbf{I} \\ \lambda_1 & \lambda_2 & \lambda_3 & \mathbf{A} \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \mathbf{A}^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \mathbf{A}^3 \end{vmatrix} \\ &\quad + f(\lambda_3) \begin{vmatrix} 1 & 1 & 1 & \mathbf{I} \\ \lambda_1 & \lambda_2 & \lambda_4 & \mathbf{A} \\ \lambda_1^2 & \lambda_2^2 & \lambda_4^2 & \mathbf{A}^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_4^3 & \mathbf{A}^3 \end{vmatrix} - f(\lambda_2) \begin{vmatrix} 1 & 1 & 1 & \mathbf{I} \\ \lambda_1 & \lambda_3 & \lambda_4 & \mathbf{A} \\ \lambda_1^2 & \lambda_3^2 & \lambda_4^2 & \mathbf{A}^2 \\ \lambda_1^3 & \lambda_3^3 & \lambda_4^3 & \mathbf{A}^3 \end{vmatrix} \\ &\quad + f(\lambda_1) \begin{vmatrix} 1 & 1 & 1 & \mathbf{I} \\ \lambda_2 & \lambda_3 & \lambda_4 & \mathbf{A} \\ \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \mathbf{A}^2 \\ \lambda_2^3 & \lambda_3^3 & \lambda_4^3 & \mathbf{A}^3 \end{vmatrix} \end{aligned}$$

Since

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 \end{vmatrix} = (\lambda_4 - \lambda_3)(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)$$

and

$$\begin{vmatrix} 1 & 1 & 1 & \mathbf{I} \\ \lambda_i & \lambda_j & \lambda_k & \mathbf{A} \\ \lambda_i^2 & \lambda_j^2 & \lambda_k^2 & \mathbf{A}^2 \\ \lambda_i^3 & \lambda_j^3 & \lambda_k^3 & \mathbf{A}^3 \end{vmatrix} = (\mathbf{A} - \lambda_k \mathbf{I})(\mathbf{A} - \lambda_j \mathbf{I})(\mathbf{A} - \lambda_i \mathbf{I})(\lambda_k - \lambda_j)(\lambda_k - \lambda_i)(\lambda_j - \lambda_i)$$

we obtain

$$\begin{aligned} \Delta &= f(\mathbf{A})[(\lambda_4 - \lambda_3)(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)] \\ &\quad - f(\lambda_4)[(\mathbf{A} - \lambda_3 \mathbf{I})(\mathbf{A} - \lambda_2 \mathbf{I})(\mathbf{A} - \lambda_1 \mathbf{I})(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)] \\ &\quad + f(\lambda_3)[(\mathbf{A} - \lambda_4 \mathbf{I})(\mathbf{A} - \lambda_2 \mathbf{I})(\mathbf{A} - \lambda_1 \mathbf{I})(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_1)(\lambda_2 - \lambda_1)] \\ &\quad - f(\lambda_2)[(\mathbf{A} - \lambda_4 \mathbf{I})(\mathbf{A} - \lambda_3 \mathbf{I})(\mathbf{A} - \lambda_1 \mathbf{I})(\lambda_4 - \lambda_3)(\lambda_4 - \lambda_1)(\lambda_3 - \lambda_1)] \\ &\quad + f(\lambda_1)[(\mathbf{A} - \lambda_4 \mathbf{I})(\mathbf{A} - \lambda_3 \mathbf{I})(\mathbf{A} - \lambda_2 \mathbf{I})(\lambda_4 - \lambda_3)(\lambda_4 - \lambda_2)(\lambda_3 - \lambda_2)] \\ &= \mathbf{0} \end{aligned}$$

Solving this last equation for $f(\mathbf{A})$, we obtain

$$\begin{aligned} f(\mathbf{A}) &= f(\lambda_1) \frac{(\mathbf{A} - \lambda_2 \mathbf{I})(\mathbf{A} - \lambda_3 \mathbf{I})(\mathbf{A} - \lambda_4 \mathbf{I})}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)} + f(\lambda_2) \frac{(\mathbf{A} - \lambda_1 \mathbf{I})(\mathbf{A} - \lambda_3 \mathbf{I})(\mathbf{A} - \lambda_4 \mathbf{I})}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)} \\ &\quad + f(\lambda_3) \frac{(\mathbf{A} - \lambda_1 \mathbf{I})(\mathbf{A} - \lambda_2 \mathbf{I})(\mathbf{A} - \lambda_4 \mathbf{I})}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_4)} + f(\lambda_4) \frac{(\mathbf{A} - \lambda_1 \mathbf{I})(\mathbf{A} - \lambda_2 \mathbf{I})(\mathbf{A} - \lambda_3 \mathbf{I})}{(\lambda_4 - \lambda_1)(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_3)} \\ &= \sum_{k=1}^m f(\lambda_k) \frac{(\mathbf{A} - \lambda_1 \mathbf{I}) \cdots (\mathbf{A} - \lambda_{k-1} \mathbf{I})(\mathbf{A} - \lambda_{k+1} \mathbf{I}) \cdots (\mathbf{A} - \lambda_m \mathbf{I})}{(\lambda_k - \lambda_1) \cdots (\lambda_k - \lambda_{k-1})(\lambda_k - \lambda_{k+1}) \cdots (\lambda_k - \lambda_m)} \end{aligned}$$

where $m = 4$. Thus, we have shown the equivalence of Equations (9-102) and (9-103). Although we assumed $m = 4$, the entire argument can be extended to an arbitrary positive integer m . (For the case when the matrix \mathbf{A} involves multiple eigenvalues, refer to Problem **A-9-13**.)

A-9-13. Consider Sylvester's interpolation formula in the form given by Equation (9-104):

$$\begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{m-1} & f(\lambda_1) \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{m-1} & f(\lambda_2) \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 1 & \lambda_m & \lambda_m^2 & \cdots & \lambda_m^{m-1} & f(\lambda_m) \\ \mathbf{I} & \mathbf{A} & \mathbf{A}^2 & \cdots & \mathbf{A}^{m-1} & f(\mathbf{A}) \end{vmatrix} = \mathbf{0}$$

This formula for the determination of $f(\mathbf{A})$ applies to the case where the minimal polynomial of \mathbf{A} involves only distinct roots.

Suppose that the minimal polynomial of \mathbf{A} involves multiple roots. Then the rows in the determinant that correspond to the multiple roots become identical, and therefore modification of the determinant in Equation (9-104) becomes necessary.

Modify the form of Sylvester's interpolation formula given by Equation (9-104) when the minimal polynomial of \mathbf{A} involves multiple roots. In deriving a modified determinant equation, assume that there are three equal roots ($\lambda_1 = \lambda_2 = \lambda_3$) in the minimal polynomial of \mathbf{A} and that there are other roots ($\lambda_4, \lambda_5, \dots, \lambda_m$) that are distinct.

Solution. Since the minimal polynomial of \mathbf{A} involves three equal roots, the minimal polynomial $\phi(\lambda)$ can be written as

$$\begin{aligned} \phi(\lambda) &= \lambda^m + a_1 \lambda^{m-1} + \cdots + a_{m-1} \lambda + a_m \\ &= (\lambda - \lambda_1)^3 (\lambda - \lambda_4) (\lambda - \lambda_5) \cdots (\lambda - \lambda_m) \end{aligned}$$

An arbitrary function $f(\mathbf{A})$ of an $n \times n$ matrix \mathbf{A} can be written as

$$f(\mathbf{A}) = g(\mathbf{A})\phi(\mathbf{A}) + \alpha(\mathbf{A})$$

where the minimal polynomial $\phi(\mathbf{A})$ is of degree m and $\alpha(\mathbf{A})$ is a polynomial in \mathbf{A} of degree $m - 1$ or less. Hence we have

$$f(\lambda) = g(\lambda)\phi(\lambda) + \alpha(\lambda)$$

where $\alpha(\lambda)$ is a polynomial in λ of degree $m - 1$ or less, which can thus be written as

$$\alpha(\lambda) = \alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2 + \cdots + \alpha_{m-1} \lambda^{m-1} \quad (9-105)$$

In the present case we have

$$\begin{aligned} f(\lambda) &= g(\lambda)\phi(\lambda) + \alpha(\lambda) \\ &= g(\lambda)[(\lambda - \lambda_1)^3(\lambda - \lambda_4)\cdots(\lambda - \lambda_m)] + \alpha(\lambda) \end{aligned} \quad (9-106)$$

By substituting $\lambda_1, \lambda_4, \dots, \lambda_m$ for λ in Equation (9-106), we obtain the following $m - 2$ equations:

$$\begin{aligned} f(\lambda_1) &= \alpha(\lambda_1) \\ f(\lambda_4) &= \alpha(\lambda_4) \\ &\vdots \\ &\vdots \\ &\vdots \\ f(\lambda_m) &= \alpha(\lambda_m) \end{aligned} \quad (9-107)$$

By differentiating Equation (9-106) with respect to λ , we obtain

$$\frac{d}{d\lambda} f(\lambda) = (\lambda - \lambda_1)^2 h(\lambda) + \frac{d}{d\lambda} \alpha(\lambda) \quad (9-108)$$

where

$$(\lambda - \lambda_1)^2 h(\lambda) = \frac{d}{d\lambda} [g(\lambda)(\lambda - \lambda_1)^3(\lambda - \lambda_4)\cdots(\lambda - \lambda_m)]$$

Substitution of λ_1 for λ in Equation (9-108) gives

$$\left. \frac{d}{d\lambda} f(\lambda) \right|_{\lambda=\lambda_1} = f'(\lambda_1) = \left. \frac{d}{d\lambda} \alpha(\lambda) \right|_{\lambda=\lambda_1}$$

Referring to Equation (9-105), this last equation becomes

$$f'(\lambda_1) = \alpha_1 + 2\alpha_2\lambda_1 + \cdots + (m-1)\alpha_{m-1}\lambda_1^{m-2} \quad (9-109)$$

Similarly, differentiating Equation (9-106) twice with respect to λ and substituting λ_1 for λ , we obtain

$$\left. \frac{d^2}{d\lambda^2} f(\lambda) \right|_{\lambda=\lambda_1} = f''(\lambda_1) = \left. \frac{d^2}{d\lambda^2} \alpha(\lambda) \right|_{\lambda=\lambda_1}$$

This last equation can be written as

$$f''(\lambda_1) = 2\alpha_2 + 6\alpha_3\lambda_1 + \cdots + (m-1)(m-2)\alpha_{m-1}\lambda_1^{m-3} \quad (9-110)$$

Rewriting Equations (9-110), (9-109), and (9-107), we get

$$\begin{aligned} \alpha_2 + 3\alpha_3\lambda_1 + \cdots + \frac{(m-1)(m-2)}{2}\alpha_{m-1}\lambda_1^{m-3} &= \frac{f''(\lambda_1)}{2} \\ \alpha_1 + 2\alpha_2\lambda_1 + \cdots + (m-1)\alpha_{m-1}\lambda_1^{m-2} &= f'(\lambda_1) \\ \alpha_0 + \alpha_1\lambda_1 + \alpha_2\lambda_1^2 + \cdots + \alpha_{m-1}\lambda_1^{m-1} &= f(\lambda_1) \\ \alpha_0 + \alpha_1\lambda_4 + \alpha_2\lambda_4^2 + \cdots + \alpha_{m-1}\lambda_4^{m-1} &= f(\lambda_4) \\ &\vdots \\ &\vdots \\ &\vdots \\ \alpha_0 + \alpha_1\lambda_m + \alpha_2\lambda_m^2 + \cdots + \alpha_{m-1}\lambda_m^{m-1} &= f(\lambda_m) \end{aligned} \quad (9-111)$$

These m simultaneous equations determine the α_k values (where $k = 0, 1, 2, \dots, m - 1$). Noting that $\phi(\mathbf{A}) = \mathbf{0}$ because it is a minimal polynomial, we have $f(\mathbf{A})$ as follows:

$$f(\mathbf{A}) = g(\mathbf{A})\phi(\mathbf{A}) + \alpha(\mathbf{A}) = \alpha(\mathbf{A})$$

Hence, referring to Equation (9-105), we have

$$f(\mathbf{A}) = \alpha(\mathbf{A}) = \alpha_0\mathbf{I} + \alpha_1\mathbf{A} + \alpha_2\mathbf{A}^2 + \cdots + \alpha_{m-1}\mathbf{A}^{m-1} \quad (9-112)$$

where the α_k values are given in terms of $f(\lambda_1), f'(\lambda_1), f''(\lambda_1), f(\lambda_4), f(\lambda_5), \dots, f(\lambda_m)$. In terms of the determinant equation, $f(\mathbf{A})$ can be obtained by solving the following equation:

$$\begin{vmatrix} 0 & 0 & 1 & 3\lambda_1 & \cdots & \frac{(m-1)(m-2)}{2}\lambda_1^{m-3} & \frac{f''(\lambda_1)}{2} \\ 0 & 1 & 2\lambda_1 & 3\lambda_1^2 & \cdots & (m-1)\lambda_1^{m-2} & f'(\lambda_1) \\ 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 & \cdots & \lambda_1^{m-1} & f(\lambda_1) \\ 1 & \lambda_4 & \lambda_4^2 & \lambda_4^3 & \cdots & \lambda_4^{m-1} & f(\lambda_4) \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 1 & \lambda_m & \lambda_m^2 & \lambda_m^3 & \cdots & \lambda_m^{m-1} & f(\lambda_m) \\ \mathbf{I} & \mathbf{A} & \mathbf{A}^2 & \mathbf{A}^3 & \cdots & \mathbf{A}^{m-1} & f(\mathbf{A}) \end{vmatrix} = \mathbf{0} \quad (9-113)$$

Equation (9-113) shows the desired modification in the form of the determinant. This equation gives the form of Sylvester's interpolation formula when the minimal polynomial of \mathbf{A} involves three equal roots. (The necessary modification of the form of the determinant for other cases will be apparent.)

A-9-14. Using Sylvester's interpolation formula, compute $e^{\mathbf{A}t}$, where

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

Solution. Referring to Problem **A-9-9**, the characteristic polynomial and the minimal polynomial are the same for this \mathbf{A} . The minimal polynomial (characteristic polynomial) is given by

$$\phi(\lambda) = (\lambda - 2)^2(\lambda - 1)$$

Note that $\lambda_1 = \lambda_2 = 2$ and $\lambda_3 = 1$. Referring to Equation (9-112) and noting that $f(\mathbf{A})$ in this problem is $e^{\mathbf{A}t}$, we have

$$e^{\mathbf{A}t} = \alpha_0(t)\mathbf{I} + \alpha_1(t)\mathbf{A} + \alpha_2(t)\mathbf{A}^2$$

where $\alpha_0(t)$, $\alpha_1(t)$, and $\alpha_2(t)$ are determined from the equations

$$\begin{aligned} \alpha_1(t) + 2\alpha_2(t)\lambda_1 &= te^{\lambda_1 t} \\ \alpha_0(t) + \alpha_1(t)\lambda_1 + \alpha_2(t)\lambda_1^2 &= e^{\lambda_1 t} \\ \alpha_0(t) + \alpha_1(t)\lambda_3 + \alpha_2(t)\lambda_3^2 &= e^{\lambda_3 t} \end{aligned}$$

Substituting $\lambda_1 = 2$, and $\lambda_3 = 1$ into these three equations gives

$$\begin{aligned}\alpha_1(t) + 4\alpha_2(t) &= te^{2t} \\ \alpha_0(t) + 2\alpha_1(t) + 4\alpha_2(t) &= e^{2t} \\ \alpha_0(t) + \alpha_1(t) + \alpha_2(t) &= e^t\end{aligned}$$

Solving for $\alpha_0(t)$, $\alpha_1(t)$, and $\alpha_2(t)$, we obtain

$$\begin{aligned}\alpha_0(t) &= 4e^t - 3e^{2t} + 2te^{2t} \\ \alpha_1(t) &= -4e^t + 4e^{2t} - 3te^{2t} \\ \alpha_2(t) &= e^t - e^{2t} + te^{2t}\end{aligned}$$

Hence,

$$\begin{aligned}e^{\mathbf{A}t} &= (4e^t - 3e^{2t} + 2te^{2t}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (-4e^t + 4e^{2t} - 3te^{2t}) \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix} \\ &\quad + (e^t - e^{2t} + te^{2t}) \begin{bmatrix} 4 & 16 & 12 \\ 0 & 4 & 0 \\ 0 & 9 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{2t} & 12e^t - 12e^{2t} + 13te^{2t} & -4e^t + 4e^{2t} \\ 0 & e^{2t} & 0 \\ 0 & -3e^t + 3e^{2t} & e^t \end{bmatrix}\end{aligned}$$

A-9-15. Show that the system described by

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \quad (9-114)$$

$$\mathbf{y} = \mathbf{Cx} \quad (9-115)$$

where \mathbf{x} = state vector (n -vector)

\mathbf{u} = control vector (r -vector)

\mathbf{y} = output vector (m -vector) ($m \leq n$)

\mathbf{A} = $n \times n$ matrix

\mathbf{B} = $n \times r$ matrix

\mathbf{C} = $m \times n$ matrix

is completely output controllable if and only if the composite $m \times nr$ matrix \mathbf{P} , where

$$\mathbf{P} = [\mathbf{CB} \mid \mathbf{CAB} \mid \mathbf{CA}^2\mathbf{B} \mid \cdots \mid \mathbf{CA}^{n-1}\mathbf{B}]$$

is of rank m . (Notice that complete state controllability is neither necessary nor sufficient for complete output controllability.)

Solution. Suppose that the system is output controllable and the output $\mathbf{y}(t)$ starting from any $\mathbf{y}(0)$, the initial output, can be transferred to the origin of the output space in a finite time interval $0 \leq t \leq T$. That is,

$$\mathbf{y}(T) = \mathbf{Cx}(T) = \mathbf{0} \quad (9-116)$$

Since the solution of Equation (9-114) is

$$\mathbf{x}(t) = e^{\mathbf{A}t} \left[\mathbf{x}(0) + \int_0^t e^{-\mathbf{A}\tau} \mathbf{B}\mathbf{u}(\tau) d\tau \right]$$

at $t = T$, we have

$$\mathbf{x}(T) = e^{\mathbf{A}T} \left[\mathbf{x}(0) + \int_0^T e^{-\mathbf{A}\tau} \mathbf{B}\mathbf{u}(\tau) d\tau \right] \quad (9-117)$$

Substituting Equation (9-117) into Equation (9-116), we obtain

$$\begin{aligned} \mathbf{y}(T) &= \mathbf{C}\mathbf{x}(T) \\ &= \mathbf{C}e^{\mathbf{A}T} \left[\mathbf{x}(0) + \int_0^T e^{-\mathbf{A}\tau} \mathbf{B}\mathbf{u}(\tau) d\tau \right] = \mathbf{0} \end{aligned} \quad (9-118)$$

On the other hand, $\mathbf{y}(0) = \mathbf{C}\mathbf{x}(0)$. Notice that the complete output controllability means that the vector $\mathbf{C}\mathbf{x}(0)$ spans the m -dimensional output space. Since $e^{\mathbf{A}T}$ is nonsingular, if $\mathbf{C}\mathbf{x}(0)$ spans the m -dimensional output space, so does $\mathbf{C}e^{\mathbf{A}T}\mathbf{x}(0)$, and vice versa. From Equation (9-118) we obtain

$$\begin{aligned} \mathbf{C}e^{\mathbf{A}T}\mathbf{x}(0) &= -\mathbf{C}e^{\mathbf{A}T} \int_0^T e^{-\mathbf{A}\tau} \mathbf{B}\mathbf{u}(\tau) d\tau \\ &= -\mathbf{C} \int_0^T e^{\mathbf{A}\tau} \mathbf{B}\mathbf{u}(T - \tau) d\tau \end{aligned}$$

Note that $\int_0^T e^{\mathbf{A}\tau} \mathbf{B}\mathbf{u}(T - \tau) d\tau$ can be expressed as the sum of $\mathbf{A}^i \mathbf{B}_j$; that is,

$$\int_0^T e^{\mathbf{A}\tau} \mathbf{B}\mathbf{u}(T - \tau) d\tau = \sum_{i=0}^{p-1} \sum_{j=1}^r \gamma_{ij} \mathbf{A}^i \mathbf{B}_j$$

where

$$\gamma_{ij} = \int_0^T \alpha_i(\tau) u_j(T - \tau) d\tau = \text{scalar}$$

and $\alpha_i(\tau)$ satisfies

$$e^{\mathbf{A}\tau} = \sum_{i=0}^{p-1} \alpha_i(\tau) \mathbf{A}^i \quad (p: \text{degree of the minimal polynomial of } \mathbf{A})$$

and \mathbf{B}_j is the j th column of \mathbf{B} . Therefore, we can write $\mathbf{C}e^{\mathbf{A}T}\mathbf{x}(0)$ as

$$\mathbf{C}e^{\mathbf{A}T}\mathbf{x}(0) = - \sum_{i=0}^{p-1} \sum_{j=1}^r \gamma_{ij} \mathbf{C}\mathbf{A}^i \mathbf{B}_j$$

From this last equation, we see that $\mathbf{C}e^{\mathbf{A}T}\mathbf{x}(0)$ is a linear combination of $\mathbf{C}\mathbf{A}^i \mathbf{B}_j$ ($i = 0, 1, 2, \dots, p-1; j = 1, 2, \dots, r$). Note that if the rank of \mathbf{Q} , where

$$\mathbf{Q} = [\mathbf{C}\mathbf{B} \mid \mathbf{C}\mathbf{A}\mathbf{B} \mid \mathbf{C}\mathbf{A}^2\mathbf{B} \mid \dots \mid \mathbf{C}\mathbf{A}^{p-1}\mathbf{B}] \quad (p \leq n)$$

is m , then so is the rank of \mathbf{P} , and vice versa. [This is obvious if $p = n$. If $p < n$, then the $\mathbf{C}\mathbf{A}^h \mathbf{B}_j$ (where $p \leq h \leq n-1$) are linearly dependent on $\mathbf{C}\mathbf{B}_j, \mathbf{C}\mathbf{A}\mathbf{B}_j, \dots, \mathbf{C}\mathbf{A}^{p-1}\mathbf{B}_j$. Hence, the rank of

\mathbf{P} is equal to that of \mathbf{Q} .] If the rank of \mathbf{P} is m , then $\mathbf{C}e^{A^T\mathbf{x}}(0)$ spans the m -dimensional output space. This means that if the rank of \mathbf{P} is m , then $\mathbf{C}\mathbf{x}(0)$ also spans the m -dimensional output space and the system is completely output controllable.

Conversely, suppose that the system is completely output controllable, but the rank of \mathbf{P} is k , where $k < m$. Then the set of all initial outputs that can be transferred to the origin is of k -dimensional space. Hence, the dimension of this set is less than m . This contradicts the assumption that the system is completely output controllable. This completes the proof.

Note that it can be immediately proved that, in the system of Equations (9–114) and (9–115), complete state controllability on $0 \leq t \leq T$ implies complete output controllability on $0 \leq t \leq T$ if and only if m rows of \mathbf{C} are linearly independent.

A–9–16. Discuss the state controllability of the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -2 & 1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \end{bmatrix} u \quad (9-119)$$

Solution. For this system,

$$\mathbf{A} = \begin{bmatrix} -3 & 1 \\ -2 & 1.5 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Since

$$\mathbf{AB} = \begin{bmatrix} -3 & 1 \\ -2 & 1.5 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

we see that vectors \mathbf{B} and \mathbf{AB} are not linearly independent and the rank of the matrix $[\mathbf{B} \ \mathbf{AB}]$ is 1. Therefore, the system is not completely state controllable. In fact, elimination of x_2 from Equation (9–119), or the following two simultaneous equations,

$$\begin{aligned} \dot{x}_1 &= -3x_1 + x_2 + u \\ \dot{x}_2 &= -2x_1 + 1.5x_2 + 4u \end{aligned}$$

yields

$$\ddot{x}_1 + 1.5\dot{x}_1 - 2.5x_1 = \dot{u} + 2.5u$$

or, in the form of a transfer function,

$$\frac{X_1(s)}{U(s)} = \frac{s + 2.5}{(s + 2.5)(s - 1)}$$

Notice that cancellation of the factor $(s + 2.5)$ occurs in the numerator and denominator of the transfer function. Because of this cancellation, this system is not completely state controllable. This is an unstable system. Remember that stability and controllability are quite different things. There are many systems that are unstable, but are completely state controllable.

A–9–17. A state-space representation of a system in the controllable canonical form is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.4 & -1.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad (9-120)$$

$$y = [0.8 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (9-121)$$

The same system may be represented by the following state-space equation, which is in the observable canonical form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -0.4 \\ 1 & -1.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.8 \\ 1 \end{bmatrix} u \quad (9-122)$$

$$y = [0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (9-123)$$

Show that the state-space representation given by Equations (9-120) and (9-121) gives a system that is state controllable, but not observable. Show, on the other hand, that the state-space representation defined by Equations (9-122) and (9-123) gives a system that is not completely state controllable, but is observable. Explain what causes the apparent difference in the controllability and observability of the same system.

Solution. Consider the system defined by Equations (9-120) and (9-121). The rank of the controllability matrix

$$[\mathbf{B} \quad \mathbf{AB}] = \begin{bmatrix} 0 & 1 \\ 1 & -1.3 \end{bmatrix}$$

is 2. Hence, the system is completely state controllable. The rank of the observability matrix

$$[\mathbf{C}^* \quad \mathbf{A}^*\mathbf{C}^*] = \begin{bmatrix} 0.8 & -0.4 \\ 1 & -0.5 \end{bmatrix}$$

is 1. Hence the system is not observable.

Next consider the system defined by Equations (9-122) and (9-123). The rank of the controllability matrix

$$[\mathbf{B} \quad \mathbf{AB}] = \begin{bmatrix} 0.8 & -0.4 \\ 1 & -0.5 \end{bmatrix}$$

is 1. Hence, the system is not completely state controllable. The rank of the observability matrix

$$[\mathbf{C}^* \quad \mathbf{A}^*\mathbf{C}^*] = \begin{bmatrix} 0 & 1 \\ 1 & -1.3 \end{bmatrix}$$

is 2. Hence, the system is observable.

The apparent difference in the controllability and observability of the same system is caused by the fact that the original system has a pole-zero cancellation in the transfer function. Referring to Equation (2-29), for $D = 0$ we have

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

If we use Equations (9-120) and (9-121), then

$$\begin{aligned} G(s) &= [0.8 \quad 1] \begin{bmatrix} s & -1 \\ 0.4 & s + 1.3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{s^2 + 1.3s + 0.4} [0.8 \quad 1] \begin{bmatrix} s + 1.3 & 1 \\ -0.4 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{s + 0.8}{(s + 0.8)(s + 0.5)} \end{aligned}$$

[Note that the same transfer function can be obtained by using Equations (9-122) and (9-123).] Clearly, cancellation occurs in this transfer function.

If a pole-zero cancellation occurs in the transfer function, then the controllability and observability vary, depending on how the state variables are chosen. Remember that, to be completely state controllable and observable, the transfer function must not have any pole-zero cancellations.

A-9-18. Prove that the system defined by

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} \\ \mathbf{y} &= \mathbf{C}\mathbf{x}\end{aligned}$$

where \mathbf{x} = state vector (n -vector)
 \mathbf{y} = output vector (m -vector) ($m \leq n$)
 \mathbf{A} = $n \times n$ matrix
 \mathbf{C} = $m \times n$ matrix

is completely observable if and only if the composite $mn \times n$ matrix \mathbf{P} , where

$$\mathbf{P} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}$$

is of rank n .

Solution. We shall first obtain the necessary condition. Suppose that

$$\text{rank } \mathbf{P} < n$$

Then there exists $\mathbf{x}(0)$ such that

$$\mathbf{P}\mathbf{x}(0) = \mathbf{0}$$

or

$$\mathbf{P}\mathbf{x}(0) = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix} \mathbf{x}(0) = \begin{bmatrix} \mathbf{C}\mathbf{x}(0) \\ \mathbf{CA}\mathbf{x}(0) \\ \vdots \\ \mathbf{CA}^{n-1}\mathbf{x}(0) \end{bmatrix} = \mathbf{0}$$

Hence, we obtain, for a certain $\mathbf{x}(0)$,

$$\mathbf{CA}^i\mathbf{x}(0) = \mathbf{0}, \quad \text{for } i = 0, 1, 2, \dots, n - 1$$

Notice that from Equation (9-48) or (9-50), we have

$$e^{\mathbf{A}t} = \alpha_0(t)\mathbf{I} + \alpha_1(t)\mathbf{A} + \alpha_2(t)\mathbf{A}^2 + \dots + \alpha_{m-1}(t)\mathbf{A}^{m-1}$$

where m ($m \leq n$) is the degree of the minimal polynomial for \mathbf{A} . Hence, for a certain $\mathbf{x}(0)$, we have

$$\mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) = \mathbf{C}[\alpha_0(t)\mathbf{I} + \alpha_1(t)\mathbf{A} + \alpha_2(t)\mathbf{A}^2 + \dots + \alpha_{m-1}(t)\mathbf{A}^{m-1}]\mathbf{x}(0) = \mathbf{0}$$

Consequently, for a certain $\mathbf{x}(0)$,

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) = \mathbf{0}$$

which implies that, for a certain $\mathbf{x}(0)$, $\mathbf{x}(0)$ cannot be determined from $\mathbf{y}(t)$. Therefore, the rank of matrix \mathbf{P} must be equal to n .

Next we shall obtain the sufficient condition. Suppose that $\text{rank } \mathbf{P} = n$. Since

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0)$$

by premultiplying both sides of this last equation by $e^{\mathbf{A}^*t}\mathbf{C}^*$, we get

$$e^{\mathbf{A}^*t}\mathbf{C}^*\mathbf{y}(t) = e^{\mathbf{A}^*t}\mathbf{C}^*\mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0)$$

If we integrate this last equation from 0 to t , we obtain

$$\int_0^t e^{\mathbf{A}^*\tau}\mathbf{C}^*\mathbf{y}(\tau) d\tau = \int_0^t e^{\mathbf{A}^*\tau}\mathbf{C}^*\mathbf{C}e^{\mathbf{A}\tau}\mathbf{x}(0) d\tau \quad (9-124)$$

Notice that the left-hand side of this equation is a known quantity. Define

$$\mathbf{Q}(t) = \int_0^t e^{\mathbf{A}^*\tau}\mathbf{C}^*\mathbf{y}(\tau) d\tau = \text{known quantity} \quad (9-125)$$

Then, from Equations (9-124) and (9-125), we have

$$\mathbf{Q}(t) = \mathbf{W}(t)\mathbf{x}(0) \quad (9-126)$$

where

$$\mathbf{W}(t) = \int_0^t e^{\mathbf{A}^*\tau}\mathbf{C}^*\mathbf{C}e^{\mathbf{A}\tau} d\tau$$

It can be established that $\mathbf{W}(t)$ is a nonsingular matrix as follows: If $|\mathbf{W}(t)|$ were equal to 0, then

$$\mathbf{x}^*\mathbf{W}(t_1)\mathbf{x} = \int_0^{t_1} \|\mathbf{C}e^{\mathbf{A}t}\mathbf{x}\|^2 dt = 0$$

which means that

$$\mathbf{C}e^{\mathbf{A}t}\mathbf{x} = \mathbf{0}, \quad \text{for } 0 \leq t \leq t_1$$

which implies that $\text{rank } \mathbf{P} < n$. Therefore, $|\mathbf{W}(t)| \neq 0$, or $\mathbf{W}(t)$ is nonsingular. Then, from Equation (9-126), we obtain

$$\mathbf{x}(0) = [\mathbf{W}(t)]^{-1}\mathbf{Q}(t) \quad (9-127)$$

and $\mathbf{x}(0)$ can be determined from Equation (9-127).

Hence, we have proved that $\mathbf{x}(0)$ can be determined from $\mathbf{y}(t)$ if and only if $\text{rank } \mathbf{P} = n$. Note that $\mathbf{x}(0)$ and $\mathbf{y}(t)$ are related by

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) = \alpha_0(t)\mathbf{C}\mathbf{x}(0) + \alpha_1(t)\mathbf{C}\mathbf{A}\mathbf{x}(0) + \cdots + \alpha_{n-1}(t)\mathbf{C}\mathbf{A}^{n-1}\mathbf{x}(0)$$

PROBLEMS

B-9-1. Consider the following transfer-function system:

$$\frac{Y(s)}{U(s)} = \frac{s + 6}{s^2 + 5s + 6}$$

Obtain the state-space representation of this system in (a) controllable canonical form and (b) observable canonical form.

B-9-2. Consider the following system:

$$\ddot{y} + 6\dot{y} + 11y + 6y = 6u$$

Obtain a state-space representation of this system in a diagonal canonical form.

B-9-3. Consider the system defined by

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x}\end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -4 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 1]$$

Transform the system equations into the controllable canonical form.

B-9-4. Consider the system defined by

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x}\end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 1 \quad 0]$$

Obtain the transfer function $Y(s)/U(s)$.

B-9-5. Consider the following matrix \mathbf{A} :

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Obtain the eigenvalues $\lambda_1, \lambda_2, \lambda_3,$ and λ_4 of the matrix \mathbf{A} . Then obtain a transformation matrix \mathbf{P} such that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$$

B-9-6. Consider the following matrix \mathbf{A} :

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

Compute $e^{\mathbf{A}t}$ by three methods.

B-9-7. Given the system equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

find the solution in terms of the initial conditions $x_1(0), x_2(0),$ and $x_3(0)$.

B-9-8. Find $x_1(t)$ and $x_2(t)$ of the system described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where the initial conditions are

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

B-9-9. Consider the following state equation and output equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -6 & 1 & 0 \\ -11 & 0 & 1 \\ -6 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Show that the state equation can be transformed into the following form by use of a proper transformation matrix:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

Then obtain the output y in terms of $z_1, z_2,$ and z_3 .

B-9-10. Obtain a state-space representation of the following system with MATLAB:

$$\frac{Y(s)}{U(s)} = \frac{10.4s^2 + 47s + 160}{s^3 + 14s^2 + 56s + 160}$$

B-9-11. Obtain a transfer-function representation of the following system with MATLAB:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = [0 \quad 0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

B-9-12. Obtain a transfer-function representation of the following system with MATLAB:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

B-9-13. Consider the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -2 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Is the system completely state controllable and completely observable?

B-9-14. Consider the system given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Is the system completely state controllable and completely observable? Is the system completely output controllable?

B-9-15. Is the following system completely state controllable and completely observable?

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [20 \quad 9 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

B-9-16. Consider the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [c_1 \quad c_2 \quad c_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Except for an obvious choice of $c_1 = c_2 = c_3 = 0$, find an example of a set of c_1, c_2, c_3 that will make the system unobservable.

B-9-17. Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

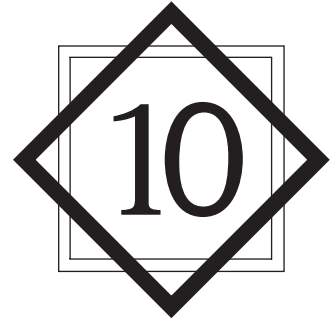
The output is given by

$$y = [1 \quad 1 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(a) Show that the system is not completely observable.

(b) Show that the system is completely observable if the output is given by

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



Control Systems Design in State Space

10-1 INTRODUCTION

This chapter discusses state-space design methods based on the pole-placement method, observers, the quadratic optimal regulator systems, and introductory aspects of robust control systems. The pole-placement method is somewhat similar to the root-locus method in that we place closed-loop poles at desired locations. The basic difference is that in the root-locus design we place only the dominant closed-loop poles at the desired locations, while in the pole-placement design we place all closed-loop poles at desired locations.

We begin by presenting the basic materials on pole placement in regulator systems. We then discuss the design of state observers, followed by the design of regulator systems and control systems using the pole-placement-with-state-observer approach. Then, we discuss the quadratic optimal regulator systems. Finally, we present an introduction to robust control systems.

Outline of the Chapter. Section 10-1 has presented introductory material. Section 10-2 discusses the pole-placement approach to the design of control systems. We begin with the derivation of the necessary and sufficient conditions for arbitrary pole placement. Then we derive equations for the state feedback gain matrix \mathbf{K} for pole placement. Section 10-3 presents the solution of the pole-placement problem with MATLAB. Section 10-4 discusses the design of servo systems using the pole-placement approach. Section 10-5 presents state observers. We discuss both full-order and minimum-order state observers. Also, transfer functions of observer controllers are derived. Section 10-6 presents the design of regulator systems with observers. Section 10-7 treats the design of control

systems with observers. Section 10–8 discusses quadratic optimal regulator systems. Note that the state feedback gain matrix \mathbf{K} can be obtained by both the pole-placement method and the quadratic optimal control method. Finally, Section 10–9 presents robust control systems. The discussions here are limited to introductory subjects only.

10–2 POLE PLACEMENT

In this section we shall present a design method commonly called the *pole-placement* or *pole-assignment technique*. We assume that all state variables are measurable and are available for feedback. It will be shown that if the system considered is completely state controllable, then poles of the closed-loop system may be placed at any desired locations by means of state feedback through an appropriate state feedback gain matrix.

The present design technique begins with a determination of the desired closed-loop poles based on the transient-response and/or frequency-response requirements, such as speed, damping ratio, or bandwidth, as well as steady-state requirements.

Let us assume that we decide that the desired closed-loop poles are to be at $s = \mu_1$, $s = \mu_2, \dots, s = \mu_n$. By choosing an appropriate gain matrix for state feedback, it is possible to force the system to have closed-loop poles at the desired locations, provided that the original system is completely state controllable.

In this chapter we limit our discussions to single-input, single-output systems. That is, we assume the control signal $u(t)$ and output signal $y(t)$ to be scalars. In the derivation in this section we assume that the reference input $r(t)$ is zero. [In Section 10–7 we discuss the case where the reference input $r(t)$ is nonzero.]

In what follows we shall prove that a necessary and sufficient condition that the closed-loop poles can be placed at any arbitrary locations in the s plane is that the system be completely state controllable. Then we shall discuss methods for determining the required state feedback gain matrix.

It is noted that when the control signal is a vector quantity, the mathematical aspects of the pole-placement scheme become complicated. We shall not discuss such a case in this book. (When the control signal is a vector quantity, the state feedback gain matrix is not unique. It is possible to choose freely more than n parameters; that is, in addition to being able to place n closed-loop poles properly, we have the freedom to satisfy some or all of the other requirements, if any, of the closed-loop system.)

Design by Pole Placement. In the conventional approach to the design of a single-input, single-output control system, we design a controller (compensator) such that the dominant closed-loop poles have a desired damping ratio ζ and a desired undamped natural frequency ω_n . In this approach, the order of the system may be raised by 1 or 2 unless pole-zero cancellation takes place. Note that in this approach we assume the effects on the responses of nondominant closed-loop poles to be negligible.

Different from specifying only dominant closed-loop poles (the conventional design approach), the present pole-placement approach specifies all closed-loop poles. (There is a cost associated with placing all closed-loop poles, however, because placing all closed-loop poles requires successful measurements of all state variables or else requires the inclusion of a state observer in the system.) There is also a requirement on the part of the system for the closed-loop poles to be placed at arbitrarily chosen locations. The requirement is that the system be completely state controllable. We shall prove this fact in this section.

Consider a control system

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x} + Du\end{aligned}\tag{10-1}$$

where \mathbf{x} = state vector (n -vector)
 y = output signal (scalar)
 u = control signal (scalar)
 \mathbf{A} = $n \times n$ constant matrix
 \mathbf{B} = $n \times 1$ constant matrix
 \mathbf{C} = $1 \times n$ constant matrix
 D = constant (scalar)

We shall choose the control signal to be

$$u = -\mathbf{K}\mathbf{x}\tag{10-2}$$

This means that the control signal u is determined by an instantaneous state. Such a scheme is called state feedback. The $1 \times n$ matrix \mathbf{K} is called the state feedback gain matrix. We assume that all state variables are available for feedback. In the following analysis we assume that u is unconstrained. A block diagram for this system is shown in Figure 10-1.

This closed-loop system has no input. Its objective is to maintain the zero output. Because of the disturbances that may be present, the output will deviate from zero. The nonzero output will be returned to the zero reference input because of the state feedback scheme of the system. Such a system where the reference input is always zero is called a regulator system. (Note that if the reference input to the system is always a nonzero constant, the system is also called a regulator system.)

Substituting Equation (10-2) into Equation (10-1) gives

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t)$$

The solution of this equation is given by

$$\mathbf{x}(t) = e^{(\mathbf{A} - \mathbf{B}\mathbf{K})t}\mathbf{x}(0)\tag{10-3}$$

where $\mathbf{x}(0)$ is the initial state caused by external disturbances. The stability and transient-response characteristics are determined by the eigenvalues of matrix $\mathbf{A} - \mathbf{B}\mathbf{K}$. If matrix

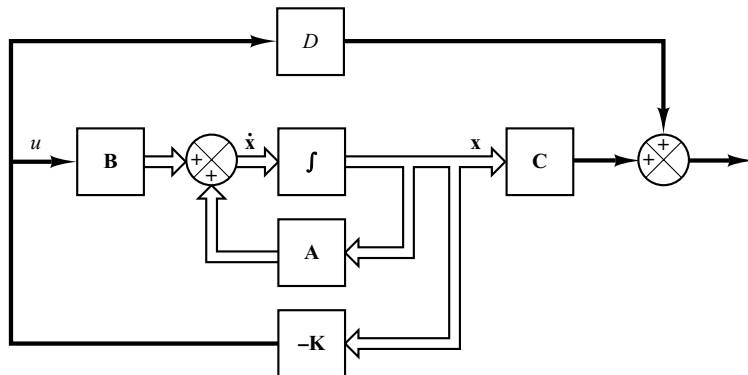


Figure 10-1
 Closed-loop control
 system with
 $u = -\mathbf{K}\mathbf{x}$.

\mathbf{K} is chosen properly, the matrix $\mathbf{A} - \mathbf{BK}$ can be made an asymptotically stable matrix, and for all $\mathbf{x}(0) \neq \mathbf{0}$, it is possible to make $\mathbf{x}(t)$ approach $\mathbf{0}$ as t approaches infinity. The eigenvalues of matrix $\mathbf{A} - \mathbf{BK}$ are called the regulator poles. If these regulator poles are placed in the left-half s plane, then $\mathbf{x}(t)$ approaches $\mathbf{0}$ as t approaches infinity. The problem of placing the regulator poles (closed-loop poles) at the desired location is called a pole-placement problem.

In what follows, we shall prove that arbitrary pole placement for a given system is possible if and only if the system is completely state controllable.

Necessary and Sufficient Condition for Arbitrary Pole Placement We shall now prove that a necessary and sufficient condition for arbitrary pole placement is that the system be completely state controllable. We shall first derive the necessary condition. We begin by proving that if the system is not completely state controllable, then there are eigenvalues of matrix $\mathbf{A} - \mathbf{BK}$ that cannot be controlled by state feedback.

Suppose that the system of Equation (10-1) is not completely state controllable. Then the rank of the controllability matrix is less than n , or

$$\text{rank}[\mathbf{B} \mid \mathbf{AB} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}] = q < n$$

This means that there are q linearly independent column vectors in the controllability matrix. Let us define such q linearly independent column vectors as $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_q$. Also, let us choose $n - q$ additional n -vectors $\mathbf{v}_{q+1}, \mathbf{v}_{q+2}, \dots, \mathbf{v}_n$ such that

$$\mathbf{P} = [\mathbf{f}_1 \mid \mathbf{f}_2 \mid \cdots \mid \mathbf{f}_q \mid \mathbf{v}_{q+1} \mid \mathbf{v}_{q+2} \mid \cdots \mid \mathbf{v}_n]$$

is of rank n . Then it can be shown that

$$\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \left[\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{0} & \mathbf{A}_{22} \end{array} \right], \quad \hat{\mathbf{B}} = \mathbf{P}^{-1}\mathbf{B} = \left[\begin{array}{c} \mathbf{B}_{11} \\ \hline \mathbf{0} \end{array} \right]$$

(See Problem A-10-1 for the derivation of these equations.) Now define

$$\hat{\mathbf{K}} = \mathbf{K}\mathbf{P} = [\mathbf{k}_1 \mid \mathbf{k}_2]$$

Then we have

$$\begin{aligned} |s\mathbf{I} - \mathbf{A} + \mathbf{BK}| &= |\mathbf{P}^{-1}(s\mathbf{I} - \mathbf{A} + \mathbf{BK})\mathbf{P}| \\ &= |s\mathbf{I} - \mathbf{P}^{-1}\mathbf{A}\mathbf{P} + \mathbf{P}^{-1}\mathbf{BK}\mathbf{P}| \\ &= |s\mathbf{I} - \hat{\mathbf{A}} + \hat{\mathbf{B}}\hat{\mathbf{K}}| \\ &= \left| s\mathbf{I} - \left[\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{0} & \mathbf{A}_{22} \end{array} \right] + \left[\begin{array}{c} \mathbf{B}_{11} \\ \hline \mathbf{0} \end{array} \right] [\mathbf{k}_1 \mid \mathbf{k}_2] \right| \\ &= \left| \begin{array}{cc} s\mathbf{I}_q - \mathbf{A}_{11} + \mathbf{B}_{11}\mathbf{k}_1 & -\mathbf{A}_{12} + \mathbf{B}_{11}\mathbf{k}_2 \\ \mathbf{0} & s\mathbf{I}_{n-q} - \mathbf{A}_{22} \end{array} \right| \\ &= |s\mathbf{I}_q - \mathbf{A}_{11} + \mathbf{B}_{11}\mathbf{k}_1| \cdot |s\mathbf{I}_{n-q} - \mathbf{A}_{22}| = 0 \end{aligned}$$

where \mathbf{I}_q is a q -dimensional identity matrix and \mathbf{I}_{n-q} is an $(n - q)$ -dimensional identity matrix.

Notice that the eigenvalues of \mathbf{A}_{22} do not depend on \mathbf{K} . Thus, if the system is not completely state controllable, then there are eigenvalues of matrix \mathbf{A} that cannot be arbitrarily placed. Therefore, to place the eigenvalues of matrix $\mathbf{A} - \mathbf{BK}$ arbitrarily, the system must be completely state controllable (necessary condition).

Next we shall prove a sufficient condition: that is, if the system is completely state controllable, then all eigenvalues of matrix \mathbf{A} can be arbitrarily placed.

In proving a sufficient condition, it is convenient to transform the state equation given by Equation (10-1) into the controllable canonical form.

Define a transformation matrix \mathbf{T} by

$$\mathbf{T} = \mathbf{M}\mathbf{W} \quad (10-4)$$

where \mathbf{M} is the controllability matrix

$$\mathbf{M} = [\mathbf{B} \mid \mathbf{AB} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}] \quad (10-5)$$

and

$$\mathbf{W} = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \quad (10-6)$$

where the a_i 's are coefficients of the characteristic polynomial

$$|s\mathbf{I} - \mathbf{A}| = s^n + a_1s^{n-1} + \cdots + a_{n-1}s + a_n$$

Define a new state vector $\hat{\mathbf{x}}$ by

$$\mathbf{x} = \mathbf{T}\hat{\mathbf{x}}$$

If the rank of the controllability matrix \mathbf{M} is n (meaning that the system is completely state controllable), then the inverse of matrix \mathbf{T} exists, and Equation (10-1) can be modified to

$$\dot{\hat{\mathbf{x}}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\hat{\mathbf{x}} + \mathbf{T}^{-1}\mathbf{B}u \quad (10-7)$$

where

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \quad (10-8)$$

$$\mathbf{T}^{-1}\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \quad (10-9)$$

[See Problems **A-10-2** and **A-10-3** for the derivation of Equations (10-8) and (10-9).] Equation (10-7) is in the controllable canonical form. Thus, given a state equation, Equation (10-1), it can be transformed into the controllable canonical form if the system is completely state controllable and if we transform the state vector \mathbf{x} into state vector $\hat{\mathbf{x}}$ by use of the transformation matrix \mathbf{T} given by Equation (10-4).

Let us choose a set of the desired eigenvalues as $\mu_1, \mu_2, \dots, \mu_n$. Then the desired characteristic equation becomes

$$(s - \mu_1)(s - \mu_2) \cdots (s - \mu_n) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1} s + \alpha_n = 0 \quad (10-10)$$

Let us write

$$\mathbf{K}\mathbf{T} = [\delta_n \quad \delta_{n-1} \quad \cdots \quad \delta_1] \quad (10-11)$$

When $u = -\mathbf{K}\mathbf{T}\hat{\mathbf{x}}$ is used to control the system given by Equation (10-7), the system equation becomes

$$\dot{\hat{\mathbf{x}}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\hat{\mathbf{x}} - \mathbf{T}^{-1}\mathbf{B}\mathbf{K}\mathbf{T}\hat{\mathbf{x}}$$

The characteristic equation is

$$|s\mathbf{I} - \mathbf{T}^{-1}\mathbf{A}\mathbf{T} + \mathbf{T}^{-1}\mathbf{B}\mathbf{K}\mathbf{T}| = 0$$

This characteristic equation is the same as the characteristic equation for the system, defined by Equation (10-1), when $u = -\mathbf{K}\mathbf{x}$ is used as the control signal. This can be seen as follows: Since

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}$$

the characteristic equation for this system is

$$|s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}| = |\mathbf{T}^{-1}(s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K})\mathbf{T}| = |s\mathbf{I} - \mathbf{T}^{-1}\mathbf{A}\mathbf{T} + \mathbf{T}^{-1}\mathbf{B}\mathbf{K}\mathbf{T}| = 0$$

Now let us simplify the characteristic equation of the system in the controllable canonical form. Referring to Equations (10–8), (10–9), and (10–11), we have

$$\begin{aligned}
 & |s\mathbf{I} - \mathbf{T}^{-1}\mathbf{A}\mathbf{T} + \mathbf{T}^{-1}\mathbf{B}\mathbf{K}\mathbf{T}| \\
 &= \left| s\mathbf{I} - \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & \cdots & -a_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} [\delta_n \ \delta_{n-1} \ \cdots \ \delta_1] \right| \\
 &= \begin{vmatrix} s & -1 & \cdots & 0 \\ 0 & s & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_n + \delta_n & a_{n-1} + \delta_{n-1} & \cdots & s + a_1 + \delta_1 \end{vmatrix} \\
 &= s^n + (a_1 + \delta_1)s^{n-1} + \cdots + (a_{n-1} + \delta_{n-1})s + (a_n + \delta_n) = 0 \quad (10-12)
 \end{aligned}$$

This is the characteristic equation for the system with state feedback. Therefore, it must be equal to Equation (10–10), the desired characteristic equation. By equating the coefficients of like powers of s , we get

$$\begin{aligned}
 a_1 + \delta_1 &= \alpha_1 \\
 a_2 + \delta_2 &= \alpha_2 \\
 &\cdot \\
 &\cdot \\
 &\cdot \\
 a_n + \delta_n &= \alpha_n
 \end{aligned}$$

Solving the preceding equations for the δ_i 's and substituting them into Equation (10–11), we obtain

$$\begin{aligned}
 \mathbf{K} &= [\delta_n \ \delta_{n-1} \ \cdots \ \delta_1]\mathbf{T}^{-1} \\
 &= [\alpha_n - a_n \mid \alpha_{n-1} - a_{n-1} \mid \cdots \mid \alpha_2 - a_2 \mid \alpha_1 - a_1]\mathbf{T}^{-1} \quad (10-13)
 \end{aligned}$$

Thus, if the system is completely state controllable, all eigenvalues can be arbitrarily placed by choosing matrix \mathbf{K} according to Equation (10–13) (sufficient condition).

We have thus proved that a necessary and sufficient condition for arbitrary pole placement is that the system be completely state controllable.

It is noted that if the system is not completely state controllable, but is stabilizable, then it is possible to make the entire system stable by placing the closed-loop poles at desired locations for q controllable modes. The remaining $n - q$ uncontrollable modes are stable. So the entire system can be made stable.

Determination of Matrix \mathbf{K} Using Transformation Matrix \mathbf{T} . Suppose that the system is defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

and the control signal is given by

$$u = -\mathbf{K}\mathbf{x}$$

The feedback gain matrix \mathbf{K} that forces the eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ to be $\mu_1, \mu_2, \dots, \mu_n$ (desired values) can be determined by the following steps (if μ_i is a complex eigenvalue, then its conjugate must also be an eigenvalue of $\mathbf{A} - \mathbf{B}\mathbf{K}$):

Step 1: Check the controllability condition for the system. If the system is completely state controllable, then use the following steps:

Step 2: From the characteristic polynomial for matrix \mathbf{A} , that is,

$$|s\mathbf{I} - \mathbf{A}| = s^n + a_1s^{n-1} + \dots + a_{n-1}s + a_n$$

determine the values of a_1, a_2, \dots, a_n .

Step 3: Determine the transformation matrix \mathbf{T} that transforms the system state equation into the controllable canonical form. (If the given system equation is already in the controllable canonical form, then $\mathbf{T} = \mathbf{I}$.) It is not necessary to write the state equation in the controllable canonical form. All we need here is to find the matrix \mathbf{T} . The transformation matrix \mathbf{T} is given by Equation (10-4), or

$$\mathbf{T} = \mathbf{M}\mathbf{W}$$

where \mathbf{M} is given by Equation (10-5) and \mathbf{W} is given by Equation (10-6).

Step 4: Using the desired eigenvalues (desired closed-loop poles), write the desired characteristic polynomial:

$$(s - \mu_1)(s - \mu_2) \cdots (s - \mu_n) = s^n + \alpha_1s^{n-1} + \dots + \alpha_{n-1}s + \alpha_n$$

and determine the values of $\alpha_1, \alpha_2, \dots, \alpha_n$.

Step 5: The required state feedback gain matrix \mathbf{K} can be determined from Equation (10-13), rewritten thus:

$$\mathbf{K} = [\alpha_n - a_n \quad \alpha_{n-1} - a_{n-1} \quad \cdots \quad \alpha_2 - a_2 \quad \alpha_1 - a_1] \mathbf{T}^{-1}$$

Determination of Matrix \mathbf{K} Using Direct Substitution Method. If the system is of low order ($n \leq 3$), direct substitution of matrix \mathbf{K} into the desired characteristic polynomial may be simpler. For example, if $n = 3$, then write the state feedback gain matrix \mathbf{K} as

$$\mathbf{K} = [k_1 \quad k_2 \quad k_3]$$

Substitute this \mathbf{K} matrix into the desired characteristic polynomial $|s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}|$ and equate it to $(s - \mu_1)(s - \mu_2)(s - \mu_3)$, or

$$|s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}| = (s - \mu_1)(s - \mu_2)(s - \mu_3)$$

Since both sides of this characteristic equation are polynomials in s , by equating the coefficients of the like powers of s on both sides, it is possible to determine the values of k_1 , k_2 , and k_3 . This approach is convenient if $n = 2$ or 3 . (For $n = 4, 5, 6, \dots$, this approach may become very tedious.)

Note that if the system is not completely controllable, matrix \mathbf{K} cannot be determined. (No solution exists.)

Determination of Matrix \mathbf{K} Using Ackermann's Formula. There is a well-known formula, known as Ackermann's formula, for the determination of the state feedback gain matrix \mathbf{K} . We shall present this formula in what follows.

Consider the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where we use the state feedback control $u = -\mathbf{K}\mathbf{x}$. We assume that the system is completely state controllable. We also assume that the desired closed-loop poles are at $s = \mu_1, s = \mu_2, \dots, s = \mu_n$.

Use of the state feedback control

$$u = -\mathbf{K}\mathbf{x}$$

modifies the system equation to

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} \quad (10-14)$$

Let us define

$$\tilde{\mathbf{A}} = \mathbf{A} - \mathbf{B}\mathbf{K}$$

The desired characteristic equation is

$$\begin{aligned} |s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}| &= |s\mathbf{I} - \tilde{\mathbf{A}}| = (s - \mu_1)(s - \mu_2) \cdots (s - \mu_n) \\ &= s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1} s + \alpha_n = 0 \end{aligned}$$

Since the Cayley–Hamilton theorem states that $\tilde{\mathbf{A}}$ satisfies its own characteristic equation, we have

$$\phi(\tilde{\mathbf{A}}) = \tilde{\mathbf{A}}^n + \alpha_1 \tilde{\mathbf{A}}^{n-1} + \cdots + \alpha_{n-1} \tilde{\mathbf{A}} + \alpha_n \mathbf{I} = \mathbf{0} \quad (10-15)$$

We shall utilize Equation (10–15) to derive Ackermann's formula. To simplify the derivation, we consider the case where $n = 3$. (For any other positive integer n , the following derivation can be easily extended.)

Consider the following identities:

$$\begin{aligned} \mathbf{I} &= \mathbf{I} \\ \tilde{\mathbf{A}} &= \mathbf{A} - \mathbf{B}\mathbf{K} \\ \tilde{\mathbf{A}}^2 &= (\mathbf{A} - \mathbf{B}\mathbf{K})^2 = \mathbf{A}^2 - \mathbf{A}\mathbf{B}\mathbf{K} - \mathbf{B}\mathbf{K}\tilde{\mathbf{A}} \\ \tilde{\mathbf{A}}^3 &= (\mathbf{A} - \mathbf{B}\mathbf{K})^3 = \mathbf{A}^3 - \mathbf{A}^2\mathbf{B}\mathbf{K} - \mathbf{A}\mathbf{B}\mathbf{K}\tilde{\mathbf{A}} - \mathbf{B}\mathbf{K}\tilde{\mathbf{A}}^2 \end{aligned}$$

Multiplying the preceding equations in order by α_3 , α_2 , α_1 , and α_0 (where $\alpha_0 = 1$), respectively, and adding the results, we obtain

$$\begin{aligned}
& \alpha_3 \mathbf{I} + \alpha_2 \tilde{\mathbf{A}} + \alpha_1 \tilde{\mathbf{A}}^2 + \tilde{\mathbf{A}}^3 \\
&= \alpha_3 \mathbf{I} + \alpha_2 (\mathbf{A} - \mathbf{BK}) + \alpha_1 (\mathbf{A}^2 - \mathbf{ABK} - \mathbf{BK}\tilde{\mathbf{A}}) + \mathbf{A}^3 - \mathbf{A}^2 \mathbf{BK} \\
&\quad - \mathbf{ABK}\tilde{\mathbf{A}} - \mathbf{BK}\tilde{\mathbf{A}}^2 \\
&= \alpha_3 \mathbf{I} + \alpha_2 \mathbf{A} + \alpha_1 \mathbf{A}^2 + \mathbf{A}^3 - \alpha_2 \mathbf{BK} - \alpha_1 \mathbf{ABK} - \alpha_1 \mathbf{BK}\tilde{\mathbf{A}} - \mathbf{A}^2 \mathbf{BK} \\
&\quad - \mathbf{ABK}\tilde{\mathbf{A}} - \mathbf{BK}\tilde{\mathbf{A}}^2
\end{aligned} \tag{10-16}$$

Referring to Equation (10-15), we have

$$\alpha_3 \mathbf{I} + \alpha_2 \tilde{\mathbf{A}} + \alpha_1 \tilde{\mathbf{A}}^2 + \tilde{\mathbf{A}}^3 = \phi(\tilde{\mathbf{A}}) = \mathbf{0}$$

Also, we have

$$\alpha_3 \mathbf{I} + \alpha_2 \mathbf{A} + \alpha_1 \mathbf{A}^2 + \mathbf{A}^3 = \phi(\mathbf{A}) \neq \mathbf{0}$$

Substituting the last two equations into Equation (10-16), we have

$$\phi(\tilde{\mathbf{A}}) = \phi(\mathbf{A}) - \alpha_2 \mathbf{BK} - \alpha_1 \mathbf{BK}\tilde{\mathbf{A}} - \mathbf{BK}\tilde{\mathbf{A}}^2 - \alpha_1 \mathbf{ABK} - \mathbf{ABK}\tilde{\mathbf{A}} - \mathbf{A}^2 \mathbf{BK}$$

Since $\phi(\tilde{\mathbf{A}}) = \mathbf{0}$, we obtain

$$\begin{aligned}
\phi(\mathbf{A}) &= \mathbf{B}(\alpha_2 \mathbf{K} + \alpha_1 \mathbf{K}\tilde{\mathbf{A}} + \mathbf{K}\tilde{\mathbf{A}}^2) + \mathbf{AB}(\alpha_1 \mathbf{K} + \mathbf{K}\tilde{\mathbf{A}}) + \mathbf{A}^2 \mathbf{BK} \\
&= [\mathbf{B} \mid \mathbf{AB} \mid \mathbf{A}^2 \mathbf{B}] \begin{bmatrix} \alpha_2 \mathbf{K} + \alpha_1 \mathbf{K}\tilde{\mathbf{A}} + \mathbf{K}\tilde{\mathbf{A}}^2 \\ \alpha_1 \mathbf{K} + \mathbf{K}\tilde{\mathbf{A}} \\ \mathbf{K} \end{bmatrix}
\end{aligned} \tag{10-17}$$

Since the system is completely state controllable, the inverse of the controllability matrix

$$[\mathbf{B} \mid \mathbf{AB} \mid \mathbf{A}^2 \mathbf{B}]$$

exists. Premultiplying both sides of Equation (10-17) by the inverse of the controllability matrix, we obtain

$$[\mathbf{B} \mid \mathbf{AB} \mid \mathbf{A}^2 \mathbf{B}]^{-1} \phi(\mathbf{A}) = \begin{bmatrix} \alpha_2 \mathbf{K} + \alpha_1 \mathbf{K}\tilde{\mathbf{A}} + \mathbf{K}\tilde{\mathbf{A}}^2 \\ \alpha_1 \mathbf{K} + \mathbf{K}\tilde{\mathbf{A}} \\ \mathbf{K} \end{bmatrix}$$

Premultiplying both sides of this last equation by $[0 \ 0 \ 1]$, we obtain

$$[0 \ 0 \ 1][\mathbf{B} \mid \mathbf{AB} \mid \mathbf{A}^2 \mathbf{B}]^{-1} \phi(\mathbf{A}) = [0 \ 0 \ 1] \begin{bmatrix} \alpha_2 \mathbf{K} + \alpha_1 \mathbf{K}\tilde{\mathbf{A}} + \mathbf{K}\tilde{\mathbf{A}}^2 \\ \alpha_1 \mathbf{K} + \mathbf{K}\tilde{\mathbf{A}} \\ \mathbf{K} \end{bmatrix} = \mathbf{K}$$

which can be rewritten as

$$\mathbf{K} = [0 \ 0 \ 1][\mathbf{B} \mid \mathbf{AB} \mid \mathbf{A}^2 \mathbf{B}]^{-1} \phi(\mathbf{A})$$

This last equation gives the required state feedback gain matrix \mathbf{K} .

For an arbitrary positive integer n , we have

$$\mathbf{K} = [0 \ 0 \ \cdots \ 0 \ 1][\mathbf{B} \mid \mathbf{AB} \mid \cdots \mid \mathbf{A}^{n-1} \mathbf{B}]^{-1} \phi(\mathbf{A}) \tag{10-18}$$

Equation (10–18) is known as Ackermann’s formula for the determination of the state feedback gain matrix \mathbf{K} .

Regulator Systems and Control Systems. Systems that include controllers can be divided into two categories: regulator systems (where the reference input is constant, including zero) and control systems (where the reference input is time varying). In what follows we shall consider regulator systems. Control systems will be treated in Section 10–7.

Choosing the Locations of Desired Closed-Loop Poles. The first step in the pole-placement design approach is to choose the locations of the desired closed-loop poles. The most frequently used approach is to choose such poles based on experience in the root-locus design, placing a dominant pair of closed-loop poles and choosing other poles so that they are far to the left of the dominant closed-loop poles.

Note that if we place the dominant closed-loop poles far from the $j\omega$ axis, so that the system response becomes very fast, the signals in the system become very large, with the result that the system may become nonlinear. This should be avoided.

Another approach is based on the quadratic optimal control approach. This approach will determine the desired closed-loop poles such that it balances between the acceptable response and the amount of control energy required. (See Section 10–8.) Note that requiring a high-speed response implies requiring large amounts of control energy. Also, in general, increasing the speed of response requires a larger, heavier actuator, which will cost more.

EXAMPLE 10–1 Consider the regulator system shown in Figure 10–2. The plant is given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The system uses the state feedback control $\mathbf{u} = -\mathbf{K}\mathbf{x}$. Let us choose the desired closed-loop poles at

$$s = -2 + j4, \quad s = -2 - j4, \quad s = -10$$

(We make such a choice because we know from experience that such a set of closed-loop poles will result in a reasonable or acceptable transient response.) Determine the state feedback gain matrix \mathbf{K} .

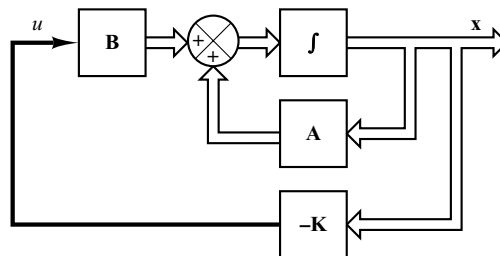


Figure 10–2
Regulator system.

First, we need to check the controllability matrix of the system. Since the controllability matrix \mathbf{M} is given by

$$\mathbf{M} = [\mathbf{B} \mid \mathbf{AB} \mid \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$$

we find that $|\mathbf{M}| = -1$, and therefore, $\text{rank } \mathbf{M} = 3$. Thus, the system is completely state controllable and arbitrary pole placement is possible.

Next, we shall solve this problem. We shall demonstrate each of the three methods presented in this chapter.

Method 1: The first method is to use Equation (10–13). The characteristic equation for the system is

$$\begin{aligned} |s\mathbf{I} - \mathbf{A}| &= \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 5 & s + 6 \end{vmatrix} \\ &= s^3 + 6s^2 + 5s + 1 \\ &= s^3 + a_1s^2 + a_2s + a_3 = 0 \end{aligned}$$

Hence,

$$a_1 = 6, \quad a_2 = 5, \quad a_3 = 1$$

The desired characteristic equation is

$$\begin{aligned} (s + 2 - j4)(s + 2 + j4)(s + 10) &= s^3 + 14s^2 + 60s + 200 \\ &= s^3 + \alpha_1s^2 + \alpha_2s + \alpha_3 = 0 \end{aligned}$$

Hence,

$$\alpha_1 = 14, \quad \alpha_2 = 60, \quad \alpha_3 = 200$$

Referring to Equation (10–13), we have

$$\mathbf{K} = [\alpha_3 - a_3 \mid \alpha_2 - a_2 \mid \alpha_1 - a_1] \mathbf{T}^{-1}$$

where $\mathbf{T} = \mathbf{I}$ for this problem because the given state equation is in the controllable canonical form. Then we have

$$\begin{aligned} \mathbf{K} &= [200 - 1 \mid 60 - 5 \mid 14 - 6] \\ &= [199 \quad 55 \quad 8] \end{aligned}$$

Method 2: By defining the desired state feedback gain matrix \mathbf{K} as

$$\mathbf{K} = [k_1 \quad k_2 \quad k_3]$$

and equating $|s\mathbf{I} - \mathbf{A} + \mathbf{BK}|$ with the desired characteristic equation, we obtain

$$\begin{aligned} |s\mathbf{I} - \mathbf{A} + \mathbf{BK}| &= \begin{vmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{vmatrix} - \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{vmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [k_1 \quad k_2 \quad k_3] \\ &= \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 + k_1 & 5 + k_2 & s + 6 + k_3 \end{vmatrix} \\ &= s^3 + (6 + k_3)s^2 + (5 + k_2)s + 1 + k_1 \\ &= s^3 + 14s^2 + 60s + 200 \end{aligned}$$

Thus,

$$6 + k_3 = 14, \quad 5 + k_2 = 60, \quad 1 + k_1 = 200$$

from which we obtain

$$k_1 = 199, \quad k_2 = 55, \quad k_3 = 8$$

or

$$\mathbf{K} = [199 \quad 55 \quad 8]$$

Method 3: The third method is to use Ackermann's formula. Referring to Equation (10–18), we have

$$\mathbf{K} = [0 \quad 0 \quad 1][\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}]^{-1}\phi(\mathbf{A})$$

Since

$$\begin{aligned} \phi(\mathbf{A}) &= \mathbf{A}^3 + 14\mathbf{A}^2 + 60\mathbf{A} + 200\mathbf{I} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}^3 + 14 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}^2 \\ &\quad + 60 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + 200 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix} \end{aligned}$$

and

$$[\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$$

we obtain

$$\begin{aligned} \mathbf{K} &= [0 \quad 0 \quad 1] \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}^{-1} \begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix} \\ &= [0 \quad 0 \quad 1] \begin{bmatrix} 5 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix} \\ &= [199 \quad 55 \quad 8] \end{aligned}$$

As a matter of course, the feedback gain matrix \mathbf{K} obtained by the three methods are the same. With this state feedback, the closed-loop poles are placed at $s = -2 \pm j4$ and $s = -10$, as desired.

It is noted that if the order n of the system were 4 or higher, methods 1 and 3 are recommended, since all matrix computations can be carried out by a computer. If method 2 is used, hand computations become necessary because a computer may not handle the characteristic equation with unknown parameters k_1, k_2, \dots, k_n .

Comments. It is important to note that matrix \mathbf{K} is not unique for a given system, but depends on the desired closed-loop pole locations (which determine the speed and damping of the response) selected. Note that the selection of the desired closed-loop poles or the desired characteristic equation is a compromise between the rapidity of the response of the error vector and the sensitivity to disturbances and measurement noises. That is, if we increase the speed of error response, then the adverse effects of disturbances and measurement noises generally increase. If the system is of second order, then the system dynamics (response characteristics) can be precisely correlated to the location of the desired closed-loop poles and the zero(s) of the plant. For higher-order systems, the location of the closed-loop poles and the system dynamics (response characteristics) are not easily correlated. Hence, in determining the state feedback gain matrix \mathbf{K} for a given system, it is desirable to examine by computer simulations the response characteristics of the system for several different matrices \mathbf{K} (based on several different desired characteristic equations) and to choose the one that gives the best overall system performance.

10-3 SOLVING POLE-PLACEMENT PROBLEMS WITH MATLAB

Pole-placement problems can be solved easily with MATLAB. MATLAB has two commands—`acker` and `place`—for the computation of feedback-gain matrix \mathbf{K} . The command `acker` is based on Ackermann's formula. This command applies to single-input systems only. The desired closed-loop poles can include multiple poles (poles located at the same place).

If the system involves multiple inputs, for a specified set of closed-loop poles the state-feedback gain matrix \mathbf{K} is not unique and we have an additional freedom (or freedoms) to choose \mathbf{K} . There are many approaches to constructively utilize this additional freedom (or freedoms) to determine \mathbf{K} . One common use is to maximize the stability margin. The pole placement based on this approach is called the robust pole placement. The MATLAB command for the robust pole placement is `place`.

Although the command `place` can be used for both single-input and multiple-input systems, this command requires that the multiplicity of poles in the desired closed-loop poles be no greater than the rank of \mathbf{B} . That is, if matrix \mathbf{B} is an $n \times 1$ matrix, the command `place` requires that there be no multiple poles in the set of desired closed-loop poles.

For single-input systems, the commands `acker` and `place` yield the same \mathbf{K} . (But for multiple-input systems, one must use the command `place` instead of `acker`.)

It is noted that when the single-input system is barely controllable, some computational problem may occur if the command `acker` is used. In such a case the use of the `place` command is preferred, provided that no multiple poles are involved in the desired set of closed-loop poles.

To use the command `acker` or `place`, we first enter the following matrices in the program:

\mathbf{A} matrix, \mathbf{B} matrix, \mathbf{J} matrix

where \mathbf{J} matrix is the matrix consisting of the desired closed-loop poles such that

$$\mathbf{J} = [\mu_1 \quad \mu_2 \quad \dots \quad \mu_n]$$

Then we enter

$$K = \text{acker}(A,B,J)$$

or

$$K = \text{place}(A,B,J)$$

It is noted that the command $\text{eig}(A-B*K)$ may be used to verify that K thus obtained gives the desired eigenvalues.

EXAMPLE 10–2 Consider the same system as treated in Example 10–1. The system equation is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

By using state feedback control $u = -\mathbf{K}\mathbf{x}$, it is desired to have the closed-loop poles at $s = \mu_i$ ($i = 1, 2, 3$), where

$$\mu_1 = -2 + j4, \quad \mu_2 = -2 - j4, \quad \mu_3 = -10$$

Determine the state feedback-gain matrix \mathbf{K} with MATLAB.

MATLAB programs that generate matrix \mathbf{K} are shown in MATLAB Programs 10–1 and 10–2. MATLAB Program 10–1 uses command `acker` and MATLAB Program 10–2 uses command `place`.

MATLAB Program 10–1

```
A = [0 1 0;0 0 1;-1 -5 -6];
B = [0;0;1];
J = [-2+j*4 -2-j*4 -10];
K = acker(A,B,J)

K =

    199    55     8
```

MATLAB Program 10–2

```
A = [0 1 0;0 0 1;-1 -5 -6];
B = [0;0;1];
J = [-2+j*4 -2-j*4 -10];
K = place(A,B,J)
place: ndigits = 15

K =

    199.0000    55.0000     8.0000
```

EXAMPLE 10–3 Consider the same system as discussed in Example 10–1. It is desired that this regulator system have closed-loop poles at

$$s = -2 + j4, \quad s = -2 - j4, \quad s = -10$$

The necessary state feedback gain matrix \mathbf{K} was obtained in Example 10–1 as follows:

$$\mathbf{K} = [199 \quad 55 \quad 8]$$

Using MATLAB, obtain the response of the system to the following initial condition:

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Response to Initial Condition: To obtain the response to the given initial condition $\mathbf{x}(0)$, we substitute $u = -\mathbf{K}\mathbf{x}$ into the plant equation to get

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

To plot the response curves (x_1 versus t , x_2 versus t , and x_3 versus t), we may use the command `initial`. We first define the state-space equations for the system as follows:

$$\begin{aligned} \dot{\mathbf{x}} &= (\mathbf{A} - \mathbf{BK})\mathbf{x} + \mathbf{I}\mathbf{u} \\ \mathbf{y} &= \mathbf{I}\mathbf{x} + \mathbf{I}\mathbf{u} \end{aligned}$$

where we included \mathbf{u} (a three-dimensional input vector). This \mathbf{u} vector is considered $\mathbf{0}$ in the computation of the response to the initial condition. Then we define

$$\text{sys} = \text{ss}(\mathbf{A} - \mathbf{BK}, \text{eye}(3), \text{eye}(3), \text{eye}(3))$$

and use the `initial` command as follows:

$$\mathbf{x} = \text{initial}(\text{sys}, [1;0;0], t)$$

where t is the time duration we want to use, such as

$$t = 0:0.01:4;$$

Then obtain x_1 , x_2 , and x_3 as follows:

$$\begin{aligned} x_1 &= [1 \quad 0 \quad 0] * \mathbf{x}'; \\ x_2 &= [0 \quad 1 \quad 0] * \mathbf{x}'; \\ x_3 &= [0 \quad 0 \quad 1] * \mathbf{x}'; \end{aligned}$$

and use the `plot` command. This program is shown in MATLAB Program 10–3. The resulting response curves are shown in Figure 10–3.

MATLAB Program 10–3

```
% Response to initial condition:  
A = [0 1 0;0 0 1;-1 -5 -6];  
B = [0;0;1];  
K = [199 55 8];  
sys = ss(A-B*K, eye(3), eye(3), eye(3));  
t = 0:0.01:4;  
x = initial(sys,[1;0;0],t);  
x1 = [1 0 0]*x';  
x2 = [0 1 0]*x';  
x3 = [0 0 1]*x';  
  
subplot(3,1,1); plot(t,x1), grid  
title('Response to Initial Condition')  
ylabel('state variable x1')  
  
subplot(3,1,2); plot(t,x2),grid  
ylabel('state variable x2')  
  
subplot(3,1,3); plot(t,x3),grid  
xlabel('t (sec)')  
ylabel('state variable x3')
```

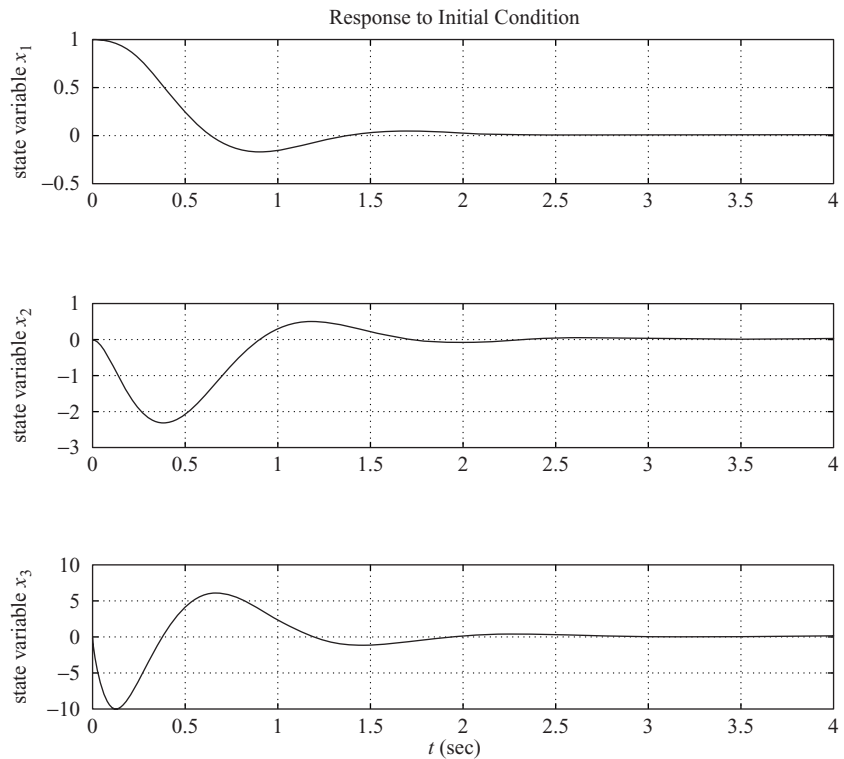


Figure 10–3
Response to initial condition.

10-4 DESIGN OF SERVO SYSTEMS

In this section we shall discuss the pole-placement approach to the design of type 1 servo systems. Here we shall limit our systems each to have a scalar control signal u and a scalar output y .

In what follows we shall first discuss a problem of designing a type 1 servo system when the plant involves an integrator. Then we shall discuss the design of a type 1 servo system when the plant has no integrator.

Design of Type 1 Servo System when the Plant Has An Integrator. Assume that the plant is defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \quad (10-19)$$

$$y = \mathbf{C}\mathbf{x} \quad (10-20)$$

where \mathbf{x} = state vector for the plant (n -vector)

u = control signal (scalar)

y = output signal (scalar)

\mathbf{A} = $n \times n$ constant matrix

\mathbf{B} = $n \times 1$ constant matrix

\mathbf{C} = $1 \times n$ constant matrix

As stated earlier, we assume that both the control signal u and the output signal y are scalars. By a proper choice of a set of state variables, it is possible to choose the output to be equal to one of the state variables. (See the method presented in Chapter 2 for obtaining a state-space representation of the transfer function system in which the output y becomes equal to x_1 .)

Figure 10-4 shows a general configuration of the type 1 servo system when the plant has an integrator. Here we assumed that $y = x_1$. In the present analysis we assume that

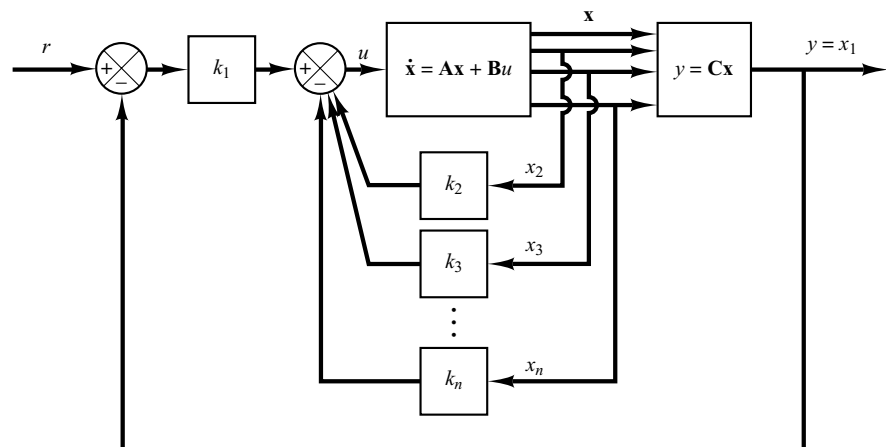


Figure 10-4
Type 1 servo system
when the plant has
an integrator.

the reference input r is a step function. In this system we use the following state-feedback control scheme:

$$u = -\begin{bmatrix} 0 & k_2 & k_3 & \cdots & k_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + k_1(r - x_1)$$

$$= -\mathbf{K}\mathbf{x} + k_1r \quad (10-21)$$

where

$$\mathbf{K} = [k_1 \quad k_2 \quad \cdots \quad k_n]$$

Assume that the reference input (step function) is applied at $t = 0$. Then, for $t > 0$, the system dynamics can be described by Equations (10-19) and (10-21), or

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}k_1r \quad (10-22)$$

We shall design the type 1 servo system such that the closed-loop poles are located at desired positions. The designed system will be an asymptotically stable system, $y(\infty)$ will approach the constant value r , and $u(\infty)$ will approach zero. (r is a step input.)

Notice that at steady state we have

$$\dot{\mathbf{x}}(\infty) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(\infty) + \mathbf{B}k_1r(\infty) \quad (10-23)$$

Noting that $r(t)$ is a step input, we have $r(\infty) = r(t) = r(\text{constant})$ for $t > 0$. By subtracting Equation (10-23) from Equation (10-22), we obtain

$$\dot{\mathbf{x}}(t) - \dot{\mathbf{x}}(\infty) = (\mathbf{A} - \mathbf{B}\mathbf{K})[\mathbf{x}(t) - \mathbf{x}(\infty)] \quad (10-24)$$

Define

$$\mathbf{x}(t) - \mathbf{x}(\infty) = \mathbf{e}(t)$$

Then Equation (10-24) becomes

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{e} \quad (10-25)$$

Equation (10-25) describes the error dynamics.

The design of the type 1 servo system here is converted to the design of an asymptotically stable regulator system such that $\mathbf{e}(t)$ approaches zero, given any initial condition $\mathbf{e}(0)$. If the system defined by Equation (10-19) is completely state controllable, then, by specifying the desired eigenvalues $\mu_1, \mu_2, \dots, \mu_n$ for the matrix $\mathbf{A} - \mathbf{B}\mathbf{K}$, matrix \mathbf{K} can be determined by the pole-placement technique presented in Section 10-2.

The steady-state values of $\mathbf{x}(t)$ and $u(t)$ can be found as follows: At steady state ($t = \infty$), we have, from Equation (10-22),

$$\dot{\mathbf{x}}(\infty) = \mathbf{0} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(\infty) + \mathbf{B}k_1r$$

Since the desired eigenvalues of $\mathbf{A} - \mathbf{BK}$ are all in the left-half s plane, the inverse of matrix $\mathbf{A} - \mathbf{BK}$ exists. Consequently, $\mathbf{x}(\infty)$ can be determined as

$$\mathbf{x}(\infty) = -(\mathbf{A} - \mathbf{BK})^{-1} \mathbf{B}k_1 r$$

Also, $u(\infty)$ can be obtained as

$$u(\infty) = -\mathbf{K}\mathbf{x}(\infty) + k_1 r = 0$$

(See Example 10–4 to verify this last equation.)

EXAMPLE 10–4 Design a type 1 servo system when the plant transfer function has an integrator. Assume that the plant transfer function is given by

$$\frac{Y(s)}{U(s)} = \frac{1}{s(s+1)(s+2)}$$

The desired closed-loop poles are $s = -2 \pm j2\sqrt{3}$ and $s = -10$. Assume that the system configuration is the same as that shown in Figure 10–4 and the reference input r is a step function. Obtain the unit-step response of the designed system.

Define state variables x_1 , x_2 , and x_3 as follows:

$$x_1 = y$$

$$x_2 = \dot{x}_1$$

$$x_3 = \dot{x}_2$$

Then the state-space representation of the system becomes

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \quad (10-26)$$

$$y = \mathbf{C}\mathbf{x} \quad (10-27)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0 \quad 0]$$

Referring to Figure 10–4 and noting that $n = 3$, the control signal u is given by

$$u = -(k_2 x_2 + k_3 x_3) + k_1(r - x_1) = -\mathbf{K}\mathbf{x} + k_1 r \quad (10-28)$$

where

$$\mathbf{K} = [k_1 \quad k_2 \quad k_3]$$

The state-feedback gain matrix \mathbf{K} can be obtained easily with MATLAB. See MATLAB Program 10–4.

MATLAB Program 10–4

```
A = [0 1 0;0 0 1;0 -2 -3];
B = [0;0;1];
J = [-2+j*2*sqrt(3) -2-j*2*sqrt(3) -10];
K = acker(A,B,J)

K =

160.0000 54.0000 11.0000
```

The state feedback gain matrix \mathbf{K} is thus

$$\mathbf{K} = [160 \ 54 \ 11]$$

Unit-Step Response of the Designed System: The unit-step response of the designed system can be obtained as follows:

Since

$$\mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [160 \ 54 \ 11] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -160 & -56 & -14 \end{bmatrix}$$

from Equation (10-22) the state equation for the designed system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -160 & -56 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 160 \end{bmatrix} r \quad (10-29)$$

and the output equation is

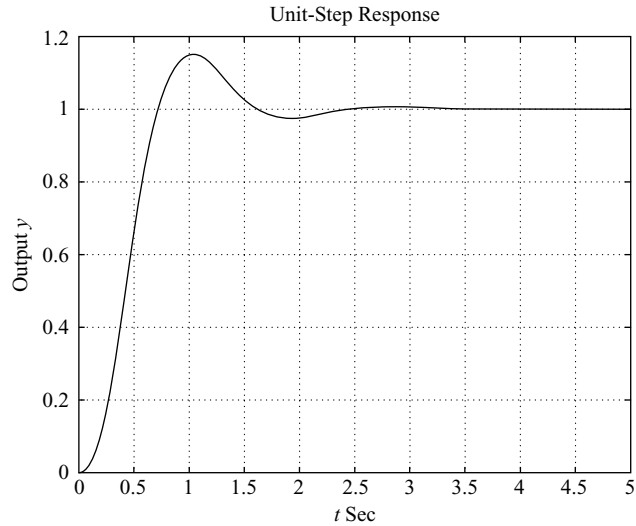
$$y = [1 \ 0 \ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (10-30)$$

Solving Equations (10-29) and (10-30) for $y(t)$ when r is a unit-step function gives the unit-step response curve $y(t)$ versus t . MATLAB Program 10-5 yields the unit-step response. The resulting unit-step response curve is shown in Figure 10-5.

MATLAB Program 10-5

```
% ----- Unit-step response -----
% ***** Enter the state matrix, control matrix, output matrix,
% and direct transmission matrix of the designed system *****
AA = [0 1 0;0 0 1;-160 -56 -14];
BB = [0;0;160];
CC = [1 0 0];
DD = [0];
% ***** Enter step command and plot command *****
t = 0:0.01:5;
y = step(AA,BB,CC,DD,1,t);
plot(t,y)
grid
title('Unit-Step Response')
xlabel('t Sec')
ylabel('Output y')
```

Figure 10-5
Unit-step response curve $y(t)$ versus t for the system designed in Example 10-4.



Note that since

$$u(\infty) = -\mathbf{K}\mathbf{x}(\infty) + k_1 r(\infty) = -\mathbf{K}\mathbf{x}(\infty) + k_1 r$$

we have

$$\begin{aligned} u(\infty) &= -\begin{bmatrix} 160 & 54 & 11 \end{bmatrix} \begin{bmatrix} x_1(\infty) \\ x_2(\infty) \\ x_3(\infty) \end{bmatrix} + 160r \\ &= -\begin{bmatrix} 160 & 54 & 11 \end{bmatrix} \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} + 160r = 0 \end{aligned}$$

At steady state the control signal u becomes zero.

Design of Type 1 Servo System when the Plant Has No Integrator. If the plant has no integrator (type 0 plant), the basic principle of the design of a type 1 servo system is to insert an integrator in the feedforward path between the error comparator and the plant, as shown in Figure 10-6. (The block diagram of Figure 10-6 is a basic form of the type 1 servo system where the plant has no integrator.) From the diagram, we obtain

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \quad (10-31)$$

$$y = \mathbf{C}\mathbf{x} \quad (10-32)$$

$$u = -\mathbf{K}\mathbf{x} + k_1 \xi \quad (10-33)$$

$$\dot{\xi} = r - y = r - \mathbf{C}\mathbf{x} \quad (10-34)$$

where \mathbf{x} = state vector of the plant (n -vector)

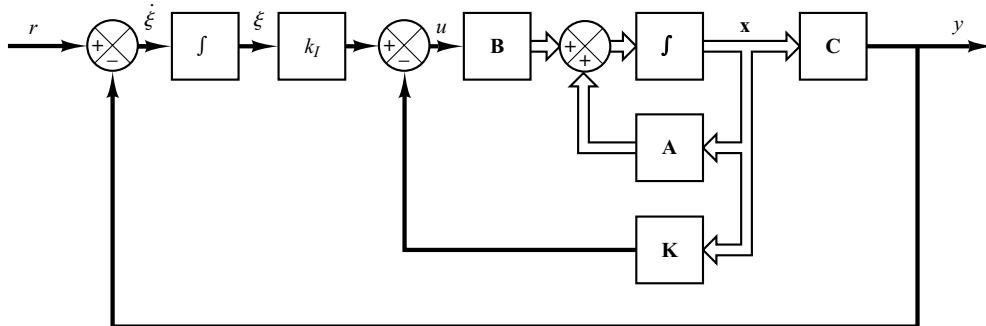


Figure 10–6
Type 1 servo system.

- u = control signal (scalar)
- y = output signal (scalar)
- ξ = output of the integrator (state variable of the system, scalar)
- r = reference input signal (step function, scalar)
- $\mathbf{A} = n \times n$ constant matrix
- $\mathbf{B} = n \times 1$ constant matrix
- $\mathbf{C} = 1 \times n$ constant matrix

We assume that the plant given by Equation (10–31) is completely state controllable. The transfer function of the plant can be given by

$$G_p(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

To avoid the possibility of the inserted integrator being canceled by the zero at the origin of the plant, we assume that $G_p(s)$ has no zero at the origin.

Assume that the reference input (step function) is applied at $t = 0$. Then, for $t > 0$, the system dynamics can be described by an equation that is a combination of Equations (10–31) and (10–34):

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\xi}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \xi(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r(t) \quad (10-35)$$

We shall design an asymptotically stable system such that $\mathbf{x}(\infty)$, $\xi(\infty)$, and $u(\infty)$ approach constant values, respectively. Then, at steady state, $\dot{\xi}(t) = 0$, and we get $y(\infty) = r$.

Notice that at steady state we have

$$\begin{bmatrix} \dot{\mathbf{x}}(\infty) \\ \dot{\xi}(\infty) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(\infty) \\ \xi(\infty) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u(\infty) + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r(\infty) \quad (10-36)$$

Noting that $r(t)$ is a step input, we have $r(\infty) = r(t) = r$ (constant) for $t > 0$. By subtracting Equation (10–36) from Equation (10–35), we obtain

$$\begin{bmatrix} \dot{\mathbf{x}}(t) - \dot{\mathbf{x}}(\infty) \\ \dot{\xi}(t) - \dot{\xi}(\infty) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) - \mathbf{x}(\infty) \\ \xi(t) - \xi(\infty) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} [u(t) - u(\infty)] \quad (10-37)$$

Define

$$\mathbf{x}(t) - \mathbf{x}(\infty) = \mathbf{x}_e(t)$$

$$\xi(t) - \xi(\infty) = \xi_e(t)$$

$$u(t) - u(\infty) = u_e(t)$$

Then Equation (10–37) can be written as

$$\begin{bmatrix} \dot{\mathbf{x}}_e(t) \\ \dot{\xi}_e(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_e(t) \\ \xi_e(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u_e(t) \quad (10-38)$$

where

$$u_e(t) = -\mathbf{K}\mathbf{x}_e(t) + k_I \xi_e(t) \quad (10-39)$$

Define a new $(n + 1)$ th-order error vector $\mathbf{e}(t)$ by

$$\mathbf{e}(t) = \begin{bmatrix} \mathbf{x}_e(t) \\ \xi_e(t) \end{bmatrix} = (n + 1)\text{-vector}$$

Then Equation (10–38) becomes

$$\dot{\mathbf{e}} = \hat{\mathbf{A}}\mathbf{e} + \hat{\mathbf{B}}u_e \quad (10-40)$$

where

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix}, \quad \hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix}$$

and Equation (10–39) becomes

$$u_e = -\hat{\mathbf{K}}\mathbf{e} \quad (10-41)$$

where

$$\hat{\mathbf{K}} = [\mathbf{K} \quad -k_I]$$

The state error equation can be obtained by substituting Equation (10–41) into Equation (10–40):

$$\dot{\mathbf{e}} = (\hat{\mathbf{A}} - \hat{\mathbf{B}}\hat{\mathbf{K}})\mathbf{e} \quad (10-42)$$

If the desired eigenvalues of matrix $\hat{\mathbf{A}} - \hat{\mathbf{B}}\hat{\mathbf{K}}$ (that is, the desired closed-loop poles) are specified as $\mu_1, \mu_2, \dots, \mu_{n+1}$, then the state-feedback gain matrix \mathbf{K} and the integral gain constant k_I can be determined by the pole-placement technique presented in Section 10–2, provided that the system defined by Equation (10–40) is completely state controllable. Note that if the matrix

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{C} & 0 \end{bmatrix}$$

has rank $n + 1$, then the system defined by Equation (10–40) is completely state controllable. (See Problem **A–10–12**.)

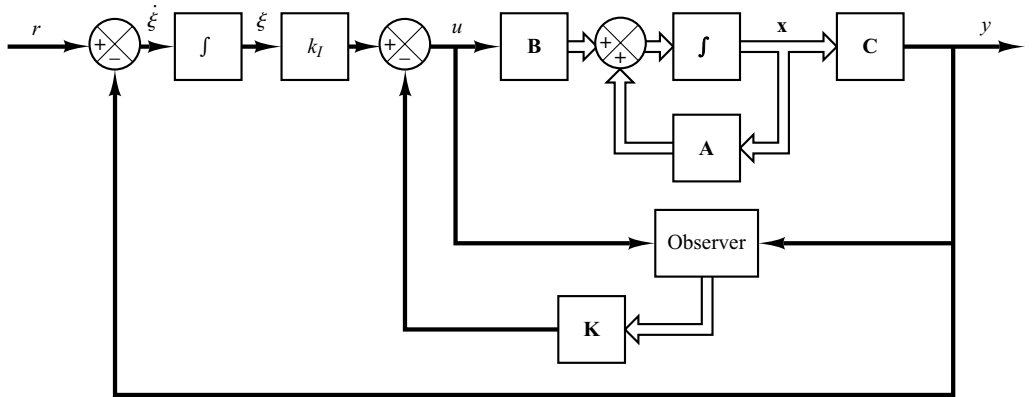


Figure 10-7
Type 1 servo system with state observer.

As is usually the case, not all state variables can be directly measurable. If this is the case, we need to use a state observer. Figure 10-7 shows a block diagram of a type 1 servo system with a state observer. [In the figure, each block with an integral symbol represents an integrator ($1/s$).] Detailed discussions of state observers are given in Section 10-5.

EXAMPLE 10-5

Consider the inverted-pendulum control system shown in Figure 10-8. In this example, we are concerned only with the motion of the pendulum and motion of the cart in the plane of the page. It is desired to keep the inverted pendulum upright as much as possible and yet control the position of the cart—for instance, move the cart in a step fashion. To control the position of the cart, we need to build a type 1 servo system. The inverted-pendulum system mounted on a cart does not have an integrator. Therefore, we feed the position signal y (which indicates the position of the cart) back to the input and insert an integrator in the feedforward path, as shown

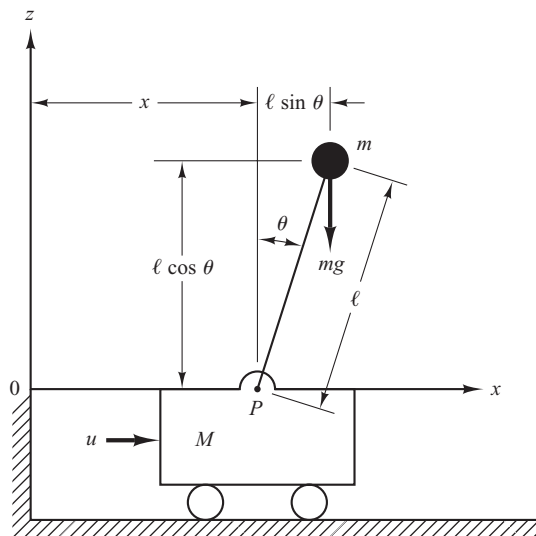


Figure 10-8
Inverted-pendulum control system.

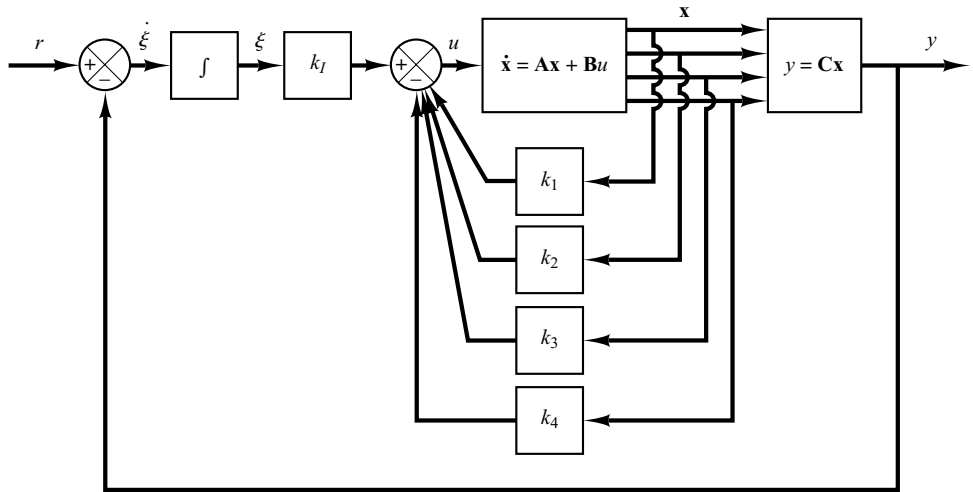


Figure 10–9
Inverted-pendulum control system. (Type 1 servo system when the plant has no integrator.)

in Figure 10–9. We assume that the pendulum angle θ and the angular velocity $\dot{\theta}$ are small, so that $\sin \theta \doteq \theta$, $\cos \theta \doteq 1$, and $\theta\dot{\theta}^2 \doteq 0$. We also assume that the numerical values for M , m , and l are given as

$$M = 2 \text{ kg}, \quad m = 0.1 \text{ kg}, \quad l = 0.5 \text{ m}$$

Earlier in Example 3–6 we derived the equations for the inverted-pendulum system shown in Figure 3–6, which is the same as that in Figure 10–8. Referring to Figure 3–6, we started with the force-balance and torque-balance equations and ended up with Equations (3–20) and (3–21) to model the inverted-pendulum system. Referring to Equations (3–20) and (3–21), the equations for the inverted-pendulum control system shown in Figure 10–8 are

$$Ml\ddot{\theta} = (M + m)g\theta - u \quad (10-43)$$

$$M\dot{x} = u - mg\theta \quad (10-44)$$

When the given numerical values are substituted, Equations (10–43) and (10–44) become

$$\ddot{\theta} = 20.601\theta - u \quad (10-45)$$

$$\dot{x} = 0.5u - 0.4905\theta \quad (10-46)$$

Let us define the state variables x_1 , x_2 , x_3 , and x_4 as

$$x_1 = \theta$$

$$x_2 = \dot{\theta}$$

$$x_3 = x$$

$$x_4 = \dot{x}$$

Then, referring to Equations (10–45) and (10–46) and Figure 10–9 and considering the cart position x as the output of the system, we obtain the equations for the system as follows:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \quad (10-47)$$

$$y = \mathbf{C}\mathbf{x} \quad (10-48)$$

$$u = -\mathbf{K}\mathbf{x} + k_I\xi \quad (10-49)$$

$$\dot{\xi} = r - y = r - \mathbf{C}\mathbf{x} \quad (10-50)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 20.601 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.4905 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0.5 \end{bmatrix}, \quad \mathbf{C} = [0 \ 0 \ 1 \ 0]$$

For the type 1 servo system, we have the state error equation as given by Equation (10-40):

$$\dot{\mathbf{e}} = \hat{\mathbf{A}}\mathbf{e} + \hat{\mathbf{B}}u_e \quad (10-51)$$

where

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 20.601 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -0.4905 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}, \quad \hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0.5 \\ 0 \end{bmatrix}$$

and the control signal is given by Equation (10-41):

$$u_e = -\hat{\mathbf{K}}\mathbf{e}$$

where

$$\hat{\mathbf{K}} = [\mathbf{K} \ \vdots \ -k_I] = [k_1 \ k_2 \ k_3 \ k_4 \ \vdots \ -k_I]$$

To obtain a reasonable speed and damping in the response of the designed system (for example, the settling time of approximately 4 ~ 5 sec and the maximum overshoot of 15% ~ 16% in the step response of the cart), let us choose the desired closed-loop poles at $s = \mu_i$ ($i = 1, 2, 3, 4, 5$), where

$$\mu_1 = -1 + j\sqrt{3}, \quad \mu_2 = -1 - j\sqrt{3}, \quad \mu_3 = -5, \quad \mu_4 = -5, \quad \mu_5 = -5$$

We shall determine the necessary state-feedback gain matrix by the use of MATLAB.

Before we proceed further, we must examine the rank of matrix \mathbf{P} , where

$$\mathbf{P} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{C} & 0 \end{bmatrix}$$

Matrix \mathbf{P} is given by

$$\mathbf{P} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{C} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 20.601 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ -0.4905 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix} \quad (10-52)$$

The rank of this matrix can be found to be 5. Therefore, the system defined by Equation (10-51) is completely state controllable, and arbitrary pole placement is possible. MATLAB Program 10-6 produces the state feedback gain matrix $\hat{\mathbf{K}}$.

MATLAB Program 10–6

```

A = [0 1 0 0; 20.601 0 0 0; 0 0 0 1; -0.4905 0 0 0];
B = [0;-1;0;0.5];
C = [0 0 1 0];
Ahat = [A zeros(4,1); -C 0];
Bhat = [B;0];
J = [-1+j*sqrt(3) -1-j*sqrt(3) -5 -5 -5];
Khat = acker(Ahat,Bhat,J)

Khat =

-157.6336 -35.3733 -56.0652 -36.7466 50.9684

```

Thus, we get

$$\mathbf{K} = [k_1 \quad k_2 \quad k_3 \quad k_4] = [-157.6336 \quad -35.3733 \quad -56.0652 \quad -36.7466]$$

and

$$k_I = -50.9684$$

Unit Step-Response Characteristics of the Designed System. Once we determine the feedback gain matrix \mathbf{K} and the integral gain constant k_I , the step response in the cart position can be obtained by solving the following equation, which is obtained by substituting Equation (10–49) into Equation (10–35):

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{B}k_I \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \xi \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r \quad (10-53)$$

The output $y(t)$ of the system is $x_3(t)$, or

$$y = [0 \quad 0 \quad 1 \quad 0 \quad 0] \begin{bmatrix} \mathbf{x} \\ \xi \end{bmatrix} + [0]r \quad (10-54)$$

Define the state matrix, control matrix, output matrix, and direct transmission matrix of the system given by Equations (10–53) and (10–54) as AA, BB, CC, and DD, respectively. MATLAB Program 10–7 may be used to obtain the step-response curves of the designed system. Notice that, to obtain the unit-step response, we entered the command

$$[y,x,t] = \text{step}(\text{AA},\text{BB},\text{CC},\text{DD},1,t)$$

Figure 10–10 shows curves x_1 versus t , x_2 versus t , x_3 (= output y) versus t , x_4 versus t , and x_5 (= ξ) versus t . Notice that $y(t)$ [= $x_3(t)$] has approximately 15% overshoot and the settling time is approximately 4.5 sec. $\xi(t)$ [= $x_5(t)$] approaches 1.1. This result can be derived as follows: Since

$$\dot{\mathbf{x}}(\infty) = \mathbf{0} = \mathbf{Ax}(\infty) + \mathbf{Bu}(\infty)$$

or

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 20.601 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.4905 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ r \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0.5 \end{bmatrix} u(\infty)$$

MATLAB Program 10–7

```
%**** The following program is to obtain step response
% of the inverted-pendulum system just designed ****

A = [0 1 0 0;20.601 0 0 0;0 0 0 1;-0.4905 0 0 0];
B = [0;-1;0;0.5];
C = [0 0 1 0]
D = [0];
K = [-157.6336 -35.3733 -56.0652 -36.7466];
KI = -50.9684;
AA = [A - B*K B*KI;-C 0];
BB = [0;0;0;0;1];
CC = [C 0];
DD = [0];

%***** To obtain response curves x1 versus t, x2 versus t,
% x3 versus t, x4 versus t, and x5 versus t, separately, enter
% the following command *****

t = 0:0.02:6;
[y,x,t] = step(AA,BB,CC,DD,1,t);

x1 = [1 0 0 0 0]*x';
x2 = [0 1 0 0 0]*x';
x3 = [0 0 1 0 0]*x';
x4 = [0 0 0 1 0]*x';
x5 = [0 0 0 0 1]*x';

subplot(3,2,1); plot(t,x1); grid
title('x1 versus t')
xlabel('t Sec'); ylabel('x1')

subplot(3,2,2); plot(t,x2); grid
title('x2 versus t')
xlabel('t Sec'); ylabel('x2')

subplot(3,2,3); plot(t,x3); grid
title('x3 versus t')
xlabel('t Sec'); ylabel('x3')

subplot(3,2,4); plot(t,x4); grid
title('x4 versus t')
xlabel('t Sec'); ylabel('x4')

subplot(3,2,5); plot(t,x5); grid
title('x5 versus t')
xlabel('t Sec'); ylabel('x5')
```

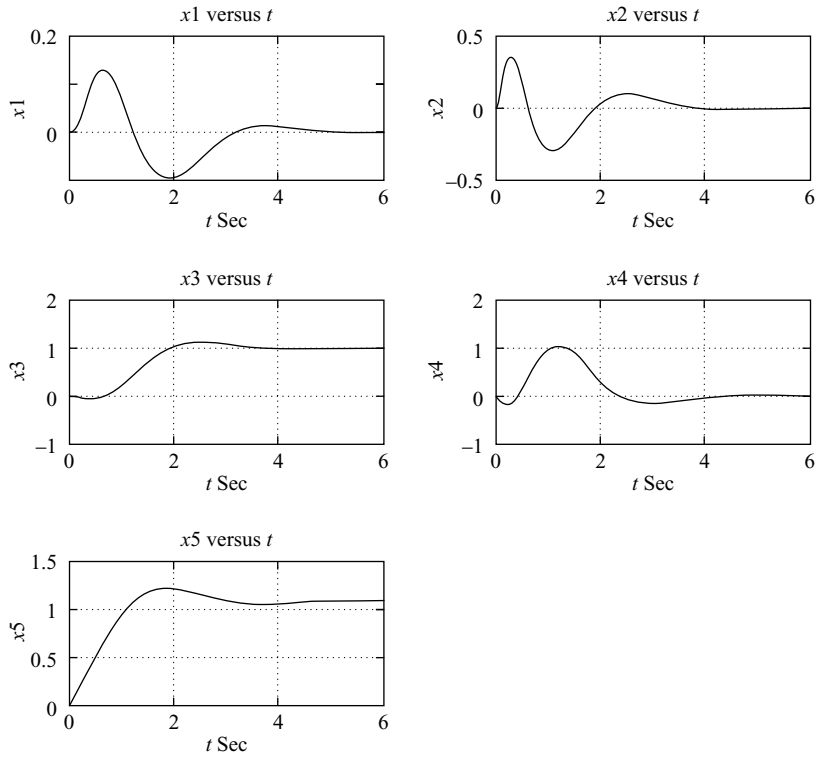


Figure 10-10
Curves x_1 versus t , x_2 versus t , x_3 (= output y) versus t , x_4 versus t , and x_5 (= ξ) versus t .

we get

$$u(\infty) = 0$$

Since $u(\infty) = 0$, we have, from Equation (10-33),

$$u(\infty) = 0 = -\mathbf{K}\mathbf{x}(\infty) + k_I \xi(\infty)$$

and so

$$\xi(\infty) = \frac{1}{k_I} [\mathbf{K}\mathbf{x}(\infty)] = \frac{1}{k_I} k_3 x_3(\infty) = \frac{-56.0652}{-50.9684} r = 1.1r$$

Hence, for $r = 1$, we have

$$\xi(\infty) = 1.1$$

It is noted that, as in any design problem, if the speed and damping are not quite satisfactory, then we must modify the desired characteristic equation and determine a new matrix $\hat{\mathbf{K}}$. Computer simulations must be repeated until a satisfactory result is obtained.

10-5 STATE OBSERVERS

In the pole-placement approach to the design of control systems, we assumed that all state variables are available for feedback. In practice, however, not all state variables are available for feedback. Then we need to estimate unavailable state variables.

Estimation of unmeasurable state variables is commonly called *observation*. A device (or a computer program) that estimates or observes the state variables is called a *state observer*, or simply an *observer*. If the state observer observes all state variables of the system, regardless of whether some state variables are available for direct measurement, it is called a *full-order state observer*. There are times when this will not be necessary, when we will need observation of only the unmeasurable state variables, but not of those that are directly measurable as well. For example, since the output variables are observable and they are linearly related to the state variables, we need not observe all state variables, but observe only $n - m$ state variables, where n is the dimension of the state vector and m is the dimension of the output vector.

An observer that estimates fewer than n state variables, where n is the dimension of the state vector, is called a *reduced-order state observer* or, simply, a *reduced-order observer*. If the order of the reduced-order state observer is the minimum possible, the observer is called a *minimum-order state observer* or *minimum-order observer*. In this section, we shall discuss both the full-order state observer and the minimum-order state observer.

State Observer. A state observer estimates the state variables based on the measurements of the output and control variables. Here the concept of observability discussed in Section 9-7 plays an important role. As we shall see later, state observers can be designed if and only if the observability condition is satisfied.

In the following discussions of state observers, we shall use the notation $\tilde{\mathbf{x}}$ to designate the observed state vector. In many practical cases, the observed state vector $\tilde{\mathbf{x}}$ is used in the state feedback to generate the desired control vector.

Consider the plant defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \quad (10-55)$$

$$y = \mathbf{C}\mathbf{x} \quad (10-56)$$

The observer is a subsystem to reconstruct the state vector of the plant. The mathematical model of the observer is basically the same as that of the plant, except that we include an additional term that includes the estimation error to compensate for inaccuracies in matrices \mathbf{A} and \mathbf{B} and the lack of the initial error. The estimation error or observation error is the difference between the measured output and the estimated output. The initial error is the difference between the initial state and the initial estimated state. Thus, we define the mathematical model of the observer to be

$$\begin{aligned} \dot{\tilde{\mathbf{x}}} &= \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}u + \mathbf{K}_e(y - \mathbf{C}\tilde{\mathbf{x}}) \\ &= (\mathbf{A} - \mathbf{K}_e\mathbf{C})\tilde{\mathbf{x}} + \mathbf{B}u + \mathbf{K}_ey \end{aligned} \quad (10-57)$$

where $\tilde{\mathbf{x}}$ is the estimated state and $\mathbf{C}\tilde{\mathbf{x}}$ is the estimated output. The inputs to the observer are the output y and the control input u . Matrix \mathbf{K}_e , which is called the observer gain matrix, is a weighting matrix to the correction term involving the difference between the measured output y and the estimated output $\mathbf{C}\tilde{\mathbf{x}}$. This term continuously corrects the model output and improves the performance of the observer. Figure 10-11 shows the block diagram of the system and the full-order state observer.

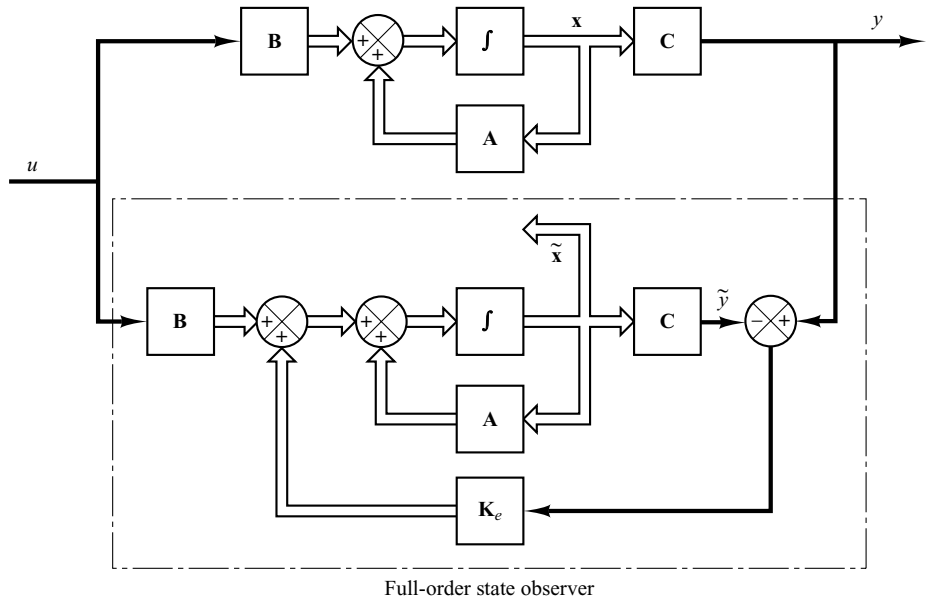


Figure 10–11
Block diagram of system and full-order state observer, when input u and output y are scalars.

Full-Order State Observer. The order of the state observer that will be discussed here is the same as that of the plant. Assume that the plant is defined by Equations (10–55) and (10–56) and the observer model is defined by Equation (10–57).

To obtain the observer error equation, let us subtract Equation (10–57) from Equation (10–55):

$$\begin{aligned}\dot{\mathbf{x}} - \dot{\tilde{\mathbf{x}}} &= \mathbf{A}\mathbf{x} - \mathbf{A}\tilde{\mathbf{x}} - \mathbf{K}_e(\mathbf{C}\mathbf{x} - \mathbf{C}\tilde{\mathbf{x}}) \\ &= (\mathbf{A} - \mathbf{K}_e\mathbf{C})(\mathbf{x} - \tilde{\mathbf{x}})\end{aligned}\quad (10-58)$$

Define the difference between \mathbf{x} and $\tilde{\mathbf{x}}$ as the error vector \mathbf{e} , or

$$\mathbf{e} = \mathbf{x} - \tilde{\mathbf{x}}$$

Then Equation (10–58) becomes

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C})\mathbf{e}\quad (10-59)$$

From Equation (10–59), we see that the dynamic behavior of the error vector is determined by the eigenvalues of matrix $\mathbf{A} - \mathbf{K}_e\mathbf{C}$. If matrix $\mathbf{A} - \mathbf{K}_e\mathbf{C}$ is a stable matrix, the error vector will converge to zero for any initial error vector $\mathbf{e}(0)$. That is, $\tilde{\mathbf{x}}(t)$ will converge to $\mathbf{x}(t)$ regardless of the values of $\mathbf{x}(0)$ and $\tilde{\mathbf{x}}(0)$. If the eigenvalues of matrix $\mathbf{A} - \mathbf{K}_e\mathbf{C}$ are chosen in such a way that the dynamic behavior of the error vector is asymptotically stable and is adequately fast, then any error vector will tend to zero (the origin) with an adequate speed.

If the plant is completely observable, then it can be proved that it is possible to choose matrix \mathbf{K}_e such that $\mathbf{A} - \mathbf{K}_e\mathbf{C}$ has arbitrarily desired eigenvalues. That is, the observer gain matrix \mathbf{K}_e can be determined to yield the desired matrix $\mathbf{A} - \mathbf{K}_e\mathbf{C}$. We shall discuss this matter in what follows.

Dual Problem. The problem of designing a full-order observer becomes that of determining the observer gain matrix \mathbf{K}_e such that the error dynamics defined by Equation (10–59) are asymptotically stable with sufficient speed of response. (The asymptotic stability and the speed of response of the error dynamics are determined by the eigenvalues of matrix $\mathbf{A} - \mathbf{K}_e \mathbf{C}$.) Hence, the design of the full-order observer becomes that of determining an appropriate \mathbf{K}_e such that $\mathbf{A} - \mathbf{K}_e \mathbf{C}$ has desired eigenvalues. Thus, the problem here becomes the same as the pole-placement problem we discussed in Section 10–2. In fact, the two problems are mathematically the same. This property is called duality.

Consider the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

In designing the full-order state observer, we may solve the dual problem, that is, solve the pole-placement problem for the dual system

$$\dot{\mathbf{z}} = \mathbf{A}^*\mathbf{z} + \mathbf{C}^*v$$

$$n = \mathbf{B}^*\mathbf{z}$$

assuming the control signal v to be

$$v = -\mathbf{K}\mathbf{z}$$

If the dual system is completely state controllable, then the state feedback gain matrix \mathbf{K} can be determined such that matrix $\mathbf{A}^* - \mathbf{C}^*\mathbf{K}$ will yield a set of the desired eigenvalues.

If $\mu_1, \mu_2, \dots, \mu_n$ are the desired eigenvalues of the state observer matrix, then by taking the same μ_i 's as the desired eigenvalues of the state-feedback gain matrix of the dual system, we obtain

$$|s\mathbf{I} - (\mathbf{A}^* - \mathbf{C}^*\mathbf{K})| = (s - \mu_1)(s - \mu_2)\cdots(s - \mu_n)$$

Noting that the eigenvalues of $\mathbf{A}^* - \mathbf{C}^*\mathbf{K}$ and those of $\mathbf{A} - \mathbf{K}^*\mathbf{C}$ are the same, we have

$$|s\mathbf{I} - (\mathbf{A}^* - \mathbf{C}^*\mathbf{K})| = |s\mathbf{I} - (\mathbf{A} - \mathbf{K}^*\mathbf{C})|$$

Comparing the characteristic polynomial $|s\mathbf{I} - (\mathbf{A} - \mathbf{K}^*\mathbf{C})|$ and the characteristic polynomial $|s\mathbf{I} - (\mathbf{A} - \mathbf{K}_e \mathbf{C})|$ for the observer system [refer to Equation (10–57)], we find that \mathbf{K}_e and \mathbf{K}^* are related by

$$\mathbf{K}_e = \mathbf{K}^*$$

Thus, using the matrix \mathbf{K} determined by the pole-placement approach in the dual system, the observer gain matrix \mathbf{K}_e for the original system can be determined by using the relationship $\mathbf{K}_e = \mathbf{K}^*$. (See Problem A–10–10 for the details.)

Necessary and Sufficient Condition for State Observation. As discussed, a necessary and sufficient condition for the determination of the observer gain matrix \mathbf{K}_e for the desired eigenvalues of $\mathbf{A} - \mathbf{K}_e \mathbf{C}$ is that the dual of the original system

$$\dot{\mathbf{z}} = \mathbf{A}^*\mathbf{z} + \mathbf{C}^*v$$

be completely state controllable. The complete state controllability condition for this dual system is that the rank of

$$[\mathbf{C}^* \mid \mathbf{A}^*\mathbf{C}^* \mid \dots \mid (\mathbf{A}^*)^{n-1}\mathbf{C}^*]$$

be n . This is the condition for complete observability of the original system defined by Equations (10–55) and (10–56). This means that a necessary and sufficient condition for the observation of the state of the system defined by Equations (10–55) and (10–56) is that the system be completely observable.

Once we select the desired eigenvalues (or desired characteristic equation), the full-order state observer can be designed, provided the plant is completely observable. The desired eigenvalues of the characteristic equation should be chosen so that the state observer responds at least two to five times faster than the closed-loop system considered. As stated earlier, the equation for the full-order state observer is

$$\dot{\tilde{\mathbf{x}}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C})\tilde{\mathbf{x}} + \mathbf{B}u + \mathbf{K}_e y \quad (10-60)$$

It is noted that thus far we have assumed the matrices \mathbf{A} , \mathbf{B} , and \mathbf{C} in the observer to be exactly the same as those of the physical plant. If there are discrepancies in \mathbf{A} , \mathbf{B} , and \mathbf{C} in the observer and in the physical plant, the dynamics of the observer error are no longer governed by Equation (10–59). This means that the error may not approach zero as expected. Therefore, we need to choose \mathbf{K}_e so that the observer is stable and the error remains acceptably small in the presence of small modeling errors.

Transformation Approach to Obtain State Observer Gain Matrix \mathbf{K}_e . By following the same approach as we used in deriving the equation for the state feedback gain matrix \mathbf{K} , we can obtain the following equation:

$$\mathbf{K}_e = \mathbf{Q} \begin{bmatrix} \alpha_n - a_n \\ \alpha_{n-1} - a_{n-1} \\ \vdots \\ \alpha_1 - a_1 \end{bmatrix} = (\mathbf{W}\mathbf{N}^*)^{-1} \begin{bmatrix} \alpha_n - a_n \\ \alpha_{n-1} - a_{n-1} \\ \vdots \\ \alpha_1 - a_1 \end{bmatrix} \quad (10-61)$$

where \mathbf{K}_e is an $n \times 1$ matrix,

$$\mathbf{Q} = (\mathbf{W}\mathbf{N}^*)^{-1}$$

and

$$\mathbf{N} = [\mathbf{C}^* \mid \mathbf{A}^*\mathbf{C}^* \mid \dots \mid (\mathbf{A}^*)^{n-1}\mathbf{C}^*]$$

$$\mathbf{W} = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

[Refer to Problem **A–10–10** for the derivation of Equation (10–61).]

Direct-Substitution Approach to Obtain State Observer Gain Matrix \mathbf{K}_e . Similar to the case of pole placement, if the system is of low order, then direct substitution of matrix \mathbf{K}_e into the desired characteristic polynomial may be simpler. For example, if \mathbf{x} is a 3-vector, then write the observer gain matrix \mathbf{K}_e as

$$\mathbf{K}_e = \begin{bmatrix} k_{e1} \\ k_{e2} \\ k_{e3} \end{bmatrix}$$

Substitute this \mathbf{K}_e matrix into the desired characteristic polynomial:

$$|s\mathbf{I} - (\mathbf{A} - \mathbf{K}_e\mathbf{C})| = (s - \mu_1)(s - \mu_2)(s - \mu_3)$$

By equating the coefficients of the like powers of s on both sides of this last equation, we can determine the values of k_{e1} , k_{e2} , and k_{e3} . This approach is convenient if $n = 1, 2$, or 3 , where n is the dimension of the state vector \mathbf{x} . (Although this approach can be used when $n = 4, 5, 6, \dots$, the computations involved may become very tedious.)

Another approach to the determination of the state observer gain matrix \mathbf{K}_e is to use Ackermann's formula. This approach is presented in the following.

Ackermann's Formula. Consider the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \quad (10-62)$$

$$y = \mathbf{C}\mathbf{x} \quad (10-63)$$

In Section 10-2 we derived Ackermann's formula for pole placement for the system defined by Equation (10-62). The result was given by Equation (10-18), rewritten thus:

$$\mathbf{K} = [0 \ 0 \ \dots \ 0 \ 1][\mathbf{B} \ \mathbf{A}\mathbf{B} \ \dots \ \mathbf{A}^{n-1}\mathbf{B}]^{-1}\phi(\mathbf{A})$$

For the dual of the system defined by Equations (10-62) and (10-63),

$$\dot{\mathbf{z}} = \mathbf{A}^*\mathbf{z} + \mathbf{C}^*v$$

$$n = \mathbf{B}^*\mathbf{z}$$

the preceding Ackermann's formula for pole placement is modified to

$$\mathbf{K} = [0 \ 0 \ \dots \ 0 \ 1][\mathbf{C}^* \ \mathbf{A}^*\mathbf{C}^* \ \dots \ (\mathbf{A}^*)^{n-1}\mathbf{C}^*]^{-1}\phi(\mathbf{A}^*) \quad (10-64)$$

As stated earlier, the state observer gain matrix \mathbf{K}_e is given by \mathbf{K}^* , where \mathbf{K} is given by Equation (10-64). Thus,

$$\mathbf{K}_e = \mathbf{K}^* = \phi(\mathbf{A}^*)^* \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \cdot \\ \cdot \\ \mathbf{C}\mathbf{A}^{n-2} \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} = \phi(\mathbf{A}) \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \cdot \\ \cdot \\ \mathbf{C}\mathbf{A}^{n-2} \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \quad (10-65)$$

where $\phi(s)$ is the desired characteristic polynomial for the state observer, or

$$\phi(s) = (s - \mu_1)(s - \mu_2) \cdots (s - \mu_n)$$

where $\mu_1, \mu_2, \dots, \mu_n$ are the desired eigenvalues. Equation (10–65) is called Ackermann's formula for the determination of the observer gain matrix \mathbf{K}_e .

Comments on Selecting the Best \mathbf{K}_e . Referring to Figure 10–11, notice that the feedback signal through the observer gain matrix \mathbf{K}_e serves as a correction signal to the plant model to account for the unknowns in the plant. If significant unknowns are involved, the feedback signal through the matrix \mathbf{K}_e should be relatively large. However, if the output signal is contaminated significantly by disturbances and measurement noises, then the output y is not reliable and the feedback signal through the matrix \mathbf{K}_e should be relatively small. In determining the matrix \mathbf{K}_e , we should carefully examine the effects of disturbances and noises involved in the output y .

Remember that the observer gain matrix \mathbf{K}_e depends on the desired characteristic equation

$$(s - \mu_1)(s - \mu_2) \cdots (s - \mu_n) = 0$$

The choice of a set of $\mu_1, \mu_2, \dots, \mu_n$ is, in many instances, not unique. As a general rule, however, the observer poles must be two to five times faster than the controller poles to make sure the observation error (estimation error) converges to zero quickly. This means that the observer estimation error decays two to five times faster than does the state vector \mathbf{x} . Such faster decay of the observer error compared with the desired dynamics makes the controller poles dominate the system response.

It is important to note that if sensor noise is considerable, we may choose the observer poles to be slower than two times the controller poles, so that the bandwidth of the system will become lower and smooth the noise. In this case the system response will be strongly influenced by the observer poles. If the observer poles are located to the right of the controller poles in the left-half s plane, the system response will be dominated by the observer poles rather than by the control poles.

In the design of the state observer, it is desirable to determine several observer gain matrices \mathbf{K}_e based on several different desired characteristic equations. For each of the several different matrices \mathbf{K}_e , simulation tests must be run to evaluate the resulting system performance. Then we select the best \mathbf{K}_e from the viewpoint of overall system performance. In many practical cases, the selection of the best matrix \mathbf{K}_e boils down to a compromise between speedy response and sensitivity to disturbances and noises.

EXAMPLE 10–6 Consider the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 20.6 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [0 \quad 1]$$

We use the observed state feedback such that

$$u = -\mathbf{K}\tilde{\mathbf{x}}$$

Design a full-order state observer, assuming that the system configuration is identical to that shown in Figure 10–11. Assume that the desired eigenvalues of the observer matrix are

$$\mu_1 = -10, \quad \mu_2 = -10$$

The design of the state observer reduces to the determination of an appropriate observer gain matrix \mathbf{K}_e .

Let us examine the observability matrix. The rank of

$$[\mathbf{C}^* \quad \mathbf{A}^*\mathbf{C}^*] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is 2. Hence, the system is completely observable and the determination of the desired observer gain matrix is possible. We shall solve this problem by three methods.

Method 1: We shall determine the observer gain matrix by use of Equation (10–61). The given system is already in the observable canonical form. Hence, the transformation matrix $\mathbf{Q} = (\mathbf{WN}^*)^{-1}$ is \mathbf{I} . Since the characteristic equation of the given system is

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s & -20.6 \\ -1 & s \end{vmatrix} = s^2 - 20.6 = s^2 + a_1s + a_2 = 0$$

we have

$$a_1 = 0, \quad a_2 = -20.6$$

The desired characteristic equation is

$$(s + 10)^2 = s^2 + 20s + 100 = s^2 + \alpha_1s + \alpha_2 = 0$$

Hence,

$$\alpha_1 = 20, \quad \alpha_2 = 100$$

Then the observer gain matrix \mathbf{K}_e can be obtained from Equation (10–61) as follows:

$$\mathbf{K}_e = (\mathbf{WN}^*)^{-1} \begin{bmatrix} \alpha_2 - a_2 \\ \alpha_1 - a_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 100 + 20.6 \\ 20 - 0 \end{bmatrix} = \begin{bmatrix} 120.6 \\ 20 \end{bmatrix}$$

Method 2: Referring to Equation (10–59):

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C})\mathbf{e}$$

the characteristic equation for the observer becomes

$$|s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C}| = 0$$

Define

$$\mathbf{K}_e = \begin{bmatrix} k_{e1} \\ k_{e2} \end{bmatrix}$$

Then the characteristic equation becomes

$$\begin{vmatrix} \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 20.6 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} k_{e1} \\ k_{e2} \end{bmatrix} [0 \quad 1] \end{vmatrix} = \begin{vmatrix} s & -20.6 + k_{e1} \\ -1 & s + k_{e2} \end{vmatrix} \\ = s^2 + k_{e2}s - 20.6 + k_{e1} = 0 \end{vmatrix} \quad (10-66)$$

Since the desired characteristic equation is

$$s^2 + 20s + 100 = 0$$

by comparing Equation (10–66) with this last equation, we obtain

$$k_{e1} = 120.6, \quad k_{e2} = 20$$

or

$$\mathbf{K}_e = \begin{bmatrix} 120.6 \\ 20 \end{bmatrix}$$

Method 3: We shall use Ackermann's formula given by Equation (10–65):

$$\mathbf{K}_e = \phi(\mathbf{A}) \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

where

$$\phi(s) = (s - \mu_1)(s - \mu_2) = s^2 + 20s + 100$$

Thus,

$$\phi(\mathbf{A}) = \mathbf{A}^2 + 20\mathbf{A} + 100\mathbf{I}$$

and

$$\begin{aligned} \mathbf{K}_e &= (\mathbf{A}^2 + 20\mathbf{A} + 100\mathbf{I}) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 120.6 & 412 \\ 20 & 120.6 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 120.6 \\ 20 \end{bmatrix} \end{aligned}$$

As a matter of course, we get the same \mathbf{K}_e regardless of the method employed.

The equation for the full-order state observer is given by Equation (10–57),

$$\dot{\tilde{\mathbf{x}}} = (\mathbf{A} - \mathbf{K}_e \mathbf{C}) \tilde{\mathbf{x}} + \mathbf{B}u + \mathbf{K}_e y$$

or

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & -100 \\ 1 & -20 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 120.6 \\ 20 \end{bmatrix} y$$

Finally, it is noted that, similar to the case of pole placement, if the system order n is 4 or higher, methods 1 and 3 are preferred, because all matrix computations can be carried out by a computer, while method 2 always requires hand computation of the characteristic equation involving unknown parameters $k_{e1}, k_{e2}, \dots, k_{en}$.

Effects of the Addition of the Observer on a Closed-Loop System. In the pole-placement design process, we assumed that the actual state $\mathbf{x}(t)$ was available for feedback. In practice, however, the actual state $\mathbf{x}(t)$ may not be measurable, so we will need to design an observer and use the observed state $\tilde{\mathbf{x}}(t)$ for feedback as shown in Figure 10–12. The design process, therefore, becomes a two-stage process, the first stage being the determination of the feedback gain matrix \mathbf{K} to yield the desired characteristic equation and the second stage being the determination of the observer gain matrix \mathbf{K}_e to yield the desired observer characteristic equation.

Let us now investigate the effects of the use of the observed state $\tilde{\mathbf{x}}(t)$, rather than the actual state $\mathbf{x}(t)$, on the characteristic equation of a closed-loop control system.

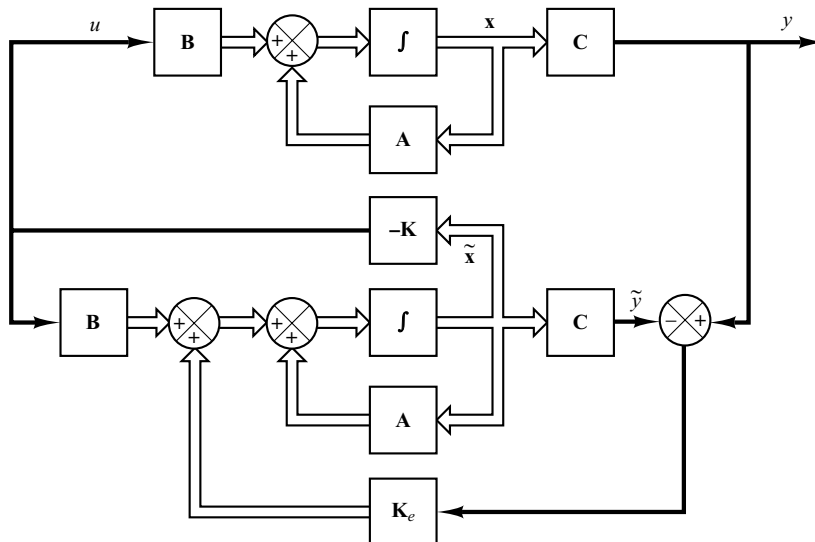


Figure 10–12
Observed-state
feedback control
system.

Consider the completely state controllable and completely observable system defined by the equations

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x}\end{aligned}$$

For the state-feedback control based on the observed state $\tilde{\mathbf{x}}$,

$$u = -\mathbf{K}\tilde{\mathbf{x}}$$

With this control, the state equation becomes

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{BK}\tilde{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x} + \mathbf{BK}(\mathbf{x} - \tilde{\mathbf{x}}) \quad (10-67)$$

The difference between the actual state $\mathbf{x}(t)$ and the observed state $\tilde{\mathbf{x}}(t)$ has been defined as the error $\mathbf{e}(t)$:

$$\mathbf{e}(t) = \mathbf{x}(t) - \tilde{\mathbf{x}}(t)$$

Substitution of the error vector $\mathbf{e}(t)$ into Equation (10–67) gives

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x} + \mathbf{BK}\mathbf{e} \quad (10-68)$$

Note that the observer error equation was given by Equation (10–59), repeated here:

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C})\mathbf{e} \quad (10-69)$$

Combining Equations (10–68) and (10–69), we obtain

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A} - \mathbf{K}_e\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} \quad (10-70)$$

Equation (10–70) describes the dynamics of the observed-state feedback control system. The characteristic equation for the system is

$$\begin{vmatrix} s\mathbf{I} - \mathbf{A} + \mathbf{BK} & -\mathbf{BK} \\ \mathbf{0} & s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C} \end{vmatrix} = 0$$

or

$$|s\mathbf{I} - \mathbf{A} + \mathbf{BK}| |s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C}| = 0$$

Notice that the closed-loop poles of the observed-state feedback control system consist of the poles due to the pole-placement design alone and the poles due to the observer design alone. This means that the pole-placement design and the observer design are independent of each other. They can be designed separately and combined to form the observed-state feedback control system. Note that, if the order of the plant is n , then the observer is also of n th order (if the full-order state observer is used), and the resulting characteristic equation for the entire closed-loop system becomes of order $2n$.

Transfer Function of the Observer-Based Controller. Consider the plant defined by

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$

$$y = \mathbf{Cx}$$

Assume that the plant is completely observable. Assume that we use observed-state feedback control $u = -\mathbf{K}\tilde{\mathbf{x}}$. Then, the equations for the observer are given by

$$\dot{\tilde{\mathbf{x}}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C} - \mathbf{BK})\tilde{\mathbf{x}} + \mathbf{K}_e y \quad (10-71)$$

$$u = -\mathbf{K}\tilde{\mathbf{x}} \quad (10-72)$$

where Equation (10–71) is obtained by substituting $u = -\mathbf{K}\tilde{\mathbf{x}}$ into Equation (10–57).

By taking the Laplace transform of Equation (10–71), assuming a zero initial condition, and solving for $\tilde{\mathbf{X}}(s)$, we obtain

$$\tilde{\mathbf{X}}(s) = (s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C} + \mathbf{BK})^{-1} \mathbf{K}_e Y(s)$$

By substituting this $\tilde{\mathbf{X}}(s)$ into the Laplace transform of Equation (10–72), we obtain

$$U(s) = -\mathbf{K}(s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C} + \mathbf{BK})^{-1} \mathbf{K}_e Y(s) \quad (10-73)$$

Then the transfer function $U(s)/Y(s)$ can be obtained as

$$\frac{U(s)}{Y(s)} = -\mathbf{K}(s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C} + \mathbf{BK})^{-1} \mathbf{K}_e$$

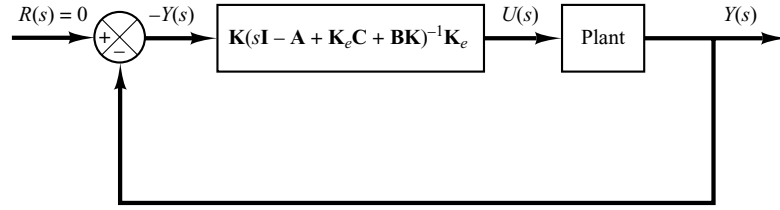
Figure 10–13 shows the block diagram representation for the system. Notice that the transfer function

$$\mathbf{K}(s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C} + \mathbf{BK})^{-1} \mathbf{K}_e$$

acts as a controller for the system. Hence, we call the transfer function

$$\frac{U(s)}{-Y(s)} = \frac{\text{num}}{\text{den}} = \mathbf{K}(s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C} + \mathbf{BK})^{-1} \mathbf{K}_e \quad (10-74)$$

Figure 10–13
Block diagram
representation of
system with a
controller-observer.



the observer-based controller transfer function or, simply, the observer-controller transfer function.

Note that the observer-controller matrix

$$\mathbf{A} - \mathbf{K}_e \mathbf{C} - \mathbf{BK}$$

may or may not be stable, although $\mathbf{A} - \mathbf{BK}$ and $\mathbf{A} - \mathbf{K}_e \mathbf{C}$ are chosen to be stable. In fact, in some cases the matrix $\mathbf{A} - \mathbf{K}_e \mathbf{C} - \mathbf{BK}$ may be poorly stable or even unstable.

EXAMPLE 10–7 Consider the design of a regulator system for the following plant:

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \quad (10-75)$$

$$y = \mathbf{Cx} \quad (10-76)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 20.6 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0]$$

Suppose that we use the pole-placement approach to the design of the system and that the desired closed-loop poles for this system are at $s = \mu_i$ ($i = 1, 2$), where $\mu_1 = -1.8 + j2.4$ and $\mu_2 = -1.8 - j2.4$. The state-feedback gain matrix \mathbf{K} for this case can be obtained as follows:

$$\mathbf{K} = [29.6 \quad 3.6]$$

Using this state-feedback gain matrix \mathbf{K} , the control signal u is given by

$$u = -\mathbf{Kx} = -[29.6 \quad 3.6] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Suppose that we use the observed-state feedback control instead of the actual-state feedback control, or

$$u = -\mathbf{K}\tilde{\mathbf{x}} = -[29.6 \quad 3.6] \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

where we choose the observer poles to be at

$$s = -8, \quad s = -8$$

Obtain the observer gain matrix \mathbf{K}_e and draw a block diagram for the observed-state feedback control system. Then obtain the transfer function $U(s)/[-Y(s)]$ for the observer controller, and draw another block diagram with the observer controller as a series controller in the feedforward path. Finally, obtain the response of the system to the following initial condition:

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}(0) = \mathbf{x}(0) - \tilde{\mathbf{x}}(0) = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

For the system defined by Equation (10-75), the characteristic polynomial is

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s & -1 \\ -20.6 & s \end{vmatrix} = s^2 - 20.6 = s^2 + a_1s + a_2$$

Thus,

$$a_1 = 0, \quad a_2 = -20.6$$

The desired characteristic polynomial for the observer is

$$\begin{aligned} (s - \mu_1)(s - \mu_2) &= (s + 8)(s + 8) = s^2 + 16s + 64 \\ &= s^2 + \alpha_1s + \alpha_2 \end{aligned}$$

Hence,

$$\alpha_1 = 16, \quad \alpha_2 = 64$$

For the determination of the observer gain matrix, we use Equation (10-61), or

$$\mathbf{K}_e = (\mathbf{W}\mathbf{N}^*)^{-1} \begin{bmatrix} \alpha_2 - a_2 \\ \alpha_1 - a_1 \end{bmatrix}$$

where

$$\begin{aligned} \mathbf{N} &= [\mathbf{C}^* \ \vdots \ \mathbf{A}^*\mathbf{C}^*] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \mathbf{W} &= \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

Hence,

$$\begin{aligned} \mathbf{K}_e &= \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}^{-1} \begin{bmatrix} 64 + 20.6 \\ 16 - 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 84.6 \\ 16 \end{bmatrix} = \begin{bmatrix} 16 \\ 84.6 \end{bmatrix} \end{aligned} \quad (10-77)$$

Equation (10-77) gives the observer gain matrix \mathbf{K}_e . The observer equation is given by Equation (10-60):

$$\dot{\tilde{\mathbf{x}}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C})\tilde{\mathbf{x}} + \mathbf{B}u + \mathbf{K}_ey \quad (10-78)$$

Since

$$u = -\mathbf{K}\tilde{\mathbf{x}}$$

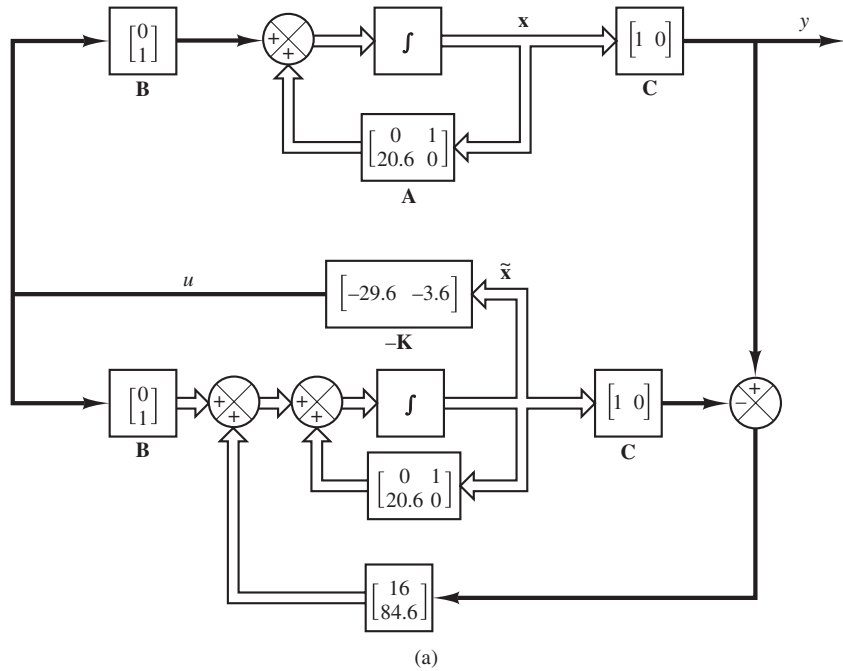
Equation (10-78) becomes

$$\dot{\tilde{\mathbf{x}}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C} - \mathbf{BK})\tilde{\mathbf{x}} + \mathbf{K}_ey$$

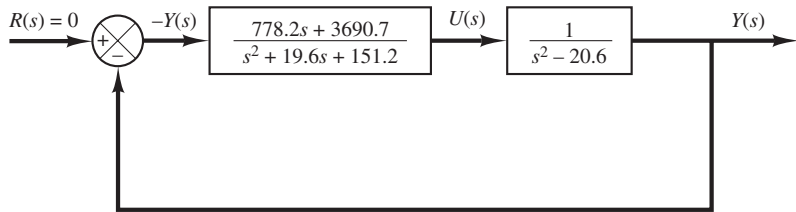
or

$$\begin{aligned} \begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} &= \left\{ \begin{bmatrix} 0 & 1 \\ 20.6 & 0 \end{bmatrix} - \begin{bmatrix} 16 \\ 84.6 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 29.6 & 3.6 \end{bmatrix} \right\} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 16 \\ 84.6 \end{bmatrix} y \\ &= \begin{bmatrix} -16 & 1 \\ -93.6 & -3.6 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 16 \\ 84.6 \end{bmatrix} y \end{aligned}$$

The block diagram of the system with observed-state feedback is shown in Figure 10-14(a).



(a)



(b)

Figure 10-14
 (a) Block diagram of system with observed-state feedback; (b) block diagram of transfer-function system.

Referring to Equation (10-74), the transfer function of the observer-controller is

$$\begin{aligned}
 \frac{U(s)}{-Y(s)} &= \mathbf{K}(s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C} + \mathbf{BK})^{-1}\mathbf{K}_e \\
 &= [29.6 \quad 3.6] \begin{bmatrix} s + 16 & -1 \\ 93.6 & s + 3.6 \end{bmatrix}^{-1} \begin{bmatrix} 16 \\ 84.6 \end{bmatrix} \\
 &= \frac{778.2s + 3690.7}{s^2 + 19.6s + 151.2}
 \end{aligned}$$

As a matter of course, the same transfer function can be obtained with MATLAB. For example, MATLAB Program 10-8 produces the transfer function of the observer controller. Figure 10-14(b) shows a block diagram of the system.

MATLAB Program 10–8

```
% Obtaining transfer function of observer controller --- full-order observer
A = [0 1;20.6 0];
B = [0;1];
C = [1 0];
K = [29.6 3.6];
Ke = [16;84.6];
AA = A-Ke*C-B*K;
BB = Ke;
CC = K;
DD = 0;
[num,den] = ss2tf(AA,BB,CC,DD)

num =
    1.0e+003*
         0    0.7782    3.6907
den =
    1.0000    19.6000    151.2000
```

The dynamics of the observed-state feedback control system just designed can be described by the following equations: For the plant,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 20.6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

For the observer,

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} -16 & 1 \\ -93.6 & -3.6 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 16 \\ 84.6 \end{bmatrix} y$$
$$u = -[29.6 \quad 3.6] \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

The system, as a whole, is of fourth order. The characteristic equation for the system is

$$\begin{aligned} |s\mathbf{I} - \mathbf{A} + \mathbf{BK}| |s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C}| &= (s^2 + 3.6s + 9)(s^2 + 16s + 64) \\ &= s^4 + 19.6s^3 + 130.6s^2 + 374.4s + 576 = 0 \end{aligned}$$

The characteristic equation can also be obtained from the block diagram for the system shown in Figure 10–14(b). Since the closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{778.2s + 3690.7}{(s^2 + 19.6s + 151.2)(s^2 - 20.6) + 778.2s + 3690.7}$$

the characteristic equation is

$$(s^2 + 19.6s + 151.2)(s^2 - 20.6) + 778.2s + 3690.7 \\ = s^4 + 19.6s^3 + 130.6s^2 + 374.4s + 576 = 0$$

As a matter of course, the characteristic equation is the same for the system in state-space representation and in transfer-function representation.

Finally, we shall obtain the response of the system to the following initial condition:

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}(0) = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

Referring to Equation (10–70), the response to the initial condition can be determined from

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A} - \mathbf{K}_e\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{x}(0) \\ \mathbf{e}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0.5 \\ 0 \end{bmatrix}$$

A MATLAB Program to obtain the response is shown in MATLAB Program 10–9. The resulting response curves are shown in Figure 10–15.

MATLAB Program 10–9

```
A = [0 1; 20.6 0];
B = [0;1];
C = [1 0];
K = [29.6 3.6];
Ke = [16; 84.6];
sys = ss([A-B*K B*K; zeros(2,2) A-Ke*C],eye(4),eye(4),eye(4));
t = 0:0.01:4;
z = initial(sys,[1;0;0.5;0],t);
x1 = [1 0 0 0]*z';
x2 = [0 1 0 0]*z';
e1 = [0 0 1 0]*z';
e2 = [0 0 0 1]*z';

subplot(2,2,1); plot(t,x1 ),grid
title('Response to Initial Condition')
ylabel('state variable x1')

subplot(2,2,2); plot(t,x2),grid
title('Response to Initial Condition')
ylabel('state variable x2')

subplot(2,2,3); plot(t,e1),grid
xlabel('t (sec)'), ylabel('error state variable e1')

subplot(2,2,4); plot(t,e2),grid
xlabel('t (sec)'), ylabel('error state variable e2')
```

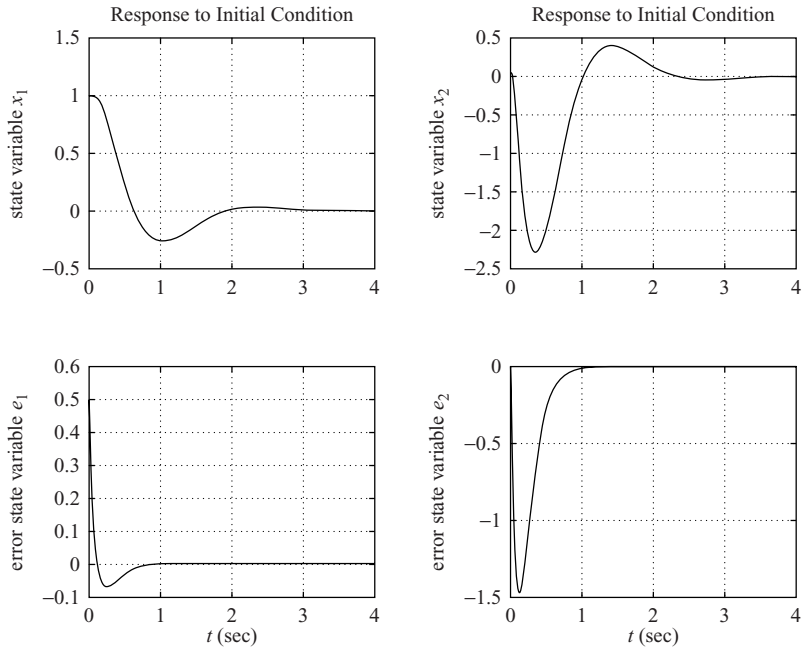


Figure 10-15
Response curves to initial condition.

Minimum-Order Observer. The observers discussed thus far are designed to reconstruct all the state variables. In practice, some of the state variables may be accurately measured. Such accurately measurable state variables need not be estimated.

Suppose that the state vector \mathbf{x} is an n -vector and the output vector \mathbf{y} is an m -vector that can be measured. Since m output variables are linear combinations of the state variables, m state variables need not be estimated. We need to estimate only $n - m$ state variables. Then the reduced-order observer becomes an $(n - m)$ th-order observer. Such an $(n - m)$ th-order observer is the minimum-order observer. Figure 10-16 shows the block diagram of a system with a minimum-order observer.

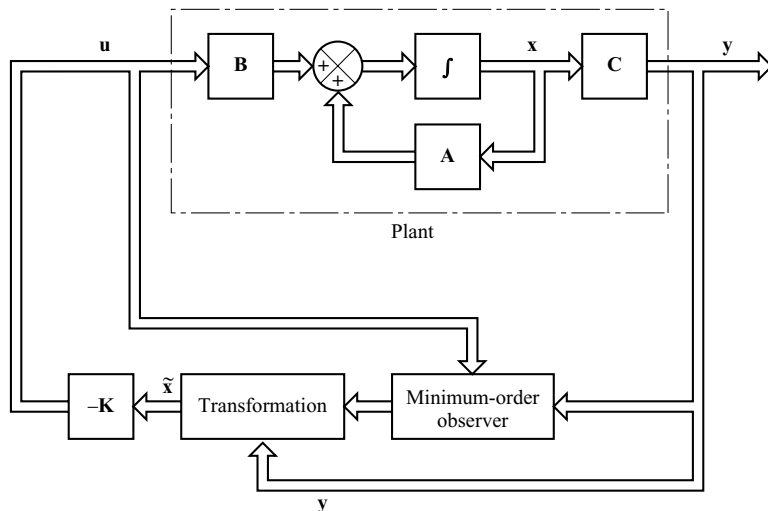


Figure 10-16
Observed-state feedback control system with a minimum-order observer.

It is important to note, however, that if the measurement of output variables involves significant noises and is relatively inaccurate, then the use of the full-order observer may result in a better system performance.

To present the basic idea of the minimum-order observer, without undue mathematical complications, we shall present the case where the output is a scalar (that is, $m = 1$) and derive the state equation for the minimum-order observer. Consider the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \quad (10-79)$$

$$y = \mathbf{C}\mathbf{x} \quad (10-80)$$

where the state vector \mathbf{x} can be partitioned into two parts x_a (a scalar) and \mathbf{x}_b [an $(n - 1)$ -vector]. Here the state variable x_a is equal to the output y and thus can be directly measured, and \mathbf{x}_b is the unmeasurable portion of the state vector. Then the partitioned state and output equations become

$$\begin{bmatrix} \dot{x}_a \\ \dot{\mathbf{x}}_b \end{bmatrix} = \begin{bmatrix} A_{aa} & \mathbf{A}_{ab} \\ \mathbf{A}_{ba} & \mathbf{A}_{bb} \end{bmatrix} \begin{bmatrix} x_a \\ \mathbf{x}_b \end{bmatrix} + \begin{bmatrix} B_a \\ \mathbf{B}_b \end{bmatrix} u \quad (10-81)$$

$$y = [1 \quad \mathbf{0}] \begin{bmatrix} x_a \\ \mathbf{x}_b \end{bmatrix} \quad (10-82)$$

where $A_{aa} = \text{scalar}$

$\mathbf{A}_{ab} = 1 \times (n - 1)$ matrix

$\mathbf{A}_{ba} = (n - 1) \times 1$ matrix

$\mathbf{A}_{bb} = (n - 1) \times (n - 1)$ matrix

$B_a = \text{scalar}$

$\mathbf{B}_b = (n - 1) \times 1$ matrix

From Equation (10-81), the equation for the measured portion of the state becomes

$$\dot{x}_a = A_{aa}x_a + \mathbf{A}_{ab}\mathbf{x}_b + B_a u$$

or

$$\dot{x}_a - A_{aa}x_a - B_a u = \mathbf{A}_{ab}\mathbf{x}_b \quad (10-83)$$

The terms on the left-hand side of Equation (10-83) can be measured. Equation (10-83) acts as the output equation. In designing the minimum-order observer, we consider the left-hand side of Equation (10-83) to be known quantities. Thus, Equation (10-83) relates the measurable quantities and unmeasurable quantities of the state.

From Equation (10-81), the equation for the unmeasured portion of the state becomes

$$\dot{\mathbf{x}}_b = \mathbf{A}_{ba}x_a + \mathbf{A}_{bb}\mathbf{x}_b + \mathbf{B}_b u \quad (10-84)$$

Noting that terms $\mathbf{A}_{ba}x_a$ and $\mathbf{B}_b u$ are known quantities, Equation (10-84) describes the dynamics of the unmeasured portion of the state.

In what follows we shall present a method for designing a minimum-order observer. The design procedure can be simplified if we utilize the design technique developed for the full-order state observer.

Let us compare the state equation for the full-order observer with that for the minimum-order observer. The state equation for the full-order observer is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

and the “state equation” for the minimum-order observer is

$$\dot{\mathbf{x}}_b = \mathbf{A}_{bb}\mathbf{x}_b + \mathbf{A}_{ba}x_a + \mathbf{B}_b u$$

The output equation for the full-order observer is

$$y = \mathbf{C}\mathbf{x}$$

and the “output equation” for the minimum-order observer is

$$\dot{x}_a - A_{aa}x_a - B_a u = \mathbf{A}_{ab}\mathbf{x}_b$$

The design of the minimum-order observer can be carried out as follows: First, note that the observer equation for the full-order observer was given by Equation (10-57), which we repeat here:

$$\dot{\tilde{\mathbf{x}}} = (\mathbf{A} - \mathbf{K}_e \mathbf{C})\tilde{\mathbf{x}} + \mathbf{B}u + \mathbf{K}_e y \quad (10-85)$$

Then, making the substitutions of Table 10-1 into Equation (10-85), we obtain

$$\dot{\tilde{\mathbf{x}}}_b = (\mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab})\tilde{\mathbf{x}}_b + \mathbf{A}_{ba}x_a + \mathbf{B}_b u + \mathbf{K}_e(\dot{x}_a - A_{aa}x_a - B_a u) \quad (10-86)$$

where the state observer gain matrix \mathbf{K}_e is an $(n - 1) \times 1$ matrix. In Equation (10-86), notice that in order to estimate $\tilde{\mathbf{x}}_b$, we need the derivative of x_a . This presents a difficulty, because differentiation amplifies noise. If $x_a (= y)$ is noisy, the use of \dot{x}_a is unacceptable.

Table 10-1 List of Necessary Substitutions for Writing the Observer Equation for the Minimum-Order State Observer

Full-Order State Observer	Minimum-Order State Observer
$\tilde{\mathbf{x}}$	$\tilde{\mathbf{x}}_b$
\mathbf{A}	\mathbf{A}_{bb}
$\mathbf{B}u$	$\mathbf{A}_{ba}x_a + \mathbf{B}_b u$
y	$\dot{x}_a - A_{aa}x_a - B_a u$
\mathbf{C}	\mathbf{A}_{ab}
\mathbf{K}_e ($n \times 1$ matrix)	\mathbf{K}_e [$(n - 1) \times 1$ matrix]

To avoid this difficulty, we eliminate \dot{x}_a in the following way. First rewrite Equation (10–86) as

$$\begin{aligned}\tilde{\mathbf{x}}_b - \mathbf{K}_e \dot{x}_a &= (\mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab}) \tilde{\mathbf{x}}_b + (\mathbf{A}_{ba} - \mathbf{K}_e A_{aa})y + (\mathbf{B}_b - \mathbf{K}_e B_a)u \\ &= (\mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab})(\tilde{\mathbf{x}}_b - \mathbf{K}_e y) \\ &\quad + [(\mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab})\mathbf{K}_e + \mathbf{A}_{ba} - \mathbf{K}_e A_{aa}]y \\ &\quad + (\mathbf{B}_b - \mathbf{K}_e B_a)u\end{aligned}\quad (10-87)$$

Define

$$\mathbf{x}_b - \mathbf{K}_e y = \mathbf{x}_b - \mathbf{K}_e x_a = \boldsymbol{\eta}$$

and

$$\tilde{\mathbf{x}}_b - \mathbf{K}_e y = \tilde{\mathbf{x}}_b - \mathbf{K}_e x_a = \tilde{\boldsymbol{\eta}} \quad (10-88)$$

Then Equation (10–87) becomes

$$\begin{aligned}\dot{\tilde{\boldsymbol{\eta}}} &= (\mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab})\tilde{\boldsymbol{\eta}} + [(\mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab})\mathbf{K}_e \\ &\quad + \mathbf{A}_{ba} - \mathbf{K}_e A_{aa}]y + (\mathbf{B}_b - \mathbf{K}_e B_a)u\end{aligned}\quad (10-89)$$

Define

$$\begin{aligned}\hat{\mathbf{A}} &= \mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab} \\ \hat{\mathbf{B}} &= \hat{\mathbf{A}}\mathbf{K}_e + \mathbf{A}_{ba} - \mathbf{K}_e A_{aa} \\ \hat{\mathbf{F}} &= \mathbf{B}_b - \mathbf{K}_e B_a\end{aligned}$$

Then Equation (10–89) becomes

$$\dot{\tilde{\boldsymbol{\eta}}} = \hat{\mathbf{A}}\tilde{\boldsymbol{\eta}} + \hat{\mathbf{B}}y + \hat{\mathbf{F}}u \quad (10-90)$$

Equation (10–90) and Equation (10–88) together define the minimum-order observer.

Since

$$\begin{aligned}y &= \begin{bmatrix} 1 & \vdots & \mathbf{0} \end{bmatrix} \begin{bmatrix} x_a \\ \mathbf{x}_b \end{bmatrix} \\ \tilde{\mathbf{x}} &= \begin{bmatrix} x_a \\ \tilde{\mathbf{x}}_b \end{bmatrix} = \begin{bmatrix} y \\ \tilde{\mathbf{x}}_b \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{n-1} \end{bmatrix} [\tilde{\mathbf{x}}_b - \mathbf{K}_e y] + \begin{bmatrix} 1 \\ \mathbf{K}_e \end{bmatrix} y\end{aligned}$$

where $\mathbf{0}$ is a row vector consisting of $(n - 1)$ zeros, if we define

$$\hat{\mathbf{C}} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{n-1} \end{bmatrix}, \quad \hat{\mathbf{D}} = \begin{bmatrix} 1 \\ \mathbf{K}_e \end{bmatrix}$$

then we can write $\tilde{\mathbf{x}}$ in terms of $\tilde{\boldsymbol{\eta}}$ and y as follows:

$$\tilde{\mathbf{x}} = \hat{\mathbf{C}}\tilde{\boldsymbol{\eta}} + \hat{\mathbf{D}}y \quad (10-91)$$

This equation gives the transformation from $\tilde{\boldsymbol{\eta}}$ to $\tilde{\mathbf{x}}$.

Figure 10–17 shows the block diagram of the observed-state feedback control system with the minimum-order observer, based on Equations (10–79), (10–80), (10–90), (10–91) and $u = -\mathbf{K}\tilde{\mathbf{x}}$.

Next we shall derive the observer error equation. Using Equation (10–83), Equation (10–86) can be modified to

$$\dot{\tilde{\mathbf{x}}}_b = (\mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab})\tilde{\mathbf{x}}_b + \mathbf{A}_{ba}x_a + \mathbf{B}_b u + \mathbf{K}_e \mathbf{A}_{ab}\mathbf{x}_b \quad (10-92)$$

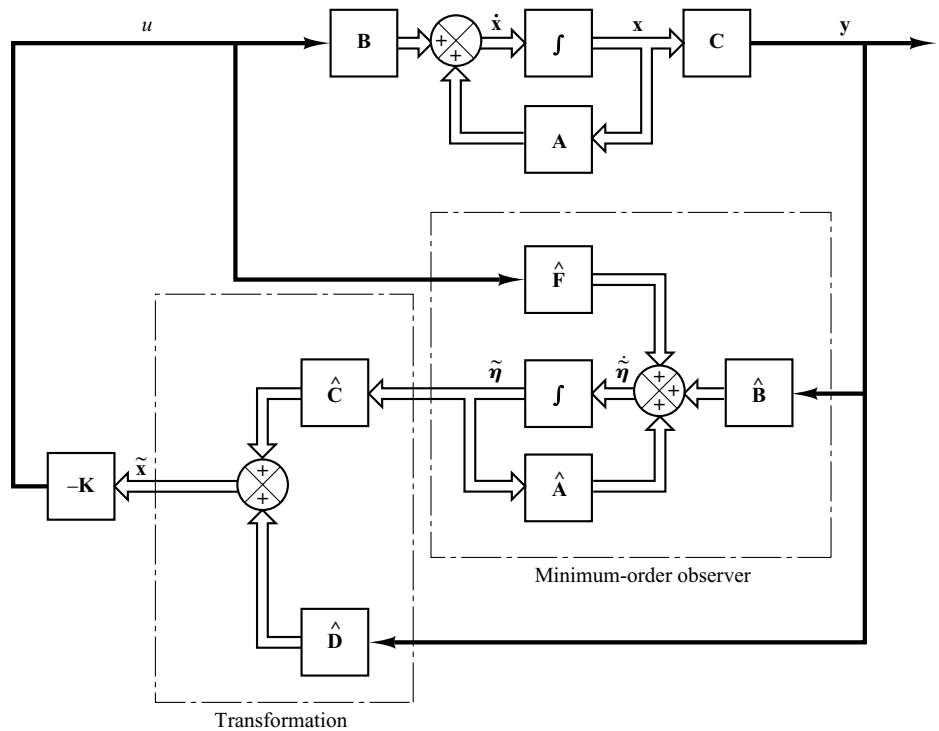


Figure 10–17
System with observed-state feedback, where the observer is the minimum-order observer.

By subtracting Equation (10–92) from Equation (10–84), we obtain

$$\dot{\mathbf{x}}_b - \dot{\tilde{\mathbf{x}}}_b = (\mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab})(\mathbf{x}_b - \tilde{\mathbf{x}}_b) \quad (10-93)$$

Define

$$\mathbf{e} = \mathbf{x}_b - \tilde{\mathbf{x}}_b = \boldsymbol{\eta} - \tilde{\boldsymbol{\eta}}$$

Then Equation (10–93) becomes

$$\dot{\mathbf{e}} = (\mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab})\mathbf{e} \quad (10-94)$$

This is the error equation for the minimum-order observer. Note that \mathbf{e} is an $(n - 1)$ -vector.

The error dynamics can be chosen as desired by following the technique developed for the full-order observer, provided that the rank of matrix

$$\begin{bmatrix} \mathbf{A}_{ab} \\ \mathbf{A}_{ab} \mathbf{A}_{bb} \\ \vdots \\ \mathbf{A}_{ab} \mathbf{A}_{bb}^{n-2} \end{bmatrix}$$

is $n - 1$. (This is the complete observability condition applicable to the minimum-order observer.)

The characteristic equation for the minimum-order observer is obtained from Equation (10–94) as follows:

$$\begin{aligned} |s\mathbf{I} - \mathbf{A}_{bb} + \mathbf{K}_e \mathbf{A}_{ab}| &= (s - \mu_1)(s - \mu_2) \cdots (s - \mu_{n-1}) \\ &= s^{n-1} + \hat{\alpha}_1 s^{n-2} + \cdots + \hat{\alpha}_{n-2} s + \hat{\alpha}_{n-1} = 0 \end{aligned} \quad (10-95)$$

where $\mu_1, \mu_2, \dots, \mu_{n-1}$ are desired eigenvalues for the minimum-order observer. The observer gain matrix \mathbf{K}_e can be determined by first choosing the desired eigenvalues for the minimum-order observer [that is, by placing the roots of the characteristic equation, Equation (10–95), at the desired locations] and then using the procedure developed for the full-order observer with appropriate modifications. For example, if the formula for determining matrix \mathbf{K}_e given by Equation (10–61) is to be used, it should be modified to

$$\mathbf{K}_e = \hat{\mathbf{Q}} \begin{bmatrix} \hat{\alpha}_{n-1} - \hat{a}_{n-1} \\ \hat{\alpha}_{n-2} - \hat{a}_{n-2} \\ \vdots \\ \hat{\alpha}_1 - \hat{a}_1 \end{bmatrix} = (\hat{\mathbf{W}}\hat{\mathbf{N}}^*)^{-1} \begin{bmatrix} \hat{\alpha}_{n-1} - \hat{a}_{n-1} \\ \hat{\alpha}_{n-2} - \hat{a}_{n-2} \\ \vdots \\ \hat{\alpha}_1 - \hat{a}_1 \end{bmatrix} \quad (10-96)$$

where \mathbf{K}_e is an $(n - 1) \times 1$ matrix and

$$\begin{aligned} \hat{\mathbf{N}} &= [\mathbf{A}_{ab}^* \mid \mathbf{A}_{bb}^* \mathbf{A}_{ab}^* \mid \cdots \mid (\mathbf{A}_{bb}^*)^{n-2} \mathbf{A}_{ab}^*] = (n - 1) \times (n - 1) \text{ matrix} \\ \hat{\mathbf{W}} &= \begin{bmatrix} \hat{a}_{n-2} & \hat{a}_{n-3} & \cdots & \hat{a}_1 & 1 \\ \hat{a}_{n-3} & \hat{a}_{n-4} & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \hat{a}_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} = (n - 1) \times (n - 1) \text{ matrix} \end{aligned}$$

Note that $\hat{a}_1, \hat{a}_2, \dots, \hat{a}_{n-2}$ are coefficients in the characteristic equation for the state equation

$$|s\mathbf{I} - \mathbf{A}_{bb}| = s^{n-1} + \hat{a}_1 s^{n-2} + \cdots + \hat{a}_{n-2} s + \hat{a}_{n-1} = 0$$

Also, if Ackermann's formula given by Equation (10–65) is to be used, then it should be modified to

$$\mathbf{K}_e = \phi(\mathbf{A}_{bb}) \begin{bmatrix} \mathbf{A}_{ab} \\ \mathbf{A}_{ab} \mathbf{A}_{bb} \\ \vdots \\ \mathbf{A}_{ab} \mathbf{A}_{bb}^{n-3} \\ \mathbf{A}_{ab} \mathbf{A}_{bb}^{n-2} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad (10-97)$$

where

$$\phi(\mathbf{A}_{bb}) = \mathbf{A}_{bb}^{n-1} + \hat{\alpha}_1 \mathbf{A}_{bb}^{n-2} + \cdots + \hat{\alpha}_{n-2} \mathbf{A}_{bb} + \hat{\alpha}_{n-1} \mathbf{I}$$

Observed-State Feedback Control System with Minimum-Order Observer. For the case of the observed-state feedback control system with full-order state observer, we have shown that the closed-loop poles of the observed-state feedback control system consist of the poles due to the pole-placement design alone, plus the poles due to the observer design alone. Hence, the pole-placement design and the full-order observer design are independent of each other.

For the observed-state feedback control system with minimum-order observer, the same conclusion applies. The system characteristic equation can be derived as

$$|s\mathbf{I} - \mathbf{A} + \mathbf{BK}| |s\mathbf{I} - \mathbf{A}_{bb} + \mathbf{K}_e \mathbf{A}_{ab}| = 0 \quad (10-98)$$

(See Problem **A-10-11** for the details.) The closed-loop poles of the observed-state feedback control system with a minimum-order observer comprise the closed-loop poles due to pole placement [the eigenvalues of matrix $(\mathbf{A} - \mathbf{BK})$] and the closed-loop poles due to the minimum-order observer [the eigenvalues of matrix $(\mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab})$]. Therefore, the pole-placement design and the design of the minimum-order observer are independent of each other.

Determining Observer Gain Matrix \mathbf{K}_e with MATLAB. Because of the duality of pole-placement and observer design, the same algorithm can be applied to both the pole-placement problem and the observer-design problem. Thus, the commands `acker` and `place` can be used to determine the observer gain matrix \mathbf{K}_e .

The closed-loop poles of the observer are the eigenvalues of matrix $\mathbf{A} - \mathbf{K}_e \mathbf{C}$. The closed-loop poles of the pole-placement are the eigenvalues of matrix $\mathbf{A} - \mathbf{BK}$.

Referring to the duality problem between the pole-placement problem and observer-design problem, we can determine \mathbf{K}_e by considering the pole-placement problem for the dual system. That is, we determine \mathbf{K}_e by placing the eigenvalues of $\mathbf{A}^* - \mathbf{C}^* \mathbf{K}_e$ at the desired place. Since $\mathbf{K}_e = \mathbf{K}^*$, for the full-order observer we use the command

$$\mathbf{K}_e = \text{acker}(\mathbf{A}', \mathbf{C}', \mathbf{L})'$$

where \mathbf{L} is the vector of the desired eigenvalues for the observer. Similarly, for the full-order observer, we may use

$$\mathbf{K}_e = \text{place}(\mathbf{A}', \mathbf{C}', \mathbf{L})'$$

provided \mathbf{L} does not include multiple poles. [In the above commands, prime (') indicates the transpose.] For the minimum-order (or reduced-order) observers, use the following commands:

$$\mathbf{K}_e = \text{acker}(\mathbf{A}_{bb}', \mathbf{A}_{ab}', \mathbf{L})'$$

or

$$\mathbf{K}_e = \text{place}(\mathbf{A}_{bb}', \mathbf{A}_{ab}', \mathbf{L})'$$

EXAMPLE 10–8 Consider the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [1 \ 0 \ 0]$$

Let us assume that we want to place the closed-loop poles at

$$s_1 = -2 + j2\sqrt{3}, \quad s_2 = -2 - j2\sqrt{3}, \quad s_3 = -6$$

Then the necessary state-feedback gain matrix \mathbf{K} can be obtained as follows:

$$\mathbf{K} = [90 \ 29 \ 4]$$

(See MATLAB Program 10–10 for a MATLAB computation of this matrix \mathbf{K} .)

Next, let us assume that the output y can be measured accurately so that state variable x_1 (which is equal to y) need not be estimated. Let us design a minimum-order observer. (The minimum-order observer is of second order.) Assume that we choose the desired observer poles to be at

$$s = -10, \quad s = -10$$

Referring to Equation (10–95), the characteristic equation for the minimum-order observer is

$$\begin{aligned} |s\mathbf{I} - \mathbf{A}_{bb} + \mathbf{K}_e\mathbf{A}_{ab}| &= (s - \mu_1)(s - \mu_2) \\ &= (s + 10)(s + 10) \\ &= s^2 + 20s + 100 = 0 \end{aligned}$$

In what follows, we shall use Ackermann's formula given by Equation (10–97).

$$\mathbf{K}_e = \phi(\mathbf{A}_{bb}) \begin{bmatrix} \mathbf{A}_{ab} \\ \dots \\ \mathbf{A}_{ab} \mathbf{A}_{bb} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (10-99)$$

where

$$\phi(\mathbf{A}_{bb}) = \mathbf{A}_{bb}^2 + \hat{\alpha}_1 \mathbf{A}_{bb} + \hat{\alpha}_2 \mathbf{I} = \mathbf{A}_{bb}^2 + 20\mathbf{A}_{bb} + 100\mathbf{I}$$

Since

$$\tilde{\mathbf{x}} = \begin{bmatrix} x_a \\ \dots \\ \tilde{\mathbf{x}}_b \end{bmatrix} = \begin{bmatrix} x_1 \\ \dots \\ \tilde{x}_2 \\ \dots \\ \tilde{x}_3 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ \dots & \dots & \dots \\ 0 & 0 & 1 \\ \dots & \dots & \dots \\ -6 & -11 & -6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ \dots \\ 0 \\ \dots \\ 1 \end{bmatrix}$$

we have

$$\begin{aligned} A_{aa} &= 0, & \mathbf{A}_{ab} &= [1 \ 0], & \mathbf{A}_{ba} &= \begin{bmatrix} 0 \\ -6 \end{bmatrix} \\ \mathbf{A}_{bb} &= \begin{bmatrix} 0 & 1 \\ -11 & -6 \end{bmatrix}, & B_a &= 0, & \mathbf{B}_b &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

Equation (10–99) now becomes

$$\begin{aligned}\mathbf{K}_e &= \left\{ \begin{bmatrix} 0 & 1 \\ -11 & -6 \end{bmatrix}^2 + 20 \begin{bmatrix} 0 & 1 \\ -11 & -6 \end{bmatrix} + 100 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 89 & 14 \\ -154 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 14 \\ 5 \end{bmatrix}\end{aligned}$$

(A MATLAB computation of this \mathbf{K}_e is given in MATLAB Program 10–10.)

MATLAB Program 10–10

```
A = [0 1 0;0 0 1;-6 -11 -6];
B = [0;0;1];
J = [-2+j*2*sqrt(3) -2-j*2*sqrt(3) -6];
K = acker(A,B,J)

K =

    90.0000    29.0000    4.0000

Abb = [0 1;-11 -6];
Aab = [1 0];
L = [-10 -10];
Ke = acker(Abb',Aab',L)'

Ke =

    14
     5
```

Referring to Equations (10–88) and (10–89), the equation for the minimum-order observer can be given by

$$\dot{\tilde{\boldsymbol{\eta}}}_2 = (\mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab}) \tilde{\boldsymbol{\eta}}_2 + [(\mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab}) \mathbf{K}_e + \mathbf{A}_{ba} - \mathbf{K}_e \mathbf{A}_{aa}] y + (\mathbf{B}_b - \mathbf{K}_e \mathbf{B}_a) u \quad (10-100)$$

where

$$\tilde{\boldsymbol{\eta}}_2 = \tilde{\mathbf{x}}_b - \mathbf{K}_e y = \tilde{\mathbf{x}}_b - \mathbf{K}_e x_1$$

Noting that

$$\mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab} = \begin{bmatrix} 0 & 1 \\ -11 & -6 \end{bmatrix} - \begin{bmatrix} 14 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -14 & 1 \\ -16 & -6 \end{bmatrix}$$

the equation for the minimum-order observer, Equation (10–100), becomes

$$\begin{aligned}\begin{bmatrix} \dot{\tilde{\eta}}_2 \\ \dot{\tilde{\eta}}_3 \end{bmatrix} &= \begin{bmatrix} -14 & 1 \\ -16 & -6 \end{bmatrix} \begin{bmatrix} \tilde{\eta}_2 \\ \tilde{\eta}_3 \end{bmatrix} + \left\{ \begin{bmatrix} -14 & 1 \\ -16 & -6 \end{bmatrix} \begin{bmatrix} 14 \\ 5 \end{bmatrix} \right. \\ &\quad \left. + \begin{bmatrix} 0 \\ -6 \end{bmatrix} - \begin{bmatrix} 14 \\ 5 \end{bmatrix} 0 \right\} y + \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 14 \\ 5 \end{bmatrix} 0 \right\} u\end{aligned}$$

or

$$\begin{bmatrix} \dot{\tilde{\eta}}_2 \\ \dot{\tilde{\eta}}_3 \end{bmatrix} = \begin{bmatrix} -14 & 1 \\ -16 & -6 \end{bmatrix} \begin{bmatrix} \tilde{\eta}_2 \\ \tilde{\eta}_3 \end{bmatrix} + \begin{bmatrix} -191 \\ -260 \end{bmatrix} y + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

where

$$\begin{bmatrix} \tilde{\eta}_2 \\ \tilde{\eta}_3 \end{bmatrix} = \begin{bmatrix} \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} - \mathbf{K}_e y$$

or

$$\begin{bmatrix} \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} = \begin{bmatrix} \tilde{\eta}_2 \\ \tilde{\eta}_3 \end{bmatrix} + \mathbf{K}_e x_1$$

If the observed-state feedback is used, then the control signal u becomes

$$u = -\mathbf{K}\tilde{\mathbf{x}} = -\mathbf{K} \begin{bmatrix} x_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix}$$

where \mathbf{K} is the state feedback gain matrix. Figure 10–18 is a block diagram showing the configuration of the system with observed-state feedback, where the observer is the minimum-order observer.

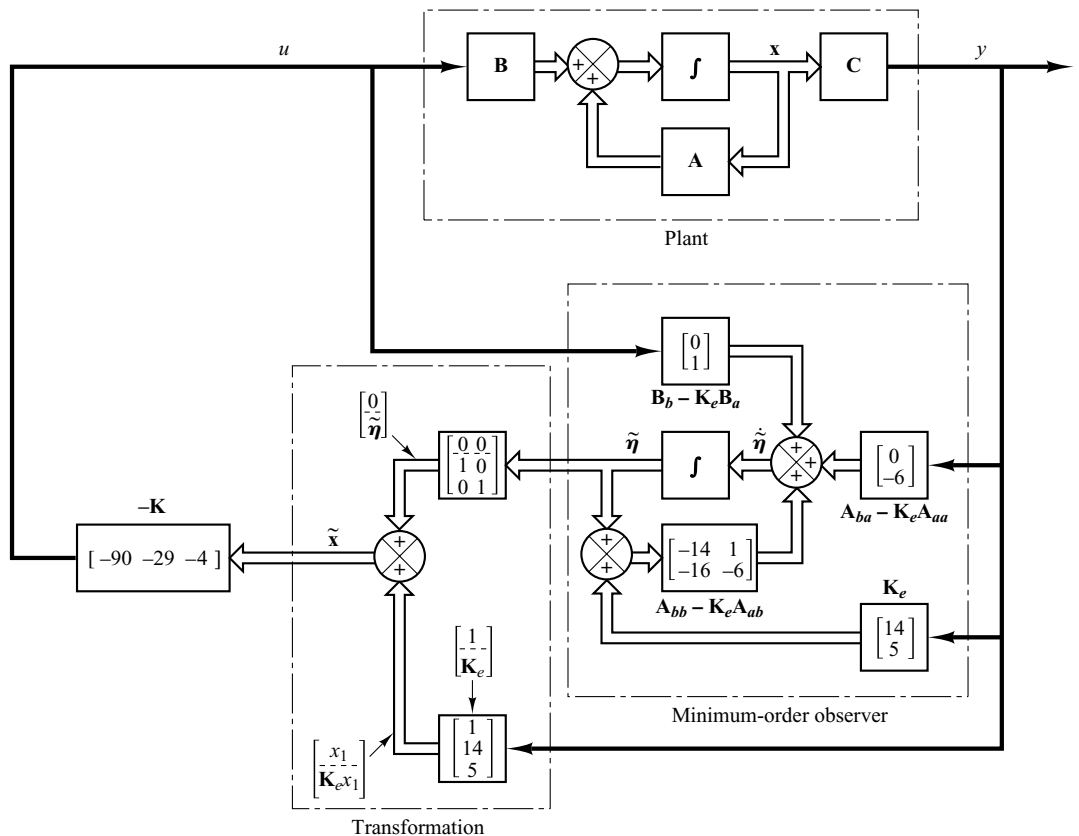


Figure 10–18

System with observed state feedback, where the observer is the minimum-order observer designed in Example 10–8.

Transfer Function of Minimum-Order Observer-Based Controller. In the minimum-order observer equation given by Equation (10–89):

$$\tilde{\boldsymbol{\eta}} = (\mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab})\tilde{\boldsymbol{\eta}} + [(\mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab})\mathbf{K}_e + \mathbf{A}_{ba} - \mathbf{K}_e \mathbf{A}_{aa}]y + (\mathbf{B}_b - \mathbf{K}_e \mathbf{B}_a)u$$

define, similar to the case of the derivation of Equation (10–90),

$$\hat{\mathbf{A}} = \mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab}$$

$$\hat{\mathbf{B}} = \hat{\mathbf{A}}\mathbf{K}_e + \mathbf{A}_{ba} - \mathbf{K}_e \mathbf{A}_{aa}$$

$$\hat{\mathbf{F}} = \mathbf{B}_b - \mathbf{K}_e \mathbf{B}_a$$

Then, the following three equations define the minimum-order observer:

$$\dot{\tilde{\boldsymbol{\eta}}} = \hat{\mathbf{A}}\tilde{\boldsymbol{\eta}} + \hat{\mathbf{B}}y + \hat{\mathbf{F}}u \quad (10-101)$$

$$\tilde{\boldsymbol{\eta}} = \tilde{\mathbf{x}}_b - \mathbf{K}_e y \quad (10-102)$$

$$u = -\mathbf{K}\tilde{\mathbf{x}} \quad (10-103)$$

Since Equation (10–103) can be rewritten as

$$\begin{aligned} u = -\mathbf{K}\tilde{\mathbf{x}} &= -[\mathbf{K}_a \quad \mathbf{K}_b] \begin{bmatrix} y \\ \tilde{\mathbf{x}}_b \end{bmatrix} = -\mathbf{K}_a y - \mathbf{K}_b \tilde{\mathbf{x}}_b \\ &= -\mathbf{K}_b \tilde{\boldsymbol{\eta}} - (\mathbf{K}_a + \mathbf{K}_b \mathbf{K}_e)y \end{aligned} \quad (10-104)$$

by substituting Equation (10–104) into Equation (10–101), we obtain

$$\begin{aligned} \dot{\tilde{\boldsymbol{\eta}}} &= \hat{\mathbf{A}}\tilde{\boldsymbol{\eta}} + \hat{\mathbf{B}}y + \hat{\mathbf{F}}[-\mathbf{K}_b \tilde{\boldsymbol{\eta}} - (\mathbf{K}_a + \mathbf{K}_b \mathbf{K}_e)y] \\ &= (\hat{\mathbf{A}} - \hat{\mathbf{F}}\mathbf{K}_b)\tilde{\boldsymbol{\eta}} + [\hat{\mathbf{B}} - \hat{\mathbf{F}}(\mathbf{K}_a + \mathbf{K}_b \mathbf{K}_e)]y \end{aligned} \quad (10-105)$$

Define

$$\tilde{\mathbf{A}} = \hat{\mathbf{A}} - \hat{\mathbf{F}}\mathbf{K}_b$$

$$\tilde{\mathbf{B}} = \hat{\mathbf{B}} - \hat{\mathbf{F}}(\mathbf{K}_a + \mathbf{K}_b \mathbf{K}_e)$$

$$\tilde{\mathbf{C}} = -\mathbf{K}_b$$

$$\tilde{\mathbf{D}} = -(\mathbf{K}_a + \mathbf{K}_b \mathbf{K}_e)$$

Then Equations (10–105) and (10–104) can be written as

$$\dot{\tilde{\boldsymbol{\eta}}} = \tilde{\mathbf{A}}\tilde{\boldsymbol{\eta}} + \tilde{\mathbf{B}}y \quad (10-106)$$

$$u = \tilde{\mathbf{C}}\tilde{\boldsymbol{\eta}} + \tilde{\mathbf{D}}y \quad (10-107)$$

Equations (10–106) and (10–107) define the minimum-order observer-based controller. By considering u as the output and $-y$ as the input, $U(s)$ can be written as

$$\begin{aligned} U(s) &= [\tilde{\mathbf{C}}(s\mathbf{I} - \tilde{\mathbf{A}})^{-1}\tilde{\mathbf{B}} + \tilde{\mathbf{D}}]Y(s) \\ &= -[\tilde{\mathbf{C}}(s\mathbf{I} - \tilde{\mathbf{A}})^{-1}\tilde{\mathbf{B}} + \tilde{\mathbf{D}}][-Y(s)] \end{aligned}$$

Since the input to the observer controller is $-Y(s)$, rather than $Y(s)$, the transfer function of the observer controller is

$$\frac{U(s)}{-Y(s)} = \frac{\text{num}}{\text{den}} = -[\tilde{\mathbf{C}}(s\mathbf{I} - \tilde{\mathbf{A}})^{-1}\tilde{\mathbf{B}} + \tilde{\mathbf{D}}] \quad (10-108)$$

This transfer function can be easily obtained by using the following MATLAB statement:

$$[\text{num}, \text{den}] = \text{ss2tf}(\text{Atilde}, \text{Btilde}, -\text{Ctilde}, -\text{Dtilde}) \quad (10-109)$$

10-6 DESIGN OF REGULATOR SYSTEMS WITH OBSERVERS

In this section we shall consider a problem of designing regulator systems by using the pole-placement-with-observer approach.

Consider the regulator system shown in Figure 10-19. (The reference input is zero.) The plant transfer function is

$$G(s) = \frac{10(s + 2)}{s(s + 4)(s + 6)}$$

Using the pole-placement approach, design a controller such that when the system is subjected to the following initial condition:

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

where \mathbf{x} is the state vector for the plant and \mathbf{e} is the observer error vector, the maximum undershoot of $y(t)$ is 25 to 35% and the settling time is about 4 sec. Assume that we use the minimum-order observer. (We assume that only the output y is measurable.)

We shall use the following design procedure:

1. Derive a state-space model of the plant.
2. Choose the desired closed-loop poles for pole placement. Choose the desired observer poles.
3. Determine the state feedback gain matrix \mathbf{K} and the observer gain matrix \mathbf{K}_e .
4. Using the gain matrices \mathbf{K} and \mathbf{K}_e obtained in step 3, derive the transfer function of the observer controller. If it is a stable controller, check the response to the given initial condition. If the response is not acceptable, adjust the closed-loop pole location and/or observer pole location until an acceptable response is obtained.

Design step 1: We shall derive the state-space representation of the plant. Since the plant transfer function is

$$\frac{Y(s)}{U(s)} = \frac{10(s + 2)}{s(s + 4)(s + 6)}$$

the corresponding differential equation is

$$\ddot{y} + 10\dot{y} + 24y = 10\dot{u} + 20u$$

Referring to Section 2-5, let us define the state variables x_1 , x_2 , and x_3 as follows:

$$\begin{aligned} x_1 &= y - \beta_0 u \\ x_2 &= \dot{x}_1 - \beta_1 u \\ x_3 &= \dot{x}_2 - \beta_2 u \end{aligned}$$

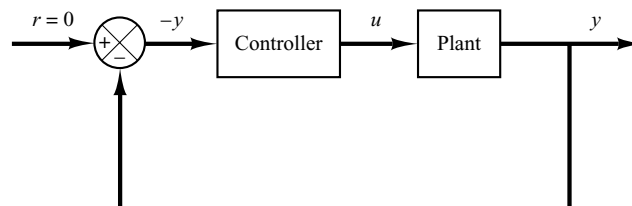


Figure 10-19
Regulator system.

Also, \dot{x}_3 is defined by

$$\begin{aligned}\dot{x}_3 &= -a_3x_1 - a_2x_2 - a_1x_3 + \beta_3u \\ &= -24x_2 - 10x_3 + \beta_3u\end{aligned}$$

where $\beta_0 = 0$, $\beta_1 = 0$, $\beta_2 = 10$, and $\beta_3 = -80$.

[See Equation (2-35) for the calculation of β 's.] Then the state-space equation and output equation can be obtained as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -24 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 10 \\ -80 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u$$

Design step 2: As the first trial, let us choose the desired closed-loop poles at

$$s = -1 + j2, \quad s = -1 - j2, \quad s = -5$$

and choose the desired observer poles at

$$s = -10, \quad s = -10$$

Design step 3: We shall use MATLAB to compute the state feedback gain matrix \mathbf{K} and the observer gain matrix \mathbf{K}_e . MATLAB Program 10-11 produces matrices \mathbf{K} and \mathbf{K}_e .

MATLAB Program 10-11

```
% Obtaining the state feedback gain matrix K
```

```
A = [0 1 0; 0 0 1; 0 -24 -10];
```

```
B = [0; 10; -80];
```

```
C = [1 0 0];
```

```
J = [-1+j*2 -1-j*2 -5];
```

```
K = acker(A,B,J)
```

```
K =
```

```
1.2500 1.2500 0.19375
```

```
% Obtaining the observer gain matrix Ke
```

```
Aaa = 0; Aab = [1 0]; Aba = [0;0]; Abb = [0 1;-24 -10]; Ba = 0; Bb = [10;-80];
```

```
L = [-10 -10];
```

```
Ke = acker(Abb',Aab',L)'
```

```
Ke =
```

```
10
```

```
-24
```

In the program, matrices **J** and **L** represent the desired closed-loop poles for pole placement and the desired poles for the observer, respectively. The matrices **K** and **K_e** are obtained as

$$\mathbf{K} = [1.25 \quad 1.25 \quad 0.19375]$$

$$\mathbf{K}_e = \begin{bmatrix} 10 \\ -24 \end{bmatrix}$$

Design step 4: We shall determine the transfer function of the observer controller. Referring to Equation (10–108), the transfer function of the observer controller can be given by

$$G_c(s) = \frac{U(s)}{-Y(s)} = \frac{\text{num}}{\text{den}} = -[\tilde{\mathbf{C}}(s\mathbf{I} - \tilde{\mathbf{A}})^{-1}\tilde{\mathbf{B}} + \tilde{\mathbf{D}}]$$

We shall use MATLAB to calculate the transfer function of the observer controller. MATLAB Program 10–12 produces this transfer function. The result is

$$G_c(s) = \frac{9.1s^2 + 73.5s + 125}{s^2 + 17s - 30}$$

$$= \frac{9.1(s + 5.6425)(s + 2.4344)}{(s + 18.6119)(s - 1.6119)}$$

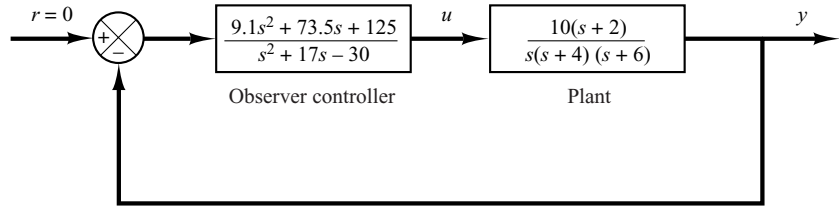
Define the system with this observer controller as System 1. Figure 10–20 shows the block diagram of System 1.

MATLAB Program 10–12

```
% Determination of transfer function of observer controller
A = [0 1 0;0 0 1;0 -24 -10];
B = [0;10;-80];
Aaa = 0; Aab = [1 0]; Aba = [0;0]; Abb = [0 1;-24 -10];
Ba = 0; Bb = [10;-80];
Ka = 1.25; Kb = [1.25 0.19375];
Ke = [10;-24];
Ahat = Abb - Ke*Aab;
Bhat = Ahat*Ke + Aba - Ke*Aaa;
Fhat = Bb - Ke*Ba;
Atilde = Ahat - Fhat*Kb;
Btilde = Bhat - Fhat*(Ka + Kb*Ke);
Ctilde = -Kb;
Dtilde = -(Ka + Kb*Ke);
[num,den] = ss2tf(Atilde, Btilde, -Ctilde, -Dtilde)

num =
    9.1000 73.5000 125.0000
den =
    1.0000 17.0000 -30.0000
```


Figure 10–20
Block diagram of
System 1.



The observer controller has a pole in the right-half s plane ($s = 1.6119$). The existence of an open-loop right-half s plane pole in the observer controller means that the system is open-loop unstable, although the closed-loop system is stable. The latter can be seen from the characteristic equation for the system:

$$\begin{aligned} |s\mathbf{I} - \mathbf{A} + \mathbf{BK}| \cdot |s\mathbf{I} - \mathbf{A}_{bb} + \mathbf{K}_e \mathbf{A}_{ab}| \\ = s^5 + 27s^4 + 255s^3 + 1025s^2 + 2000s + 2500 \\ = (s + 1 + j2)(s + 1 - j2)(s + 5)(s + 10)(s + 10) = 0 \end{aligned}$$

(See MATLAB Program 10–13 for the calculation of the characteristic equation.)

A disadvantage of using an unstable controller is that the system becomes unstable if the dc gain of the system becomes small. Such a control system is neither desirable nor acceptable. Hence, to get a satisfactory system, we need to modify the closed-loop pole location and/or observer pole location.

MATLAB Program 10–13

```
% Obtaining the characteristic equation
[num1,den1] = ss2tf(A-B*K,eye(3),eye(3),eye(3),1);
[num2,den2] = ss2tf(Abb-Ke*Aab,eye(2),eye(2),eye(2),1);
charact_eq = conv(den1,den2)

charact_eq =
    1.0e+003*
    0.0010  0.0270  0.2550  1.0250  2.0000  2.5000
```

Second trial: Let us keep the desired closed-loop poles for pole placement as before, but modify the observer pole locations as follows:

$$s = -4.5, \quad s = -4.5$$

Thus,

$$\mathbf{L} = [-4.5 \quad -4.5]$$

Using MATLAB, we find the new \mathbf{K}_e to be

$$\mathbf{K}_e = \begin{bmatrix} -1 \\ 6.25 \end{bmatrix}$$

Next, we shall obtain the transfer function of the observer controller. MATLAB Program 10–14 produces this transfer function as follows:

$$G_c(s) = \frac{1.2109s^2 + 11.2125s + 25.3125}{s^2 + 6s + 2.1406}$$

$$= \frac{1.2109(s + 5.3582)(s + 3.9012)}{(s + 5.619)(s + 0.381)}$$

MATLAB Program 10–14

```
% Determination of transfer function of observer controller.
A = [0 1 0;0 0 1;0 -24 -10];
B = [0;10;-80];
Aaa = 0; Aab = [1 0]; Aba = [0;0]; Abb = [0 1;-24 -10];
Ba = 0; Bb = [10;-80];
Ka = 1.25; Kb = [1.25 0.19375];
Ke = [-1;6.25];
Ahat = Abb - Ke*Aab;
Bhat = Ahat*Ke + Aba - Ke*Aaa;
Fhat = Bb - Ke*Ba;
Atilde = Ahat - Fhat*Kb;
Btilde = Bhat - Fhat*(Ka + Kb*Ke);
Ctilde = -Kb;
Dtilde = -(Ka + Kb*Ke);
[num,den] = ss2tf(Atilde,Btilde,-Ctilde,-Dtilde)

num =
    1.2109  11.2125  25.3125

den =
    1.0000  6.0000  2.1406
```

Notice that this is a stable controller. Define the system with this observer controller as System 2. We shall proceed to obtain the response of System 2 to the given initial condition:

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

By substituting $u = -\tilde{\mathbf{K}}\tilde{\mathbf{x}}$ into the state-space equation for the plant, we obtain

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{Ax} - \mathbf{BK}\tilde{\mathbf{x}} = \mathbf{Ax} - \mathbf{BK} \begin{bmatrix} x_a \\ \tilde{\mathbf{x}}_b \end{bmatrix} = \mathbf{Ax} - \mathbf{BK} \begin{bmatrix} x_a \\ \mathbf{x}_b - \mathbf{e} \end{bmatrix} \\ &= \mathbf{Ax} - \mathbf{BK} \left\{ \mathbf{x} - \begin{bmatrix} 0 \\ \mathbf{e} \end{bmatrix} \right\} = \mathbf{Ax} - \mathbf{BK}\mathbf{x} + \mathbf{B} \begin{bmatrix} K_a & \mathbf{K}_b \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{e} \end{bmatrix} \end{aligned} \quad (10-110)$$

The error equation for the minimum-order observer is

$$\dot{\mathbf{e}} = (\mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab})\mathbf{e} \quad (10-111)$$

By combining Equations (10-110) and (10-111), we get

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BK}_b \\ \mathbf{0} & \mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}$$

with the initial condition

$$\begin{bmatrix} \mathbf{x}(0) \\ \mathbf{e}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

MATLAB Program 10-15 produces the response to the given initial condition. The response curves are shown in Figure 10-21. They seem to be acceptable.

MATLAB Program 10-15

```
% Response to initial condition.
A = [0 1 0;0 0 1;0 -24 -10];
B = [0;10;-80];
K = [1.25 1.25 0.19375];
Kb = [1.25 0.19375];
Ke = [-1;6.25];
Aab = [1 0]; Abb = [0 1;-24 -10];
AA = [A-B*K B*Kb; zeros(2,3) Abb-Ke*Aab];
sys = ss(AA,eye(5),eye(5),eye(5));
t = 0:0.01:8;
x = initial(sys,[1;0;0;1;0],t);
x1 = [1 0 0 0 0]*x';
x2 = [0 1 0 0 0]*x';
x3 = [0 0 1 0 0]*x';
e1 = [0 0 0 1 0]*x';
e2 = [0 0 0 0 1]*x';

subplot(3,2,1); plot(t,x1); grid
xlabel('t (sec)'); ylabel('x1')

subplot(3,2,2); plot(t,x2); grid
xlabel('t (sec)'); ylabel('x2')

subplot(3,2,3); plot(t,x3); grid
xlabel('t (sec)'); ylabel('x3')

subplot(3,2,4); plot(t,e1); grid
xlabel('t (sec)'); ylabel('e1')

subplot(3,2,5); plot(t,e2); grid
xlabel('t (sec)'); ylabel('e2')
```

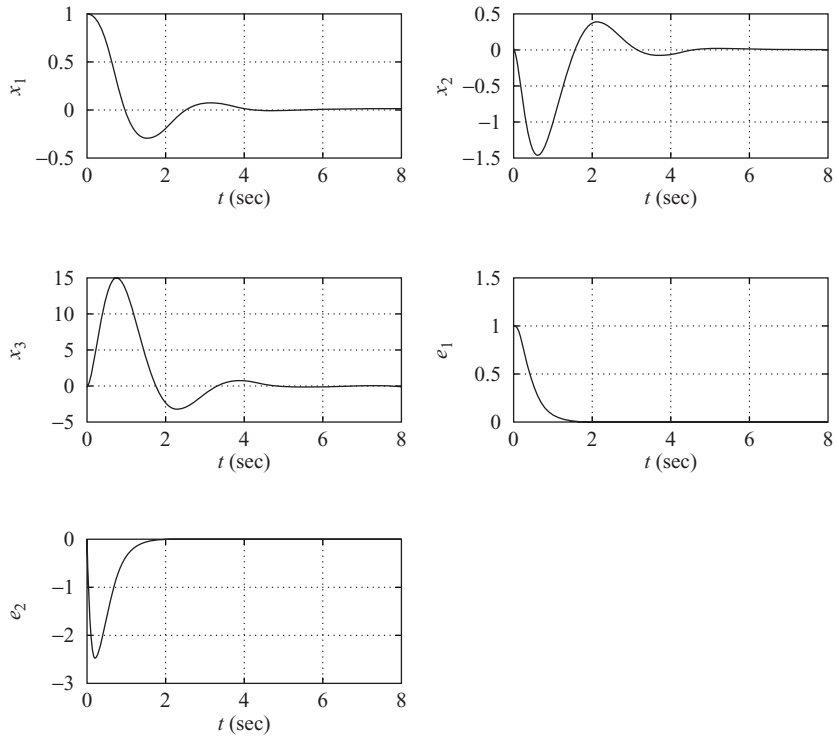


Figure 10-21
 Response to the given initial condition; $x_1(0) = 1$, $x_2(0) = 0$, $x_3(0) = 0$, $e_1(0) = 1$, $e_2(0) = 0$.

Next, we shall check the frequency-response characteristics. The Bode diagram of the open-loop system just designed is shown in Figure 10-22. The phase margin is about 40° and the gain margin is $+\infty$ dB. The Bode diagram of the closed-loop system is shown in Figure 10-23. The bandwidth of the system is approximately 3.8 rad/sec.

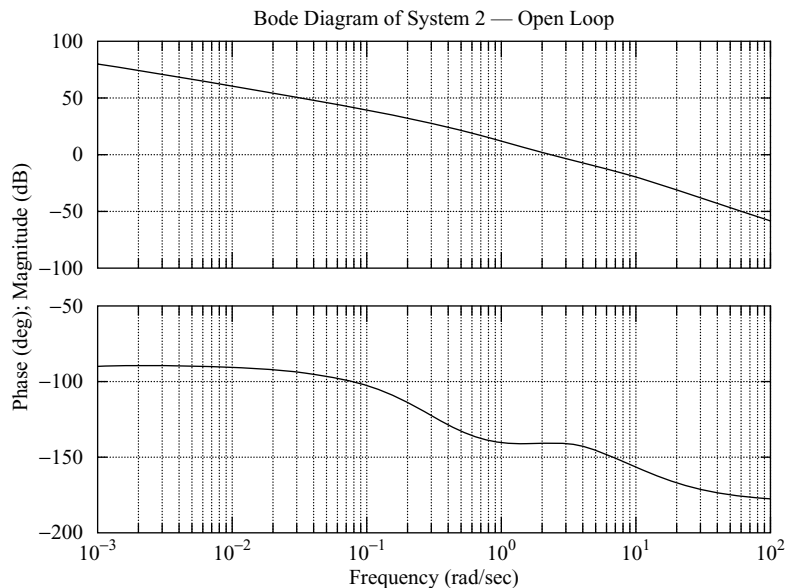


Figure 10-22
 Bode diagram for the open-loop transfer function of System 2.

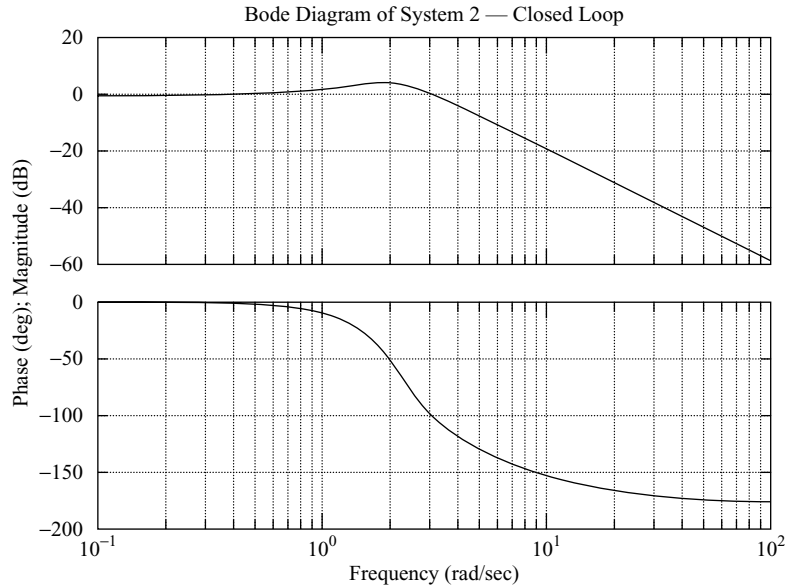


Figure 10–23
Bode diagram for the closed-loop transfer function of System 2.

Finally, we shall compare the root-locus plots of the first system with $L = [-10 \ -10]$ and the second system with $L = [-4.5 \ -4.5]$. The plot for the first system given in Figure 10–24(a) shows that the system is unstable for small dc gain and becomes stable for large dc gain. The plot for the second system given in Figure 10–24(b), on the other hand, shows that the system is stable for any positive dc gain.

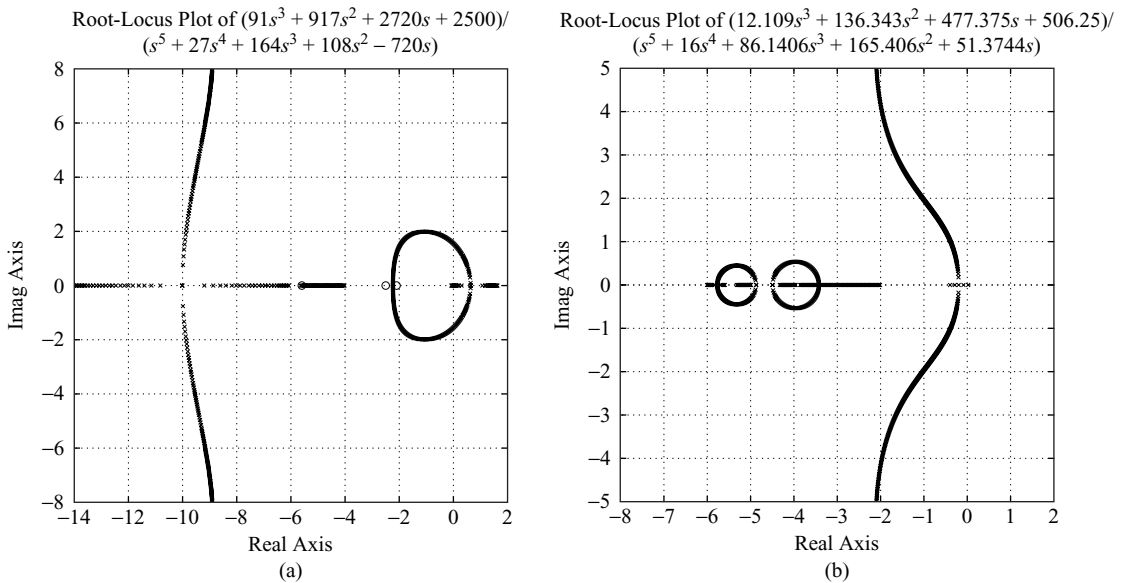


Figure 10–24
(a) Root-locus plot of the system with observer poles at $s = -10$ and $s = -10$; (b) root-locus plot of the system with observer poles at $s = -4.5$ and $s = -4.5$.

Comments

1. In designing regulator systems, note that if the dominant controller poles are placed far to the left of the $j\omega$ axis, the elements of the state feedback gain matrix \mathbf{K} will become large. Large gain values will make the actuator output become large, so that saturation may take place. Then the designed system will not behave as designed.
2. Also, by placing the observer poles far to the left of the $j\omega$ axis, the observer controller becomes unstable, although the closed-loop system is stable. An unstable observer controller is not acceptable.
3. If the observer controller becomes unstable, move the observer poles to the right in the left-half s plane until the observer controller becomes stable. Also, the desired closed-loop pole locations may need to be modified.
4. Note that if the observer poles are placed far to the left of the $j\omega$ axis, the bandwidth of the observer will increase and will cause noise problems. If there is a serious noise problem, the observer poles should not be placed too far to the left of the $j\omega$ axis. The general requirement is that the bandwidth should be sufficiently low so that the sensor noise will not become a problem.
5. The bandwidth of the system with the minimum-order observer is higher than that of the system with the full-order observer, provided that the multiple observer poles are placed at the same place for both observers. If the sensor noise is a serious problem, use of a full-order observer is recommended.

10-7 DESIGN OF CONTROL SYSTEMS WITH OBSERVERS

In Section 10-6 we discussed the design of regulator systems with observers. (The systems did not have reference or command inputs.) In this section we consider the design of control systems with observers when the systems have reference inputs or command inputs. The output of the control system must follow the input that is time varying. In following the command input, the system must exhibit satisfactory performance (a reasonable rise time, overshoot, settling time, and so on).

In this section we consider control systems that are designed by use of the pole-placement-with-observer approach. Specifically, we consider control systems using observer controllers. In Section 10-6 we discussed regulator systems, whose block diagram is shown in Figure 10-25. This system has no reference input, or $r = 0$. When the system has a reference input, several different block diagram configurations are conceivable, each having an observer controller. Two of these configurations are shown in Figures 10-26 (a) and (b); we shall consider them in this section.

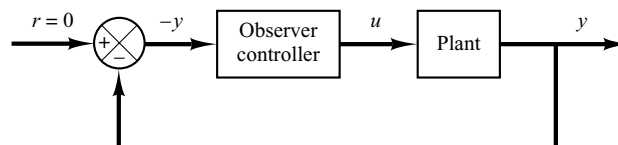
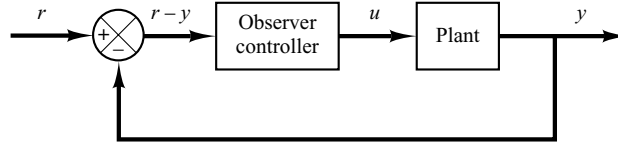


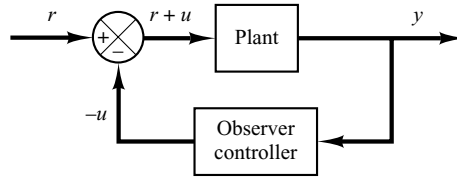
Figure 10-25
Regulator system.



(a)

Figure 10–26

(a) Control system with observer controller in the feedforward path; (b) Control system with observer controller in the feedback path.



(b)

Configuration 1: Consider the system shown in Figure 10–27. In this system the reference input is simply added at the summing point. We would like to design the observer controller such that in the unit-step response the maximum overshoot is less than 30% and the settling time is about 5 sec.

In what follows we first design a regulator system. Then, using the observer controller designed, we simply add the reference input r at the summing point.

Before we design the observer controller, we need to obtain a state-space representation of the plant. Since

$$\frac{Y(s)}{U(s)} = \frac{1}{s(s^2 + 1)}$$

we obtain

$$\ddot{y} + \dot{y} = u$$

By choosing the state variables as

$$x_1 = y$$

$$x_2 = \dot{y}$$

$$x_3 = \ddot{y}$$

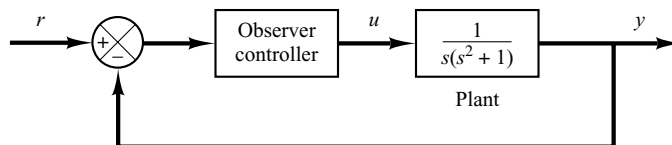
we get

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$

$$y = \mathbf{Cx}$$

Figure 10–27

Control system with observer controller in the feedforward path.



where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [1 \ 0 \ 0]$$

Next, we choose the desired closed-loop poles for pole placement at

$$s = -1 + j, \quad s = -1 - j, \quad s = -8$$

and the desired observer poles at

$$s = -4, \quad s = -4$$

The state feedback gain matrix \mathbf{K} and the observer gain matrix \mathbf{K}_e can be obtained as follows:

$$\mathbf{K} = [16 \ 17 \ 10]$$

$$\mathbf{K}_e = \begin{bmatrix} 8 \\ 15 \end{bmatrix}$$

See MATLAB Program 10–16.

MATLAB Program 10–16
<pre> A = [0 1 0;0 0 1;0 -1 0]; B = [0;0;1]; J = [-1+j -1-j -8]; K = acker(A,B,J) K = 16 17 10 Aab = [1 0]; Abb = [0 1;-1 0]; L = [-4 -4]; Ke = acker(Abb',Aab',L)' Ke = 8 15 </pre>

The transfer function of the observer controller is obtained by use of MATLAB Program 10–17. The result is

$$\begin{aligned} G_c(s) &= \frac{302s^2 + 303s + 256}{s^2 + 18s + 113} \\ &= \frac{302(s + 0.5017 + j0.772)(s + 0.5017 - j0.772)}{(s + 9 + j5.6569)(s + 9 - j5.6569)} \end{aligned}$$

MATLAB Program 10-17

```

% Determination of transfer function of observer controller
A = [0 1 0;0 0 1;0 -1 0];
B = [0;0;1];
Aaa = 0; Aab = [1 0]; Aba = [0;0]; Abb = [0 1;-1 0];
Ba = 0; Bb = [0;1];
Ka = 16; Kb=[17 10];
Ke = [8;15];
Ahat = Abb - Ke*Aab;
Bhat = Ahat*Ke + Aba - Ke*Aaa;
Fhat = Bb - Ke*Ba;
Atilde = Ahat - Fhat*Kb;
Btilde = Bhat - Fhat*(Ka + Kb*Ke);
Ctilde = -Kb;
Dtilde = -(Ka + Kb*Ke);
[num,den] = ss2tf(Atilde,Btilde,-Ctilde,-Dtilde)

num =
    302.0000  303.0000  256.0000
den =
    1  18  113

```

Figure 10-28 shows the block diagram of the regulator system just designed. Figure 10-29 shows the block diagram of a possible configuration of the control system based on the regulator system shown in Figure 10-28. The unit-step response curve for this control system is shown in Figure 10-30. The maximum overshoot is about 28% and the settling time is about 4.5 sec. Thus, the designed system satisfies the design requirements.

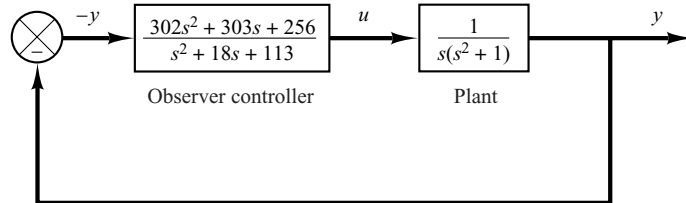


Figure 10-28
Regulator system with observer controller.

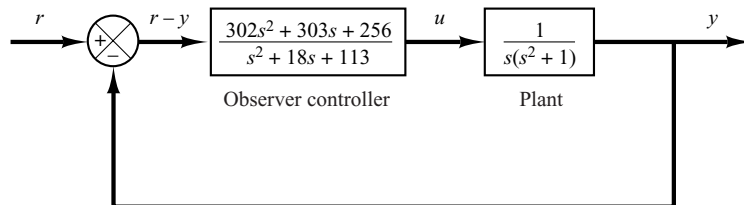


Figure 10-29
Control system with observer controller in the feedforward path.

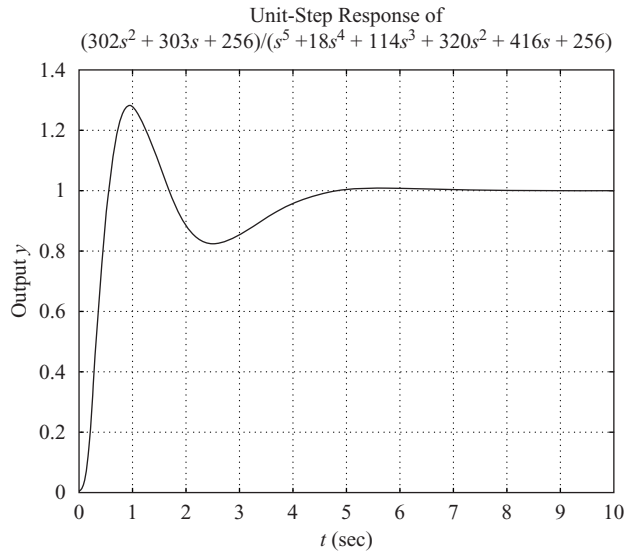


Figure 10–30
 Unit-step response of
 the control system
 shown in Figure
 10–29.

Configuration 2: A different configuration of the control system is shown in Figure 10–31. The observer controller is placed in the feedback path. The input r is introduced into the closed-loop system through the box with gain N . From this block diagram, the closed-loop transfer function is obtained as

$$\frac{Y(s)}{R(s)} = \frac{N(s^2 + 18s + 113)}{s(s^2 + 1)(s^2 + 18s + 113) + 302s^2 + 303s + 256}$$

We determine the value of constant N such that for a unit-step input r , the output y is unity as t approaches infinity. Thus we choose

$$N = \frac{256}{113} = 2.2655$$

The unit-step response of the system is shown in Figure 10–32. Notice that the maximum overshoot is very small, approximately 4%. The settling time is about 5 sec.

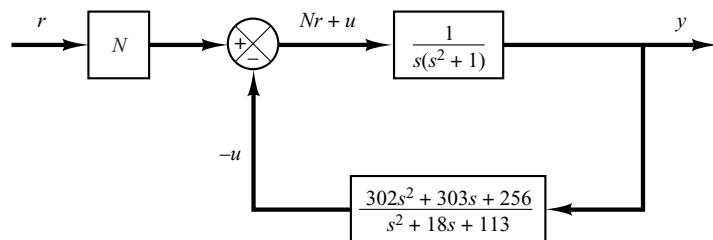
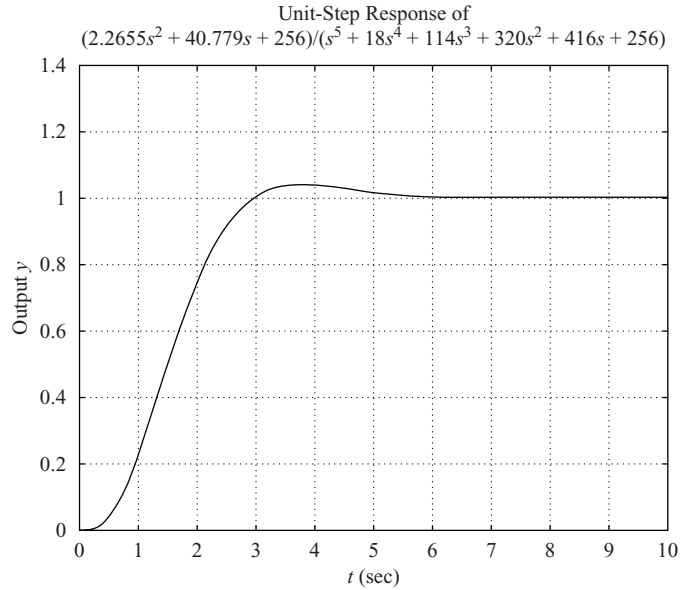


Figure 10–31
 Control system with
 observer controller
 in the feedback path.

Figure 10–32
 The unit-step response of the system shown in Figure 10–31. (The closed-loop poles for pole placement are at $s = -1 \pm j$, $s = -8$. The observer poles are at $s = -4$, $s = -4$.)

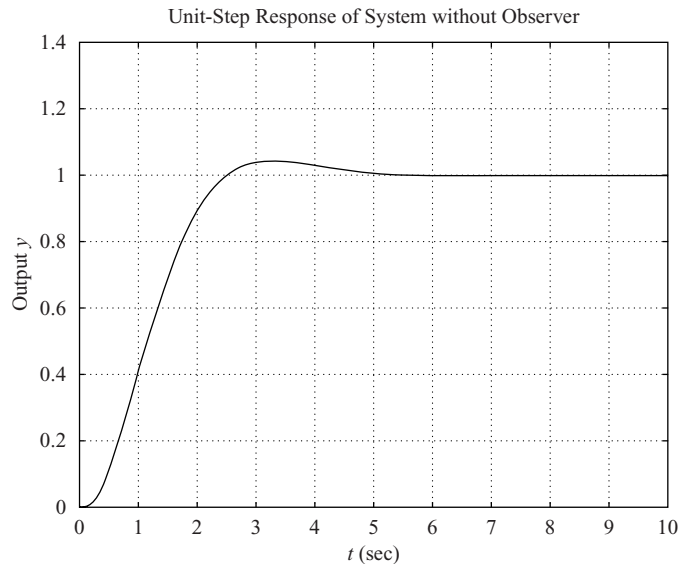


Comments. We considered two possible configurations for the closed-loop control systems using observer controllers. As stated earlier, other configurations are possible.

The first configuration, which places the observer controller in the feedforward path, generally gives a fairly large overshoot. The second configuration, which places the observer controller in the feedback path, gives a smaller overshoot. This response curve is quite similar to that of the system designed by the pole-placement approach without using the observer controller. See the unit-step response curve of the system, shown in Figure 10–33, designed by the pole-placement approach without observer. Here the desired closed-loop poles used are

$$s = -1 + j, \quad s = -1 - j, \quad s = -8$$

Figure 10–33
 The unit-step response of the control system designed by the pole placement approach without observer. (The closed-loop poles are at $s = -1 \pm j$, $s = -8$.)



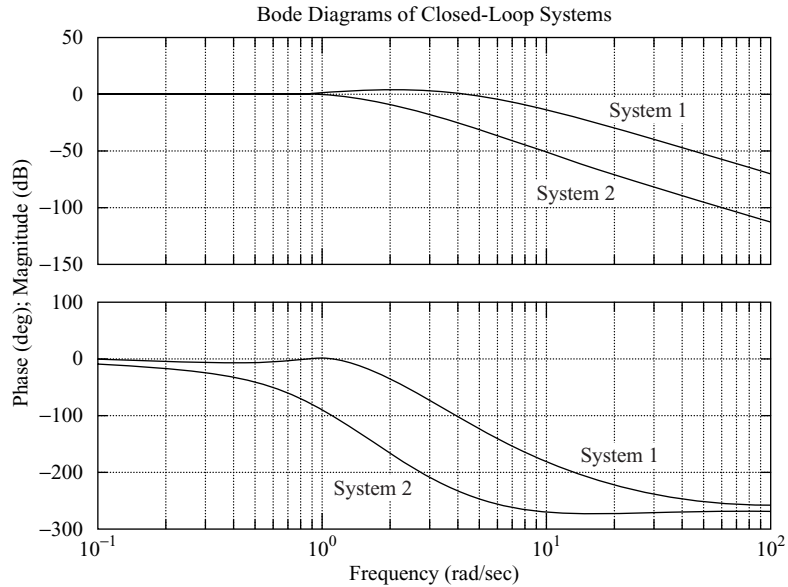


Figure 10–34
Bode diagrams of closed-loop system 1 (shown in Figure 10–29) and closed-loop system 2 (shown in Figure 10–31).

Note that, in these two systems, the rise time and settling time are determined primarily by the desired closed-loop poles for pole placement. (See Figures 10–32 and 10–33.)

The Bode diagrams of closed-loop system 1 (shown in Figure 10–29) and closed-loop system 2 (shown in Figure 10–31) are shown in Figure 10–34. From this figure, we find that the bandwidth of system 1 is 5 rad/sec and that of system 2 is 1.3 rad/sec.

Summary of State-Space Design Method

1. The state-space design method based on the pole-placement-combined-with-observer approach is very powerful. It is a time-domain method. The desired closed-loop poles can be arbitrarily placed, provided the plant is completely state controllable.
2. If not all state variables can be measured, an observer must be incorporated to estimate the unmeasurable state variables.
3. In designing a system using the pole-placement approach, several different sets of desired closed-loop poles need be considered, the response characteristics compared, and the best one chosen.
4. The bandwidth of the observer controller is generally large, because we choose observer poles far to the left in the s plane. A large bandwidth passes high-frequency noises and causes the noise problem.
5. Adding an observer to the system generally reduces the stability margin. In some cases, an observer controller may have zero(s) in the right-half s plane, which means that the controller may be stable but of nonminimum phase. In other cases, the controller may have pole(s) in the right-half s plane—that is, the controller is unstable. Then the designed system may become conditionally stable.
6. When the system is designed by the pole-placement-with-observer approach, it is advisable to check the stability margins (phase margin and gain margin), using a

frequency-response method. If the system designed has poor stability margins, it is possible that the designed system may become unstable if the mathematical model involves uncertainties.

7. Note that for n th-order systems, classical design methods (root-locus and frequency-response methods) yield low-order compensators (first or second order). Since the observer-based controllers are n th-order [or $(N - m)$ th-order if the minimum-order observer is used] for an n th-order system, the designed system will become $2n$ th order [or $(2n - m)$ th order]. Since lower-order compensators are cheaper than higher-order ones, the designer should first apply classical methods and, if no suitable compensators can be determined, then try the pole-placement-with-observer design approach presented in this chapter.

10-8 QUADRATIC OPTIMAL REGULATOR SYSTEMS

An advantage of the quadratic optimal control method over the pole-placement method is that the former provides a systematic way of computing the state feedback control gain matrix.

Quadratic Optimal Regulator Problems. We shall now consider the optimal regulator problem that, given the system equation

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \quad (10-112)$$

determines the matrix \mathbf{K} of the optimal control vector

$$\mathbf{u}(t) = -\mathbf{Kx}(t) \quad (10-113)$$

so as to minimize the performance index

$$J = \int_0^{\infty} (\mathbf{x}^* \mathbf{Qx} + \mathbf{u}^* \mathbf{Ru}) dt \quad (10-114)$$

where \mathbf{Q} is a positive-definite (or positive-semidefinite) Hermitian or real symmetric matrix and \mathbf{R} is a positive-definite Hermitian or real symmetric matrix. Note that the second term on the right-hand side of Equation (10-114) accounts for the expenditure of the energy of the control signals. The matrices \mathbf{Q} and \mathbf{R} determine the relative importance of the error and the expenditure of this energy. In this problem, we assume that the control vector $\mathbf{u}(t)$ is unconstrained.

As will be seen later, the linear control law given by Equation (10-113) is the optimal control law. Therefore, if the unknown elements of the matrix \mathbf{K} are determined so as to minimize the performance index, then $\mathbf{u}(t) = -\mathbf{Kx}(t)$ is optimal for any initial state $\mathbf{x}(0)$. The block diagram showing the optimal configuration is shown in Figure 10-35.

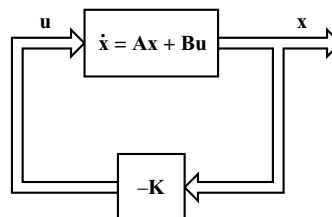


Figure 10-35
Optimal regulator system.

Now let us solve the optimization problem. Substituting Equation (10–113) into Equation (10–112), we obtain

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\mathbf{x} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}$$

In the following derivations, we assume that the matrix $\mathbf{A} - \mathbf{B}\mathbf{K}$ is stable, or that the eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ have negative real parts.

Substituting Equation (10–113) into Equation (10–114) yields

$$\begin{aligned} J &= \int_0^{\infty} (\mathbf{x}^* \mathbf{Q} \mathbf{x} + \mathbf{x}^* \mathbf{K}^* \mathbf{R} \mathbf{K} \mathbf{x}) dt \\ &= \int_0^{\infty} \mathbf{x}^* (\mathbf{Q} + \mathbf{K}^* \mathbf{R} \mathbf{K}) \mathbf{x} dt \end{aligned}$$

Let us set

$$\mathbf{x}^* (\mathbf{Q} + \mathbf{K}^* \mathbf{R} \mathbf{K}) \mathbf{x} = - \frac{d}{dt} (\mathbf{x}^* \mathbf{P} \mathbf{x})$$

where \mathbf{P} is a positive-definite Hermitian or real symmetric matrix. Then we obtain

$$\mathbf{x}^* (\mathbf{Q} + \mathbf{K}^* \mathbf{R} \mathbf{K}) \mathbf{x} = -\dot{\mathbf{x}}^* \mathbf{P} \mathbf{x} - \mathbf{x}^* \dot{\mathbf{P}} \mathbf{x} = -\mathbf{x}^* [(\mathbf{A} - \mathbf{B}\mathbf{K})^* \mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{B}\mathbf{K})] \mathbf{x}$$

Comparing both sides of this last equation and noting that this equation must hold true for any \mathbf{x} , we require that

$$(\mathbf{A} - \mathbf{B}\mathbf{K})^* \mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{B}\mathbf{K}) = -(\mathbf{Q} + \mathbf{K}^* \mathbf{R} \mathbf{K}) \quad (10-115)$$

It can be proved that if $\mathbf{A} - \mathbf{B}\mathbf{K}$ is a stable matrix, there exists a positive-definite matrix \mathbf{P} that satisfies Equation (10–115). (See Problem A–10–15.)

Hence our procedure is to determine the elements of \mathbf{P} from Equation (10–115) and see if it is positive definite. (Note that more than one matrix \mathbf{P} may satisfy this equation. If the system is stable, there always exists one positive-definite matrix \mathbf{P} to satisfy this equation. This means that, if we solve this equation and find one positive-definite matrix \mathbf{P} , the system is stable. Other \mathbf{P} matrices that satisfy this equation are not positive definite and must be discarded.)

The performance index J can be evaluated as

$$J = \int_0^{\infty} \mathbf{x}^* (\mathbf{Q} + \mathbf{K}^* \mathbf{R} \mathbf{K}) \mathbf{x} dt = -\mathbf{x}^* \mathbf{P} \mathbf{x} \Big|_0^{\infty} = -\mathbf{x}^*(\infty) \mathbf{P} \mathbf{x}(\infty) + \mathbf{x}^*(0) \mathbf{P} \mathbf{x}(0)$$

Since all eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ are assumed to have negative real parts, we have $\mathbf{x}(\infty) \rightarrow \mathbf{0}$. Therefore, we obtain

$$J = \mathbf{x}^*(0) \mathbf{P} \mathbf{x}(0) \quad (10-116)$$

Thus, the performance index J can be obtained in terms of the initial condition $\mathbf{x}(0)$ and \mathbf{P} .

To obtain the solution to the quadratic optimal control problem, we proceed as follows: Since \mathbf{R} has been assumed to be a positive-definite Hermitian or real symmetric matrix, we can write

$$\mathbf{R} = \mathbf{T}^* \mathbf{T}$$

where \mathbf{T} is a nonsingular matrix. Then Equation (10–115) can be written as

$$(\mathbf{A}^* - \mathbf{K}^*\mathbf{B}^*)\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{BK}) + \mathbf{Q} + \mathbf{K}^*\mathbf{T}^*\mathbf{TK} = \mathbf{0}$$

which can be rewritten as

$$\mathbf{A}^*\mathbf{P} + \mathbf{PA} + [\mathbf{TK} - (\mathbf{T}^*)^{-1}\mathbf{B}^*\mathbf{P}]^*[\mathbf{TK} - (\mathbf{T}^*)^{-1}\mathbf{B}^*\mathbf{P}] - \mathbf{PBR}^{-1}\mathbf{B}^*\mathbf{P} + \mathbf{Q} = \mathbf{0}$$

The minimization of J with respect to \mathbf{K} requires the minimization of

$$\mathbf{x}^*[\mathbf{TK} - (\mathbf{T}^*)^{-1}\mathbf{B}^*\mathbf{P}]^*[\mathbf{TK} - (\mathbf{T}^*)^{-1}\mathbf{B}^*\mathbf{P}]\mathbf{x}$$

with respect to \mathbf{K} . (See Problem A–10–16.) Since this last expression is nonnegative, the minimum occurs when it is zero, or when

$$\mathbf{TK} = (\mathbf{T}^*)^{-1}\mathbf{B}^*\mathbf{P}$$

Hence,

$$\mathbf{K} = \mathbf{T}^{-1}(\mathbf{T}^*)^{-1}\mathbf{B}^*\mathbf{P} = \mathbf{R}^{-1}\mathbf{B}^*\mathbf{P} \quad (10-117)$$

Equation (10–117) gives the optimal matrix \mathbf{K} . Thus, the optimal control law to the quadratic optimal control problem when the performance index is given by Equation (10–114) is linear and is given by

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) = -\mathbf{R}^{-1}\mathbf{B}^*\mathbf{P}\mathbf{x}(t)$$

The matrix \mathbf{P} in Equation (10–117) must satisfy Equation (10–115) or the following reduced equation:

$$\mathbf{A}^*\mathbf{P} + \mathbf{PA} - \mathbf{PBR}^{-1}\mathbf{B}^*\mathbf{P} + \mathbf{Q} = \mathbf{0} \quad (10-118)$$

Equation (10–118) is called the reduced-matrix Riccati equation. The design steps may be stated as follows:

1. Solve Equation (10–118), the reduced-matrix Riccati equation, for the matrix \mathbf{P} . [If a positive-definite matrix \mathbf{P} exists (certain systems may not have a positive-definite matrix \mathbf{P}), the system is stable, or matrix $\mathbf{A} - \mathbf{BK}$ is stable.]
2. Substitute this matrix \mathbf{P} into Equation (10–117). The resulting matrix \mathbf{K} is the optimal matrix.

A design example based on this approach is given in Example 10–9. Note that if the matrix $\mathbf{A} - \mathbf{BK}$ is stable, the present method always gives the correct result.

Finally, note that if the performance index is given in terms of the output vector rather than the state vector, that is,

$$J = \int_0^{\infty} (\mathbf{y}^*\mathbf{Q}\mathbf{y} + \mathbf{u}^*\mathbf{R}\mathbf{u}) dt$$

then the index can be modified by using the output equation

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

to

$$J = \int_0^{\infty} (\mathbf{x}^*\mathbf{C}^*\mathbf{Q}\mathbf{C}\mathbf{x} + \mathbf{u}^*\mathbf{R}\mathbf{u}) dt \quad (10-119)$$

and the design steps presented in this section can be applied to obtain the optimal matrix \mathbf{K} .

EXAMPLE 10–9 Consider the system shown in Figure 10–36. Assuming the control signal to be

$$u(t) = -\mathbf{K}\mathbf{x}(t)$$

determine the optimal feedback gain matrix \mathbf{K} such that the following performance index is minimized:

$$J = \int_0^{\infty} (\mathbf{x}^T \mathbf{Q} \mathbf{x} + u^2) dt$$

where

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix} \quad (\mu \geq 0)$$

From Figure 10–36, we find that the state equation for the plant is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We shall demonstrate the use of the reduced-matrix Riccati equation in the design of the optimal control system. Let us solve Equation (10–118), rewritten as

$$\mathbf{A}^* \mathbf{P} + \mathbf{P} \mathbf{A} - \mathbf{P} \mathbf{B} \mathbf{R}^{-1} \mathbf{B}^* \mathbf{P} + \mathbf{Q} = \mathbf{0}$$

Noting that matrix \mathbf{A} is real and matrix \mathbf{Q} is real symmetric, we see that matrix \mathbf{P} is a real symmetric matrix. Hence, this last equation can be written as

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} [1] [0 \quad 1] \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This equation can be simplified to

$$\begin{bmatrix} 0 & 0 \\ p_{11} & p_{12} \end{bmatrix} + \begin{bmatrix} 0 & p_{11} \\ 0 & p_{12} \end{bmatrix} - \begin{bmatrix} p_{12}^2 & p_{12} p_{22} \\ p_{12} p_{22} & p_{22}^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

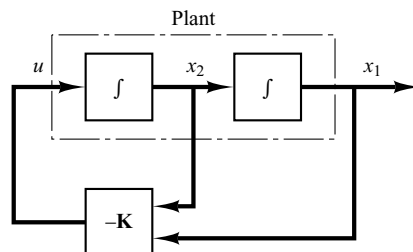
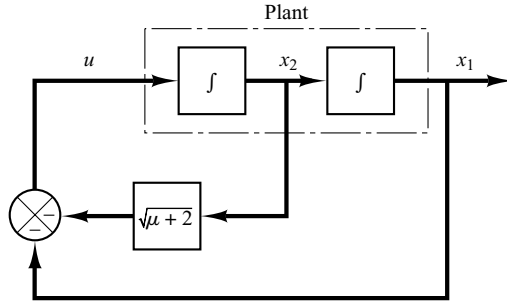


Figure 10–36
Control system.

Figure 10–37
Optimal control of
the plant shown in
Figure 10–36.



from which we obtain the following three equations:

$$\begin{aligned} 1 - p_{12}^2 &= 0 \\ p_{11} - p_{12}p_{22} &= 0 \\ \mu + 2p_{12} - p_{22}^2 &= 0 \end{aligned}$$

Solving these three simultaneous equations for p_{11} , p_{12} , and p_{22} , requiring \mathbf{P} to be positive definite, we obtain

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} \sqrt{\mu + 2} & 1 \\ 1 & \sqrt{\mu + 2} \end{bmatrix}$$

Referring to Equation (10–117), the optimal feedback gain matrix \mathbf{K} is obtained as

$$\begin{aligned} \mathbf{K} &= \mathbf{R}^{-1}\mathbf{B}^*\mathbf{P} \\ &= [1][0 \quad 1] \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \\ &= [p_{12} \quad p_{22}] \\ &= [1 \quad \sqrt{\mu + 2}] \end{aligned}$$

Thus, the optimal control signal is

$$u = -\mathbf{K}\mathbf{x} = -x_1 - \sqrt{\mu + 2} x_2 \quad (10-120)$$

Note that the control law given by Equation (10–120) yields an optimal result for any initial state under the given performance index. Figure 10–37 is the block diagram for this system.

Since the characteristic equation is

$$|s\mathbf{I} - \mathbf{A} + \mathbf{BK}| = s^2 + \sqrt{\mu + 2} s + 1 = 0$$

if $\mu = 1$, the two closed-loop poles are located at

$$s = -0.866 + j 0.5, \quad s = -0.866 - j 0.5$$

These correspond to the desired closed-loop poles when $\mu = 1$.

Solving Quadratic Optimal Regulator Problems with MATLAB. In MATLAB, the command

$$\text{lqr}(\mathbf{A}, \mathbf{B}, \mathbf{Q}, \mathbf{R})$$

solves the continuous-time, linear, quadratic regulator problem and the associated Riccati equation. This command calculates the optimal feedback gain matrix \mathbf{K} such that the feedback control law

$$u = -\mathbf{K}\mathbf{x}$$

minimizes the performance index

$$J = \int_0^{\infty} (\mathbf{x}^* \mathbf{Q} \mathbf{x} + \mathbf{u}^* \mathbf{R} \mathbf{u}) dt$$

subject to the constraint equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

Another command

$$[\mathbf{K}, \mathbf{P}, \mathbf{E}] = \text{lqr}(\mathbf{A}, \mathbf{B}, \mathbf{Q}, \mathbf{R})$$

returns the gain matrix \mathbf{K} , eigenvalue vector \mathbf{E} , and matrix \mathbf{P} , the unique positive-definite solution to the associated matrix Riccati equation:

$$\mathbf{P}\mathbf{A} + \mathbf{A}^* \mathbf{P} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1} \mathbf{B}^* \mathbf{P} + \mathbf{Q} = \mathbf{0}$$

If matrix $\mathbf{A} - \mathbf{B}\mathbf{K}$ is a stable matrix, such a positive-definite solution \mathbf{P} always exists. The eigenvalue vector \mathbf{E} gives the closed-loop poles of $\mathbf{A} - \mathbf{B}\mathbf{K}$.

It is important to note that for certain systems matrix $\mathbf{A} - \mathbf{B}\mathbf{K}$ cannot be made a stable matrix, whatever \mathbf{K} is chosen. In such a case, there does not exist a positive-definite matrix \mathbf{P} for the matrix Riccati equation. For such a case, the commands

$$\mathbf{K} = \text{lqr}(\mathbf{A}, \mathbf{B}, \mathbf{Q}, \mathbf{R})$$

$$[\mathbf{K}, \mathbf{P}, \mathbf{E}] = \text{lqr}(\mathbf{A}, \mathbf{B}, \mathbf{Q}, \mathbf{R})$$

do not give the solution. See MATLAB Program 10–18.

EXAMPLE 10–10 Consider the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

Show that the system cannot be stabilized by the state-feedback control scheme

$$u = -\mathbf{K}\mathbf{x}$$

whatever matrix \mathbf{K} is chosen. (Notice that this system is not state controllable.) Define

$$\mathbf{K} = [k_1 \quad k_2]$$

Then

$$\begin{aligned} \mathbf{A} - \mathbf{B}\mathbf{K} &= \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} [k_1 \quad k_2] \\ &= \begin{bmatrix} -1 - k_1 & 1 - k_2 \\ 0 & 2 \end{bmatrix} \end{aligned}$$

Hence, the characteristic equation becomes

$$\begin{aligned} |s\mathbf{I} - \mathbf{A} + \mathbf{BK}| &= \begin{vmatrix} s + 1 + k_1 & -1 + k_2 \\ 0 & s - 2 \end{vmatrix} \\ &= (s + 1 + k_1)(s - 2) = 0 \end{aligned}$$

The closed-loop poles are located at

$$s = -1 - k_1, \quad s = 2$$

Since the pole at $s = 2$ is in the right-half s plane, the system is unstable whatever \mathbf{K} matrix is chosen. Hence, quadratic optimal control techniques cannot be applied to this system.

Let us assume that matrices \mathbf{Q} and \mathbf{R} in the quadratic performance index are given by

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = [1]$$

and that we write MATLAB Program 10–18. The resulting MATLAB solution is

$$\mathbf{K} = [\text{NaN} \quad \text{NaN}]$$

(NaN means ‘not a number.’) Whenever the solution to a quadratic optimal control problem does not exist, MATLAB tells us that matrix \mathbf{K} consists of NaN.

MATLAB Program 10–18

```
% ----- Design of quadratic optimal regulator system -----
A = [-1 1;0 2];
B = [1;0];
Q = [1 0;0 1];
R = [1];
K = lqr(A,B,Q,R)
Warning: Matrix is singular to working precision.
K =
    NaN    NaN
% ***** If we enter the command [K,P,E] = lqr(A,B,Q,R), then *****
[K,P,E] = lqr(A,B,Q,R)
Warning: Matrix is singular to working precision.
K =
    NaN    NaN
P =
    -Inf    -Inf
    -Inf    -Inf
E =
    -2.0000
    -1.4142
```

EXAMPLE 10–11 Consider the system described by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The performance index J is given by

$$J = \int_0^{\infty} (\mathbf{x}'\mathbf{Q}\mathbf{x} + u'Ru) dt$$

where

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = [1]$$

Assume that the following control u is used.

$$u = -\mathbf{K}\mathbf{x}$$

Determine the optimal feedback gain matrix \mathbf{K} .

The optimal feedback gain matrix \mathbf{K} can be obtained by solving the following Riccati equation for a positive-definite matrix \mathbf{P} :

$$\mathbf{A}'\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}R^{-1}\mathbf{B}'\mathbf{P} + \mathbf{Q} = \mathbf{0}$$

The result is

$$\mathbf{P} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Substituting this \mathbf{P} matrix into the following equation gives the optimal \mathbf{K} matrix:

$$\begin{aligned} \mathbf{K} &= R^{-1}\mathbf{B}'\mathbf{P} \\ &= [1][0 \quad 1] \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = [1 \quad 1] \end{aligned}$$

Thus, the optimal control signal is given by

$$u = -\mathbf{K}\mathbf{x} = -x_1 - x_2$$

MATLAB 10–19 also yields the solution to this problem.

MATLAB Program 10–19

```
% ----- Design of quadratic optimal regulator system -----
A = [0 1;0 -1];
B = [0;1];
Q = [1 0;0 1];
R = [1];
K = lqr(A,B,Q,R)
K =
    1.0000    1.0000
```

EXAMPLE 10–12 Consider the system given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -35 & -27 & -9 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The performance index J is given by

$$J = \int_0^{\infty} (\mathbf{x}'\mathbf{Q}\mathbf{x} + u'Ru) dt$$

where

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R = [1]$$

Obtain the positive-definite solution matrix \mathbf{P} of the Riccati equation, the optimal feedback gain matrix \mathbf{K} , and the eigenvalues of matrix $\mathbf{A} - \mathbf{B}\mathbf{K}$.

MATLAB Program 10–20 will solve this problem.

MATLAB Program 10–20

```
% ----- Design of quadratic optimal regulator system -----
A = [0 1 0;0 0 1;-35 -27 -9];
B = [0;0;1];
Q = [1 0 0;0 1 0;0 0 1];
R = [1];
[K,P,E] = lqr(A,B,Q,R)
K =
    0.0143    0.1107    0.0676
P =
    4.2625    2.4957    0.0143
    2.4957    2.8150    0.1107
    0.0143    0.1107    0.0676
E =
   -5.0958
  -1.9859 + 1.7110i
  -1.9859 - 1.7110i
```

Next, let us obtain the response \mathbf{x} of the regulator system to the initial condition $\mathbf{x}(0)$, where

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

With state feedback $u = -\mathbf{K}\mathbf{x}$, the state equation for the system becomes

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}$$

Then the system, or sys, can be given by

$$\text{sys} = \text{ss}(\mathbf{A} - \mathbf{B}\mathbf{K}, \text{eye}(3), \text{eye}(3), \text{eye}(3))$$

MATLAB Program 10–21 produces the response to the given initial condition. The response curves are shown in Figure 10–38.

MATLAB Program 10–21

```
% Response to initial condition.
A = [0 1 0;0 0 1;-35 -27 -9];
B = [0;0;1];
K = [0.0143 0.1107 0.0676];
sys = ss(A-B*K, eye(3),eye(3),eye(3));
t = 0:0.01:8;
x = initial(sys,[1;0;0],t);
x1 = [1 0 0]*x';
x2 = [0 1 0]*x';
x3 = [0 0 1]*x';

subplot(2,2,1); plot(t,x1); grid
xlabel('t (sec)'); ylabel('x1')

subplot(2,2,2); plot(t,x2); grid
xlabel('t (sec)'); ylabel('x2')

subplot(2,2,3); plot(t,x3); grid
xlabel('t (sec)'); ylabel('x3')
```

EXAMPLE 10–13 Consider the system shown in Figure 10–39. The plant is defined by the following state-space equations:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x} + Du \end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [1 \ 0 \ 0], \quad D = [0]$$

The control signal u is given by

$$u = k_1(r - x_1) - (k_2x_2 + k_3x_3) = k_1r - (k_1x_1 + k_2x_2 + k_3x_3)$$

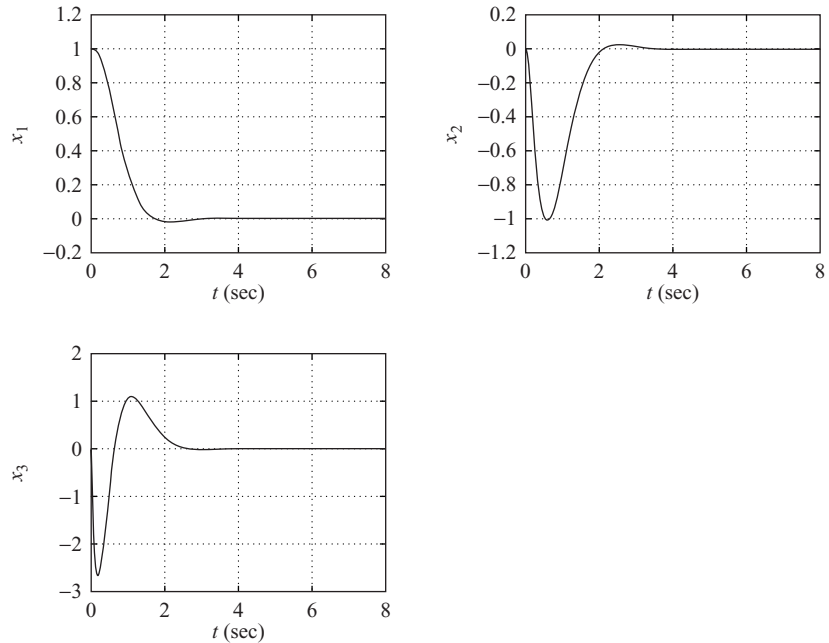


Figure 10-38
Response curves to
initial condition.

In determining an optimal control law, we assume that the input is zero, or $r = 0$.

Let us determine the state-feedback gain matrix \mathbf{K} , where

$$\mathbf{K} = [k_1 \quad k_2 \quad k_3]$$

such that the following performance index is minimized:

$$J = \int_0^{\infty} (\mathbf{x}'\mathbf{Q}\mathbf{x} + u'Ru) dt$$

where

$$\mathbf{Q} = \begin{bmatrix} q_{11} & 0 & 0 \\ 0 & q_{22} & 0 \\ 0 & 0 & q_{33} \end{bmatrix}, \quad R = 1, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \end{bmatrix}$$

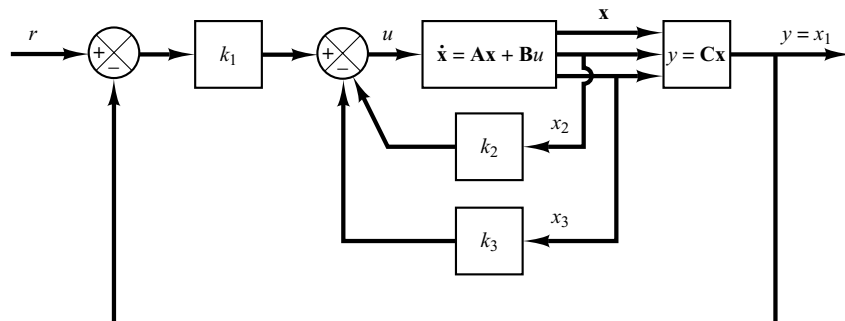


Figure 10-39
Control system.

To get a fast response, q_{11} must be sufficiently large compared with q_{22} , q_{33} , and R . In this problem, we choose

$$q_{11} = 100, \quad q_{22} = q_{33} = 1, \quad R = 0.01$$

To solve this problem with MATLAB, we use the command

$$\mathbf{K} = \text{lqr}(\mathbf{A}, \mathbf{B}, \mathbf{Q}, \mathbf{R})$$

MATLAB Program 10–22 yields the solution to this problem.

MATLAB Program 10–22

```
% ----- Design of quadratic optimal control system -----
A = [0 1 0; 0 0 1; 0 -2 -3];
B = [0; 0; 1];
Q = [100 0 0; 0 1 0; 0 0 1];
R = [0.01];
K = lqr(A,B,Q,R)
K =
    100.0000    53.1200    11.6711
```

Next we shall investigate the step-response characteristics of the designed system using the matrix \mathbf{K} thus determined. The state equation for the designed system is

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ &= \mathbf{A}\mathbf{x} + \mathbf{B}(-\mathbf{K}\mathbf{x} + k_1 r) \\ &= (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}k_1 r \end{aligned}$$

and the output equation is

$$y = \mathbf{C}\mathbf{x} = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

To obtain the unit-step response, use the following command:

$$[y, x, t] = \text{step}(\mathbf{AA}, \mathbf{BB}, \mathbf{CC}, \mathbf{DD})$$

where

$$\mathbf{AA} = \mathbf{A} - \mathbf{B}\mathbf{K}, \quad \mathbf{BB} = \mathbf{B}k_1, \quad \mathbf{CC} = \mathbf{C}, \quad \mathbf{DD} = D$$

MATLAB Program 10–23 produces the unit-step response of the designed system. Figure 10–40 shows the response curves x_1 , x_2 , and x_3 versus t on one diagram.

MATLAB Program 10-23

```
% ----- Unit-step response of designed system -----  
A = [0 1 0;0 0 1;0 -2 -3];  
B = [0;0;1]  
C = [1 0 0];  
D = [0];  
K = [100.0000 53.1200 11.6711];  
k1 = K(1); k2 = K(2); k3 = K(3);  
  
% ***** Define the state matrix, control matrix, output matrix,  
% and direct transmission matrix of the designed systems as AA,  
% BB, CC, and DD *****  
  
AA = A - B*K;  
BB = B*k1;  
CC = C;  
DD = D;  
t = 0:0.01:8;  
[y,x,t] = step (AA,BB,CC,DD,1,t);  
  
plot(t,x)  
grid  
title('Response Curves x1, x2, x3, versus t')  
xlabel('t Sec')  
ylabel('x1,x2,x3')  
text(2.6,1.35,'x1')  
text(1.2,1.5,'x2')  
text(0.6,3.5,'x3')
```

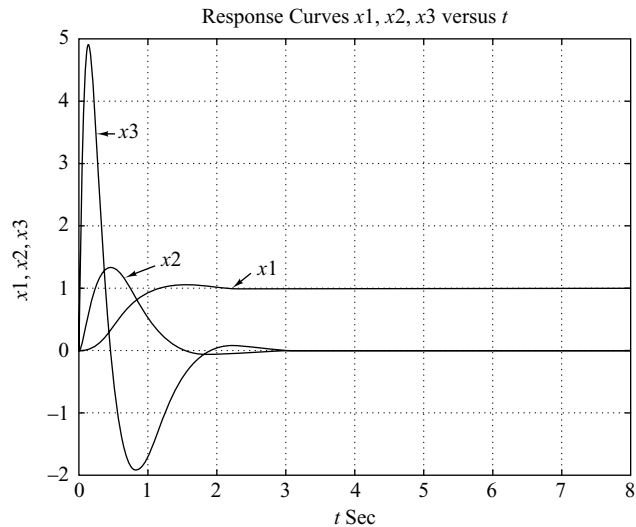


Figure 10-40
Response curves x_1
versus t , x_2 versus t ,
and x_3 versus t .

Concluding Comments on Optimal Regulator Systems

1. Given any initial state $\mathbf{x}(t_0)$, the optimal regulator problem is to find an allowable control vector $\mathbf{u}(t)$ that transfers the state to the desired region of the state space and for which the performance index is minimized. For the existence of an optimal control vector $\mathbf{u}(t)$, the system must be completely state controllable.
2. The system that minimizes (or maximizes, as the case may be) the selected performance index is, by definition, optimal. Although the controller may have nothing to do with “optimality” in many practical applications, the important point is that the design based on the quadratic performance index yields a stable control system.
3. The characteristic of an optimal control law based on a quadratic performance index is that it is a linear function of the state variables, which implies that we need to feed back all state variables. This requires that all such variables be available for feedback. If not all state variables are available for feedback, then we need to employ a state observer to estimate unmeasurable state variables and use the estimated values to generate optimal control signals.

Note that the closed-loop poles of the system designed by the use of the quadratic optimal regulator approach can be found from

$$|s\mathbf{I} - \mathbf{A} + \mathbf{BK}| = 0$$

Since these closed-loop poles correspond to the desired closed-loop poles in the pole-placement approach, the transfer functions of the observer controllers can be obtained from either Equation (10–74) if the observer is of full-order type or Equation (10–108) if the observer is of minimum-order type.

4. When the optimal control system is designed in the time domain, it is desirable to investigate the frequency-response characteristics to compensate for noise effects. The system frequency-response characteristics must be such that the system attenuates highly in the frequency range where noise and resonance of components are expected. (To compensate for noise effects, we must in some cases either modify the optimal configuration and accept suboptimal performance or modify the performance index.)
5. If the upper limit of integration in the performance index J given by Equation (10–114) is finite, then it can be shown that the optimal control vector is still a linear function of the state variables, but with time-varying coefficients. (Therefore, the determination of the optimal control vector involves that of optimal time-varying matrices.)

10–9 ROBUST CONTROL SYSTEMS

Suppose that given a control object (i.e., a system with a flexible arm) we wish to design a control system. The first step in the design of a control system is to obtain a mathematical model of the control object based on the physical law. Quite often the model may be nonlinear and possibly with distributed parameters. Such a model may be difficult to analyze. It is desirable to approximate it by a linear constant-coefficient system that will approximate the actual object fairly well. Note that even though the

model to be used for design purposes may be a simplified one, it is necessary that such a model must include any intrinsic character of the actual object. Assuming that we can get a model that approximates the actual system quite well, we must get a simplified model for the purpose of designing the control system that will require a compensator of lowest order possible. Thus, a model of a control object (whatever it may be) will probably include an error in the modeling process. Note that in the frequency-response approach to control systems design, we use phase and gain margins to take care of the modeling errors. However, in the state-space approach, which is based on the differential equations of the plant dynamics, no such “margins” are involved in the design process.

Since the actual plant differs from the model used in the design, a question arises whether the controller designed using a model will work satisfactorily with the actual plant. To ensure that it will do so, robust control theory has been developed since around 1980.

Robust control theory uses the assumption that the models we use in designing control systems have modeling errors. We shall present an introduction to this theory in this section. Basically, the theory assumes that there is an uncertainty or error between the actual plant and its mathematical model and includes such uncertainty or error in the design process of the control system.

Systems designed based on the robust control theory will possess the following properties:

- (1) *Robust stability.* The control system designed is stable in the presence of perturbation.
- (2) *Robust performance.* The control system exhibits predetermined response characteristics in the presence of perturbation.

This theory requires considerations based on frequency-response analysis and time-domain analysis. Because of the mathematical complications associated with robust control theory, detailed discussion of robust control theory is beyond the scope of the senior engineering student. In this section, only introductory discussion of robust control theory is presented.

Uncertain Elements in Plant Dynamics. The term *uncertainty* refers to the differences or errors between the model of the plant and the actual plant.

Uncertain elements that may appear in practical systems may be classified as *structured* uncertainty and *unstructured* uncertainty. An example of structured uncertainty is any parametric variation in the plant dynamics, such as variations in poles and zeros of the plant transfer function. Examples of unstructured uncertainty include frequency-dependent uncertainty, such as high-frequency modes that we normally neglect in modeling plant dynamics. For example, in the modeling of a flexible-arm system, the model may include a finite number of modes of oscillation. The modes of oscillation that are not included in the modeling behave as uncertainty of the system. Another example of uncertainty occurs in the linearization of a nonlinear plant. If the actual plant is nonlinear and its model is linear, then the difference acts as unstructured uncertainty.

In this section we consider the case where the uncertainty is unstructured. In addition we assume that the plant involves only one uncertainty. (Some plants may involve multiple uncertain elements.)

In the robust control theory, we define unstructured uncertainty as $\Delta(s)$. Since the exact description of $\Delta(s)$ is unknown, we use an estimate of $\Delta(s)$ (as to the magnitude and phase characteristics) and use this estimate in the design of the controller that stabilizes the control system. Stability of a system with unstructured uncertainty can then be examined by use of the small gain theorem to be given following the definition of the H_∞ norm.

H_∞ Norm. The H_∞ norm of a stable single-input–single-output system is the largest possible amplification factor of the steady-state response to sinusoidal excitation.

For a scalar $\Phi(s)$, $\|\Phi\|_\infty$ gives the maximum value of $|\Phi(j\omega)|$. It is called the H_∞ norm. See Figure 10–41.

In robust control theory we measure the magnitude of the transfer function by the H_∞ norm. Assume that the transfer function $\Phi(s)$ is proper and stable. [Note that a transfer function $\Phi(s)$ is called proper if $\Phi(\infty)$ is limited and definite. If $\Phi(\infty) = 0$, it is called strictly proper.] The H_∞ norm of $\Phi(s)$ is defined by

$$\|\Phi\|_\infty = \bar{\sigma} [\Phi(j\omega)]$$

$\bar{\sigma} [\Phi(j\omega)]$ means the maximum singular value of $[\Phi(j\omega)]$. ($\bar{\sigma}$ means σ_{\max} .) Note that the singular value of a transfer function Φ is defined by

$$\sigma_i(\Phi) = \sqrt{\lambda_i(\Phi^*\Phi)}$$

where $\lambda_i(\Phi^*\Phi)$ is the i th largest eigenvalue of $\Phi^*\Phi$ and it is always a non-negative real value. By making $\|\Phi\|_\infty$ smaller, we make the effect of input w on the output z smaller. It is frequently the case that instead of using the maximum singular value $\|\Phi\|_\infty$, we use the inequality

$$\|\Phi\|_\infty < \gamma$$

and limit the magnitude of $\Phi(s)$ by γ . To make the magnitude of $\|\Phi\|_\infty$ small, we choose a small γ and require that $\|\Phi\|_\infty < \gamma$.

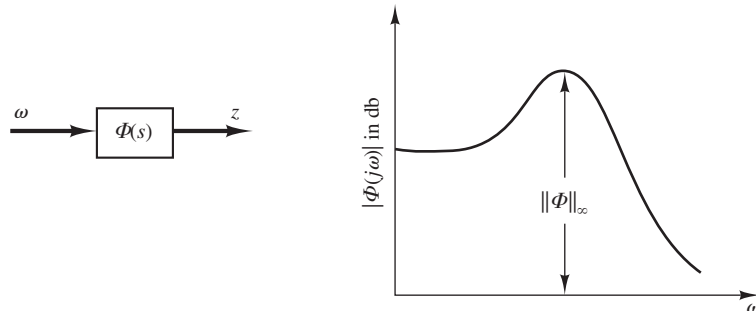
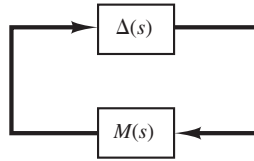


Figure 10–41
Bode diagram and
the H_∞ norm $\|\Phi\|_\infty$.

Figure 10–42
Closed-loop system.



Small-Gain Theorem. Consider the closed-loop system shown in Figure 10–42. In the figure $\Delta(s)$ and $M(s)$ are stable and proper transfer functions. The small-gain theorem states that if

$$\|\Delta(s)M(s)\|_{\infty} < 1$$

then this closed-loop system is stable. That is, if the H_{∞} norm of $\Delta(s)M(s)$ is smaller than 1, this closed-loop system is stable. This theorem is an extension of the Nyquist stability criterion.

It is important to note that the small-gain theorem gives a sufficient condition for stability. That is, a system may be stable even if it does not satisfy this theorem. However, if a system satisfies the small-gain theorem, it is always stable.

System with Unstructured Uncertainty. In some cases an unstructured uncertainty error may be considered multiplicative such that

$$\tilde{G} = G(1 + \Delta_m)$$

where \tilde{G} is the true plant dynamics and G is the model plant dynamics. In other cases an unstructured uncertainty error may be considered additive such that

$$\tilde{G} = G + \Delta_a$$

In either case we assume that the norm of Δ_m or Δ_a is bounded such that

$$\|\Delta_m\| < \gamma_m, \quad \|\Delta_a\| < \gamma_a$$

where γ_m and γ_a are positive constants.

EXAMPLE 10–14 Consider a control system with unstructured multiplicative uncertainty. We shall consider robust stability and robust performance of the system. (A system with unstructured additive uncertainty will be discussed in Problem A–10–18.)

Robust Stability. Let us define

\tilde{G} = true plant dynamics

G = model of plant dynamics

Δ_m = unstructured multiplicative uncertainty

We assume that Δ_m is stable and its upper bound is known. We also assume that \tilde{G} and G are related by

$$\tilde{G} = G(I + \Delta_m)$$

Consider the system shown in Figure 10–43(a). Let us examine the transfer function between point A and point B . Notice that Figure 10–43(a) can be redrawn as shown in Figure 10–43(b). The transfer function between point A and point B can be given by

$$\frac{KG}{1 + KG} = (1 + KG)^{-1} KG$$

Define

$$(1 + KG)^{-1} KG = T \quad (10-121)$$

Using Equation (10–121) we can redraw Figure 10–43(b) as Figure 10–43(c). Applying the small-gain theorem to the system consisting of Δ_m and T as shown in Figure 10–43(c), we obtain the condition for stability to be

$$\|\Delta_m T\|_\infty < 1 \quad (10-122)$$

In general, it is impossible to precisely model Δ_m . Therefore, let us use a scalar transfer function $W_m(j\omega)$ such that

$$\bar{\sigma}\{\Delta_m(j\omega)\} < |W_m(j\omega)|$$

where $\bar{\sigma}\{\Delta_m(j\omega)\}$ is the largest singular value of $\Delta_m(j\omega)$.

Consider, instead of Inequality (10–122), the following inequality:

$$\|W_m T\|_\infty < 1 \quad (10-123)$$

If Inequality (10–123) holds true, Inequality (10–122) will always be satisfied. By making the H_∞ norm of $W_m T$ to be less than 1, we obtain the controller K that will make the system stable.

Suppose that we cut the line at point A in Figure 10–43(a). Then we obtain Figure 10–43(d). Replacing Δ_m by $W_m I$, we obtain Figure 10–43(e). Redrawing Figure 10–43(e), we obtain Figure 10–43(f). Figure 10–43(f) is called a *generalized plant diagram*.

Referring to Equation (10–121), T is given by

$$T = \frac{KG}{1 + KG} \quad (10-124)$$

Then Inequality (10–123) can be rewritten as

$$\left\| \frac{W_m K(s)G(s)}{1 + K(s)G(s)} \right\|_\infty < 1 \quad (10-125)$$

Clearly, for a stable plant model $G(s)$, $K(s) = 0$ will satisfy Inequality (10–125). However, $K(s) = 0$ is not the desirable transfer function for the controller. To find an acceptable transfer function for $K(s)$, we may add another condition—for example, that the resulting system will have robust performance such that the system output follows the input with minimum error, or another reasonable condition. In what follows we shall obtain the condition for robust performance.

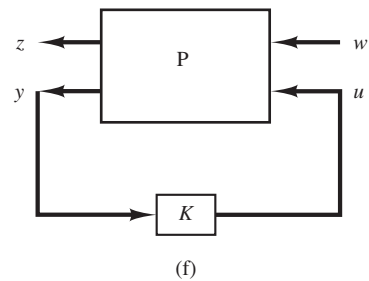
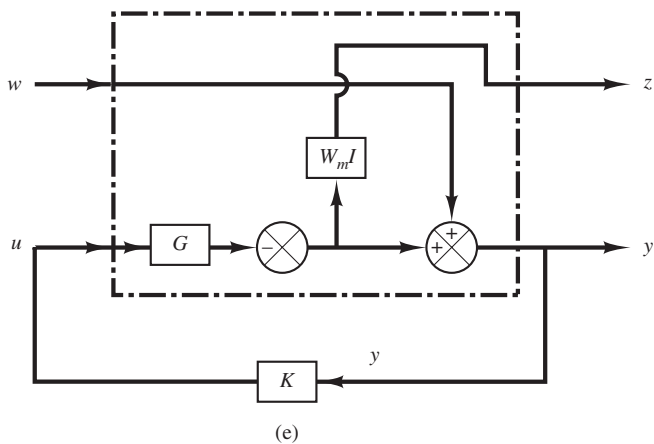
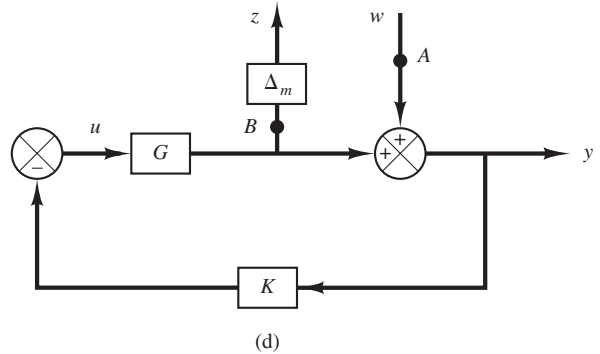
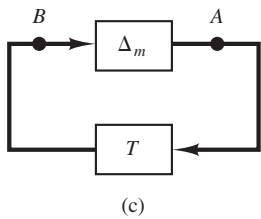
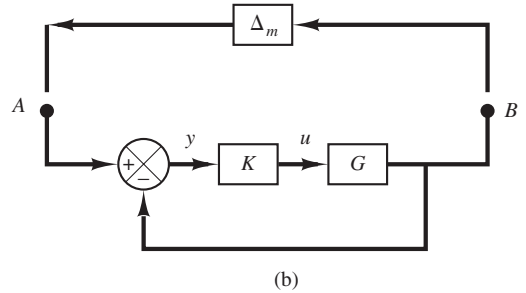
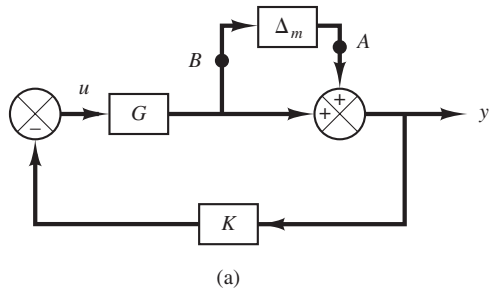


Figure 10-43
 (a) Block diagram of a system with unstructured multiplicative uncertainty;
 (b)–(d) successive modifications of the block diagram of (a);
 (e) block diagram showing a generalized plant with unstructured multiplicative uncertainty;
 (f) generalized plant diagram.

Robust Performance. Consider the system shown in Figure 10–44. Suppose that we want the output $y(t)$ to follow the input $r(t)$ as closely as possible, or we wish to have

$$\lim_{t \rightarrow \infty} [r(t) - y(t)] = \lim_{t \rightarrow \infty} e(t) \rightarrow 0$$

Since the transfer function $Y(s)/R(s)$ is

$$\frac{Y(s)}{R(s)} = \frac{KG}{1 + KG}$$

we have

$$\frac{E(s)}{R(s)} = \frac{R(s) - Y(s)}{R(s)} = 1 - \frac{Y(s)}{R(s)} = \frac{1}{1 + KG}$$

Define

$$\frac{1}{1 + KG} = S$$

where S is commonly called the sensitivity function and T defined by Equation (10–124) is called the complementary sensitivity function. In this robust performance problem we want to make the H_∞ norm of S smaller than the desired transfer function W_s^{-1} or $\|S\|_\infty < W_s^{-1}$ which can be written as

$$\|W_s S\|_\infty < 1 \quad (10-126)$$

Combining Inequalities (10–123) and (10–126), we get

$$\left\| \frac{W_m T}{W_s S} \right\|_\infty < 1$$

where $T + S = 1$, or

$$\left\| \frac{W_m(s) \frac{K(s)G(s)}{1 + K(s)G(s)}}{W_s(s) \frac{1}{1 + K(s)G(s)}} \right\|_\infty < 1 \quad (10-127)$$

Our problem then becomes to find $K(s)$ that will satisfy Inequality (10–127). Note that depending on the chosen $W_m(s)$ and $W_s(s)$ there may be many $K(s)$ that satisfy Inequality (10–127), or may be no $K(s)$ that satisfies Inequality (10–127). Such a robust control problem using Inequality (10–127) is called a mixed-sensitivity problem.

Figure 10–45(a) is a generalized plant diagram, where two conditions (robust stability and robust performance) are specified. A simplified version of this diagram is shown in Figure 10–45(b).

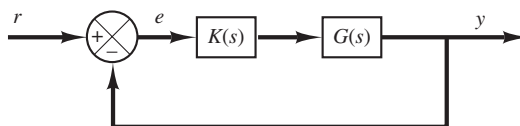


Figure 10–44
Closed-loop system.

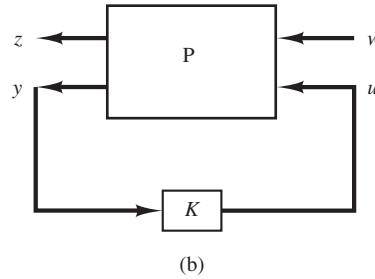
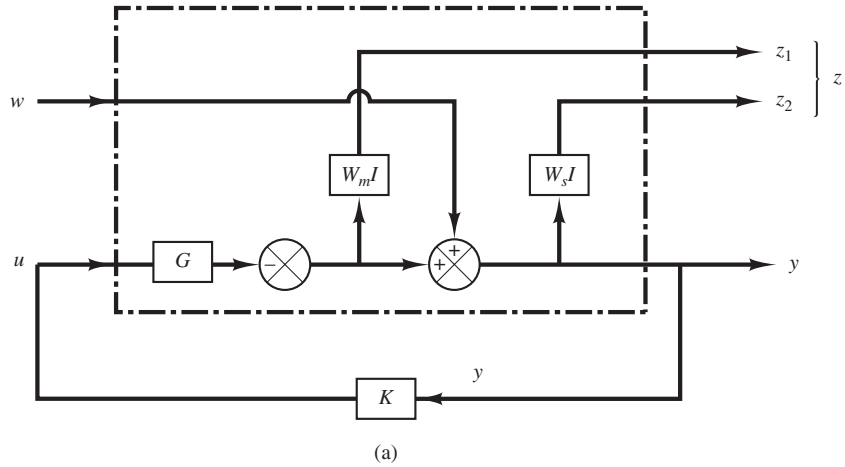


Figure 10-45
 (a) Generalized plant diagram;
 (b) simplified version of the generalized plant diagram shown in (a).

Finding Transfer Function $z(s)/w(s)$ from a Generalized Plant Diagram. Consider the generalized plant diagram shown in Figure 10-46.

In this diagram $w(s)$ is the exogenous disturbance and $u(s)$ is the manipulated variable. $z(s)$ is the controlled variable and $y(s)$ is the observed variable.

Consider this control system consisting of the generalized plant $P(s)$ and the controller $K(s)$. The equation that relates the outputs $z(s)$ and $y(s)$ and the inputs $w(s)$ and $u(s)$ of the generalized plant $P(s)$ is

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w(s) \\ u(s) \end{bmatrix}$$

The equation that relates $u(s)$ and $y(s)$ is given by

$$u(s) = K(s)y(s)$$

Define the transfer function that relates the controlled variable $z(s)$ to the exogenous disturbance $w(s)$ as $\Phi(s)$. Then

$$z(s) = \Phi(s)w(s)$$

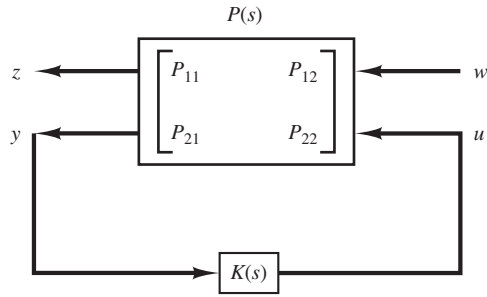


Figure 10–46
A generalized plant diagram.

Note that $\Phi(s)$ can be determined as follows: Since

$$z(s) = P_{11}w(s) + P_{12}u(s)$$

$$y(s) = P_{21}w(s) + P_{22}u(s)$$

$$u(s) = K(s)y(s)$$

we obtain

$$y(s) = P_{21}w(s) + P_{22}K(s)y(s)$$

Hence

$$[I - P_{22}K(s)]y(s) = P_{21}w(s)$$

or

$$y(s) = [I - P_{22}K(s)]^{-1}P_{21}w(s)$$

Therefore,

$$\begin{aligned} z(s) &= P_{11}w(s) + P_{12}K(s)[I - P_{22}K(s)]^{-1}P_{21}w(s) \\ &= \{P_{11} + P_{12}K(s)[I - P_{22}K(s)]^{-1}P_{21}\}w(s) \end{aligned}$$

Hence,

$$\Phi(s) = P_{11} + P_{12}K(s)[I - P_{22}K(s)]^{-1}P_{21} \quad (10-128)$$

EXAMPLE 10–15 Let us determine the P matrix in the generalized plant diagram of the control system considered in Example 10–14. We derived Inequality (10–125) for the control system to be robust stable. Rewriting Inequality (10–125), we have

$$\left\| \frac{W_m K G}{1 + K G} \right\|_{\infty} < 1 \quad (10-129)$$

If we define

$$\Phi_1 = \frac{W_m KG}{1 + KG} \quad (10-130)$$

then Inequality (10-129) can be written as

$$\|\Phi_1\|_\infty < 1$$

Referring to Equation (10-128), rewritten as

$$\Phi = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

notice that if we choose the generalized plant P matrix as

$$P = \begin{bmatrix} 0 & W_m G \\ I & -G \end{bmatrix} \quad (10-131)$$

Then we obtain

$$\begin{aligned} \Phi &= P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \\ &= W_m KG(I + KG)^{-1} \end{aligned}$$

which is exactly the same as Φ_1 in Equation (10-130).

We derived in Example 10-14 that if we wished to have the output y follow the input r as close as possible, we needed to make the H_∞ norm of $\Phi_2(s)$, where

$$\Phi_2 = \frac{W_s}{I + KG} \quad (10-132)$$

less than 1. [See Inequality (10-126).]

Note that the controlled variable z is related to the exogenous disturbance w by

$$z = \Phi(s)w$$

and referring to Equation (10-128)

$$\Phi(s) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

Notice that if we choose the P matrix as

$$P = \begin{bmatrix} W_s & -W_s G \\ I & -G \end{bmatrix} \quad (10-133)$$

then we obtain

$$\begin{aligned} \Phi &= P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \\ &= W_s - W_s KG(I + KG)^{-1} \\ &= W_s \left[1 - \frac{KG}{1 + KG} \right] \\ &= W_s \left[\frac{1}{1 + KG} \right] \end{aligned}$$

which is the same as Φ_2 in Equation (10-132).

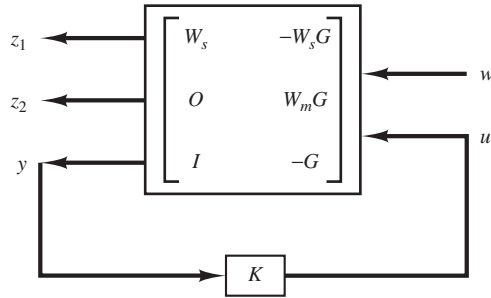


Figure 10–47
Generalized plant of
the system discussed
in Example 10–15.

If both the robust stability and robust performance conditions are required, the control system must satisfy the condition given by Inequality (10–127), rewritten as

$$\left\| \begin{array}{c} W_m \frac{KG}{1 + KG} \\ W_s \frac{1}{1 + KG} \end{array} \right\| < 1 \quad (10-134)$$

For the P matrix, we combine Equations (10–133) and (10–131) and get

$$P = \begin{bmatrix} W_s & -W_s G \\ 0 & W_m G \\ I & -G \end{bmatrix} \quad (10-135)$$

If we construct $P(s)$ as given by Equation (10–135), then the problem of designing a control system to satisfy both robust stability and robust performance conditions can be formulated by using the generalized plant represented by Equation (10–135). As mentioned earlier, such a problem is called a mixed-sensitivity problem. By using the generalized plant given by Equation (10–135) we are able to determine the controller $K(s)$ that satisfies Inequality (10–134). The generalized plant diagram for the system considered in Example 10–14 becomes as shown in Figure 10–47.

H Infinity Control Problem. To design a controller K of a control system to satisfy various stability and performance specifications, we utilize the concept of the generalized plant.

As mentioned earlier a generalized plant is a linear model consisting of a model of the plant and weighting functions corresponding to the specifications for the required performance. Referring to the generalized plant shown in Figure 10–48, the H infinity control problem is a problem to design a controller K that will make the H_∞ norm of the transfer function from the exogenous disturbance w to the controlled variable z less than a specified value.

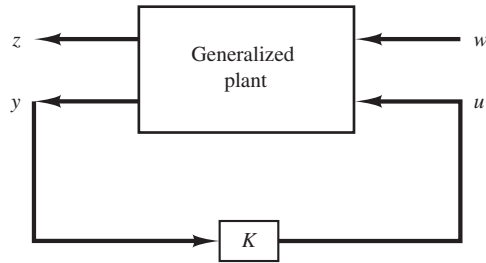


Figure 10–48
A generalized plant diagram.

The reason to use generalized plants, rather than individual block diagrams of control systems, is that a number of control systems with uncertain elements have been designed using generalized plants and, consequently, established design approaches using such plants are available.

Note that any weighting function, such as $W(s)$, is an important parameter to influence the resulting controller $K(s)$. In fact, the goodness of the resulting designed system depends on the choice of the weighting function or functions used in the design process.

Note that the controller that is the solution to the H infinity control problem is commonly called the H infinity controller.

Solving Robust Control Problems. There are three established approaches to solve robust control problems. They are

1. Solve robust control problems by deriving the Riccati equations and solving them.
2. Solve robust control problems by using the linear matrix inequality approach.
3. Solve robust control problems that involve structural uncertainties by using the μ analysis and μ synthesis approach.

Solving robust control problems by use of any of the above methods requires a broad mathematical background.

In this section we have presented only an introduction to the robust control theory. Solving any robust control problem requires mathematical background beyond the scope of the senior engineering student. Therefore, an interested reader may take a graduate-level control course at an established college or university and study this subject in detail.

EXAMPLE PROBLEMS AND SOLUTIONS

A–10–1. Consider the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

Suppose that this system is not completely state controllable. Then the rank of the controllability matrix is less than n , or

$$\text{rank} [\mathbf{B} \mid \mathbf{A}\mathbf{B} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}] = q < n \quad (10-136)$$

This means that there are q linearly independent column vectors in the controllability matrix. Let us define such q linearly independent column vectors as $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_q$. Also, let us choose $n - q$ additional n -vectors $\mathbf{v}_{q+1}, \mathbf{v}_{q+2}, \dots, \mathbf{v}_n$ such that

$$\mathbf{P} = [\mathbf{f}_1 \mid \mathbf{f}_2 \mid \cdots \mid \mathbf{f}_q \mid \mathbf{v}_{q+1} \mid \mathbf{v}_{q+2} \mid \cdots \mid \mathbf{v}_n]$$

is of rank n . By using matrix \mathbf{P} as the transformation matrix, define

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \hat{\mathbf{A}}, \quad \mathbf{P}^{-1}\mathbf{B} = \hat{\mathbf{B}}$$

Show that $\hat{\mathbf{A}}$ can be given by

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix}$$

where \mathbf{A}_{11} is a $q \times q$ matrix, \mathbf{A}_{12} is a $q \times (n - q)$ matrix, \mathbf{A}_{22} is an $(n - q) \times (n - q)$ matrix, and $\mathbf{0}$ is an $(n - q) \times q$ matrix. Show also that matrix $\hat{\mathbf{B}}$ can be given by

$$\hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B}_{11} \\ \mathbf{0} \end{bmatrix}$$

where \mathbf{B}_{11} is a $q \times 1$ matrix and $\mathbf{0}$ is an $(n - q) \times 1$ matrix.

Solution. Notice that

$$\mathbf{A}\mathbf{P} = \mathbf{P}\hat{\mathbf{A}}$$

or

$$\begin{aligned} & [\mathbf{A}\mathbf{f}_1 \mid \mathbf{A}\mathbf{f}_2 \mid \cdots \mid \mathbf{A}\mathbf{f}_q \mid \mathbf{A}\mathbf{v}_{q+1} \mid \cdots \mid \mathbf{A}\mathbf{v}_n] \\ &= [\mathbf{f}_1 \mid \mathbf{f}_2 \mid \cdots \mid \mathbf{f}_q \mid \mathbf{v}_{q+1} \mid \cdots \mid \mathbf{v}_n] \hat{\mathbf{A}} \end{aligned} \quad (10-137)$$

Also,

$$\mathbf{B} = \mathbf{P}\hat{\mathbf{B}} \quad (10-138)$$

Since we have q linearly independent column vectors $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_q$, we can use the Cayley–Hamilton theorem to express vectors $\mathbf{A}\mathbf{f}_1, \mathbf{A}\mathbf{f}_2, \dots, \mathbf{A}\mathbf{f}_q$ in terms of these q vectors. That is,

$$\mathbf{A}\mathbf{f}_1 = a_{11}\mathbf{f}_1 + a_{21}\mathbf{f}_2 + \cdots + a_{q1}\mathbf{f}_q$$

$$\mathbf{A}\mathbf{f}_2 = a_{12}\mathbf{f}_1 + a_{22}\mathbf{f}_2 + \cdots + a_{q2}\mathbf{f}_q$$

.

.

.

$$\mathbf{A}\mathbf{f}_q = a_{1q}\mathbf{f}_1 + a_{2q}\mathbf{f}_2 + \cdots + a_{qq}\mathbf{f}_q$$

Hence, Equation (10–137) may be written as follows:

$$\begin{aligned}
 & [\mathbf{A}\mathbf{f}_1 \mid \mathbf{A}\mathbf{f}_2 \mid \cdots \mid \mathbf{A}\mathbf{f}_q \mid \mathbf{A}\mathbf{v}_{q+1} \mid \cdots \mid \mathbf{A}\mathbf{v}_n] \\
 &= [\mathbf{f}_1 \mid \mathbf{f}_2 \mid \cdots \mid \mathbf{f}_q \mid \mathbf{v}_{q+1} \mid \cdots \mid \mathbf{v}_n] \begin{bmatrix} a_{11} & \cdots & a_{1q} & a_{1q+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2q} & a_{2q+1} & \cdots & a_{2n} \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ a_{q1} & \cdots & a_{qq} & a_{qq+1} & \cdots & a_{qn} \\ \hline 0 & \cdots & 0 & a_{q+1q+1} & \cdots & a_{q+1n} \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ 0 & \cdots & 0 & a_{nq+1} & \cdots & a_{nn} \end{bmatrix}
 \end{aligned}$$

Define

$$\begin{aligned}
 & \begin{bmatrix} a_{11} & \cdots & a_{1q} \\ a_{21} & \cdots & a_{2q} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{q1} & \cdots & a_{qq} \end{bmatrix} = \mathbf{A}_{11} \\
 & \begin{bmatrix} a_{1q+1} & \cdots & a_{1n} \\ a_{2q+1} & \cdots & a_{2n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{qq+1} & \cdots & a_{qn} \end{bmatrix} = \mathbf{A}_{12} \\
 & \begin{bmatrix} 0 & \cdots & 0 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 0 & \cdots & 0 \end{bmatrix} = \mathbf{A}_{21} = (n - q) \times q \text{ zero matrix} \\
 & \begin{bmatrix} a_{q+1q+1} & \cdots & a_{q+1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{nq+1} & \cdots & a_{nn} \end{bmatrix} = \mathbf{A}_{22}
 \end{aligned}$$

Then Equation (10–137) can be written as

$$\begin{aligned}
 & [\mathbf{A}\mathbf{f}_1 \mid \mathbf{A}\mathbf{f}_2 \mid \cdots \mid \mathbf{A}\mathbf{f}_q \mid \mathbf{A}\mathbf{v}_{q+1} \mid \cdots \mid \mathbf{A}\mathbf{v}_n] \\
 &= [\mathbf{f}_1 \mid \mathbf{f}_2 \mid \cdots \mid \mathbf{f}_q \mid \mathbf{v}_{q+1} \mid \cdots \mid \mathbf{v}_n] \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix}
 \end{aligned}$$

Thus,

$$\mathbf{AP} = \mathbf{P} \left[\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{0} & \mathbf{A}_{22} \end{array} \right]$$

Hence,

$$\mathbf{P}^{-1}\mathbf{AP} = \hat{\mathbf{A}} = \left[\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{0} & \mathbf{A}_{22} \end{array} \right]$$

Next, referring to Equation (10-138), we have

$$\mathbf{B} = [\mathbf{f}_1 \mid \mathbf{f}_2 \mid \cdots \mid \mathbf{f}_q \mid \mathbf{v}_{q+1} \mid \cdots \mid \mathbf{v}_n] \hat{\mathbf{B}} \quad (10-139)$$

Referring to Equation (10-136), notice that vector \mathbf{B} can be written in terms of q linearly independent column vectors $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_q$. Thus, we have

$$\mathbf{B} = b_{11}\mathbf{f}_1 + b_{21}\mathbf{f}_2 + \cdots + b_{q1}\mathbf{f}_q$$

Consequently, Equation (10-139) may be written as follows:

$$b_{11}\mathbf{f}_1 + b_{21}\mathbf{f}_2 + \cdots + b_{q1}\mathbf{f}_q = [\mathbf{f}_1 \mid \mathbf{f}_2 \mid \cdots \mid \mathbf{f}_q \mid \mathbf{v}_{q+1} \mid \cdots \mid \mathbf{v}_n] \begin{bmatrix} b_{11} \\ b_{21} \\ \cdot \\ \cdot \\ \cdot \\ b_{q1} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

Thus,

$$\hat{\mathbf{B}} = \left[\begin{array}{c} \mathbf{B}_{11} \\ \hline \mathbf{0} \end{array} \right]$$

where

$$\mathbf{B}_{11} = \begin{bmatrix} b_{11} \\ b_{21} \\ \cdot \\ \cdot \\ \cdot \\ b_{q1} \end{bmatrix}$$

A-10-2. Consider a completely state controllable system

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$

Define the controllability matrix as \mathbf{M} :

$$\mathbf{M} = [\mathbf{B} \mid \mathbf{AB} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}]$$

Show that

$$\mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & \cdots & 0 & -a_{n-2} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix}$$

where a_1, a_2, \dots, a_n are the coefficients of the characteristic polynomial

$$|s\mathbf{I} - \mathbf{A}| = s^n + a_1s^{n-1} + \cdots + a_{n-1}s + a_n$$

Solution. Let us consider the case where $n = 3$. We shall show that

$$\mathbf{A}\mathbf{M} = \mathbf{M} \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} \quad (10-140)$$

The left-hand side of Equation (10-140) is

$$\mathbf{A}\mathbf{M} = \mathbf{A}[\mathbf{B} \mid \mathbf{A}\mathbf{B} \mid \mathbf{A}^2\mathbf{B}] = [\mathbf{A}\mathbf{B} \mid \mathbf{A}^2\mathbf{B} \mid \mathbf{A}^3\mathbf{B}]$$

The right-hand side of Equation (10-140) is

$$[\mathbf{B} \mid \mathbf{A}\mathbf{B} \mid \mathbf{A}^2\mathbf{B}] \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} = [\mathbf{A}\mathbf{B} \mid \mathbf{A}^2\mathbf{B} \mid -a_3\mathbf{B} - a_2\mathbf{A}\mathbf{B} - a_1\mathbf{A}^2\mathbf{B}] \quad (10-141)$$

The Cayley–Hamilton theorem states that matrix \mathbf{A} satisfies its own characteristic equation or, in the case of $n = 3$,

$$\mathbf{A}^3 + a_1\mathbf{A}^2 + a_2\mathbf{A} + a_3\mathbf{I} = \mathbf{0} \quad (10-142)$$

Using Equation (10-142), the third column of the right-hand side of Equation (10-141) becomes

$$-a_3\mathbf{B} - a_2\mathbf{A}\mathbf{B} - a_1\mathbf{A}^2\mathbf{B} = (-a_3\mathbf{I} - a_2\mathbf{A} - a_1\mathbf{A}^2)\mathbf{B} = \mathbf{A}^3\mathbf{B}$$

Thus, Equation (10-141) becomes

$$[\mathbf{B} \mid \mathbf{A}\mathbf{B} \mid \mathbf{A}^2\mathbf{B}] \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} = [\mathbf{A}\mathbf{B} \mid \mathbf{A}^2\mathbf{B} \mid \mathbf{A}^3\mathbf{B}]$$

Hence, the left-hand side and the right-hand side of Equation (10-140) are the same. We have thus shown that Equation (10-140) is true. Consequently,

$$\mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix}$$

The preceding derivation can be easily extended to the general case of any positive integer n .

A-10-3. Consider a completely state controllable system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

Define

$$\mathbf{M} = [\mathbf{B} \mid \mathbf{A}\mathbf{B} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}]$$

and

$$\mathbf{W} = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

where the a_i 's are coefficients of the characteristic polynomial

$$|s\mathbf{I} - \mathbf{A}| = s^n + a_1s^{n-1} + \cdots + a_{n-1}s + a_n$$

Define also

$$\mathbf{T} = \mathbf{M}\mathbf{W}$$

Show that

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \quad \mathbf{T}^{-1}\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}$$

Solution. Let us consider the case where $n = 3$. We shall show that

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = (\mathbf{M}\mathbf{W})^{-1}\mathbf{A}(\mathbf{M}\mathbf{W}) = \mathbf{W}^{-1}(\mathbf{M}^{-1}\mathbf{A}\mathbf{M})\mathbf{W} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \quad (10-143)$$

Referring to Problem **A-10-2**, we have

$$\mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix}$$

Hence, Equation (10-143) can be rewritten as

$$\mathbf{W}^{-1} \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} \mathbf{W} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$$

Therefore, we need to show that

$$\begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} \mathbf{W} = \mathbf{W} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \quad (10-144)$$

The left-hand side of Equation (10-144) is

$$\begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} \begin{bmatrix} a_2 & a_1 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -a_3 & 0 & 0 \\ 0 & a_1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The right-hand side of Equation (10–144) is

$$\begin{bmatrix} a_2 & a_1 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} = \begin{bmatrix} -a_3 & 0 & 0 \\ 0 & a_1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Clearly, Equation (10–144) holds true. Thus, we have shown that

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$$

Next, we shall show that

$$\mathbf{T}^{-1}\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (10-145)$$

Note that Equation (10–145) can be written as

$$\mathbf{B} = \mathbf{T} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{M}\mathbf{W} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Noting that

$$\mathbf{T} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = [\mathbf{B} \ \mathbf{A}\mathbf{B} \ \mathbf{A}^2\mathbf{B}] \begin{bmatrix} a_2 & a_1 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = [\mathbf{B} \ \mathbf{A}\mathbf{B} \ \mathbf{A}^2\mathbf{B}] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{B}$$

we have

$$\mathbf{T}^{-1}\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The derivation shown here can be easily extended to the general case of any positive integer n .

A-10-4. Consider the state equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -4 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

The rank of the controllability matrix \mathbf{M} ,

$$\mathbf{M} = [\mathbf{B} \ \mathbf{A}\mathbf{B}] = \begin{bmatrix} 0 & 2 \\ 2 & -6 \end{bmatrix}$$

is 2. Thus, the system is completely state controllable. Transform the given state equation into the controllable canonical form.

Solution. Since

$$\begin{aligned} |s\mathbf{I} - \mathbf{A}| &= \begin{vmatrix} s-1 & -1 \\ 4 & s+3 \end{vmatrix} = (s-1)(s+3) + 4 \\ &= s^2 + 2s + 1 = s^2 + a_1s + a_2 \end{aligned}$$

we have

$$a_1 = 2, \quad a_2 = 1$$

Define

$$\mathbf{T} = \mathbf{M}\mathbf{W}$$

where

$$\mathbf{M} = \begin{bmatrix} 0 & 2 \\ 2 & -6 \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

Then

$$\mathbf{T} = \begin{bmatrix} 0 & 2 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -2 & 2 \end{bmatrix}$$

and

$$\mathbf{T}^{-1} = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0.5 \end{bmatrix}$$

Define

$$\mathbf{x} = \mathbf{T}\hat{\mathbf{x}}$$

Then the state equation becomes

$$\dot{\hat{\mathbf{x}}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\hat{\mathbf{x}} + \mathbf{T}^{-1}\mathbf{B}u$$

Since

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -4 & -3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

and

$$\mathbf{T}^{-1}\mathbf{B} = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

we have

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

which is in the controllable canonical form.

A-10-5. Consider a system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0]$$

The characteristic equation of the system is

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s & -1 \\ 2 & s + 3 \end{vmatrix} = s^2 + 3s + 2 = (s + 1)(s + 2) = 0$$

The eigenvalues of matrix \mathbf{A} are -1 and -2 .

It is desired to have eigenvalues at -3 and -5 by using a state-feedback control $u = -\mathbf{K}\mathbf{x}$. Determine the necessary feedback gain matrix \mathbf{K} and the control signal u .

Solution. The given system is completely state controllable, since the rank of

$$\mathbf{M} = [\mathbf{B} \ \vdots \ \mathbf{AB}] = \begin{bmatrix} 0 & 2 \\ 2 & -6 \end{bmatrix}$$

is 2. Hence, arbitrary pole placement is possible.

Since the characteristic equation of the original system is

$$s^2 + 3s + 2 = s^2 + a_1s + a_2 = 0$$

we have

$$a_1 = 3, \quad a_2 = 2$$

The desired characteristic equation is

$$(s + 3)(s + 5) = s^2 + 8s + 15 = s^2 + \alpha_1s + \alpha_2 = 0$$

Hence,

$$\alpha_1 = 8, \quad \alpha_2 = 15$$

It is important to point out that the original state equation is not in the controllable canonical form, because matrix \mathbf{B} is not

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Hence, the transformation matrix \mathbf{T} must be determined.

$$\mathbf{T} = \mathbf{M}\mathbf{W} = [\mathbf{B} \ \vdots \ \mathbf{AB}] \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Hence,

$$\mathbf{T}^{-1} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

Referring to Equation (10–13), the necessary feedback gain matrix is given by

$$\begin{aligned} \mathbf{K} &= [\alpha_2 - a_2 \ \vdots \ \alpha_1 - a_1] \mathbf{T}^{-1} \\ &= [15 - 2 \ \vdots \ 8 - 3] \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} = [6.5 \ 2.5] \end{aligned}$$

Thus, the control signal u becomes

$$u = -\mathbf{K}\mathbf{x} = -[6.5 \ 2.5] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

A-10-6. A regulator system has a plant

$$\frac{Y(s)}{U(s)} = \frac{10}{(s+1)(s+2)(s+3)}$$

Define state variables as

$$x_1 = y$$

$$x_2 = \dot{x}_1$$

$$x_3 = \dot{x}_2$$

By use of the state-feedback control $u = -\mathbf{K}\mathbf{x}$, it is desired to place the closed-loop poles at

$$s = -2 + j2\sqrt{3}, \quad s = -2 - j2\sqrt{3}, \quad s = -10$$

Obtain the necessary state-feedback gain matrix \mathbf{K} with MATLAB.

Solution. The state-space equations for the system become

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u$$

$$y = [1 \ 0 \ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0u$$

Hence,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$$

$$\mathbf{C} = [1 \ 0 \ 0], \quad D = [0]$$

(Note that, for the pole placement, matrices \mathbf{C} and D do not affect the state-feedback gain matrix \mathbf{K} .)

Two MATLAB programs for obtaining state-feedback gain matrix \mathbf{K} are given in MATLAB Programs 10-24 and 10-25.

MATLAB Program 10-24
<pre>A = [0 1 0;0 0 1;-6 -11 -6]; B = [0;0;10]; J = [-2+j*2*sqrt(3) -2-j*2*sqrt(3) -10]; K = acker(A,B,J) K = 15.4000 4.5000 0.8000</pre>

MATLAB Program 10–25

```
A = [0 1 0;0 0 1; -6 -11 -6];
B = [0;0;10];
J = [-2+j*2*sqrt(3) -2-j*2*sqrt(3) -10];
K = place(A,B,J)
place: ndigits= 15
K =
    15.4000    4.5000    0.8000
```

A–10–7. Consider a completely observable system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

$$y = \mathbf{C}\mathbf{x}$$

Define the observability matrix as \mathbf{N} :

$$\mathbf{N} = [\mathbf{C}^* \mid \mathbf{A}^*\mathbf{C}^* \mid \cdots \mid (\mathbf{A}^*)^{n-1}\mathbf{C}^*]$$

Show that

$$\mathbf{N}^*\mathbf{A}(\mathbf{N}^*)^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \quad (10-146)$$

where a_1, a_2, \dots, a_n are the coefficients of the characteristic polynomial

$$|s\mathbf{I} - \mathbf{A}| = s^n + a_1s^{n-1} + \cdots + a_{n-1}s + a_n$$

Solution. Let us consider the case where $n = 3$. Then Equation (10–146) can be written as

$$\mathbf{N}^*\mathbf{A}(\mathbf{N}^*)^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \quad (10-147)$$

Equation (10–147) may be rewritten as

$$\mathbf{N}^*\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \mathbf{N}^* \quad (10-148)$$

We shall show that Equation (10–148) holds true. The left-hand side of Equation (10–148) is

$$\mathbf{N}^*\mathbf{A} = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \mathbf{C}\mathbf{A}^2 \end{bmatrix} \mathbf{A} = \begin{bmatrix} \mathbf{C}\mathbf{A} \\ \mathbf{C}\mathbf{A}^2 \\ \mathbf{C}\mathbf{A}^3 \end{bmatrix} \quad (10-149)$$

The right-hand side of Equation (10-148) is

$$\begin{aligned} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \mathbf{N}^* &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{CA} \\ \mathbf{CA}^2 \\ -a_3 \mathbf{C} - a_2 \mathbf{CA} - a_1 \mathbf{CA}^2 \end{bmatrix} \end{aligned} \quad (10-150)$$

The Cayley-Hamilton theorem states that matrix \mathbf{A} satisfies its own characteristic equation, or

$$\mathbf{A}^3 + a_1 \mathbf{A}^2 + a_2 \mathbf{A} + a_3 \mathbf{I} = \mathbf{0}$$

Hence,

$$-a_1 \mathbf{CA}^2 - a_2 \mathbf{CA} - a_3 \mathbf{C} = \mathbf{CA}^3$$

Thus, the right-hand side of Equation (10-150) becomes the same as the right-hand side of Equation (10-149). Consequently,

$$\mathbf{N}^* \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \mathbf{N}^*$$

which is Equation (10-148). This last equation can be modified to

$$\mathbf{N}^* \mathbf{A} (\mathbf{N}^*)^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$$

The derivation presented here can be extended to the general case of any positive integer n .

A-10-8. Consider a completely observable system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \quad (10-151)$$

$$y = \mathbf{C}\mathbf{x} + Du \quad (10-152)$$

Define

$$\mathbf{N} = [\mathbf{C}^* \mid \mathbf{A}^* \mathbf{C}^* \mid \cdots \mid (\mathbf{A}^*)^{n-1} \mathbf{C}^*]$$

and

$$\mathbf{W} = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

where the a 's are coefficients of the characteristic polynomial

$$|s\mathbf{I} - \mathbf{A}| = s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n$$

Define also

$$\mathbf{Q} = (\mathbf{W}\mathbf{N}^*)^{-1}$$

Show that

$$\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & \cdots & 0 & -a_{n-2} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix}$$

$$\mathbf{C}\mathbf{Q} = [0 \ 0 \ \cdots \ 0 \ 1]$$

$$\mathbf{Q}^{-1}\mathbf{B} = \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \cdot \\ \cdot \\ b_1 - a_1 b_0 \end{bmatrix}$$

where the b_k 's ($k = 0, 1, 2, \dots, n$) are those coefficients appearing in the numerator of the transfer function when $\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$ is written as follows:

$$\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

where $D = b_0$.

Solution. Let us consider the case where $n = 3$. We shall show that

$$\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = (\mathbf{W}\mathbf{N}^*)\mathbf{A}(\mathbf{W}\mathbf{N}^*)^{-1} = \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} \quad (10-153)$$

Note that, by referring to Problem **A-10-7**, we have

$$(\mathbf{W}\mathbf{N}^*)\mathbf{A}(\mathbf{W}\mathbf{N}^*)^{-1} = \mathbf{W}[\mathbf{N}^*\mathbf{A}(\mathbf{N}^*)^{-1}]\mathbf{W}^{-1} = \mathbf{W} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \mathbf{W}^{-1}$$

Hence, we need to show that

$$\mathbf{W} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \mathbf{W}^{-1} = \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix}$$

or

$$\mathbf{W} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} \mathbf{W} \quad (10-154)$$

The left-hand side of Equation (10–154) is

$$\begin{aligned} \mathbf{W} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} &= \begin{bmatrix} a_2 & a_1 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \\ &= \begin{bmatrix} -a_3 & 0 & 0 \\ 0 & a_1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

The right-hand side of Equation (10–154) is

$$\begin{aligned} \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} \mathbf{W} &= \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} \begin{bmatrix} a_2 & a_1 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -a_3 & 0 & 0 \\ 0 & a_1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

Thus, we see that Equation (10–154) holds true. Hence, we have proved Equation (10–153).
Next we shall show that

$$\mathbf{CQ} = [0 \ 0 \ 1]$$

or

$$\mathbf{C}(\mathbf{WN}^*)^{-1} = [0 \ 0 \ 1]$$

Notice that

$$\begin{aligned} [0 \ 0 \ 1](\mathbf{WN}^*) &= [0 \ 0 \ 1] \begin{bmatrix} a_2 & a_1 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} \\ &= [1 \ 0 \ 0] \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \mathbf{C} \end{aligned}$$

Hence, we have shown that

$$[0 \ 0 \ 1] = \mathbf{C}(\mathbf{WN}^*)^{-1} = \mathbf{CQ}$$

Next define

$$\mathbf{x} = \mathbf{Q}\hat{\mathbf{x}}$$

Then Equation (10–151) becomes

$$\dot{\hat{\mathbf{x}}} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}\hat{\mathbf{x}} + \mathbf{Q}^{-1}\mathbf{B}u \quad (10-155)$$

and Equation (10–152) becomes

$$y = \mathbf{CQ}\hat{\mathbf{x}} + Du \quad (10-156)$$

Referring to Equation (10–153), Equation (10–155) becomes

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} + \begin{bmatrix} \gamma_3 \\ \gamma_2 \\ \gamma_1 \end{bmatrix} u$$

where

$$\begin{bmatrix} \gamma_3 \\ \gamma_2 \\ \gamma_1 \end{bmatrix} = \mathbf{Q}^{-1}\mathbf{B}$$

The transfer function $G(s)$ for the system defined by Equations (10–155) and (10–156) is

$$G(s) = \mathbf{CQ}(s\mathbf{I} - \mathbf{Q}^{-1}\mathbf{AQ})^{-1}\mathbf{Q}^{-1}\mathbf{B} + D$$

Noting that

$$\mathbf{CQ} = [0 \quad 0 \quad 1]$$

we have

$$G(s) = [0 \quad 0 \quad 1] \begin{bmatrix} s & 0 & a_3 \\ -1 & s & a_2 \\ 0 & -1 & s + a_1 \end{bmatrix}^{-1} \begin{bmatrix} \gamma_3 \\ \gamma_2 \\ \gamma_1 \end{bmatrix} + D$$

Note that $D = b_0$. Since

$$\begin{bmatrix} s & 0 & a_3 \\ -1 & s & a_2 \\ 0 & -1 & s + a_1 \end{bmatrix}^{-1} = \frac{1}{s^3 + a_1s^2 + a_2s + a_3} \begin{bmatrix} s^2 + a_1s + a_2 & -a_3 & -a_3s \\ s + a_1 & s^2 + a_1s & -a_2s - a_3 \\ 1 & s & s^2 \end{bmatrix}$$

we have

$$\begin{aligned} G(s) &= \frac{1}{s^3 + a_1s^2 + a_2s + a_3} [1 \quad s \quad s^2] \begin{bmatrix} \gamma_3 \\ \gamma_2 \\ \gamma_1 \end{bmatrix} + D \\ &= \frac{\gamma_1s^2 + \gamma_2s + \gamma_3}{s^3 + a_1s^2 + a_2s + a_3} + b_0 \\ &= \frac{b_0s^3 + (\gamma_1 + a_1b_0)s^2 + (\gamma_2 + a_2b_0)s + \gamma_3 + a_3b_0}{s^3 + a_1s^2 + a_2s + a_3} \\ &= \frac{b_0s^3 + b_1s^2 + b_2s + b_3}{s^3 + a_1s^2 + a_2s + a_3} \end{aligned}$$

Hence,

$$\gamma_1 = b_1 - a_1b_0, \quad \gamma_2 = b_2 - a_2b_0, \quad \gamma_3 = b_3 - a_3b_0$$

Thus, we have shown that

$$\mathbf{Q}^{-1}\mathbf{B} = \begin{bmatrix} \gamma_3 \\ \gamma_2 \\ \gamma_1 \end{bmatrix} = \begin{bmatrix} b_3 - a_3b_0 \\ b_2 - a_2b_0 \\ b_1 - a_1b_0 \end{bmatrix}$$

Note that what we have derived here can be easily extended to the case when n is any positive integer.

A–10–9. Consider a system defined by

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$

$$y = \mathbf{Cx}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -4 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 1]$$

The rank of the observability matrix \mathbf{N} ,

$$\mathbf{N} = [\mathbf{C}^* \mid \mathbf{A}^* \mathbf{C}^*] = \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix}$$

is 2. Hence, the system is completely observable. Transform the system equations into the observable canonical form.

Solution. Since

$$|s\mathbf{I} - \mathbf{A}| = s^2 + 2s + 1 = s^2 + a_1s + a_2$$

we have

$$a_1 = 2, \quad a_2 = 1$$

Define

$$\mathbf{Q} = (\mathbf{W}\mathbf{N}^*)^{-1}$$

where

$$\mathbf{N} = \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

Then

$$\mathbf{Q} = \left\{ \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix} \right\}^{-1} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

and

$$\mathbf{Q}^{-1} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

Define

$$\mathbf{x} = \mathbf{Q}\hat{\mathbf{x}}$$

Then the state equation becomes

$$\dot{\hat{\mathbf{x}}} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}\hat{\mathbf{x}} + \mathbf{Q}^{-1}\mathbf{B}u$$

or

$$\begin{aligned} \begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -4 & -3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} u \\ &= \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u \end{aligned} \quad (10-157)$$

The output equation becomes

$$y = \mathbf{C}\mathbf{Q}\hat{\mathbf{x}}$$

or

$$y = [1 \quad 1] \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = [0 \quad 1] \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} \quad (10-158)$$

Equations (10-157) and (10-158) are in the observable canonical form.

A-10-10. For the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

consider the problem of designing a state observer such that the desired eigenvalues for the observer gain matrix are $\mu_1, \mu_2, \dots, \mu_n$.

Show that the observer gain matrix given by Equation (10-61), rewritten as

$$\mathbf{K}_e = (\mathbf{W}\mathbf{N}^*)^{-1} \begin{bmatrix} \alpha_n - a_n \\ \alpha_{n-1} - a_{n-1} \\ \vdots \\ \alpha_1 - a_1 \end{bmatrix} \quad (10-159)$$

can be obtained from Equation (10-13) by considering the dual problem. That is, the matrix \mathbf{K}_e can be determined by considering the pole-placement problem for the dual system, obtaining the state-feedback gain matrix \mathbf{K} , and taking its conjugate transpose, or $\mathbf{K}_e = \mathbf{K}^*$.

Solution. The dual of the given system is

$$\dot{\mathbf{z}} = \mathbf{A}^*\mathbf{z} + \mathbf{C}^*v \quad (10-160)$$

$$n = \mathbf{B}^*\mathbf{z}$$

Using the state-feedback control

$$v = -\mathbf{K}\mathbf{z}$$

Equation (10-160) becomes

$$\dot{\mathbf{z}} = (\mathbf{A}^* - \mathbf{C}^*\mathbf{K})\mathbf{z}$$

Equation (10-13), which is rewritten here, is

$$\mathbf{K} = [\alpha_n - a_n \mid \alpha_{n-1} - a_{n-1} \mid \cdots \mid \alpha_2 - a_2 \mid \alpha_1 - a_1] \mathbf{T}^{-1} \quad (10-161)$$

where

$$\mathbf{T} = \mathbf{M}\mathbf{W} = [\mathbf{C}^* \mid \mathbf{A}^*\mathbf{C}^* \mid \cdots \mid (\mathbf{A}^*)^{n-1}\mathbf{C}^*] \mathbf{W}$$

For the original system, the observability matrix is

$$[\mathbf{C}^* \mid \mathbf{A}^*\mathbf{C}^* \mid \cdots \mid (\mathbf{A}^*)^{n-1}\mathbf{C}^*] = \mathbf{N}$$

Hence, matrix \mathbf{T} can also be written as

$$\mathbf{T} = \mathbf{N}\mathbf{W}$$

Since $\mathbf{W} = \mathbf{W}^*$, we have

$$\mathbf{T}^* = \mathbf{W}^*\mathbf{N}^* = \mathbf{W}\mathbf{N}^*$$

and

$$(\mathbf{T}^*)^{-1} = (\mathbf{W}\mathbf{N}^*)^{-1}$$

Taking the conjugate transpose of both sides of Equation (10-146), we have

$$\mathbf{K}^* = (\mathbf{T}^{-1})^* \begin{bmatrix} \alpha_n - a_n \\ \alpha_{n-1} - a_{n-1} \\ \vdots \\ \alpha_1 - a_1 \end{bmatrix} = (\mathbf{T}^*)^{-1} \begin{bmatrix} \alpha_n - a_n \\ \alpha_{n-1} - a_{n-1} \\ \vdots \\ \alpha_1 - a_1 \end{bmatrix} = (\mathbf{W}\mathbf{N}^*)^{-1} \begin{bmatrix} \alpha_n - a_n \\ \alpha_{n-1} - a_{n-1} \\ \vdots \\ \alpha_1 - a_1 \end{bmatrix}$$

Since $\mathbf{K}_e = \mathbf{K}^*$, this last equation is the same as Equation (10-159). Thus, we obtained Equation (10-159) by considering the dual problem.

A-10-11. Consider an observed-state feedback control system with a minimum-order observer described by the following equations:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \quad (10-162)$$

$$y = \mathbf{C}\mathbf{x}$$

$$u = -\mathbf{K}\tilde{\mathbf{x}} \quad (10-163)$$

where

$$\mathbf{x} = \begin{bmatrix} x_a \\ \mathbf{x}_b \end{bmatrix}, \quad \tilde{\mathbf{x}} = \begin{bmatrix} x_a \\ \tilde{\mathbf{x}}_b \end{bmatrix}$$

(x_a is the state variable that can be directly measured, and $\tilde{\mathbf{x}}_b$ corresponds to the observed state variables.)

Show that the closed-loop poles of the system comprise the closed-loop poles due to pole placement [the eigenvalues of matrix $(\mathbf{A} - \mathbf{B}\mathbf{K})$] and the closed-loop poles due to the minimum-order observer [the eigenvalues of matrix $(\mathbf{A}_{bb} - \mathbf{K}_e\mathbf{A}_{ab})$]

Solution. The error equation for the minimum-order observer may be derived as given by Equation (10-94), rewritten thus:

$$\dot{\mathbf{e}} = (\mathbf{A}_{bb} - \mathbf{K}_e\mathbf{A}_{ab})\mathbf{e} \quad (10-164)$$

where

$$\mathbf{e} = \mathbf{x}_b - \tilde{\mathbf{x}}_b$$

From Equations (10-162) and (10-163), we obtain

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\tilde{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K} \begin{bmatrix} x_a \\ \tilde{\mathbf{x}}_b \end{bmatrix} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K} \begin{bmatrix} x_a \\ \mathbf{x}_b - \mathbf{e} \end{bmatrix} \\ &= \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K} \left\{ \mathbf{x} - \begin{bmatrix} 0 \\ \mathbf{e} \end{bmatrix} \right\} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}\mathbf{K} \begin{bmatrix} 0 \\ \mathbf{e} \end{bmatrix} \end{aligned} \quad (10-165)$$

Combining Equations (10-164) and (10-165) and writing

$$\mathbf{K} = [\mathbf{K}_a \quad \mathbf{K}_b]$$

we obtain

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K} & \mathbf{B}\mathbf{K}_b \\ \mathbf{0} & \mathbf{A}_{bb} - \mathbf{K}_e\mathbf{A}_{ab} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} \quad (10-166)$$

Equation (10-166) describes the dynamics of the observed-state feedback control system with a minimum-order observer. The characteristic equation for this system is

$$\begin{vmatrix} s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K} & -\mathbf{B}\mathbf{K}_b \\ \mathbf{0} & s\mathbf{I} - \mathbf{A}_{bb} + \mathbf{K}_e\mathbf{A}_{ab} \end{vmatrix} = 0$$

or

$$|s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}| |s\mathbf{I} - \mathbf{A}_{bb} + \mathbf{K}_e\mathbf{A}_{ab}| = 0$$

The closed-loop poles of the observed-state feedback control system with a minimum-order observer consist of the closed-loop poles due to pole placement and the closed-loop poles due to the minimum-order observer. (Therefore, the pole-placement design and the design of the minimum-order observer are independent of each other.)

A-10-12. Consider a completely state controllable system defined by

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x}\end{aligned}\tag{10-167}$$

where \mathbf{x} = state vector (n -vector)
 u = control signal (scalar)
 y = output signal (scalar)
 \mathbf{A} = $n \times n$ constant matrix
 \mathbf{B} = $n \times 1$ constant matrix
 \mathbf{C} = $1 \times n$ constant matrix

Suppose that the rank of the following $(n + 1) \times (n + 1)$ matrix

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{C} & 0 \end{bmatrix}$$

is $n + 1$. Show that the system defined by

$$\dot{\mathbf{e}} = \hat{\mathbf{A}}\mathbf{e} + \hat{\mathbf{B}}u_e\tag{10-168}$$

where

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix}, \quad \hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix}, \quad u_e = u(t) - u(\infty)$$

is completely state controllable.

Solution. Define

$$\mathbf{M} = [\mathbf{B} \mid \mathbf{A}\mathbf{B} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}]$$

Because the system given by Equation (10-167) is completely state controllable, the rank of matrix \mathbf{M} is n . Then the rank of

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}$$

is $n + 1$. Consider the following equation:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A}\mathbf{M} & \mathbf{B} \\ -\mathbf{C}\mathbf{M} & 0 \end{bmatrix}\tag{10-169}$$

Since matrix

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{C} & 0 \end{bmatrix}$$

is of rank $n + 1$, the left-hand side of Equation (10-169) is of rank $n + 1$. Therefore, the right-hand side of Equation (10-169) is also of rank $n + 1$. Since

$$\begin{aligned}\begin{bmatrix} \mathbf{A}\mathbf{M} & \mathbf{B} \\ -\mathbf{C}\mathbf{M} & 0 \end{bmatrix} &= \begin{bmatrix} \mathbf{A}[\mathbf{B} \mid \mathbf{A}\mathbf{B} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}] & \mathbf{B} \\ -\mathbf{C}[\mathbf{B} \mid \mathbf{A}\mathbf{B} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}] & 0 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{A}\mathbf{B} \mid \mathbf{A}^2\mathbf{B} \mid \cdots \mid \mathbf{A}^n\mathbf{B} & \mathbf{B} \\ -\mathbf{C}\mathbf{B} \mid -\mathbf{C}\mathbf{A}\mathbf{B} \mid \cdots \mid -\mathbf{C}\mathbf{A}^{n-1}\mathbf{B} & 0 \end{bmatrix} \\ &= [\hat{\mathbf{A}}\hat{\mathbf{B}} \mid \hat{\mathbf{A}}^2\hat{\mathbf{B}} \mid \cdots \mid \hat{\mathbf{A}}^n\hat{\mathbf{B}} \mid \hat{\mathbf{B}}]\end{aligned}$$

we find that the rank of

$$[\hat{\mathbf{B}} \mid \hat{\mathbf{A}}\hat{\mathbf{B}} \mid \hat{\mathbf{A}}^2\hat{\mathbf{B}} \mid \cdots \mid \hat{\mathbf{A}}^{n-1}\hat{\mathbf{B}}]$$

is $n + 1$. Thus, the system defined by Equation (10-168) is completely state controllable.

- A-10-13.** Consider the system shown in Figure 10-49. Using the pole-placement-with-observer approach, design a regulator system such that the system will maintain the zero position ($y_1 = 0$ and $y_2 = 0$) in the presence of disturbances. Choose the desired closed-loop poles for the pole-placement part to be

$$s = -2 + j2\sqrt{3}, \quad s = -2 - j2\sqrt{3}, \quad s = -10, \quad s = -10$$

and the desired poles for the minimum-order observer to be

$$s = -15, \quad s = -16$$

First, determine the state feedback gain matrix \mathbf{K} and observer gain matrix \mathbf{K}_e . Then, obtain the response of the system to an arbitrary initial condition—for example,

$$y_1(0) = 0.1, \quad y_2(0) = 0, \quad \dot{y}_1(0) = 0, \quad \dot{y}_2(0) = 0$$

$$e_1(0) = 0.1, \quad e_2(0) = 0.05$$

where e_1 and e_2 are defined by

$$e_1 = y_1 - \tilde{y}_1$$

$$e_2 = y_2 - \tilde{y}_2$$

Assume that $m_1 = 1$ kg, $m_2 = 2$ kg, $k = 36$ N/m, and $b = 0.6$ N-s/m.

Solution. The equations for the system are

$$m_1\ddot{y}_1 = k(y_2 - y_1) + b(\dot{y}_2 - \dot{y}_1) + u$$

$$m_2\ddot{y}_2 = k(y_1 - y_2) + b(\dot{y}_1 - \dot{y}_2)$$

By substituting the given numerical values for m_1 , m_2 , k , and b and simplifying, we obtain

$$\ddot{y}_1 = -36y_1 + 36y_2 - 0.6\dot{y}_1 + 0.6\dot{y}_2 + u$$

$$\ddot{y}_2 = 18y_1 - 18y_2 + 0.3\dot{y}_1 - 0.3\dot{y}_2$$

Let us choose the state variables as follows:

$$x_1 = y_1$$

$$x_2 = y_2$$

$$x_3 = \dot{y}_1$$

$$x_4 = \dot{y}_2$$

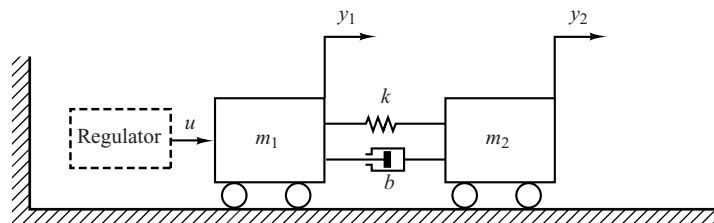


Figure 10-49
Mechanical system.

Then, the state-space equations become

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -36 & 36 & -0.6 & 0.6 \\ 18 & -18 & 0.3 & -0.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Define

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -36 & 36 & -0.6 & 0.6 \\ 18 & -18 & 0.3 & -0.3 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{aa} & \mathbf{A}_{ab} \\ \mathbf{A}_{ba} & \mathbf{A}_{bb} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{B}_a \\ \mathbf{B}_b \end{bmatrix}$$

The state feedback gain matrix \mathbf{K} and observer gain matrix \mathbf{K}_e can be obtained easily by use of MATLAB as follows:

$$\mathbf{K} = [130.4444 \quad -41.5556 \quad 23.1000 \quad 15.4185]$$

$$\mathbf{K}_e = \begin{bmatrix} 14.4 & 0.6 \\ 0.3 & 15.7 \end{bmatrix}$$

(See MATLAB Program 10–26.)

MATLAB Program 10–26

```
A = [0 0 1 0;0 0 0 1;-36 36 -0.6 0.6;18 -18 0.3 -0.3];
B = [0;0;1;0];
J = [-2+j*2*sqrt(3) -2-j*2*sqrt(3) -10 -10];
K = acker(A,B,J)
K =
    130.4444   -41.5556    23.1000    15.4185
Aab = [1 0;0 1];
Abb = [-0.6 0.6;0.3 -0.3];
L = [-15 -16];
Ke = place(Abb',Aab',L)'
place: ndigits= 15
Ke =
    14.4000    0.6000
    0.3000    15.7000
```

Response to Initial Condition: Next, we obtain the response of the designed system to the given initial condition. Since

$$\begin{aligned} \dot{\tilde{\mathbf{x}}} &= \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}u \\ u &= -\mathbf{K}\tilde{\mathbf{x}} \\ \tilde{\mathbf{x}} &= \begin{bmatrix} \mathbf{x}_a \\ \tilde{\mathbf{x}}_b \end{bmatrix} = \begin{bmatrix} \mathbf{y} \\ \tilde{\mathbf{x}}_b \end{bmatrix} \end{aligned}$$

we have

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{BK}\tilde{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x} + \mathbf{BK}(\mathbf{x} - \tilde{\mathbf{x}}) \quad (10-170)$$

Note that

$$\mathbf{x} - \tilde{\mathbf{x}} = \begin{bmatrix} \mathbf{x}_a \\ \mathbf{x}_b \end{bmatrix} - \begin{bmatrix} \mathbf{x}_a \\ \tilde{\mathbf{x}}_b \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{x}_b - \tilde{\mathbf{x}}_b \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{e} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \mathbf{e} = \mathbf{F}\mathbf{e}$$

where

$$\mathbf{F} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}$$

Then, Equation (10-170) can be written as

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x} + \mathbf{BK}\mathbf{F}\mathbf{e} \quad (10-171)$$

Since, from Equation (10-94), we have

$$\dot{\mathbf{e}} = (\mathbf{A}_{bb} - \mathbf{K}_e\mathbf{A}_{ab})\mathbf{e} \quad (10-172)$$

by combining Equations (10-171) and (10-172) into one equation, we have

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BK}\mathbf{F} \\ \mathbf{0} & \mathbf{A}_{bb} - \mathbf{K}_e\mathbf{A}_{ab} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}$$

The state matrix here is a 6×6 matrix. The response of the system to the given initial condition can be obtained easily with MATLAB. (See MATLAB Program 10-27.) The resulting response curves are shown in Figure 10-50. The response curves seem to be acceptable.

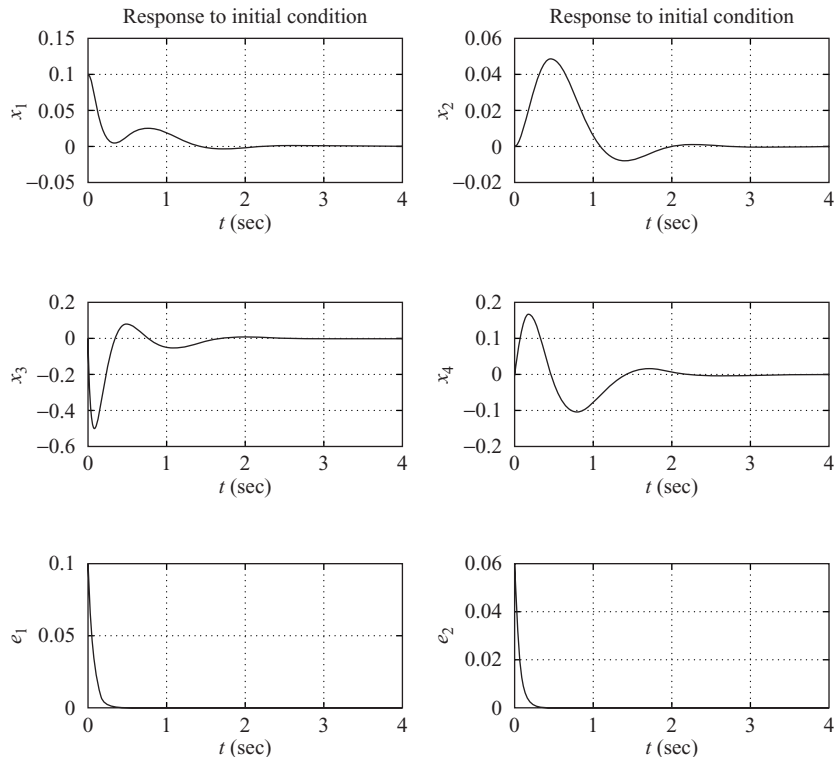


Figure 10-50
Response curves to
initial condition.

MATLAB Program 10–27

```

% Response to initial condition
A = [0 0 1 0;0 0 0 1;-36 36 -0.6 0.6;18 -18 0.3 -0.3];
B = [0;0;1;0];
K = [130.4444 -41.5556 23.1000 15.4185];
Ke = [14.4 0.6;0.3 15.7];
F = [0 0;0 0;1 0;0 1];
Aab = [1 0;0 1];
Abb = [-0.6 0.6;0.3 -0.3];
AA = [A-B*K B*K*F; zeros(2,4) Abb-Ke*Aab];
sys = ss(AA,eye(6),eye(6),eye(6));
t = 0:0.01:4;
y = initial(sys,[0.1;0;0;0;0.1;0.05],t);
x1 = [1 0 0 0 0 0]*y';
x2 = [0 1 0 0 0 0]*y';
x3 = [0 0 1 0 0 0]*y';
x4 = [0 0 0 1 0 0]*y';
e1 = [0 0 0 0 1 0]*y';
e2 = [0 0 0 0 0 1]*y';

subplot(3,2,1); plot(t,x1); grid; title('Response to initial condition'),
xlabel('t (sec)'); ylabel('x1')
subplot(3,2,2); plot(t,x2); grid; title('Response to initial condition'),
xlabel('t (sec)'); ylabel('x2')
subplot(3,2,3); plot(t,x3); grid; xlabel('t (sec)'); ylabel('x3')
subplot(3,2,4); plot(t,x4); grid; xlabel('t (sec)'); ylabel('x4')
subplot(3,2,5); plot(t,e1); grid; xlabel('t (sec)'); ylabel('e1')
subplot(3,2,6); plot(t,e2); grid; xlabel('t (sec)'); ylabel('e2')

```

- A–10–14.** Consider the system shown in Figure 10–51. Design both the full-order and minimum-order observers for the plant. Assume that the desired closed-loop poles for the pole-placement part are located at

$$s = -2 + j2\sqrt{3}, \quad s = -2 - j2\sqrt{3}$$

Assume also that the desired observer poles are located at

- $s = -8, s = -8$ for the full-order observer
- $s = -8$ for the minimum-order observer

Compare the responses to the initial conditions specified below:

- for the full-order observer:

$$x_1(0) = 1, \quad x_2(0) = 0, \quad e_1(0) = 1, \quad e_2(0) = 0$$

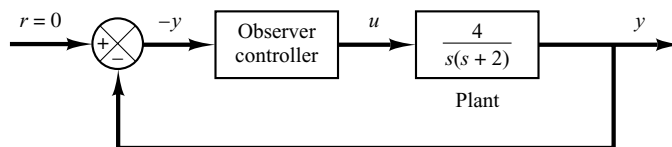


Figure 10–51
Regulator system.

(b) for the minimum-order observer:

$$x_1(0) = 1, \quad x_2(0) = 0, \quad e_1(0) = 1$$

Also, compare the bandwidths of both systems.

Solution. We first determine the state-space representation of the system. By defining state variables x_1 and x_2 as

$$x_1 = y$$

$$x_2 = \dot{y}$$

we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix} u$$

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

For the pole-placement part, we determine the state feedback gain matrix \mathbf{K} . Using MATLAB, we find \mathbf{K} to be

$$\mathbf{K} = [4 \quad 0.5]$$

(See MATLAB Program 10–28.)

Next, we determine the observer gain matrix \mathbf{K}_e for the full-order observer. Using MATLAB, we find \mathbf{K}_e to be

$$\mathbf{K}_e = \begin{bmatrix} 14 \\ 36 \end{bmatrix}$$

(See MATLAB Program 10–28.)

MATLAB Program 10–28

```
% Obtaining matrices K and Ke.
A = [0 1;0 -2];
B = [0;4];
C = [1 0];
J = [-2+j*2*sqrt(3) -2-j*2*sqrt(3)];
L = [-8 -8];
K = acker(A,B,J)

K =

    4.0000    0.5000

Ke = acker(A',C',L)'

Ke =

    14
    36
```

Now we find the response of this system to the given initial condition. Referring to Equation (10–70), we have

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A} - \mathbf{K}_e \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}$$

This equation defines the dynamics of the designed system using the full-order observer. MATLAB Program 10–29 produces the response to the given initial condition. The resulting response curves are shown in Figure 10–52.

MATLAB Program 10–29

```
% Response to initial condition ---- full-order observer
A = [0 1;0 -2];
B = [0;4];
C = [1 0];
K = [4 0.5];
Ke = [14;36];
AA = [A-B*K B*K; zeros(2,2) A-Ke*C];
sys = ss(AA, eye(4), eye(4), eye(4));
t = 0:0.01:8;
x = initial(sys, [1;0;1;0],t);
x1 = [1 0 0 0]*x';
x2 = [0 1 0 0]*x';
e1 = [0 0 1 0]*x';
e2 = [0 0 0 1]*x';

subplot(2,2,1); plot(t,x1); grid
xlabel('t (sec)'); ylabel('x1')

subplot(2,2,2); plot(t,x2); grid
xlabel('t (sec)'); ylabel('x2')

subplot(2,2,3); plot(t,e1); grid
xlabel('t (sec)'); ylabel('e1')

subplot(2,2,4); plot(t,e2); grid
xlabel('t (sec)'); ylabel('e2')
```

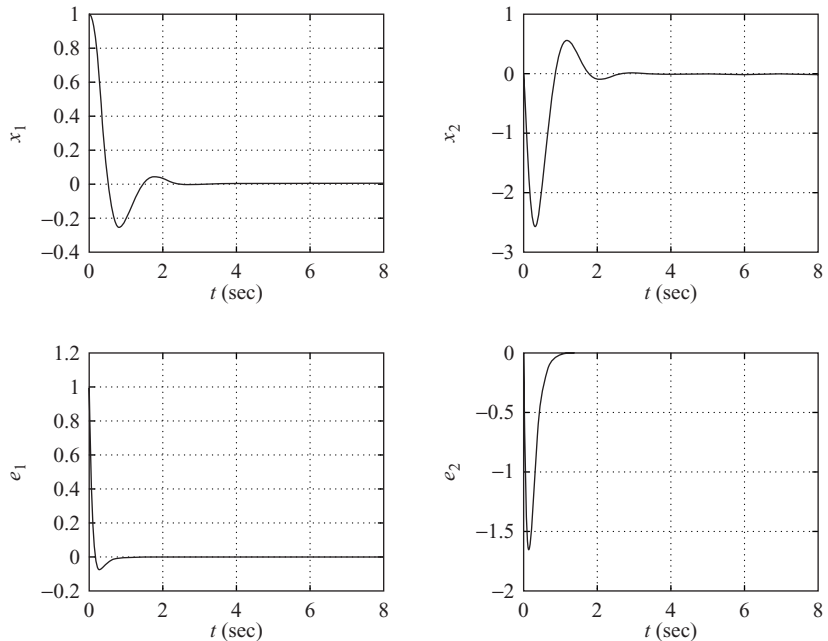


Figure 10–52
Response curves to
initial condition.

To obtain the transfer function of the observer controller, we use MATLAB. MATLAB Program 10–30 produces this transfer function. The result is

$$\frac{\text{num}}{\text{den}} = \frac{74s + 256}{s^2 + 18s + 108} = \frac{74(s + 3.4595)}{(s + 9 + j5.1962)(s + 9 - j5.1962)}$$

MATLAB Program 10–30

```
% Determination of transfer function of observer controller ---- full-order observer
A = [0 1;0 -2];
B = [0;4];
C = [1 0];
K = [4 0.5];
Ke = [14;36];
[num,den] = ss2tf(A-Ke*C-B*K, Ke,K,0)

num =
    0 74.0000 256.0000

den =
    1 18 108
```

Next, we obtain the observer gain matrix K_e for the minimum-order observer. MATLAB Program 10–31 produces K_e . The result is

$$K_e = 6$$

MATLAB Program 10–31

```
% Obtaining Ke ---- minimum-order observer
Aab = [1];
Abb = [-2];
LL = [-8];
Ke = acker(Abb',Aab',LL)'

Ke =
    6
```

The response of the system with minimum-order observer to the initial condition can be obtained as follows: By substituting $u = -\mathbf{K}\tilde{\mathbf{x}}$ into the plant equation given by Equation (10–79)

we find

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} - \mathbf{BK}\tilde{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{BK}\mathbf{x} + \mathbf{BK}(\mathbf{x} - \tilde{\mathbf{x}}) \\ &= (\mathbf{A} - \mathbf{BK})\mathbf{x} + \mathbf{B}\begin{bmatrix} K_a & K_b \end{bmatrix} \begin{bmatrix} 0 \\ e \end{bmatrix}\end{aligned}$$

or

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x} + \mathbf{BK}_b e$$

The error equation is

$$\dot{e} = (A_{bb} - K_e A_{ab})e$$

Hence the system dynamics are defined by

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BK}_b \\ 0 & A_{bb} - K_e A_{ab} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ e \end{bmatrix}$$

Based on this last equation, MATLAB Program 10–32 produces the response to the given initial condition. The resulting response curves are shown in Figure 10–53.

MATLAB Program 10–32

```
% Response to initial condition ---- minimum-order observer
A = [0 1;0 -2];
B = [0;4];
K = [4 0.5];
Kb = 0.5;
Ke = 6;
Aab = 1; Abb = -2;
AA = [A-B*K B*Kb; zeros(1,2) Abb-Ke*Aab];
sys = ss(AA,eye(3),eye(3),eye(3));
t = 0:0.01:8;
x = initial(sys,[1;0;1],t);
x1 = [1 0 0]*x';
x2 = [0 1 0]*x';
e = [0 0 1]*x';

subplot(2,2,1); plot(t,x1); grid
xlabel('t (sec)'); ylabel('x1')

subplot(2,2,2); plot(t,x2); grid
xlabel('t (sec)'); ylabel('x2')

subplot(2,2,3); plot(t,e); grid
xlabel('t (sec)'); ylabel('e')
```

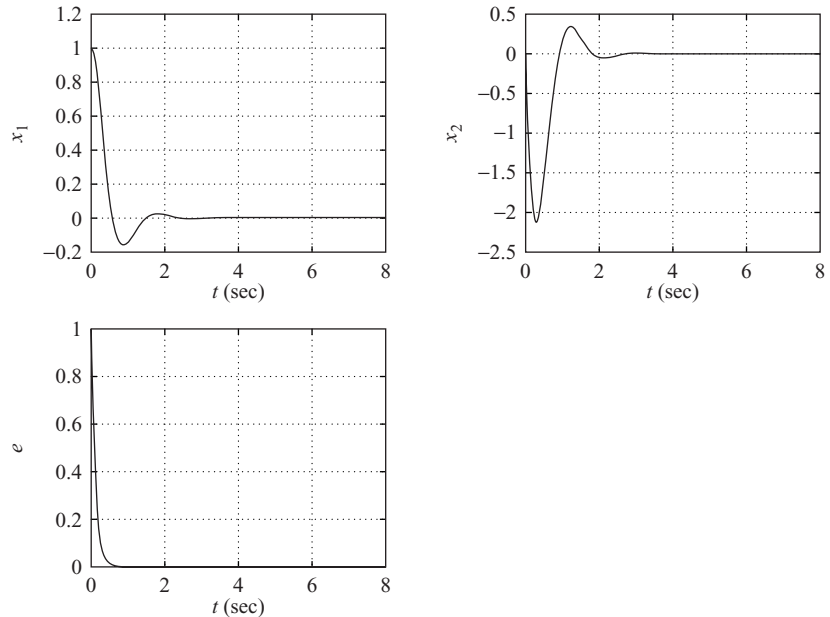


Figure 10-53
Response curves to
initial condition.

The transfer function of the observer controller, when the system uses the minimum-order observer, can be obtained by use of MATLAB Program 10-33. The result is

$$\frac{\text{num}}{\text{den}} = \frac{7s + 32}{s + 10} = \frac{7(s + 4.5714)}{s + 10}$$

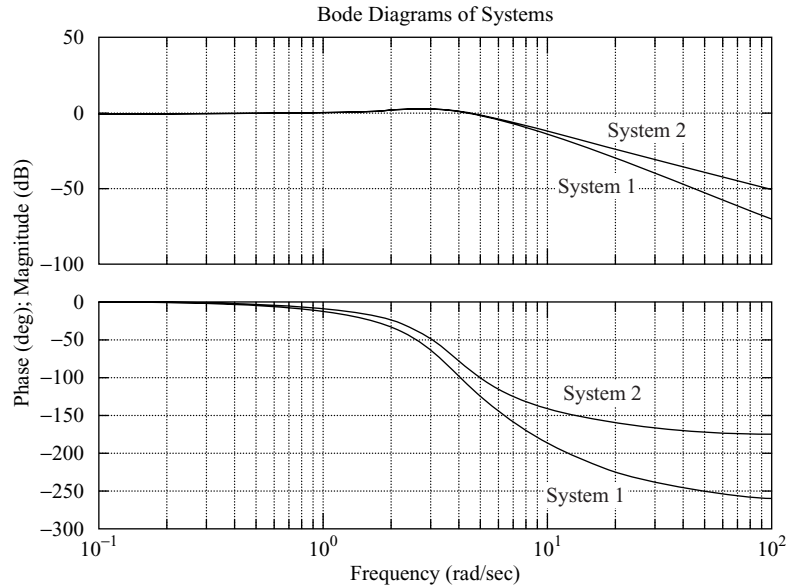
MATLAB Program 10-33

```
% Determination of transfer function of observer controller ---- minimum-order observer
A = [0 1;0 -2];
B = [0;4];
Aaa = 0; Aab = 1; Aba = 0; Abb = -2;
Ba = 0; Bb = 4;
Ka = 4; Kb = 0.5;
Ke = 6;
Ahat = Abb - Ke*Aab;
Bhat = Ahat*Ke + Aba - Ke*Aaa;
Fhat = Bb - Ke*Ba;
Atilde = Ahat - Fhat*Kb;
Btilde = Bhat - Fhat*(Ka + Kb*Ke);
Ctilde = -Kb;
Dtilde = -(Ka + Kb*Ke);
[num,den] = ss2tf(Atilde, Btilde, -Ctilde, -Dtilde)
num =
    7  32
den =
    1  10
```


Figure 10-54

Bode diagrams of System 1 (system with full-order observer) and System 2 (system with minimum-order observer).

$$\begin{aligned} \text{System 1} &= \frac{(296s + 1024)}{(s^4 + 20s^3 + 144s^2 + 512s + 1024)}; \\ \text{System 2} &= \frac{(28s + 128)}{(s^3 + 12s^2 + 48s + 128)}. \end{aligned}$$



The observer controller is clearly a lead compensator.

The Bode diagrams of System 1 (closed-loop system with full-order observer) and of System 2 (closed-loop system with minimum-order observer) are shown in Figure 10-54. Clearly, the bandwidth of System 2 is wider than that of System 1. System 1 has a better high-frequency noise-rejection characteristic than System 2.

A-10-15. Consider the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

where \mathbf{x} is a state vector (n -vector) and \mathbf{A} is an $n \times n$ constant matrix. We assume that \mathbf{A} is non-singular. Prove that if the equilibrium state $\mathbf{x} = \mathbf{0}$ of the system is asymptotically stable (that is, if \mathbf{A} is a stable matrix), then there exists a positive-definite Hermitian matrix \mathbf{P} such that

$$\mathbf{A}^*\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q}$$

where \mathbf{Q} is a positive-definite Hermitian matrix.

Solution. The matrix differential equation,

$$\dot{\mathbf{X}} = \mathbf{A}^*\mathbf{X} + \mathbf{X}\mathbf{A}, \quad \mathbf{X}(0) = \mathbf{Q}$$

has the solution

$$\mathbf{X} = e^{\mathbf{A}^*t}\mathbf{Q}e^{\mathbf{A}t}$$

Integrating both sides of this matrix differential equation from $t = 0$ to $t = \infty$, we obtain

$$\mathbf{X}(\infty) - \mathbf{X}(0) = \mathbf{A}^* \left(\int_0^\infty \mathbf{X} dt \right) + \left(\int_0^\infty \mathbf{X} dt \right) \mathbf{A}$$

Noting that \mathbf{A} is a stable matrix and, therefore, $\mathbf{X}(\infty) = \mathbf{0}$, we obtain

$$-\mathbf{X}(0) = -\mathbf{Q} = \mathbf{A}^* \left(\int_0^\infty \mathbf{X} dt \right) + \left(\int_0^\infty \mathbf{X} dt \right) \mathbf{A}$$

Let us put

$$\mathbf{P} = \int_0^\infty \mathbf{X} dt = \int_0^\infty e^{\mathbf{A}^*t} \mathbf{Q} e^{\mathbf{A}t} dt$$

Note that the elements of $e^{\mathbf{A}t}$ are finite sums of terms like $e^{\lambda_i t}$, $t e^{\lambda_i t}$, ..., $t^{m_i-1} e^{\lambda_i t}$, where the λ_i are the eigenvalues of \mathbf{A} and m_i is the multiplicity of λ_i . Since the λ_i possess negative real parts,

$$\int_0^\infty e^{\mathbf{A}^*t} \mathbf{Q} e^{\mathbf{A}t} dt$$

exists. Note that

$$\mathbf{P}^* = \int_0^\infty e^{\mathbf{A}^*t} \mathbf{Q} e^{\mathbf{A}t} dt = \mathbf{P}$$

Thus \mathbf{P} is Hermitian (or symmetric if \mathbf{P} is a real matrix). We have thus shown that for a stable \mathbf{A} and for a positive-definite Hermitian matrix \mathbf{Q} , there exists a Hermitian matrix \mathbf{P} such that $\mathbf{A}^* \mathbf{P} + \mathbf{P} \mathbf{A} = -\mathbf{Q}$. We now need to prove that \mathbf{P} is positive definite. Consider the following Hermitian form:

$$\begin{aligned} \mathbf{x}^* \mathbf{P} \mathbf{x} &= \mathbf{x}^* \int_0^\infty e^{\mathbf{A}^*t} \mathbf{Q} e^{\mathbf{A}t} dt \mathbf{x} \\ &= \int_0^\infty (e^{\mathbf{A}t} \mathbf{x})^* \mathbf{Q} (e^{\mathbf{A}t} \mathbf{x}) dt > 0, \quad \text{for } \mathbf{x} \neq \mathbf{0} \\ &= 0, \quad \text{for } \mathbf{x} = \mathbf{0} \end{aligned}$$

Hence, \mathbf{P} is positive definite. This completes the proof.

A-10-16. Consider the control system described by

$$\dot{\mathbf{x}} = \mathbf{A} \mathbf{x} + \mathbf{B} u \tag{10-173}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Assuming the linear control law

$$u = -\mathbf{K} \mathbf{x} = -k_1 x_1 - k_2 x_2 \tag{10-174}$$

determine the constants k_1 and k_2 so that the following performance index is minimized:

$$J = \int_0^\infty \mathbf{x}^T \mathbf{x} dt$$

Consider only the case where the initial condition is

$$\mathbf{x}(0) = \begin{bmatrix} c \\ 0 \end{bmatrix}$$

Choose the undamped natural frequency to be 2 rad/sec.

Solution. Substituting Equation (10–174) into Equation (10–173), we obtain

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\mathbf{x}$$

or

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [-k_1 x_1 - k_2 x_2] \\ &= \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \quad (10-175)$$

Thus,

$$\mathbf{A} - \mathbf{B}\mathbf{K} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}$$

Elimination of x_2 from Equation (10–175) yields

$$\ddot{x}_1 + k_2 \dot{x}_1 + k_1 x_1 = 0$$

Since the undamped natural frequency is specified as 2 rad/sec, we obtain

$$k_1 = 4$$

Therefore,

$$\mathbf{A} - \mathbf{B}\mathbf{K} = \begin{bmatrix} 0 & 1 \\ -4 & -k_2 \end{bmatrix}$$

$\mathbf{A} - \mathbf{B}\mathbf{K}$ is a stable matrix if $k_2 > 0$. Our problem now is to determine the value of k_2 so that the performance index

$$J = \int_0^{\infty} \mathbf{x}^T \mathbf{x} dt = \mathbf{x}^T(0) \mathbf{P}(0) \mathbf{x}(0)$$

is minimized, where the matrix \mathbf{P} is determined from Equation (10–115), rewritten

$$(\mathbf{A} - \mathbf{B}\mathbf{K})^* \mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{B}\mathbf{K}) = -(\mathbf{Q} + \mathbf{K}^* \mathbf{R}\mathbf{K})$$

Since in this system $\mathbf{Q} = \mathbf{I}$ and $\mathbf{R} = \mathbf{0}$, this last equation can be simplified to

$$(\mathbf{A} - \mathbf{B}\mathbf{K})^* \mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{B}\mathbf{K}) = -\mathbf{I} \quad (10-176)$$

Since the system involves only real vectors and real matrices, \mathbf{P} becomes a real symmetric matrix. Then Equation (10–176) can be written as

$$\begin{bmatrix} 0 & -4 \\ 1 & -k_2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -4 & -k_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Solving for the matrix \mathbf{P} , we obtain

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} \frac{5}{2k_2} + \frac{k_2}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{5}{8k_2} \end{bmatrix}$$

The performance index is then

$$\begin{aligned} J &= \mathbf{x}^T(0) \mathbf{P} \mathbf{x}(0) \\ &= [c \quad 0] \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} c \\ 0 \end{bmatrix} = p_{11} c^2 \\ &= \left(\frac{5}{2k_2} + \frac{k_2}{8} \right) c^2 \end{aligned} \quad (10-177)$$

To minimize J , we differentiate J with respect to k_2 and set $\partial J / \partial k_2$ equal to zero as follows:

$$\frac{\partial J}{\partial k_2} = \left(\frac{-5}{2k_2^2} + \frac{1}{8} \right) c^2 = 0$$

Hence,

$$k_2 = \sqrt{20}$$

With this value of k_2 , we have $\partial^2 J / \partial k_2^2 > 0$. Thus, the minimum value of J is obtained by substituting $k_2 = \sqrt{20}$ into Equation (10-177), or

$$J_{\min} = \frac{\sqrt{5}}{2} c^2$$

The designed system has the control law

$$u = -4x_1 - \sqrt{20}x_2$$

The designed system is optimal in that it results in a minimum value for the performance index J under the assumed initial condition.

A-10-17. Consider the same inverted-pendulum system as discussed in Example 10-5. The system is shown in Figure 10-8, where $M = 2$ kg, $m = 0.1$ kg, and $l = 0.5$ m. The block diagram for the system is shown in Figure 10-9. The system equations are given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

$$u = -\mathbf{K}\mathbf{x} + k_I \xi$$

$$\dot{\xi} = r - y = r - \mathbf{C}\mathbf{x}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 20.601 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.4905 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0.5 \end{bmatrix}, \quad \mathbf{C} = [0 \ 0 \ 1 \ 0]$$

Referring to Equation (10-51), the error equation for the system is given by

$$\dot{\mathbf{e}} = \hat{\mathbf{A}}\mathbf{e} + \hat{\mathbf{B}}u_e$$

where

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 20.601 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -0.4905 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}, \quad \hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0.5 \\ 0 \end{bmatrix}$$

and the control signal is given by Equation (10-41):

$$u_e = -\hat{\mathbf{K}}\mathbf{e}$$

where

$$\hat{\mathbf{K}} = [\mathbf{K} \ \vdots \ -k_I] = [k_1 \ k_2 \ k_3 \ k_4 \ \vdots \ -k_I]$$

$$\mathbf{e} = \begin{bmatrix} \mathbf{x}_e \\ \xi_e \end{bmatrix} = \begin{bmatrix} \mathbf{x}(t) - \mathbf{x}(\infty) \\ \xi(t) - \xi(\infty) \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix}$$

Using MATLAB, determine the state feedback gain matrix $\hat{\mathbf{K}}$ such that the following performance index J is minimized:

$$J = \int_0^{\infty} (\mathbf{e}^* \mathbf{Q} \mathbf{e} + u^* R u) dt$$

where

$$\mathbf{Q} = \begin{bmatrix} 100 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad R = 0.01$$

Obtain the unit-step response of the system designed.

Solution. A MATLAB program to determine $\hat{\mathbf{K}}$ is given in MATLAB Program 10–34. The result is $k_1 = -188.0799$, $k_2 = -37.0738$, $k_3 = -26.6767$, $k_4 = -30.5824$, $k_I = -10.0000$

MATLAB Program 10–34

```
% Design of quadratic optimal control system
A = [0 1 0 0;20.601 0 0 0;0 0 0 1;-0.4905 0 0 0];
B = [0;-1;0;0.5];
C = [0 0 1 0];
D = [0];
Ahat = [A zeros(4,1);-C 0];
Bhat = [B;0];
Q = [100 0 0 0 0;0 1 0 0 0;0 0 1 0 0;0 0 0 1 0;0 0 0 0 1];
R = [0.01];
Khat = lqr(Ahat,Bhat,Q,R)

Khat =
-188.0799 -37.0738 -26.6767 -30.5824 10.0000
```

Unit-Step Response. Once we have determined the feedback gain matrix \mathbf{K} and the integral gain constant k_I , we can determine the unit-step response of the designed system. The system equation is

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \xi \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r \quad (10-178)$$

[Refer to Equation (10–35).] Since

$$u = -\mathbf{K}\mathbf{x} + k_I\xi$$

Equation (10–178) can be written as follows:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{B}k_I \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \xi \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r \quad (10-179)$$

The output equation is

$$y = [\mathbf{C} \ 0] \begin{bmatrix} \mathbf{x} \\ \xi \end{bmatrix} + [0]r$$

MATLAB Program 10–35 gives the unit-step response of the system given by Equation (10–179). The resulting response curves are presented in Figure 10–55. It shows response curves $\theta [= x_1(t)]$ versus t , $\dot{\theta} [= x_2(t)]$ versus t , $y [= x_3(t)]$ versus t , $\dot{y} [= x_4(t)]$ versus t , and $\xi [= x_5(t)]$ versus t , where the input $r(t)$ to the cart is a unit-step function [$r(t) = 1$ m]. All initial conditions are set equal to zero. Figure 10–56 is an enlarged version of the cart position $y [= x_3(t)]$ versus t . The cart moves backward a very small amount for the first 0.6 sec or so. (Notice that the cart velocity is negative for the first 0.4 sec.) This is due to the fact that the inverted-pendulum-on-the-cart system is a nonminimum-phase system.

MATLAB Program 10–35

```
% Unit-step response
A = [0 1 0 0;20.601 0 0 0;0 0 0 1;-0.4905 0 0 0];
B = [0;-1;0;0.5];
C = [0 0 1 0];
D = [0];
K = [-188.0799 -37.0738 -26.6767 -30.5824];
kl = -10.0000;
AA = [A-B*K B*kl; -C 0];
BB = [0;0;0;0;1];
CC = [C 0];
DD = D;
t = 0:0.01:10;
[y,x,t] = step(AA,BB,CC,DD,1,t);
x1 = [1 0 0 0 0]*x';
x2 = [0 1 0 0 0]*x';
x3 = [0 0 1 0 0]*x';
x4 = [0 0 0 1 0]*x';
x5 = [0 0 0 0 1]*x';

subplot(3,2,1); plot(t,x1); grid;
xlabel('t (sec)'); ylabel('x1')

subplot(3,2,2); plot(t,x2); grid;
xlabel('t (sec)'); ylabel('x2')

subplot(3,2,3); plot(t,x3); grid;
xlabel('t (sec)'); ylabel('x3')

subplot(3,2,4); plot(t,x4); grid;
xlabel('t (sec)'); ylabel('x4')

subplot(3,2,5); plot(t,x5); grid;
xlabel('t (sec)'); ylabel('x5')
```

Comparing the step-response characteristics of this system with those of Example 10–5, we notice that the response of the present system is less oscillatory and exhibits less maximum overshoot in the position response (x_3 versus t). The system designed by use of the quadratic optimal regulator approach generally gives such characteristics—less oscillatory and well damped.

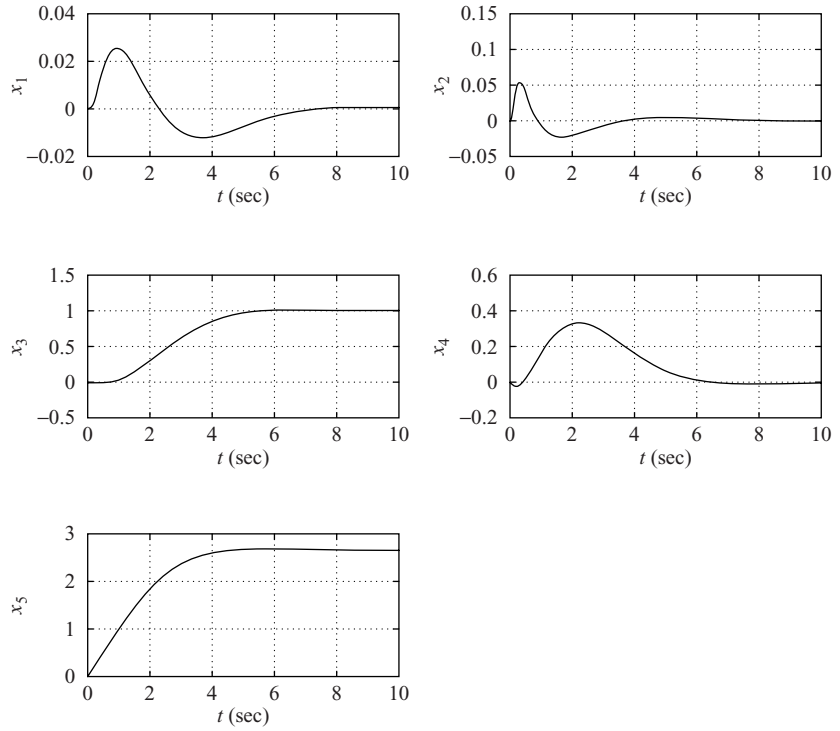


Figure 10-55
Response curves to a unit-step input.

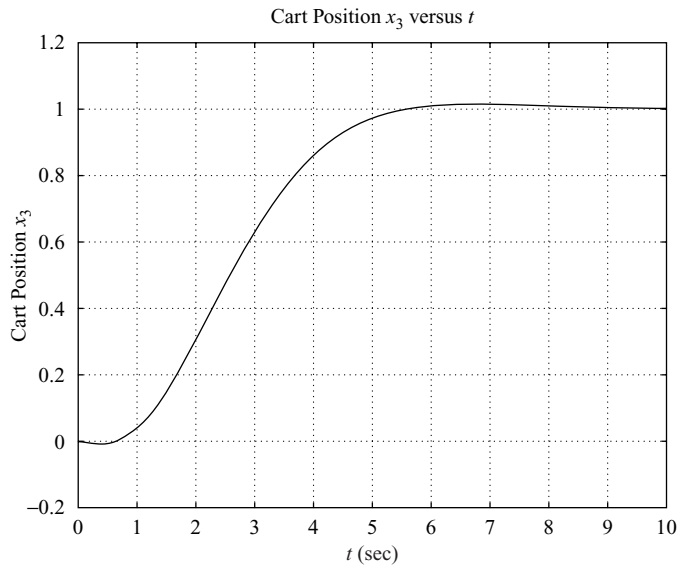
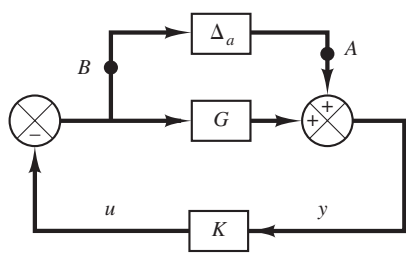


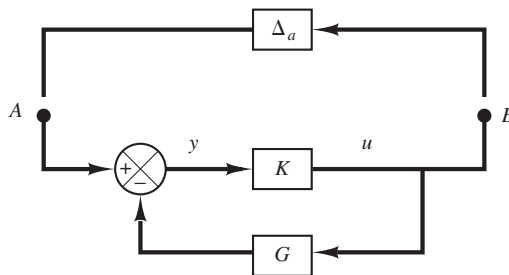
Figure 10-56
Cart position versus t curve.

A-10-18. Consider the stability of a system with unstructured additive uncertainty as shown in Figure 10-57(a). Define

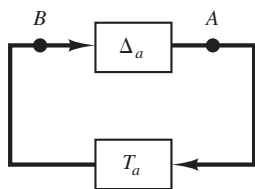
- \tilde{G} = true plant dynamics
- G = model of plant dynamics
- Δ_a = unstructured additive uncertainty



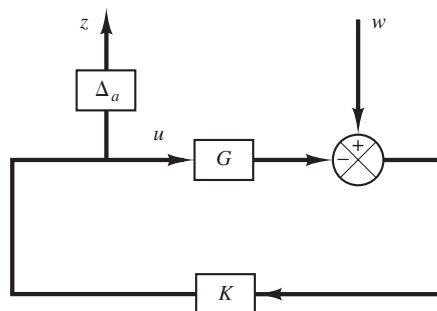
(a)



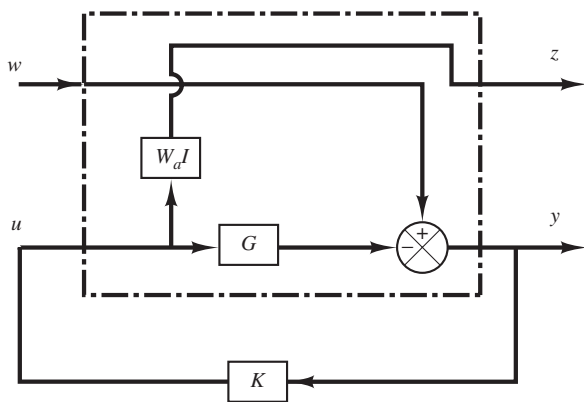
(b)



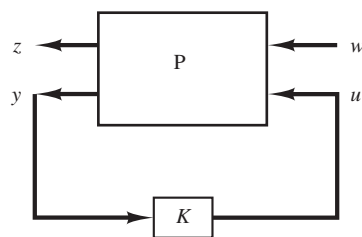
(c)



(d)



(e)



(f)

Figure 10–57

(a) Block diagram of a system with unstructured additive uncertainty;

(b)–(d) successive modifications of the block diagram of (a);

(e) block diagram showing a generalized plant with unstructured additive uncertainty;

(f) generalized plant diagram.

Assume that Δ_a is stable and its upper bound is known. Assume also that \tilde{G} and G are related by

$$\tilde{G} = G + \Delta_a$$

Obtain the condition that the controller K must satisfy for robust stability. Also, obtain a generalized plant diagram for this system.

Solution. Let us obtain the transfer function between point A and point B in Figure 10–57(a). Redrawing Figure 10–57(a), we obtain Figure 10–57(b). Then the transfer function between points A and B can be obtained as

$$\frac{K}{1 + GK} = K(1 + GK)^{-1}$$

Define

$$K(1 + GK)^{-1} = T_a$$

Then Figure 10–57(b) can be redrawn as Figure 10–57(c). By using the small-gain theorem, the condition for the robust stability of the closed-loop system can be obtained as

$$\|\Delta_a T_a\|_\infty < 1 \quad (10-180)$$

Since it is impossible to model Δ_a precisely, we need to find a scalar transfer function $W_a(j\omega)$ such that

$$\bar{\sigma}\{\Delta_a(j\omega)\} < |W_a(j\omega)| \quad \text{for all } \omega$$

and use this $W_a(j\omega)$ instead of Δ_a . Then, the condition for the robust stability of the closed-loop system can be given by

$$\|W_a T_a\|_\infty < 1 \quad (10-181)$$

If Inequality (10–181) holds true, then it is evident that Inequality (10–180) also holds true. So this is the condition to guarantee the robust stability of the designed system. In Figure 10–57(e), Δ_a in Figure 10–57(d) was replaced by $W_a I$.

To summarize, if we make the H_∞ norm of the transfer function from w to z to be less than 1, the controller K that satisfies Inequality (10–181) can be determined.

Figure 10–57(e) can be redrawn as that shown in Figure 10–57(f), which is the generalized plant diagram for the system considered.

Note that for this problem the Φ matrix that relates the controlled variable z and the exogenous disturbance w is given by

$$z = \Phi(s)w = (W_a T_a)w = [W_a K(I + GK)^{-1}]w$$

Noting that $u(s) = K(s)y(s)$ and referring to Equation (10–128), $\Phi(s)$ is given by the elements of the P matrix as follows:

$$\Phi(s) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

To make this $\Phi(s)$ equal to $W_a K(I + GK)^{-1}$, we may choose $P_{11} = 0$, $P_{12} = W_a$, $P_{21} = I$, and $P_{22} = -G$. Then, the P matrix for this problem can be obtained as

$$P = \begin{bmatrix} 0 & W_a \\ I & -G \end{bmatrix}$$

PROBLEMS

B-10-1. Consider the system defined by

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x}\end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 1 \quad 0]$$

Transform the system equations into (a) controllable canonical form and (b) observable canonical form.

B-10-2. Consider the system defined by

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x}\end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 1 \quad 1]$$

Transform the system equations into the observable canonical form.

B-10-3. Consider the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

By using the state-feedback control $u = -\mathbf{K}\mathbf{x}$, it is desired to have the closed-loop poles at $s = -2 \pm j4$, $s = -10$. Determine the state-feedback gain matrix \mathbf{K} .

B-10-4. Solve Problem B-10-3 with MATLAB.

B-10-5. Consider the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

Show that this system cannot be stabilized by the state-feedback control $u = -\mathbf{K}\mathbf{x}$, whatever matrix \mathbf{K} is chosen.

B-10-6. A regulator system has a plant

$$\frac{Y(s)}{U(s)} = \frac{10}{(s+1)(s+2)(s+3)}$$

Define state variables as

$$x_1 = y$$

$$x_2 = \dot{x}_1$$

$$x_3 = \dot{x}_2$$

By use of the state-feedback control $u = -\mathbf{K}\mathbf{x}$, it is desired to place the closed-loop poles at

$$s = -2 + j2\sqrt{3}, \quad s = -2 - j2\sqrt{3}, \quad s = -10$$

Determine the necessary state-feedback gain matrix \mathbf{K} .

B-10-7. Solve Problem B-10-6 with MATLAB.

B-10-8. Consider the type 1 servo system shown in Figure 10-58. Matrices \mathbf{A} , \mathbf{B} , and \mathbf{C} in Figure 10-58 are given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -5 & -6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0 \quad 0]$$

Determine the feedback gain constants k_1 , k_2 , and k_3 such that the closed-loop poles are located at

$$s = -2 + j4, \quad s = -2 - j4, \quad s = -10$$

Obtain the unit-step response and plot the output $y(t)$ -versus- t curve.

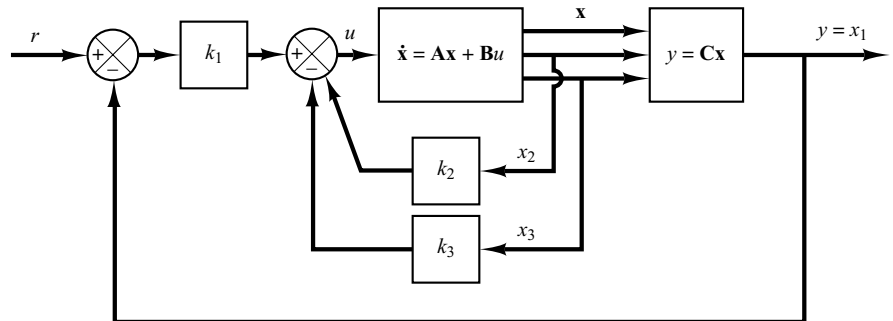


Figure 10-58
Type 1 servo system.

B-10-9. Consider the inverted-pendulum system shown in Figure 10-59. Assume that

$$M = 2 \text{ kg}, \quad m = 0.5 \text{ kg}, \quad l = 1 \text{ m}$$

Define state variables as

$$x_1 = \theta, \quad x_2 = \dot{\theta}, \quad x_3 = x, \quad x_4 = \dot{x}$$

and output variables as

$$y_1 = \theta = x_1, \quad y_2 = x = x_3$$

Derive the state-space equations for this system.

It is desired to have closed-loop poles at

$$s = -4 + j4, \quad s = -4 - j4, \quad s = -20, \quad s = -20$$

Determine the state-feedback gain matrix \mathbf{K} .

Using the state-feedback gain matrix \mathbf{K} thus determined, examine the performance of the system by computer simulation. Write a MATLAB program to obtain the response of the system to an arbitrary initial condition. Obtain the response curves $x_1(t)$ versus t , $x_2(t)$ versus t , $x_3(t)$ versus t , and $x_4(t)$ versus t for the following set of initial condition:

$$x_1(0) = 0, \quad x_2(0) = 0, \quad x_3(0) = 0, \quad x_4(0) = 1 \text{ m/s}$$

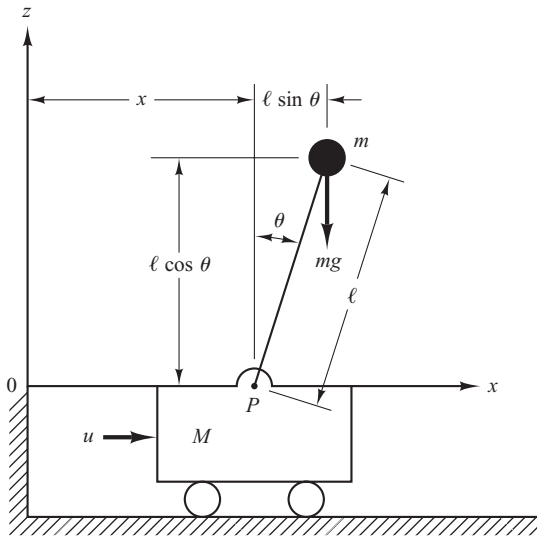


Figure 10-59
Inverted-pendulum system.

B-10-10. Consider the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

$$y = \mathbf{C}\mathbf{x}$$

where

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0]$$

Design a full-order state observer. The desired observer poles are $s = -5$ and $s = -5$.

B-10-11. Consider the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -6 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0 \quad 0]$$

Design a full-order state observer, assuming that the desired poles for the observer are located at

$$s = -10, \quad s = -10, \quad s = -15$$

B-10-12. Consider the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1.244 & 0.3956 & -3.145 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1.244 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Given the set of desired poles for the observer to be

$$s = -5 + j5\sqrt{3}, \quad s = -5 - j5\sqrt{3}, \quad s = -10$$

design a full-order observer.

B-10-13. Consider the double integrator system defined by

$$\ddot{y} = u$$

If we choose the state variables as

$$x_1 = y$$

$$x_2 = \dot{y}$$

then the state-space representation for the system becomes as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

It is desired to design a regulator for this system. Using the pole-placement-with-observer approach, design an observer controller.

Choose the desired closed-loop poles for the pole-placement part to be

$$s = -0.7071 + j0.7071, \quad s = -0.7071 - j0.7071$$

and assuming that we use a minimum-order observer, choose the desired observer pole at

$$s = -5$$

B-10-14. Consider the system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x} \end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0 \quad 0]$$

Design a regulator system by the pole-placement-with-observer approach. Assume that the desired closed-loop poles for pole placement are located at

$$s = -1 + j, \quad s = -1 - j, \quad s = -5$$

The desired observer poles are located at

$$s = -6, \quad s = -6, \quad s = -6$$

Also, obtain the transfer function of the observer controller.

B-10-15. Using the pole-placement-with-observer approach, design observer controllers (one with a full-order observer and the other with a minimum-order observer) for the system shown in Figure 10-60. The desired closed-loop poles for the pole-placement part are

$$s = -1 + j2, \quad s = -1 - j2, \quad s = -5$$

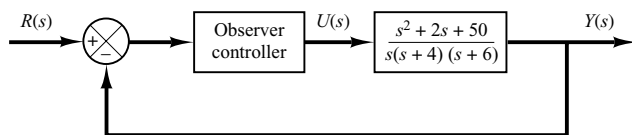


Figure 10-60

Control system with observer controller in the feedforward path.

The desired observer poles are

$$s = -10, \quad s = -10, \quad s = -10 \quad \text{for the full-order observer}$$

$$s = -10, \quad s = -10 \quad \text{for the minimum-order observer.}$$

Compare the unit-step responses of the designed systems. Compare also the bandwidths of both systems.

B-10-16. Using the pole-placement-with-observer approach, design the control systems shown in Figures 10-61(a) and (b). Assume that the desired closed-loop poles for the pole placement are located at

$$s = -2 + j2, \quad s = -2 - j2$$

and the desired observer poles are located at

$$s = -8, \quad s = -8$$

Obtain the transfer function of the observer controller. Compare the unit-step responses of both systems. [In System (b), determine the constant N so that the steady-state output $y(\infty)$ is unity when the input is a unit-step input.]

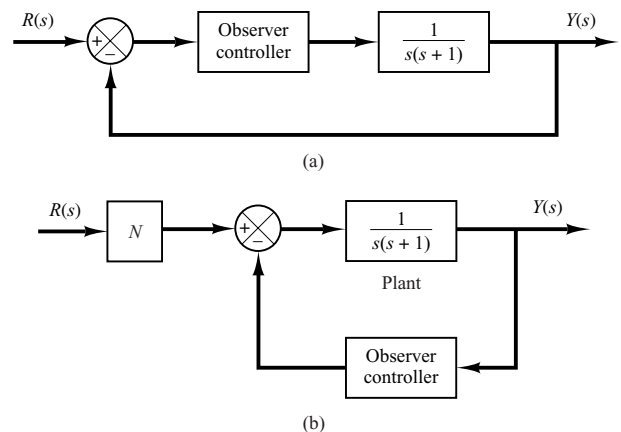


Figure 10-61

Control systems with observer controller: (a) observer controller in the feedforward path; (b) observer controller in the feedback path.

B-10-17. Consider the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -a \end{bmatrix}$$

$a =$ adjustable parameter > 0

Determine the value of the parameter a so as to minimize the following performance index:

$$J = \int_0^{\infty} \mathbf{x}^T \mathbf{x} dt$$

Assume that the initial state $\mathbf{x}(0)$ is given by

$$\mathbf{x}(0) = \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}$$

B-10-18. Consider the system shown in Figure 10-62. Determine the value of the gain K so that the damping ratio ζ of the closed-loop system is equal to 0.5. Then determine also the undamped natural frequency ω_n of the closed-loop system. Assuming that $e(0) = 1$ and $\dot{e}(0) = 0$, evaluate

$$\int_0^{\infty} e^2(t) dt$$

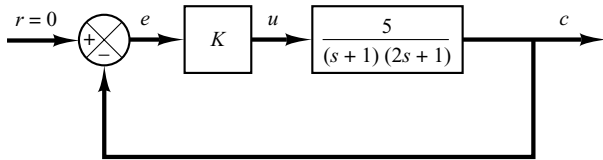


Figure 10-62
Control system.

B-10-19. Determine the optimal control signal u for the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

such that the following performance index is minimized:

$$J = \int_0^{\infty} (\mathbf{x}^T \mathbf{Q} \mathbf{x} + u^2) dt$$

B-10-20. Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

It is desired to find the optimal control signal u such that the performance index

$$J = \int_0^{\infty} (\mathbf{x}^T \mathbf{Q} \mathbf{x} + u^2) dt, \quad \mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix}$$

is minimized. Determine the optimal signal $u(t)$.

B-10-21. Consider the inverted-pendulum system shown in Figure 10-59. It is desired to design a regulator system that will maintain the inverted pendulum in a vertical position in the presence of disturbances in terms of angle θ and/or angular velocity $\dot{\theta}$. The regulator system is required to return the cart to its reference position at the end of each control process. (There is no reference input to the cart.)

The state-space equation for the system is given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 20.601 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.4905 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0.5 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix}$$

We shall use the state-feedback control scheme

$$u = -\mathbf{K}\mathbf{x}$$

Using MATLAB, determine the state-feedback gain matrix $\mathbf{K} = [k_1 \ k_2 \ k_3 \ k_4]$ such that the following performance index J is minimized:

$$J = \int_0^{\infty} (\mathbf{x}^* \mathbf{Q} \mathbf{x} + u^* R u) dt$$

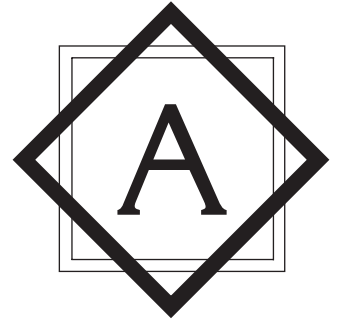
where

$$\mathbf{Q} = \begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad R = 1$$

Then obtain the system response to the following initial condition:

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Plot response curves θ versus t , $\dot{\theta}$ versus t , x versus t , and \dot{x} versus t .



Appendix

Laplace Transform Tables

Appendix A first presents the complex variable and complex function. Then it presents tables of Laplace transform pairs and properties of Laplace transforms. Finally, it presents frequently used Laplace transform theorems and Laplace transforms of pulse function and impulse function.

Complex Variable. A complex number has a real part and an imaginary part, both of which are constant. If the real part and/or imaginary part are variables, a complex quantity is called a *complex variable*. In the Laplace transformation we use the notation s as a complex variable; that is,

$$s = \sigma + j\omega$$

where σ is the real part and ω is the imaginary part.

Complex Function. A complex function $G(s)$, a function of s , has a real part and an imaginary part or

$$G(s) = G_x + jG_y$$

where G_x and G_y are real quantities. The magnitude of $G(s)$ is $\sqrt{G_x^2 + G_y^2}$, and the angle θ of $G(s)$ is $\tan^{-1}(G_y/G_x)$. The angle is measured counterclockwise from the positive real axis. The complex conjugate of $G(s)$ is $\bar{G}(s) = G_x - jG_y$.

Complex functions commonly encountered in linear control systems analysis are single-valued functions of s and are uniquely determined for a given value of s .

A complex function $G(s)$ is said to be *analytic* in a region if $G(s)$ and all its derivatives exist in that region. The derivative of an analytic function $G(s)$ is given by

$$\frac{d}{ds} G(s) = \lim_{\Delta s \rightarrow 0} \frac{G(s + \Delta s) - G(s)}{\Delta s} = \lim_{\Delta s \rightarrow 0} \frac{\Delta G}{\Delta s}$$

Since $\Delta s = \Delta\sigma + j\Delta\omega$, Δs can approach zero along an infinite number of different paths. It can be shown, but is stated without a proof here, that if the derivatives taken along two particular paths, that is, $\Delta s = \Delta\sigma$ and $\Delta s = j\Delta\omega$, are equal, then the derivative is unique for any other path $\Delta s = \Delta\sigma + j\Delta\omega$ and so the derivative exists.

For a particular path $\Delta s = \Delta\sigma$ (which means that the path is parallel to the real axis),

$$\frac{d}{ds} G(s) = \lim_{\Delta\sigma \rightarrow 0} \left(\frac{\Delta G_x}{\Delta\sigma} + j \frac{\Delta G_y}{\Delta\sigma} \right) = \frac{\partial G_x}{\partial\sigma} + j \frac{\partial G_y}{\partial\sigma}$$

For another particular path $\Delta s = j\Delta\omega$ (which means that the path is parallel to the imaginary axis),

$$\frac{d}{ds} G(s) = \lim_{j\Delta\omega \rightarrow 0} \left(\frac{\Delta G_x}{j\Delta\omega} + j \frac{\Delta G_y}{j\Delta\omega} \right) = -j \frac{\partial G_x}{\partial\omega} + \frac{\partial G_y}{\partial\omega}$$

If these two values of the derivative are equal,

$$\frac{\partial G_x}{\partial\sigma} + j \frac{\partial G_y}{\partial\sigma} = \frac{\partial G_y}{\partial\omega} - j \frac{\partial G_x}{\partial\omega}$$

or if the following two conditions

$$\frac{\partial G_x}{\partial\sigma} = \frac{\partial G_y}{\partial\omega} \quad \text{and} \quad \frac{\partial G_y}{\partial\sigma} = - \frac{\partial G_x}{\partial\omega}$$

are satisfied, then the derivative $dG(s)/ds$ is uniquely determined. These two conditions are known as the Cauchy–Riemann conditions. If these conditions are satisfied, the function $G(s)$ is analytic.

As an example, consider the following $G(s)$:

$$G(s) = \frac{1}{s + 1}$$

Then

$$G(\sigma + j\omega) = \frac{1}{\sigma + j\omega + 1} = G_x + jG_y$$

where

$$G_x = \frac{\sigma + 1}{(\sigma + 1)^2 + \omega^2} \quad \text{and} \quad G_y = \frac{-\omega}{(\sigma + 1)^2 + \omega^2}$$

It can be seen that, except at $s = -1$ (that is, $\sigma = -1$, $\omega = 0$), $G(s)$ satisfies the Cauchy–Riemann conditions:

$$\begin{aligned} \frac{\partial G_x}{\partial \sigma} &= \frac{\partial G_y}{\partial \omega} = \frac{\omega^2 - (\sigma + 1)^2}{[(\sigma + 1)^2 + \omega^2]^2} \\ \frac{\partial G_y}{\partial \sigma} &= -\frac{\partial G_x}{\partial \omega} = \frac{2\omega(\sigma + 1)}{[(\sigma + 1)^2 + \omega^2]^2} \end{aligned}$$

Hence $G(s) = 1/(s + 1)$ is analytic in the entire s plane except at $s = -1$. The derivative $dG(s)/ds$, except at $s = -1$, is found to be

$$\begin{aligned} \frac{d}{ds} G(s) &= \frac{\partial G_x}{\partial \sigma} + j \frac{\partial G_y}{\partial \sigma} = \frac{\partial G_y}{\partial \omega} - j \frac{\partial G_x}{\partial \omega} \\ &= -\frac{1}{(\sigma + j\omega + 1)^2} = -\frac{1}{(s + 1)^2} \end{aligned}$$

Note that the derivative of an analytic function can be obtained simply by differentiating $G(s)$ with respect to s . In this example,

$$\frac{d}{ds} \left(\frac{1}{s + 1} \right) = -\frac{1}{(s + 1)^2}$$

Points in the s plane at which the function $G(s)$ is analytic are called *ordinary* points, while points in the s plane at which the function $G(s)$ is not analytic are called *singular* points. Singular points at which the function $G(s)$ or its derivatives approach infinity are called *poles*. Singular points at which the function $G(s)$ equals zero are called *zeros*.

If $G(s)$ approaches infinity as s approaches $-p$ and if the function

$$G(s)(s + p)^n, \quad \text{for } n = 1, 2, 3, \dots$$

has a finite, nonzero value at $s = -p$, then $s = -p$ is called a pole of order n . If $n = 1$, the pole is called a simple pole. If $n = 2, 3, \dots$, the pole is called a second-order pole, a third-order pole, and so on.

To illustrate, consider the complex function

$$G(s) = \frac{K(s + 2)(s + 10)}{s(s + 1)(s + 5)(s + 15)^2}$$

$G(s)$ has zeros at $s = -2, s = -10$, simple poles at $s = 0, s = -1, s = -5$, and a double pole (multiple pole of order 2) at $s = -15$. Note that $G(s)$ becomes zero at $s = \infty$. Since for large values of s

$$G(s) \doteq \frac{K}{s^3}$$

$G(s)$ possesses a triple zero (multiple zero of order 3) at $s = \infty$. If points at infinity are included, $G(s)$ has the same number of poles as zeros. To summarize, $G(s)$ has five zeros ($s = -2, s = -10, s = \infty, s = \infty, s = \infty$) and five poles ($s = 0, s = -1, s = -5, s = -15, s = -15$).

Laplace Transformation. Let us define

$f(t)$ = a function of time t such that $f(t) = 0$ for $t < 0$

s = a complex variable

\mathcal{L} = an operational symbol indicating that the quantity that it prefixes is to be transformed by the Laplace integral $\int_0^\infty e^{-st} dt$

$F(s)$ = Laplace transform of $f(t)$

Then the Laplace transform of $f(t)$ is given by

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st} dt [f(t)] = \int_0^\infty f(t)e^{-st} dt$$

The reverse process of finding the time function $f(t)$ from the Laplace transform $F(s)$ is called the *inverse Laplace transformation*. The notation for the inverse Laplace transformation is \mathcal{L}^{-1} , and the inverse Laplace transform can be found from $F(s)$ by the following inversion integral:

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)e^{st} ds, \quad \text{for } t > 0$$

where c , the abscissa of convergence, is a real constant and is chosen larger than the real parts of all singular points of $F(s)$. Thus, the path of integration is parallel to the $j\omega$ axis and is displaced by the amount c from it. This path of integration is to the right of all singular points.

Evaluating the inversion integral appears complicated. In practice, we seldom use this integral for finding $f(t)$. We frequently use the partial-fraction expansion method given in Appendix B.

In what follows we give Table A-1, which presents Laplace transform pairs of commonly encountered functions, and Table A-2, which presents properties of Laplace transforms.

Table A-1 Laplace Transform Pairs

	$f(t)$	$F(s)$
1	Unit impulse $\delta(t)$	1
2	Unit step $1(t)$	$\frac{1}{s}$
3	t	$\frac{1}{s^2}$
4	$\frac{t^{n-1}}{(n-1)!} \quad (n = 1, 2, 3, \dots)$	$\frac{1}{s^n}$
5	$t^n \quad (n = 1, 2, 3, \dots)$	$\frac{n!}{s^{n+1}}$
6	e^{-at}	$\frac{1}{s+a}$
7	te^{-at}	$\frac{1}{(s+a)^2}$
8	$\frac{1}{(n-1)!} t^{n-1} e^{-at} \quad (n = 1, 2, 3, \dots)$	$\frac{1}{(s+a)^n}$
9	$t^n e^{-at} \quad (n = 1, 2, 3, \dots)$	$\frac{n!}{(s+a)^{n+1}}$
10	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
11	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
12	$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2}$
13	$\cosh \omega t$	$\frac{s}{s^2 - \omega^2}$
14	$\frac{1}{a}(1 - e^{-at})$	$\frac{1}{s(s+a)}$
15	$\frac{1}{b-a}(e^{-at} - e^{-bt})$	$\frac{1}{(s+a)(s+b)}$
16	$\frac{1}{b-a}(be^{-bt} - ae^{-at})$	$\frac{s}{(s+a)(s+b)}$
17	$\frac{1}{ab} \left[1 + \frac{1}{a-b}(be^{-at} - ae^{-bt}) \right]$	$\frac{1}{s(s+a)(s+b)}$

(continues on next page)

Table A-1 (continued)

18	$\frac{1}{a^2}(1 - e^{-at} - ate^{-at})$	$\frac{1}{s(s+a)^2}$
19	$\frac{1}{a^2}(at - 1 + e^{-at})$	$\frac{1}{s^2(s+a)}$
20	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$
21	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2 + \omega^2}$
22	$\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2} t \quad (0 < \zeta < 1)$	$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
23	$-\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2} t - \phi)$ $\phi = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}$ $(0 < \zeta < 1, \quad 0 < \phi < \pi/2)$	$\frac{s}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
24	$1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2} t + \phi)$ $\phi = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}$ $(0 < \zeta < 1, \quad 0 < \phi < \pi/2)$	$\frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$
25	$1 - \cos \omega t$	$\frac{\omega^2}{s(s^2 + \omega^2)}$
26	$\omega t - \sin \omega t$	$\frac{\omega^3}{s^2(s^2 + \omega^2)}$
27	$\sin \omega t - \omega t \cos \omega t$	$\frac{2\omega^3}{(s^2 + \omega^2)^2}$
28	$\frac{1}{2\omega} t \sin \omega t$	$\frac{s}{(s^2 + \omega^2)^2}$
29	$t \cos \omega t$	$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
30	$\frac{1}{\omega_2^2 - \omega_1^2} (\cos \omega_1 t - \cos \omega_2 t) \quad (\omega_1^2 \neq \omega_2^2)$	$\frac{s}{(s^2 + \omega_1^2)(s^2 + \omega_2^2)}$
31	$\frac{1}{2\omega} (\sin \omega t + \omega t \cos \omega t)$	$\frac{s^2}{(s^2 + \omega^2)^2}$

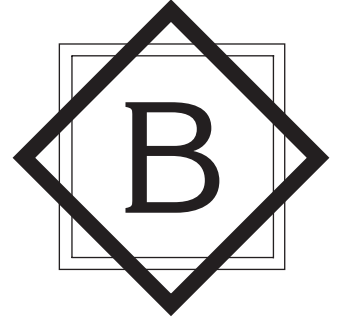
Table A-2 Properties of Laplace Transforms

1	$\mathcal{L}[Af(t)] = AF(s)$
2	$\mathcal{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$
3	$\mathcal{L}_{\pm}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0_{\pm})$
4	$\mathcal{L}_{\pm}\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s) - sf(0_{\pm}) - \dot{f}(0_{\pm})$
5	$\mathcal{L}_{\pm}\left[\frac{d^n}{dt^n}f(t)\right] = s^nF(s) - \sum_{k=1}^n s^{n-k}f^{(k-1)}(0_{\pm})$ where $f^{(k-1)}(t) = \frac{d^{k-1}}{dt^{k-1}}f(t)$
6	$\mathcal{L}_{\pm}\left[\int f(t)dt\right] = \frac{F(s)}{s} + \frac{1}{s}\left[\int f(t)dt\right]_{t=0_{\pm}}$
7	$\mathcal{L}_{\pm}\left[\int \cdots \int f(t)(dt)^n\right] = \frac{F(s)}{s^n} + \sum_{k=1}^n \frac{1}{s^{n-k+1}}\left[\int \cdots \int f(t)(dt)^k\right]_{t=0_{\pm}}$
8	$\mathcal{L}\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s}$
9	$\int_0^{\infty} f(t)dt = \lim_{s \rightarrow 0} F(s)$ if $\int_0^{\infty} f(t)dt$ exists
10	$\mathcal{L}[e^{-\alpha t}f(t)] = F(s + \alpha)$
11	$\mathcal{L}[f(t - \alpha)1(t - \alpha)] = e^{-\alpha s}F(s) \quad \alpha \geq 0$
12	$\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}$
13	$\mathcal{L}[t^2f(t)] = \frac{d^2}{ds^2}F(s)$
14	$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s) \quad (n = 1, 2, 3, \dots)$
15	$\mathcal{L}\left[\frac{1}{t}f(t)\right] = \int_s^{\infty} F(s)ds$ if $\lim_{t \rightarrow 0} \frac{1}{t}f(t)$ exists
16	$\mathcal{L}\left[f\left(\frac{1}{a}\right)\right] = aF(as)$
17	$\mathcal{L}\left[\int_0^t f_1(t - \tau)f_2(\tau)d\tau\right] = F_1(s)F_2(s)$
18	$\mathcal{L}[f(t)g(t)] = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(p)G(s - p)dp$

Finally, we present two frequently used theorems, together with Laplace transforms of the pulse function and impulse function.

Initial value theorem	$f(0+) = \lim_{t \rightarrow 0^+} f(t) = \lim_{s \rightarrow \infty} sF(s)$
Final value theorem	$f(\infty) = \lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$
Pulse function $f(t) = \frac{A}{t_0} 1(t) - \frac{A}{t_0} 1(t - t_0)$	$\mathcal{L}[f(t)] = \frac{A}{t_0 s} - \frac{A}{t_0 s} e^{-st_0}$
Impulse function $g(t) = \lim_{t_0 \rightarrow 0} \frac{A}{t_0}, \quad \text{for } 0 < t < t_0$ $= 0, \quad \text{for } t < 0, t_0 < t$	$\begin{aligned} \mathcal{L}[g(t)] &= \lim_{t_0 \rightarrow 0} \left[\frac{A}{t_0 s} (1 - e^{-st_0}) \right] \\ &= \lim_{t_0 \rightarrow 0} \frac{\frac{d}{dt_0} [A(1 - e^{-st_0})]}{\frac{d}{dt_0} (t_0 s)} \\ &= \frac{As}{s} = A \end{aligned}$

Appendix



Partial-Fraction Expansion

Before we present MATLAB approach to the partial-fraction expansions of transfer functions, we discuss the manual approach to the partial-fraction expansions of transfer functions.

Partial-Fraction Expansion when $F(s)$ Involves Distinct Poles Only. Consider $F(s)$ written in the factored form

$$F(s) = \frac{B(s)}{A(s)} = \frac{K(s + z_1)(s + z_2)\cdots(s + z_m)}{(s + p_1)(s + p_2)\cdots(s + p_n)}, \quad \text{for } m < n$$

where p_1, p_2, \dots, p_n and z_1, z_2, \dots, z_m are either real or complex quantities, but for each complex p_i or z_j there will occur the complex conjugate of p_i or z_j , respectively. If $F(s)$ involves distinct poles only, then it can be expanded into a sum of simple partial fractions as follows:

$$F(s) = \frac{B(s)}{A(s)} = \frac{a_1}{s + p_1} + \frac{a_2}{s + p_2} + \cdots + \frac{a_n}{s + p_n} \quad (\text{B-1})$$

where a_k ($k = 1, 2, \dots, n$) are constants. The coefficient a_k is called the *residue* at the pole at $s = -p_k$. The value of a_k can be found by multiplying both sides of Equation (B-1) by $(s + p_k)$ and letting $s = -p_k$, which gives

$$\begin{aligned} \left[(s + p_k) \frac{B(s)}{A(s)} \right]_{s=-p_k} &= \left[\frac{a_1}{s + p_1} (s + p_k) + \frac{a_2}{s + p_2} (s + p_k) \right. \\ &\quad \left. + \cdots + \frac{a_k}{s + p_k} (s + p_k) + \cdots + \frac{a_n}{s + p_n} (s + p_k) \right]_{s=-p_k} \\ &= a_k \end{aligned}$$

We see that all the expanded terms drop out with the exception of a_k . Thus the residue a_k is found from

$$a_k = \left[(s + p_k) \frac{B(s)}{A(s)} \right]_{s=-p_k}$$

Note that, since $f(t)$ is a real function of time, if p_1 and p_2 are complex conjugates, then the residues a_1 and a_2 are also complex conjugates. Only one of the conjugates, a_1 or a_2 , needs to be evaluated, because the other is known automatically.

Since

$$\mathcal{L}^{-1} \left[\frac{a_k}{s + p_k} \right] = a_k e^{-p_k t}$$

$f(t)$ is obtained as

$$f(t) = \mathcal{L}^{-1}[F(s)] = a_1 e^{-p_1 t} + a_2 e^{-p_2 t} + \cdots + a_n e^{-p_n t}, \quad \text{for } t \geq 0$$

EXAMPLE B-1 Find the inverse Laplace transform of

$$F(s) = \frac{s + 3}{(s + 1)(s + 2)}$$

The partial-fraction expansion of $F(s)$ is

$$F(s) = \frac{s + 3}{(s + 1)(s + 2)} = \frac{a_1}{s + 1} + \frac{a_2}{s + 2}$$

where a_1 and a_2 are found as

$$a_1 = \left[(s + 1) \frac{s + 3}{(s + 1)(s + 2)} \right]_{s=-1} = \left[\frac{s + 3}{s + 2} \right]_{s=-1} = 2$$

$$a_2 = \left[(s + 2) \frac{s + 3}{(s + 1)(s + 2)} \right]_{s=-2} = \left[\frac{s + 3}{s + 1} \right]_{s=-2} = -1$$

Thus

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[F(s)] \\ &= \mathcal{L}^{-1} \left[\frac{2}{s + 1} \right] + \mathcal{L}^{-1} \left[\frac{-1}{s + 2} \right] \\ &= 2e^{-t} - e^{-2t}, \quad \text{for } t \geq 0 \end{aligned}$$

EXAMPLE B-2 Obtain the inverse Laplace transform of

$$G(s) = \frac{s^3 + 5s^2 + 9s + 7}{(s + 1)(s + 2)}$$

Here, since the degree of the numerator polynomial is higher than that of the denominator polynomial, we must divide the numerator by the denominator.

$$G(s) = s + 2 + \frac{s + 3}{(s + 1)(s + 2)}$$

Note that the Laplace transform of the unit-impulse function $\delta(t)$ is 1 and that the Laplace transform of $d\delta(t)/dt$ is s . The third term on the right-hand side of this last equation is $F(s)$ in Example B-1. So the inverse Laplace transform of $G(s)$ is given as

$$g(t) = \frac{d}{dt}\delta(t) + 2\delta(t) + 2e^{-t} - e^{-2t}, \quad \text{for } t \geq 0-$$

EXAMPLE B-3 Find the inverse Laplace transform of

$$F(s) = \frac{2s + 12}{s^2 + 2s + 5}$$

Notice that the denominator polynomial can be factored as

$$s^2 + 2s + 5 = (s + 1 + j2)(s + 1 - j2)$$

If the function $F(s)$ involves a pair of complex-conjugate poles, it is convenient not to expand $F(s)$ into the usual partial fractions but to expand it into the sum of a damped sine and a damped cosine function.

Noting that $s^2 + 2s + 5 = (s + 1)^2 + 2^2$ and referring to the Laplace transforms of $e^{-\alpha t} \sin \omega t$ and $e^{-\alpha t} \cos \omega t$, rewritten thus,

$$\mathcal{L}[e^{-\alpha t} \sin \omega t] = \frac{\omega}{(s + \alpha)^2 + \omega^2}$$

$$\mathcal{L}[e^{-\alpha t} \cos \omega t] = \frac{s + \alpha}{(s + \alpha)^2 + \omega^2}$$

the given $F(s)$ can be written as a sum of a damped sine and a damped cosine function:

$$\begin{aligned} F(s) &= \frac{2s + 12}{s^2 + 2s + 5} = \frac{10 + 2(s + 1)}{(s + 1)^2 + 2^2} \\ &= 5 \frac{2}{(s + 1)^2 + 2^2} + 2 \frac{s + 1}{(s + 1)^2 + 2^2} \end{aligned}$$

It follows that

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[F(s)] \\ &= 5\mathcal{L}^{-1}\left[\frac{2}{(s + 1)^2 + 2^2}\right] + 2\mathcal{L}^{-1}\left[\frac{s + 1}{(s + 1)^2 + 2^2}\right] \\ &= 5e^{-t} \sin 2t + 2e^{-t} \cos 2t, \quad \text{for } t \geq 0 \end{aligned}$$

Partial-Fraction Expansion when $F(s)$ Involves Multiple Poles. Instead of discussing the general case, we shall use an example to show how to obtain the partial-fraction expansion of $F(s)$.

Consider the following $F(s)$:

$$F(s) = \frac{s^2 + 2s + 3}{(s + 1)^3}$$

The partial-fraction expansion of this $F(s)$ involves three terms,

$$F(s) = \frac{B(s)}{A(s)} = \frac{b_1}{s + 1} + \frac{b_2}{(s + 1)^2} + \frac{b_3}{(s + 1)^3}$$

where b_3 , b_2 , and b_1 are determined as follows. By multiplying both sides of this last equation by $(s + 1)^3$, we have

$$(s + 1)^3 \frac{B(s)}{A(s)} = b_1(s + 1)^2 + b_2(s + 1) + b_3 \quad (\text{B-2})$$

Then letting $s = -1$, Equation (B-2) gives

$$\left[(s + 1)^3 \frac{B(s)}{A(s)} \right]_{s=-1} = b_3$$

Also, differentiation of both sides of Equation (B-2) with respect to s yields

$$\frac{d}{ds} \left[(s + 1)^3 \frac{B(s)}{A(s)} \right] = b_2 + 2b_1(s + 1) \quad (\text{B-3})$$

If we let $s = -1$ in Equation (B-3), then

$$\frac{d}{ds} \left[(s + 1)^3 \frac{B(s)}{A(s)} \right]_{s=-1} = b_2$$

By differentiating both sides of Equation (B-3) with respect to s , the result is

$$\frac{d^2}{ds^2} \left[(s + 1)^3 \frac{B(s)}{A(s)} \right] = 2b_1$$

From the preceding analysis it can be seen that the values of b_3 , b_2 , and b_1 are found systematically as follows:

$$\begin{aligned} b_3 &= \left[(s + 1)^3 \frac{B(s)}{A(s)} \right]_{s=-1} \\ &= (s^2 + 2s + 3)_{s=-1} \\ &= 2 \\ b_2 &= \left\{ \frac{d}{ds} \left[(s + 1)^3 \frac{B(s)}{A(s)} \right] \right\}_{s=-1} \\ &= \left[\frac{d}{ds} (s^2 + 2s + 3) \right]_{s=-1} \\ &= (2s + 2)_{s=-1} \\ &= 0 \\ b_1 &= \frac{1}{2!} \left\{ \frac{d^2}{ds^2} \left[(s + 1)^3 \frac{B(s)}{A(s)} \right] \right\}_{s=-1} \\ &= \frac{1}{2!} \left[\frac{d^2}{ds^2} (s^2 + 2s + 3) \right]_{s=-1} \\ &= \frac{1}{2} (2) = 1 \end{aligned}$$

We thus obtain

$$\begin{aligned}
 f(t) &= \mathcal{L}^{-1}[F(s)] \\
 &= \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{0}{(s+1)^2}\right] + \mathcal{L}^{-1}\left[\frac{2}{(s+1)^3}\right] \\
 &= e^{-t} + 0 + t^2 e^{-t} \\
 &= (1 + t^2)e^{-t}, \quad \text{for } t \geq 0
 \end{aligned}$$

Comments. For complicated functions with denominators involving higher-order polynomials, partial-fraction expansion may be quite time consuming. In such a case, use of MATLAB is recommended.

Partial-Fraction Expansion with MATLAB. MATLAB has a command to obtain the partial-fraction expansion of $B(s)/A(s)$. Consider the following function $B(s)/A(s)$:

$$\frac{B(s)}{A(s)} = \frac{\text{num}}{\text{den}} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

where some of a_i and b_j may be zero. In MATLAB row vectors num and den specify the coefficients of the numerator and denominator of the transfer function. That is,

$$\begin{aligned}
 \text{num} &= [b_0 \ b_1 \ \dots \ b_n] \\
 \text{den} &= [1 \ a_1 \ \dots \ a_n]
 \end{aligned}$$

The command

$$[r,p,k] = \text{residue}(\text{num},\text{den})$$

finds the residues (r), poles (p), and direct terms (k) of a partial-fraction expansion of the ratio of two polynomials $B(s)$ and $A(s)$.

The partial-fraction expansion of $B(s)/A(s)$ is given by

$$\frac{B(s)}{A(s)} = \frac{r(1)}{s-p(1)} + \frac{r(2)}{s-p(2)} + \cdots + \frac{r(n)}{s-p(n)} + k(s) \quad (\text{B-4})$$

Comparing Equations (B-1) and (B-4), we note that $p(1) = -p_1$, $p(2) = -p_2, \dots$, $p(n) = -p_n$; $r(1) = a_1$, $r(2) = a_2, \dots$, $r(n) = a_n$. [$k(s)$ is a direct term.]

EXAMPLE B-4 Consider the following transfer function,

$$\frac{B(s)}{A(s)} = \frac{2s^3 + 5s^2 + 3s + 6}{s^3 + 6s^2 + 11s + 6}$$

For this function,

$$\begin{aligned}\text{num} &= [2 \ 5 \ 3 \ 6] \\ \text{den} &= [1 \ 6 \ 11 \ 6]\end{aligned}$$

The command

$$[r,p,k] = \text{residue}(\text{num},\text{den})$$

gives the following result:

```
[r,p,k] = residue(num,den)
r =
   -6.0000
   -4.0000
    3.0000
p =
   -3.0000
   -2.0000
   -1.0000
k =
     2
```

(Note that the residues are returned in column vector r , the pole locations in column vector p , and the direct term in row vector k .) This is the MATLAB representation of the following partial-fraction expansion of $B(s)/A(s)$:

$$\begin{aligned}\frac{B(s)}{A(s)} &= \frac{2s^3 + 5s^2 + 3s + 6}{s^3 + 6s^2 + 11s + 6} \\ &= \frac{-6}{s + 3} + \frac{-4}{s + 2} + \frac{3}{s + 1} + 2\end{aligned}$$

Note that if $p(j) = p(j + 1) = \dots = p(j + m - 1)$ [that is, $p_j = p_{j+1} = \dots = p_{j+m-1}$], the pole $p(j)$ is a pole of multiplicity m . In such a case, the expansion includes terms of the form

$$\frac{r(j)}{s - p(j)} + \frac{r(j + 1)}{[s - p(j)]^2} + \dots + \frac{r(j + m - 1)}{[s - p(j)]^m}$$

For details, see Example B-5.

EXAMPLE B-5 Expand the following $B(s)/A(s)$ into partial fractions with MATLAB.

$$\frac{B(s)}{A(s)} = \frac{s^2 + 2s + 3}{(s + 1)^3} = \frac{s^2 + 2s + 3}{s^3 + 3s^2 + 3s + 1}$$

For this function, we have

$$\begin{aligned}\text{num} &= [1 \ 2 \ 3] \\ \text{den} &= [1 \ 3 \ 3 \ 1]\end{aligned}$$

The command

$$[r,p,k] = \text{residue}(\text{num},\text{den})$$

gives the result shown next:

```
num = [1 2 3];
den = [1 3 3 1];
[r,p,k] = residue(num,den)

r =

    1.0000
    0.0000
    2.0000

p =

   -1.0000
   -1.0000
   -1.0000

k =

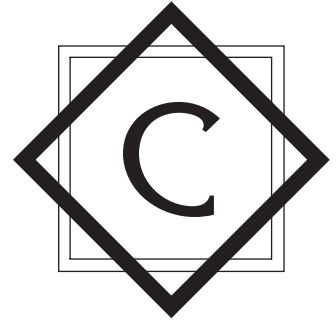
    []
```

It is the MATLAB representation of the following partial-fraction expansion of $B(s)/A(s)$:

$$\frac{B(s)}{A(s)} = \frac{1}{s + 1} + \frac{0}{(s + 1)^2} + \frac{2}{(s + 1)^3}$$

Note that the direct term k is zero.

Appendix



Vector-Matrix Algebra

In this appendix we first review the determinant of a matrix, then we define the adjoint matrix, the inverse of a matrix, and the derivative and integral of a matrix.

Determinant of a Matrix. For each square matrix, there exists a determinant. The determinant of a square matrix \mathbf{A} is usually written as $|\mathbf{A}|$ or $\det \mathbf{A}$. The determinant has the following properties:

1. If any two consecutive rows or columns are interchanged, the determinant changes its sign.
2. If any row or any column consists only of zeros, then the value of the dererminant is zero.
3. If the elements of any row (or any column) are exactly k times those of another row (or another column), then the value of the determinant is zero.
4. If, to any row (or any column), any constant times another row (or column) is added, the value of the determinant remains unchanged.
5. If a determinant is multiplied by a constant, then only one row (or one column) is multiplied by that constant. Note, however, that the determinant of k times an $n \times n$ matrix \mathbf{A} is k^n times the determinant of \mathbf{A} , or

$$|k\mathbf{A}| = k^n|\mathbf{A}|$$

This is because

$$k\mathbf{A} = \begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1m} \\ ka_{21} & ka_{22} & \dots & ka_{2m} \\ \vdots & \vdots & \dots & \vdots \\ ka_{n1} & ka_{n2} & \dots & ka_{nm} \end{bmatrix}$$

6. The determinant of the product of two square matrices \mathbf{A} and \mathbf{B} is the product of determinants, or

$$|\mathbf{AB}| = |\mathbf{A}| |\mathbf{B}|$$

If $\mathbf{B} = n \times m$ matrix and $\mathbf{C} = m \times n$ matrix, then

$$\det(\mathbf{I}_n + \mathbf{BC}) = \det(\mathbf{I}_m + \mathbf{CB})$$

If $\mathbf{A} \neq \mathbf{0}$ and $\mathbf{D} = m \times m$ matrix, then

$$\det \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \det \mathbf{A} \cdot \det \mathbf{S}$$

where $\mathbf{S} = \mathbf{D} - \mathbf{CA}^{-1}\mathbf{B}$.

If $\mathbf{D} \neq \mathbf{0}$, then

$$\det \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \det \mathbf{D} \cdot \det \mathbf{T}$$

where $\mathbf{T} = \mathbf{A} - \mathbf{BD}^{-1}\mathbf{C}$.

If $\mathbf{B} = \mathbf{0}$ or $\mathbf{C} = \mathbf{0}$, then

$$\det \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \det \mathbf{A} \cdot \det \mathbf{D}$$

$$\det \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{D} \end{bmatrix} = \det \mathbf{A} \cdot \det \mathbf{D}$$

Rank of Matrix. A matrix \mathbf{A} is said to have rank m if there exists an $m \times m$ submatrix \mathbf{M} of \mathbf{A} such that the determinant of \mathbf{M} is nonzero and the determinant of every $r \times r$ submatrix (where $r \geq m + 1$) of \mathbf{A} is zero.

As an example, consider the following matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 2 \end{bmatrix}$$

Note that $|\mathbf{A}| = 0$. One of a number of largest submatrices whose determinant is not equal to zero is

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

Hence, the rank of the matrix \mathbf{A} is 3.

Minor M_{ij} . If the i th row and j th column are deleted from an $n \times n$ matrix \mathbf{A} , the resulting matrix is an $(n - 1) \times (n - 1)$ matrix. The determinant of this $(n - 1) \times (n - 1)$ matrix is called the minor M_{ij} of the matrix \mathbf{A} .

Cofactor A_{ij} . The cofactor A_{ij} of the element a_{ij} of the $n \times n$ matrix \mathbf{A} is defined by the equation

$$A_{ij} = (-1)^{i+j} M_{ij}$$

That is, the cofactor A_{ij} of the element a_{ij} is $(-1)^{i+j}$ times the determinant of the matrix formed by deleting the i th row and the j th column from \mathbf{A} . Note that the cofactor A_{ij} of the element a_{ij} is the coefficient of the term a_{ij} in the expansion of the determinant $|\mathbf{A}|$, since it can be shown that

$$a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} = |\mathbf{A}|$$

If $a_{i1}, a_{i2}, \dots, a_{in}$ are replaced by $a_{j1}, a_{j2}, \dots, a_{jn}$, then

$$a_{j1}A_{i1} + a_{j2}A_{i2} + \cdots + a_{jn}A_{in} = 0 \quad i \neq j$$

because the determinant of \mathbf{A} in this case possesses two identical rows. Hence, we obtain

$$\sum_{k=1}^n a_{jk}A_{ik} = \delta_{ji}|\mathbf{A}|$$

Similarly,

$$\sum_{k=1}^n a_{ki}A_{kj} = \delta_{ij}|\mathbf{A}|$$

Adjoint Matrix. The matrix \mathbf{B} whose element in the i th row and j th column equals A_{ji} is called the adjoint of \mathbf{A} and is denoted by $\text{adj } \mathbf{A}$, or

$$\mathbf{B} = (b_{ij}) = (A_{ji}) = \text{adj } \mathbf{A}$$

That is, the adjoint of \mathbf{A} is the transpose of the matrix whose elements are the cofactors of \mathbf{A} , or

$$\text{adj } \mathbf{A} = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

Note that the element of the j th row and i th column of the product $\mathbf{A}(\text{adj } \mathbf{A})$ is

$$\sum_{k=1}^n a_{jk}b_{ki} = \sum_{k=1}^n a_{jk}A_{ik} = \delta_{ji}|\mathbf{A}|$$

Hence, $\mathbf{A}(\text{adj } \mathbf{A})$ is a diagonal matrix with diagonal elements equal to $|\mathbf{A}|$, or

$$\mathbf{A}(\text{adj } \mathbf{A}) = |\mathbf{A}|\mathbf{I}$$

Similarly, the element in the j th row and i th column of the product $(\text{adj } \mathbf{A})\mathbf{A}$ is

$$\sum_{k=1}^n b_{jk}a_{ki} = \sum_{k=1}^n A_{kj}a_{ki} = \delta_{ij}|\mathbf{A}|$$

Hence, we have the relationship

$$\mathbf{A}(\text{adj } \mathbf{A}) = (\text{adj } \mathbf{A})\mathbf{A} = |\mathbf{A}|\mathbf{I} \quad (\text{C-1})$$

Thus

$$\mathbf{A}^{-1} = \frac{\text{adj } \mathbf{A}}{|\mathbf{A}|} = \begin{bmatrix} \frac{A_{11}}{|\mathbf{A}|} & \frac{A_{21}}{|\mathbf{A}|} & \cdots & \frac{A_{n1}}{|\mathbf{A}|} \\ \frac{A_{12}}{|\mathbf{A}|} & \frac{A_{22}}{|\mathbf{A}|} & \cdots & \frac{A_{n2}}{|\mathbf{A}|} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{A_{1n}}{|\mathbf{A}|} & \frac{A_{2n}}{|\mathbf{A}|} & \cdots & \frac{A_{nn}}{|\mathbf{A}|} \end{bmatrix}$$

where A_{ij} is the cofactor of a_{ij} of the matrix \mathbf{A} . Thus, the terms in the i th column of \mathbf{A}^{-1} are $1/|\mathbf{A}|$ times the cofactors of the i th row of the original matrix \mathbf{A} . For example, if

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & -2 \\ 1 & 0 & -3 \end{bmatrix}$$

then the adjoint of \mathbf{A} and the determinant $|\mathbf{A}|$ are respectively found to be

$$\begin{aligned} \text{adj } \mathbf{A} &= \begin{bmatrix} \begin{vmatrix} -1 & -2 \\ 0 & -3 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 0 & -3 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ -1 & -2 \end{vmatrix} \\ -\begin{vmatrix} 3 & -2 \\ 1 & -3 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 1 & -3 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 3 & -2 \end{vmatrix} \\ \begin{vmatrix} 3 & -1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 3 & 6 & -4 \\ 7 & -3 & 2 \\ 1 & 2 & -7 \end{bmatrix} \end{aligned}$$

and

$$|\mathbf{A}| = 17$$

Hence, the inverse of \mathbf{A} is

$$\mathbf{A}^{-1} = \frac{\text{adj } \mathbf{A}}{|\mathbf{A}|} = \begin{bmatrix} \frac{3}{17} & \frac{6}{17} & -\frac{4}{17} \\ \frac{7}{17} & -\frac{3}{17} & \frac{2}{17} \\ \frac{1}{17} & \frac{2}{17} & -\frac{7}{17} \end{bmatrix}$$

In what follows, we give formulas for finding inverse matrices for the 2×2 matrix and the 3×3 matrix. For the 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{where } ad - bc \neq 0$$

the inverse matrix is given by

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

For the 3×3 matrix

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad \text{where } |\mathbf{A}| \neq 0$$

the inverse matrix is given by

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & \begin{vmatrix} b & c \\ e & f \end{vmatrix} \\ -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & \begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} \\ \begin{vmatrix} d & e \\ g & h \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$$

Note that

$$\begin{aligned} (\mathbf{A}^{-1})^{-1} &= \mathbf{A} \\ (\mathbf{A}^{-1})' &= (\mathbf{A}')^{-1} \\ (\mathbf{A}^{-1})^* &= (\mathbf{A}^*)^{-1} \end{aligned}$$

There are several more useful formulas available. Assume that $\mathbf{A} = n \times n$ matrix, $\mathbf{B} = n \times m$ matrix, $\mathbf{C} = m \times n$ matrix, and $\mathbf{D} = m \times m$ matrix. Then

$$[\mathbf{A} + \mathbf{BC}]^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}[\mathbf{I}_m + \mathbf{CA}^{-1}\mathbf{B}]^{-1}\mathbf{CA}^{-1}$$

If $|\mathbf{A}| \neq 0$ and $|\mathbf{D}| \neq 0$, then

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}\mathbf{D}^{-1} \\ \mathbf{0} & \mathbf{D}^{-1} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ -\mathbf{D}^{-1}\mathbf{C}\mathbf{A}^{-1} & \mathbf{D}^{-1} \end{bmatrix}$$

If $|\mathbf{A}| \neq 0$, $\mathbf{S} = \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$, $|\mathbf{S}| \neq 0$, then

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}\mathbf{S}^{-1}\mathbf{C}\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}\mathbf{S}^{-1} \\ -\mathbf{S}^{-1}\mathbf{C}\mathbf{A}^{-1} & \mathbf{S}^{-1} \end{bmatrix}$$

If $|\mathbf{D}| \neq 0$, $\mathbf{T} = \mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}$, and $|\mathbf{T}| \neq 0$, then

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{T}^{-1} & -\mathbf{T}^{-1}\mathbf{B}\mathbf{D}^{-1} \\ -\mathbf{D}^{-1}\mathbf{C}\mathbf{T}^{-1} & \mathbf{D}^{-1} + \mathbf{D}^{-1}\mathbf{C}\mathbf{T}^{-1}\mathbf{B}\mathbf{D}^{-1} \end{bmatrix}$$

Finally, we present the MATLAB approach to obtain the inverse of a square matrix. If all elements of the matrix are given as numerical values, this approach is best.

MATLAB Approach to Obtain the Inverse of a Square Matrix. The inverse of a square matrix \mathbf{A} can be obtained with the command

inv(A)

For example, if matrix \mathbf{A} is given by

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & 0 \\ 1 & 2 & 5 \end{bmatrix}$$

then the inverse of matrix \mathbf{A} is obtained as follows:

```
A = [1 1 2;3 4 0;1 2 5];
inv(A)
ans =
    2.2222    -0.1111   -0.8889
   -1.6667     0.3333    0.6667
    0.2222    -0.1111    0.1111
```

That is

$$\mathbf{A}^{-1} = \begin{bmatrix} 2.2222 & -0.1111 & -0.8889 \\ -1.6667 & 0.3333 & 0.6667 \\ 0.2222 & -0.1111 & 0.1111 \end{bmatrix}$$

MATLAB Is Case Sensitive. It is important to note that MATLAB is case sensitive. That is, MATLAB distinguishes between upper- and lowercase letters. Thus, x and X are not the same variable. All function names must be in lowercase, such as $\text{inv}(A)$, $\text{eig}(A)$, and $\text{poly}(A)$.

Differentiation and Integration of Matrices. The derivative of an $n \times m$ matrix $\mathbf{A}(t)$ is defined to be the $n \times m$ matrix, each element of which is the derivative of the corresponding element of the original matrix, provided that all the elements $a_{ij}(t)$ have derivatives with respect to t . That is,

$$\frac{d}{dt} \mathbf{A}(t) = \left(\frac{d}{dt} a_{ij}(t) \right) = \begin{bmatrix} \frac{d}{dt} a_{11}(t) & \frac{d}{dt} a_{12}(t) & \dots & \frac{d}{dt} a_{1m}(t) \\ \frac{d}{dt} a_{21}(t) & \frac{d}{dt} a_{22}(t) & \dots & \frac{d}{dt} a_{2m}(t) \\ \vdots & \vdots & & \vdots \\ \frac{d}{dt} a_{n1}(t) & \frac{d}{dt} a_{n2}(t) & \dots & \frac{d}{dt} a_{nm}(t) \end{bmatrix}$$

Similarly, the integral of an $n \times m$ matrix $\mathbf{A}(t)$ is defined to be

$$\int \mathbf{A}(t) dt = \left(\int a_{ij}(t) dt \right) = \begin{bmatrix} \int a_{11}(t) dt & \int a_{12}(t) dt & \dots & \int a_{1m}(t) dt \\ \int a_{21}(t) dt & \int a_{22}(t) dt & \dots & \int a_{2m}(t) dt \\ \vdots & \vdots & & \vdots \\ \int a_{n1}(t) dt & \int a_{n2}(t) dt & \dots & \int a_{nm}(t) dt \end{bmatrix}$$

Differentiation of the Product of Two Matrices. If the matrices $\mathbf{A}(t)$ and $\mathbf{B}(t)$ can be differentiated with respect to t , then

$$\frac{d}{dt} [\mathbf{A}(t)\mathbf{B}(t)] = \frac{d\mathbf{A}(t)}{dt} \mathbf{B}(t) + \mathbf{A}(t) \frac{d\mathbf{B}(t)}{dt}$$

Here again the multiplication of $\mathbf{A}(t)$ and $d\mathbf{B}(t)/dt$ [or $d\mathbf{A}(t)/dt$ and $\mathbf{B}(t)$] is, in general, not commutative.

Differentiation of $\mathbf{A}^{-1}(t)$. If a matrix $\mathbf{A}(t)$ and its inverse $\mathbf{A}^{-1}(t)$ are differentiable with respect to t , then the derivative of $\mathbf{A}^{-1}(t)$ is given by

$$\frac{d\mathbf{A}^{-1}(t)}{dt} = -\mathbf{A}^{-1}(t) \frac{d\mathbf{A}(t)}{dt} \mathbf{A}^{-1}(t)$$

The derivative may be obtained by differentiating $\mathbf{A}(t)\mathbf{A}^{-1}(t)$ with respect to t . Since

$$\frac{d}{dt} [\mathbf{A}(t)\mathbf{A}^{-1}(t)] = \frac{d\mathbf{A}(t)}{dt} \mathbf{A}^{-1}(t) + \mathbf{A}(t) \frac{d\mathbf{A}^{-1}(t)}{dt}$$

and

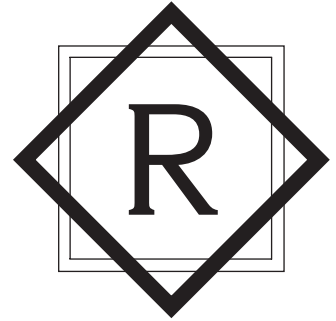
$$\frac{d}{dt} [\mathbf{A}(t)\mathbf{A}^{-1}(t)] = \frac{d}{dt} \mathbf{I} = \mathbf{0}$$

we obtain

$$\mathbf{A}(t) \frac{d\mathbf{A}^{-1}(t)}{dt} = -\frac{d\mathbf{A}(t)}{dt} \mathbf{A}^{-1}(t)$$

or

$$\frac{d\mathbf{A}^{-1}(t)}{dt} = -\mathbf{A}^{-1}(t) \frac{d\mathbf{A}(t)}{dt} \mathbf{A}^{-1}(t)$$



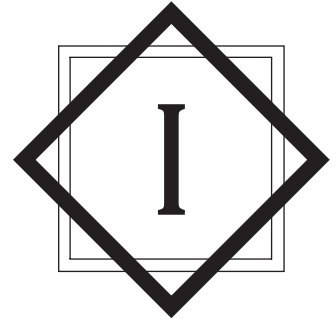
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Index

A

Absolute stability, 160
Ackermann's formula:
 for observer gain matrix, 756–57
 for pole placement, 730–31
Actuating error, 8
Actuator, 21–22
Adjoint matrix, 876
Air heating system, 150
Aircraft elevator control system, 156
Analytic function, 860
Angle:
 of arrival, 286
 of departure, 280, 286
Angle condition, 271
Asymptotes:
 Bode diagram, 406–07
 root loci, 274–75, 284–85
Attenuation, 165
Attitude-rate control system, 386
Automatic controller, 21
Automobile suspension system, 86
Auxiliary polynomial, 216

B

Back emf, 95
 constant, 95

Bandwidth, 474, 539
Basic control actions:
 integral, 24
 on-off, 22
 proportional, 24
 proportional-plus-derivative, 25
 proportional-plus-integral, 24
 proportional-plus-integral-plus-
 derivative, 35
 two-position, 22–23
Bleed-type relay, 111
Block, 17
Block diagram, 17–18
 reduction, 27–28, 48
Bode diagram, 403
 error in asymptotic expression of, 403
 of first-order factors, 406–07, 409
 general procedure for plotting, 413
 plotting with MATLAB, 422–25
 of quadratic factors, 410–12
 of system defined in state space,
 426–27
Branch point, 18
Break frequency, 406
Breakaway point, 275–76, 285–86, 351
Break-in point, 276, 281, 285–86, 351
Bridged-T networks, 90, 520
Business system, 5

C

Canonical forms:
controllable, 649
diagonal, 650
Jordan, 651, 653
observable, 650

Capacitance:
of pressure system, 107–09
of thermal system, 137
of water tank, 103

Cancellation of poles and zeros, 288

Cascaded system, 20

Cascaded transfer function, 20

Cauchy–Riemann conditions, 860–61

Cauchy’s theorem, 526

Cayley–Hamilton theorem, 668, 701

Characteristic equation, 652

Characteristic polynomial, 34

Characteristic roots, 652

Circular root locus, 282

Classical control theory, 2

Classification of control systems, 225

Closed-loop control system, 8

Closed-loop system, 20

Closed-loop frequency response, 477

Closed-loop frequency response curves:
desirable shapes of, 492
undesirable shapes of, 492

Closed-loop transfer function, 19–20

Cofactor, 876

Command compensation, 630

Compensation:
feedback, 308
parallel, 308
series, 308

Compensator:
lag, 323, 503–04
lag–lead, 332–34, 511–13
lead, 312–13, 495–96

Complete observability, 683–84
conditions for, 684–85
in the s plane, 684

Complete output controllability, 714

Complete state controllability, 676–81
in the s plane, 680–81

Complex-conjugate poles:
cancellation of undesirable, 520

Complex function, 859

Complex impedance, 75

Complex variable, 859

Computational optimization approach to
design PID controller, 583–89

Conditional stability, 299–300, 510–11

Conditionally stable system, 299–300,
458, 510–11

Conduction heat transfer, 137

Conformal mapping, 447, 462–64

Conical water tank system, 152

Constant-gain loci, 302–03

Constant-magnitude loci (M circles),
478–79

Constant phase-angle loci (N circles),
480–81

Constant ω_n loci, 296

Constant ζ lines, 298

Constant ζ loci, 296

Control actions, 21

Control signal, 3

Controllability, 675–81
matrix, 677
output, 681

Controllable canonical form, 649, 688

Controlled variable, 3

Controller, 22

Convection heat transfer, 137

Conventional control theory, 29

Convolution, integral, 16

Corner frequency, 406

Critically damped system, 167

Cutoff frequency, 474

Cutoff rate, 475

D

Damped natural frequency, 167

Damper, 64, 132

Damping ratio, 165
lines of constant, 296

Dashpot, 64, 132–33

Dead space, 43

Decade, 405

Decibel, 403

Delay time, 169–70

Derivative control action, 118–20, 222

Derivative gain, 84

Derivative time, 25, 61

Detectability, 688

Determinant, 874

Diagonal canonical form, 694

Diagonalization of $n \times n$ matrix, 652

Differential amplifier, 78

Differential gap, 23, 24

Differentiating system, 231

Differentiation:
of inverse matrix, 881
of matrix, 880
of product of two matrices, 880

Differentiator:
approximate, 617

Direct transmission matrix, 31

Disturbance, 3, 26

Dominant closed-loop poles, 182

Duality, 754

- E*
- e^{At} :
 - computation of, 670–71
 - Eigenvalue, 652
 - invariance of, 655
 - Electromagnetic valve, 23
 - Electronic controller, 77, 83
 - Engineering organizational system, 5–6
 - Equivalent moment of inertia, 234
 - Equivalent spring constant, 64
 - Equivalent viscous-friction coefficient, 65, 234
 - Evans, W. R., 2, 11, 269
 - Exponential response curve, 162
- F*
- Feedback compensation, 308–09, 342, 519
 - Feedback control, 3
 - Feedback control system, 7
 - Feedback system, 20
 - Feedforward transfer function, 19
 - Final value theorem, 866
 - First-order lag circuit, 80
 - First-order system, 161–64
 - unit-impulse response of, 163
 - unit-ramp response of, 162–63
 - unit-step response of, 161–62
 - Flapper, 110
 - valve, 156
 - Fluid systems:
 - mathematical modeling of, 100
 - Free-body diagram, 69–70
 - Frequency response, 398
 - correlation between step response and, 471–74
 - lag compensation based on, 502–11
 - lag–lead compensation based on, 511–17
 - lead compensation based on, 493–502
 - Full-order state observer, 752–53
 - Functional block, 17
- G*
- Gain crossover frequency, 467–69
 - Gain margin, 464–67
 - Gas constant, 108
 - for air, 142
 - universal, 108
 - Gear train, 232
 - system, 232–34
 - Generalized plant, 813, 815–17
 - diagram, 810–16, 853–54
- H*
- H infinity control problem, 816
 - H infinity norm, 6, 808
 - Hazen, 2, 11
 - High-pass filter, 495
 - Higher-order systems, 179
 - transient response of, 180–81
 - Hurwitz determinants, 252–58
 - Hurwitz stability criterion, 252–53, 255–58
 - equivalence of Routh’s stability criterion and, 255–57
 - Hydraulic controller:
 - integral, 130
 - jet-pipe, 147
 - proportional, 131
 - proportional-plus-derivative, 134–35
 - proportional-plus-integral, 133–34
 - proportional-plus-integral-plus-derivative, 135–36
 - Hydraulic servo system, 124–25
 - Hydraulic servomotor, 128, 130, 156
 - Hydraulic system, 106, 123–39, 149
 - advantages and disadvantages of, 124
 - compared with pneumatic system, 106
- I*
- Ideal gas law, 108
 - Impedance:
 - approach to obtain transfer function, 75–76
 - Impulse function, 866
 - Impulse response, 163, 178–79, 195–97
 - function, 16–17
 - Industrial controllers, 22
 - Initial condition:
 - response to, 203–11
 - Initial value theorem, 866
 - Input filter, 261, 630
 - Input matrix, 31
 - Integral control, 220
 - Integral control action, 24–25, 218
 - Integral controller, 22
 - Integral gain, 61
 - Integral time, 25, 61
 - Integration of matrix, 880
 - Inverse Laplace transform:
 - partial-fraction expansion method for obtaining, 867–73
 - Inverse Laplace transformation, 862
 - Inverse of a matrix:
 - MATLAB approach to obtain, 879
 - Inverse polar plot, 461–62, 537–38
 - Inverted-pendulum system, 68–72, 98
 - Inverted-pendulum control system, 746–51
 - Inverting amplifier, 78
 - I-PD control, 591–92
 - I-PD-controlled system, 592, 628–29, 643
 - with feedforward control, 642

J

Jet-pipe controller, 146–47
Jordan blocks, 679
Jordan canonical form, 651, 695, 706–07

K

Kalman, R. E., 12, 675
Kirchhoff's current law, 72
Kirchhoff's loop law, 72
Kirchhoff's node law, 72
Kirchhoff's voltage law, 72

L

Lag compensation, 321
Lag compensator, 311, 321, 502
 Bode diagram of, 503
 design by frequency-response method, 502–11
 design by root-locus method, 321, 323
 polar plot of, 503
Lag network, 82, 542
Lag–lead compensation, 330, 335, 338, 377, 511–18
Lag–lead compensator:
 Bode diagram of, 558
 design by frequency-response method, 513–17
 design by root-locus method, 331–32, 380–82
 electronic, 330–32
 polar plot of, 512
Lag–lead network:
 electronic, 330–32
 mechanical, 366
Lagrange polynomial, 708
Lagrange's interpolation formula, 708
Laminar-flow resistance, 102
Laplace transform, 862
 properties of, 865
 table of, 863–64
Lead compensator, 311, 493
 Bode diagram of, 494
 design by frequency-response method, 493–502
 design by root-locus method, 311–18
 polar plot of, 494
Lead, lag, and lag–lead compensators:
 comparison of, 517–18
Lead network, 542
 electronic, 82
 mechanical, 365
Lead time, 5
Linear approximation:
 of nonlinear mathematical models, 43
Linear system, 14
 constant coefficient, 14

Linear time-invariant system, 14, 164
Linear time-varying system, 14
Linearization:
 of nonlinear systems, 43
Liquid-level control system, 157
Liquid-level systems, 101, 103–04, 140–41
Log-magnitude curves of quadratic transfer function, 411
Logarithmic decrement, 237
Logarithmic plot, 403
Log-magnitude versus phase plot, 403, 443–44
LRC circuit, 72–73

M

M circles, 478–79
 a family of constant, 479
Magnitude condition, 271
Manipulated variable, 3
Mapping theorem, 448–49
Mathematical model, 13

MATLAB commands:

MATLAB:
 obtaining maximum overshoot with, 194
 obtaining peak time with, 194
 obtaining response to initial condition with, 266
 partial-fraction expansion with, 871–73
 plotting Bode diagram with, 422–23
 plotting root loci with, 290–91
 writing text in diagrams with, 188–89
[A,B,C,D] = tf2ss(num,den), 40, 656, 698
bode(A,B,C,D), 422, 426
bode(A,B,C,D,iu), 426–27
bode(A,B,C,D,iu,w), 422
bode(A,B,C,D,w), 422
bode(num,den), 422
bode(num,den,w), 422, 425, 551
bode(sys), 422
bode(sys,w), 552
c = step(num,den,t), 190
for loop, 243, 249, 584
[Gm,pm,wcp,wcg,] = margin(sys), 468–69
gtext('text'), 189
impulse(A,B,C,D), 195
impulse(num, den), 195
initial(A,B,C,D,[initial condition],t), 209
inv(A), 879
K = acker(A,B,J), 736
K = lqr(A,B,Q,R), 798
K = place(A,B,J), 736

MATLAB commands (Cont.)

$K_e = \text{acker}(A',C',L)'$, 773
 $K_e = \text{acker}(\text{Abb},\text{Aab},L)'$, 773
 $K_e = \text{place}(A',C',L)'$, 773
 $K_e = \text{place}(\text{Abb}',\text{Aab}',L)'$, 773
 $[K,P,E] = \text{lqr}(A,B,Q,R)$, 798
 $[K,r] = \text{rlocfind}(\text{num},\text{den})$, 303
 $\text{logspace}(d1,d2)$, 422
 $\text{logspace}(d1,d2,n)$, 422–23
 $\text{lqr}(A,B,Q,R)$, 797
 $\text{lsim}(A,B,C,D,u,t)$, 201
 $\text{lsim}(\text{num},\text{den},r,t)$, 201
 $\text{magdB} = 20*\text{log10}(\text{mag})$, 422
 $[\text{mag},\text{phase},w] = \text{bode}(A,B,C,D)$, 422
 $[\text{mag},\text{phase},w] = \text{bode}(A,B,C,D,iu,w)$, 422
 $[\text{mag},\text{phase},w] = \text{bode}(A,B,C,D,w)$, 422
 $[\text{mag},\text{phase},w] = \text{bode}(\text{num},\text{den})$, 422
 $[\text{mag},\text{phase},w] = \text{bode}(\text{num},\text{den},w)$, 422, 476
 $[\text{mag},\text{phase},w] = \text{bode}(\text{sys})$, 422
 $[\text{mag},\text{phase},w] = \text{bode}(\text{sys},w)$, 476
 mesh , 192
 $\text{mesh}(y)$, 192, 249
 $\text{mesh}(y')$, 192, 249
 $[\text{Mp},k] = \text{max}(\text{mag})$, 476
 NaN , 799
 $[\text{num},\text{den}] = \text{feedback}(\text{num1},\text{den1}, \text{num2},\text{den2})$, 20–21
 $[\text{num},\text{den}] = \text{parallel}(\text{num1},\text{den1}, \text{num2},\text{den2})$, 20–21
 $[\text{num},\text{den}] = \text{series}(\text{num1},\text{den1}, \text{num2},\text{den2})$, 20–21
 $[\text{num},\text{den}] = \text{ss2tf}(A,B,C,D)$, 41, 657
 $[\text{num},\text{den}] = \text{ss2tf}(A,B,C,D,iu)$, 41–42, 58, 657
 $[\text{NUM},\text{den}] = \text{ss2tf}(A,B,C,D,iu)$, 59, 659
 $\text{nyquist}(A,B,C,D)$, 436, 441–42
 $\text{nyquist}(A,B,C,D,iu)$, 441
 $\text{nyquist}(A,B,C,D,iu,w)$, 436, 441
 $\text{nyquist}(A,B,C,D,w)$, 436
 $\text{nyquist}(\text{num},\text{den})$, 436
 $\text{nyquist}(\text{num}, \text{den},w)$, 436
 $\text{nyquist}(\text{sys})$, 436
 $\text{polar}(\text{theta},r)$, 545
 $\text{printsys}(\text{num},\text{den})$, 20–21, 189
 $\text{printsys}(\text{num},\text{den},\text{'s'})$, 189
 $r = \text{abs}(z)$, 544
 $[r,p,k] = \text{residue}(\text{num},\text{den})$, 239, 871–72
 $[\text{re},\text{im},w] = \text{nyquist}(A,B,C,D)$, 436
 $[\text{re},\text{im},w] = \text{nyquist}(A,B,C,D,iu,w)$, 436
 $[\text{re},\text{im},w] = \text{nyquist}(A,B,C,D,w)$, 436
 $[\text{re},\text{im},w] = \text{nyquist}(\text{num},\text{den})$, 436
 $[\text{re},\text{im},w] = \text{nyquist}(\text{num},\text{den},w)$, 436
 $[\text{re},\text{im},w] = \text{nyquist}(\text{sys})$, 436
 residue , 867
 $\text{resonant_frequency} = w(k)$, 476
 $\text{resonant_peak} = 20*\text{log10}(\text{Mp})$, 476
 rlocfind , 303
 $\text{rlocus}(A,B,C,D)$, 295
 $\text{rlocus}(A,B,C,D,K)$, 290, 295
 $\text{rlocus}(\text{num},\text{den})$, 290–91
 $\text{rlocus}(\text{num},\text{den},K)$, 290
 sgrid , 297
 sortsolution , 584
 $\text{step}(A,B,C,D)$, 184, 186
 $\text{step}(A,B,C,D,iu)$, 184
 $\text{step}(\text{num},\text{den})$, 184
 $\text{step}(\text{num},\text{den},t)$, 184
 $\text{step}(\text{sys})$, 184
 $\text{sys} = \text{ss}(A,B,C,D)$, 184
 $\text{sys} = \text{tf}(\text{num},\text{den})$, 184
 text , 188
 $\text{theta} = \text{angle}(z)$, 544
 $w = \text{logspace}(d2,d3,100)$, 425
 $y = \text{lsim}(A,B,C,D,u,t)$, 201
 $y = \text{lsim}(\text{num},\text{den},r,t)$, 201
 $[y, x, t] = \text{impulse}(A,B,C,D)$, 195
 $[y, x, t] = \text{impulse}(A,B,C,D,iu)$, 195
 $[y, x, t] = \text{impulse}(A,B,C,D,iu,t)$, 195
 $[y, x, t] = \text{impulse}(\text{num},\text{den})$, 195
 $[y, x, t] = \text{impulse}(\text{num},\text{den},t)$, 195
 $[y, x, t] = \text{step}(A,B,C,D,iu)$, 184
 $[y, x, t] = \text{step}(A,B,C,D,iu,t)$, 184
 $[y, x, t] = \text{step}(\text{num},\text{den},t)$, 184, 190
 $z = \text{re}+j*\text{im}$, 544

End of MATLAB commands

Matrix exponential, 661, 669–674
closed solution for, 663
Matrix Riccati equation, 798, 800
Maximum overshoot:
in unit-impulse response, 179
in unit-step response, 170, 172
versus ζ curve, 174
Maximum percent overshoot, 170
Maximum phase lead angle, 494, 498
Measuring element, 21
Mechanical lag–lead system, 366
Mechanical lead system, 365
Mechanical vibratory system, 236
Mercury thermometer system, 151
Minimal polynomial, 669, 704–06
Minimum-order observer, 767–77
based controller, 777
Minimum-order state observer, 752
Minimum-phase system, 415–16
Minimum-phase transfer function, 415
Minor, 876
Modern control theory, 7, 29
versus conventional control theory, 29

- Motor torque constant, 95
- Motorcycle suspension system, 87
- Multiple-loop system, 458–59
- N**
- N circles, 480–81
 - a family of constant, 481
- Newton's second law, 66
- Nichols, 2, 11, 398
- Nichols chart, 482–85
- Nichols plots, 403
- Nonbleed-type relay, 111
- Nonhomogeneous state equation:
 - solution of, 666–67
- Noninverting amplifier, 79
- Nonlinear mathematical models:
 - linear approximation of, 43–45
- Nonlinear system, 43
- Nonminimum-phase systems, 300–01, 415, 417
- Nonminimum-phase transfer function, 415, 488
- Nonuniqueness:
 - of a set of state variables, 655
- Nozzle-flapper amplifier, 110
- Number-decibel conversion line, 404
- Nyquist, H., 2, 11, 398
- Nyquist path, 545
- Nyquist plot, 403, 439–40, 443
 - of positive-feedback system, 535–37
 - of system defined in state space, 440–43
- Nyquist stability analysis, 454–62
- Nyquist stability criterion, 445–54
 - applied to inverse polar plots, 461–62
- O**
- Observability, 675, 682–88
 - complete, 683–85
 - matrix, 653
- Observable canonical form, 650, 692
- Observation, 752
- Observed-state feedback control system, 761
- Observer, 753
 - design of control system with, 786–93
 - full-order, 753
 - mathematical model of, 752
 - minimum-order, 767–73
- Observer-based controller:
 - transfer function of, 761
- Observer controller:
 - in the feedback path of control system, 787, 790–93
 - in the feedforward path of control system, 787–90
- Observer-controller matrix, 762
- Observer-controller transfer function, 761–62
- Observer error equation, 753
- Observer gain matrix, 755
 - MATLAB determination of, 773
- Octave, 405
- Offset, 258
- On-off control action, 22–23
- On-off controller, 22
- One-degree-of-freedom control system, 593
 - op amps, 78
- Open-loop control system, 8
 - advantages of, 9
 - disadvantages of, 9
- Open-loop frequency response curves:
 - reshaping of, 493
- Open-loop transfer function, 19
- Operational amplifier, 78
- Operational amplifier circuits, 93–94
 - for lead or lag compensator:
 - table of, 85
- Optimal regulator problem, 806
- Ordinary point, 861
- Orthogonality:
 - of root loci and constant gain loci, 301–02
- Output controllability, 681
- Output equation, 31
- Output matrix, 31
- Overdamped system, 168–69
- Overlapped spool valve, 146
- Overlapped valve, 130
- P**
- Parallel compensation, 308–09, 342–43
- Partial-fraction expansion, 867–73
 - with MATLAB, 871–73
- PD control, 373
- PD controller, 614–15
- Peak time, 170, 172, 193
- Performance index, 793
- Performance specifications, 9
- Phase crossover frequency, 467–69
- Phase margin, 464–67
 - versus ζ curve, 472
- PI controller, 2, 614–15
- PI-D control, 590–92
- PID control system, 572–77, 583, 587, 617–21, 628–29, 642–43
 - basic, 590
 - with input filter, 629
 - two-degrees-of-freedom, 592–95
- PID controller, 567, 577, 614–16, 620, 632
 - modified, 616
 - using operational amplifiers, 83–84

- Pilot valve, 124, 130
- PI-PD control, 592
- PID-PD control, 592
- Plant, 3
- Pneumatic actuating valve, 117–18
- Pneumatic controllers, 144–45, 154–55
- Pneumatic nozzle-flapper amplifier, 110
- Pneumatic on-off controller, 115
- Pneumatic pressure system, 142
- Pneumatic proportional controller, 112–16
 - force-balance type, 115–16
 - force-distance type, 112–15
- Pneumatic proportional-plus-derivative controller, 119–20
- Pneumatic proportional-plus-integral control action, 120–22
- Pneumatic proportional-plus-integral-plus-derivative control action, 122–23
- Pneumatic relay, 111
 - bleed type, 111
 - nonbleed type, 111
 - reverse acting, 112
- Pneumatic systems, 106–23, 153
 - compared with hydraulic system, 106
- Pneumatic two-position controller, 115
- Polar grids, 297
- Polar plot, 403, 427–28, 430, 432
- Pole: 861
 - of order n , 861
 - simple, 861
- Pole assignment technique, 723
- Pole-placement:
 - necessary and sufficient conditions for arbitrary, 725
- Pole placement problem, 723–35
 - solving with MATLAB, 735–36
- Positive-feedback system:
 - Nyquist plot for, 536–37
 - root loci for, 303–07
- Positional servo system, 95–97
- Pressure system, 107, 109
- Principle of duality, 687
- Principle of superposition, 43
- Process, 3
- Proportional control, 219
- Proportional control action, 24
- Proportional controller, 22
- Proportional gain, 25, 61
- Proportional-plus-derivative control:
 - of second-order system, 224
 - of system with inertia load, 223
- Proportional-plus-derivative control action, 25
- Proportional-plus-derivative controller, 22, 542

- Proportional-plus-integral control action, 24
- Proportional-plus-integral controller, 22, 121, 542
- Proportional-plus-integral-plus-derivative control action, 25
- Proportional-plus-integral-plus-derivative controller, 22
- Pulse function, 866

Q

- Quadratic factor, 410
 - log-magnitude curves of, 411
 - phase-angle curves of, 411
- Quadratic optimal control problem:
 - MATLAB solution of, 804
- Quadratic optimal regulator system, 793–95
 - MATLAB design of, 797

R

- Ramp response, 197
- Rank of matrix, 875
- Reduced-matrix Riccati equation, 795–97
- Reduced-order observer, 752
- Reduced-order state observer, 752
- Reference input, 21
- Regulator system with observer
 - controller, 778–86, 789
- Relative stability, 160, 217, 462
- Residue, 867
- Residue theorem, 527
- Resistance:
 - gas-flow, 107
 - laminar-flow, 101–02
 - of pressure system, 107, 109
 - of thermal system, 137
 - turbulent-flow, 102
- Resonant frequency, 430, 470
- Resonant peak, 413, 430, 470
 - versus ζ curve, 413
- Resonant peak magnitude, 413, 470
- Response:
 - to arbitrary input, 201
 - to initial condition, 203–11
 - to torque disturbance, 221
- Reverse-acting relay, 112
- Riccati equation, 795
- Rise time, 169–171
 - obtaining with MATLAB, 193–94
- Robust control:
 - system, 16, 806–17
 - theory, 2, 7
- Robust performance, 7, 807, 812
- Robust pole placement, 735
- Robust stability, 7, 807, 809

- Root loci:
 - general rules for constructing, 283–87
 - for positive-feedback system, 303–07
- Root locus, 271
- method, 269–70
- Routh's stability criterion, 212–18
- S**
- Schwarz matrix, 268
- Second-order system, 164
 - impulse response of, 178–79
 - standard form of, 166
 - step response of, 165–75
 - transient-response specification of, 171
 - unit-step response curves of, 169
- Sensor, 21
- Series compensation, 308–09, 342
- Servo system, 95, 164–65
 - design of, 739–51
 - with tachometer feedback, 268
 - with velocity feedback, 175–77
- Servomechanism, 2
- Set point, 21
- Set-point kick, 590
- Settling time, 170, 172–73
 - obtaining with MATLAB, 194
 - versus ζ curve, 174
- Sign inverter, 79
- Simple pole, 861
- Singular points, 861
- Sinusoidal signal generator, 486
- Sinusoidal transfer function, 401
- Small gain theorem, 809
- Space vehicle control system, 367, 538–39
- Speed control system, 4, 148
- Spool valve:
 - linearized mathematical model of, 127
- Spring-loaded pendulum system, 98
- Spring-mass-dashpot system, 66
- Square-law nonlinearity, 43
- S-shaped curve, 569
- Stability analysis, 454–62
 - in the complex plane, 182
- Stabilizability, 688
- Stack controller, 115
- Standard second-order system, 189
- State, 29
- State controllability:
 - complete, 676, 678, 680
- State equation, 31
 - solution of homogeneous, 660
 - solution of nonhomogeneous, 666–67
 - Laplace transform solution of, 663
- State-feedback gain matrix, 724
 - MATLAB approach to determine, 735–36
- State matrix, 31
- State observation:
 - necessary and sufficient conditions for, 754–55
- State observer, 751–77
 - design with MATLAB, 773
 - type 1 servo system with, 746
- State observer gain matrix: 755
 - Ackermann's formula to obtain, 756–57
 - direct substitution approach to obtain, 756
 - transformation approach to obtain, 755
- State space, 30
- State-space equation, 30
 - correlation between transfer function and, 649, 656
 - solution of, 660
- State-space representation:
 - in canonical forms, 649
 - of n th order system, 36–39
- State-transition matrix, 664
 - properties of, 665
- State variable, 29
- State vector, 30
- Static acceleration error constant, 228, 421
 - determination of, 421–22
- Static position error constant, 226, 419
- Static velocity error constant, 227, 420
- Steady-state error, 160, 226
 - for unit parabolic input, 229
 - for unit ramp input, 228
 - in terms of gain K , 230
- Steady-state response, 160
- Step response, 699–700
 - of second-order system, 165–69
- Summing point, 18
- Suspension system:
 - automobile, 86–87
 - motorcycle, 87
- Sylvester's interpolation formula, 673, 709–713
- System, 3
- System types, 419
 - type 0, 225, 230, 419, 433, 487–88
 - type 1, 225, 230, 420, 433, 487–88
 - type 2, 225, 230, 421, 433, 487–88
- System response to initial condition:
 - MATLAB approach to obtain, 203–11
- T**
- Tachometer, 176
 - feedback, 343
- Taylor series expansion, 43–45

- Temperature control systems, 4–5
 - Test signals, 159
 - Text:
 - writing on the graphic screen, 188
 - Thermal capacitance, 137
 - Thermal resistance, 137
 - Thermal systems, 100, 136–39
 - Thermometer system, 151–52
 - Three-degrees-of-freedom system, 645
 - Three-dimensional plot, 192
 - of unit-step response curves with MATLAB, 191–93
 - Traffic control system, 8
 - Transfer function, 15
 - of cascaded elements, 73–74
 - of cascaded systems, 20
 - closed-loop, 20
 - of closed-loop system, 20
 - experimental determination of, 489–90
 - expression in terms of **A**, **B**, **C**, and **D**, 34
 - of feedback system, 19
 - feedforward, 19
 - of minimum-order observer-based controller, 777
 - of nonloading cascaded elements, 77
 - observer-controller, 762, 780–82
 - open-loop, 19
 - of parallel systems, 20
 - sinusoidal, 401
 - Transfer matrix, 35
 - Transformation:
 - from state space to transfer function, 41–42, 657
 - from transfer function to state space, 40–41, 656
 - Transient response, 160
 - analysis with MATLAB, 183–211
 - of higher-order system, 180
 - specifications, 169, 171
 - Transport lag, 417
 - phase angle characteristics of, 417
 - Turbulent-flow resistance, 102
 - Two-degrees-of-freedom control system, 593–95, 599–614, 636–41, 646–47
 - Two-position control action, 22–23
 - Two-position controller, 22
 - Type 0 system, 225, 230, 488
 - log-magnitude curve for, 419, 488
 - polar plot of, 433
 - Type 1 servo system:
 - design of, 743–51
 - pole-placement design of, 739–46
 - Type 1 system, 420
 - log-magnitude curve for, 420, 488
 - polar plot of, 433
 - Type 2 system, 421
 - log-magnitude curve for, 421, 488
 - polar plot of, 433
- ## U
- Uncontrollable system, 681
 - Undamped natural frequency, 165
 - Underdamped system, 166–67
 - Underlapped spool valve, 146
 - Unit acceleration input, 247
 - Unit-impulse response:
 - of first-order system, 163
 - of second-order system, 178
 - Unit-impulse response curves:
 - a family of, 178
 - obtained by use of MATLAB, 196–97
 - Unit-ramp response:
 - of first-order system, 162–63
 - of second-order system, 197–200
 - of system defined in state space, 199–200
 - Unit-step response:
 - of first-order system, 161
 - of second-order system, 163, 167, 169
 - Universal gas constant, 108
 - Unstructured uncertainty:
 - additive, 852–53
 - multiplicative, 809
 - system with, 809
- ## V
- Valve:
 - overlapped, 130
 - underlapped, 130
 - zero-lapped, 130
 - Valve coefficient, 127
 - Vectors:
 - linear dependence of, 674
 - linear independence of, 674
 - Velocity error, 227
 - Velocity feedback, 176, 343, 519
- ## W
- Watt's speed governor, 4
 - Weighting function, 17
- ## Z
- Zero, 861
 - of order m , 862
 - Zero-lapped valve, 130
 - Zero placement, 595, 597, 612
 - approach to improve response characteristics, 595–97
 - Ziegler–Nichols tuning rules, 11, 568–77
 - first method, 569–70
 - second method, 570–71